Precision Modeling and Bit-width Optimization of Floating-Point Applications

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1. Introduction

We present a floating-point precision modeling methodology that can be used to develop application adaptive arithmetic precision models for variable bit-width floating-point computing. We also developed optimization algorithms that minimize the total bit-width for the application such that the output accuracy meets user-defined requirements. The methodology supports different bit-widths for different variables in the datapath.

Computing using floating-point (FP) representations provides a wide dynamic range of real numbers, freeing programmers from writing the manual scaling code required for fixed-point representation. Nevertheless, floating-point operations have always been considered beyond the capabilities of custom or re-configurable hardware implementation. IEEE standard precision floating-point operations cost too much in power and area to be practical on many devices.

A promising solution to reduce the cost of FP implementation is to reduce the bit-width of the FP representation. Research results show that it is feasible and beneficial to use reduced bit-width FP representation in modern multimedia and streaming application workloads [1]. By taking advantage of bit-width information during architectural synthesis, area is reduced by 15-86%, clock speed improved by 3-249%, and power consumption reduced by 46-73% [2].

The optimal bit-widths are the smallest bit-widths that satisfy the accuracy requirement. They can be obtained through simulation-based searching or model-based optimization. Simulation-based bit-width searching is a process that simulates using all possible bit-widths, and finds the best solution. It is a straight-forward method to determine the minimal bit-width, but it does not provide any intelligent optimization, and it can consume enormous computation time, especially when the target applications are large designs or a large input space is involved. As a better approach, model-based bit-width optimization eliminates the need for exhaustive simulation, and automatically analyzes and adapts the level of precision according to the need of an application.

The FP precision modeling methodology presented in this paper is an application-adaptive arithmetic model in the form of a function between the relative error in the output and the mantissa bit-widths (one for each FP variable) used in the FP datapath. The model constructed using this methodology can estimate the output error range given the custom FP bit-widths used. The optimization algorithm developed to optimize the bit-width is a combination of the popular Steepest Descent method and the unique characteristics of the bit-width optimization problem.

2. A Methodology for FP Precision Modeling

An arithmetic precision model can be built, based on the application's function and data, to represent the relationship between output precision and bit-widths used in the FP application. Our experimental results prove that the precision model constructed via this method gives reasonable estimates of the output accuracy. We have successfully used the precision model in a bit-width optimization program and obtained optimized reduced bit-width. The methodology for developing such a model is presented in this section.

The FP application being analyzed is represented in a graphical intermediate format: the Control and Data Flow Graph (CDFG). The CDFG is commonly used in high-level synthesis and can effectively represent the functional and the structural description of an application. A CDFG representation of a differential equation solver is shown in Figure 1.

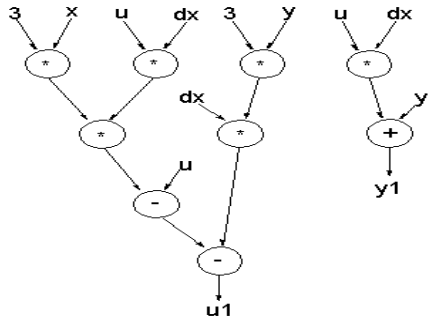

Figure 1. CDFG of a differential equation solver

The first process in the precision modeling is called behavioral profiling. Behavioral profiling is analogous to

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14. ABSTRACT

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software profiling. Given a behavioral specification (CDFG) of an application and a set of input vectors, behavioral profiling involves gathering pertinent profile data such as number of times an operation node is visited, number of times a conditional branch is taken, and number of times a loop or subprogram is executed. The behavioral profiling process involves a one-time simulation prior to constructing the model.

The behavioral profiler does the following. For each CDFG node *n*, determine the number of times the node is executed for the given profiling stimuli, and record each bit of the result of the operation. Bit probabilities (probability of the bit being "1") of the result, *Pn(i)* can be calculated based on this information.

The profile data is used to construct a precision model that best reflects the functional relationship between the bit-width of floating-point operations and the output precision. The model constructed using the methodology is an arithmetic model that describes the function between the output precision and the bit-widths used in the FP application. The overall error at the output of an operation is composed of propagation error, which is determined by the errors of input data and the operation type only, and rounding error, which is caused by the rounding of the operation result.

There are many ways to estimate the bounds of the rounding error. With the profile data available, the most accurate and convenient method for this research is presented in formula (1).

$$\varepsilon_{z\_r}=\frac{z-\hat{z}}{z}=\frac{(a_{b+1}2^{-b-1}+\cdots+a_{22}2^{-22}+a_{23}2^{-23})}{(1.0+a_1 2^{-1}+a_2 2^{-2}+\cdots+a_b 2^{-23})}\approx\frac{p_{b+1}2^{-b-1}+\cdots+p_{22}2^{-22}+p_{23}2^{-23}}{1.0+p_1 2^{-1}+p_2 2^{-2}+\cdots+p_b 2^{-23}} \quad (1)$$

The propagation error is derived based-on the Mean-value Theorem. The result is presented in formula (2).

$$\varepsilon_f=\frac{|f(x,y)-f(\hat{x},\hat{y})|}{f(x,y)}\approx\frac{f'(\hat{x},\hat{y})\hat{x}}{f(\hat{x},\hat{y})}\varepsilon_x+\frac{f'(\hat{x},\hat{y})}{f(\hat{x},\hat{y})}\varepsilon_y=k_x\varepsilon_x+k_y\varepsilon_y \quad (2)$$

The *k*s depend on the operation type, and also the profile data. For operation MULTIPLY ($z = x * y$),

$$k_x=\frac{f'(\hat{x},\hat{y})\hat{x}}{f(\hat{x},\hat{y})}=\frac{\hat{x}\hat{y}}{\hat{x}\hat{y}}=1.0,\ k_y=\frac{f'(\hat{x},\hat{y})\hat{y}}{f(\hat{x},\hat{y})}=\frac{\hat{x}\hat{y}}{\hat{x}\hat{y}}=1.0 \quad (3)$$

For operation DIVIDE ($z = x / y$),

$$k_x=\frac{f'(\hat{x},\hat{y})\hat{x}}{f(\hat{x},\hat{y})}=\frac{\hat{x}\hat{y}}{\hat{x}\hat{y}}=1.0,\ k_y=\frac{f'(\hat{x},\hat{y})\hat{y}}{f(\hat{x},\hat{y})}=\frac{-\hat{x}\hat{y}}{\hat{x}\hat{y}}=-1.0 \quad (4)$$

The arithmetic model for any FP operation is the sum of these two errors. The precision model for the entire application whose structural information is represented in the CDFG can be easily derived.

3. Experiment Results

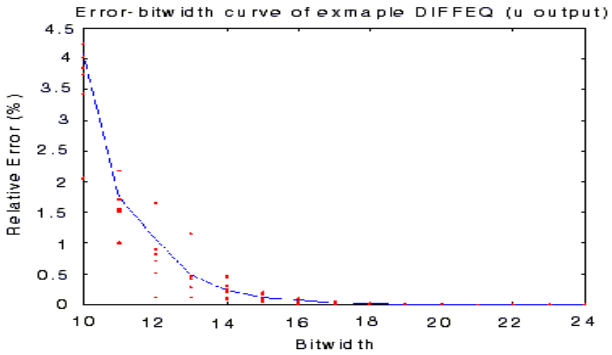

Figure 2. Comparison of estimated error and actual error

Figure 2 shows the comparison of the error estimated using the precision model (dashed line) and the actual error (dots) for the DIFFEQ example (shown in Figure 1). The result proves that the precision models developed using the methodology can effectively estimate the error range.

The bit-width optimization problem is solved by a Grid Steepest Descent (GSD) method derived for this specific precision modeling methodology and this bit-width optimization problem. Optimization results of 2 examples are shown in Table 1 – the DIFFEQ example and a simple three-multiplication-operation example.

Number of Total Bit-Width in Datapath

	IEEE 754	1%	5%	10%
DIFFEQ	**230**	**162**	**132**	**124**
3 MULT	**69**	**43**	**37**	**34**

Table 1. Optimization results for different precision targets

The results demonstrate that the GSD optimization method can be successfully used with the precision models to calculate the minimal bit-widths that satisfy the user-defined precision requirement of the application. The minimal bit-widths can be the same bit-width for all operations, or one for each individual operation.

References

[1] J.Y.F. Tong, D. Nagle, and R. Rutenbar, "Reducing Power by Optimizing the Necessary Precision Range of Floating Point Arithmetic," in *IEEE Transactions on VLSI systems*, Vol. 8, No.3, pp 273-286, June 2000.

[2] M. Stevenson, J. Babb, and S. Armarasinghe, "Bitwidth Analysis with Application to Silicon Compilation," in *ACM SIGPLAN Conference on Programming Language Design and Implementation*, June 2000, pp. 108-120.

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Outline

- Variable Bitwidth Computing
- Precision Modeling Methodology
  - Behavioral Profiling
  - Error Modeling
  - Verification
- Bitwidth Optimization
  - Problem Formulation
  - Optimization Algorithms
  - Optimization Results
- Future Work
- Conclusion

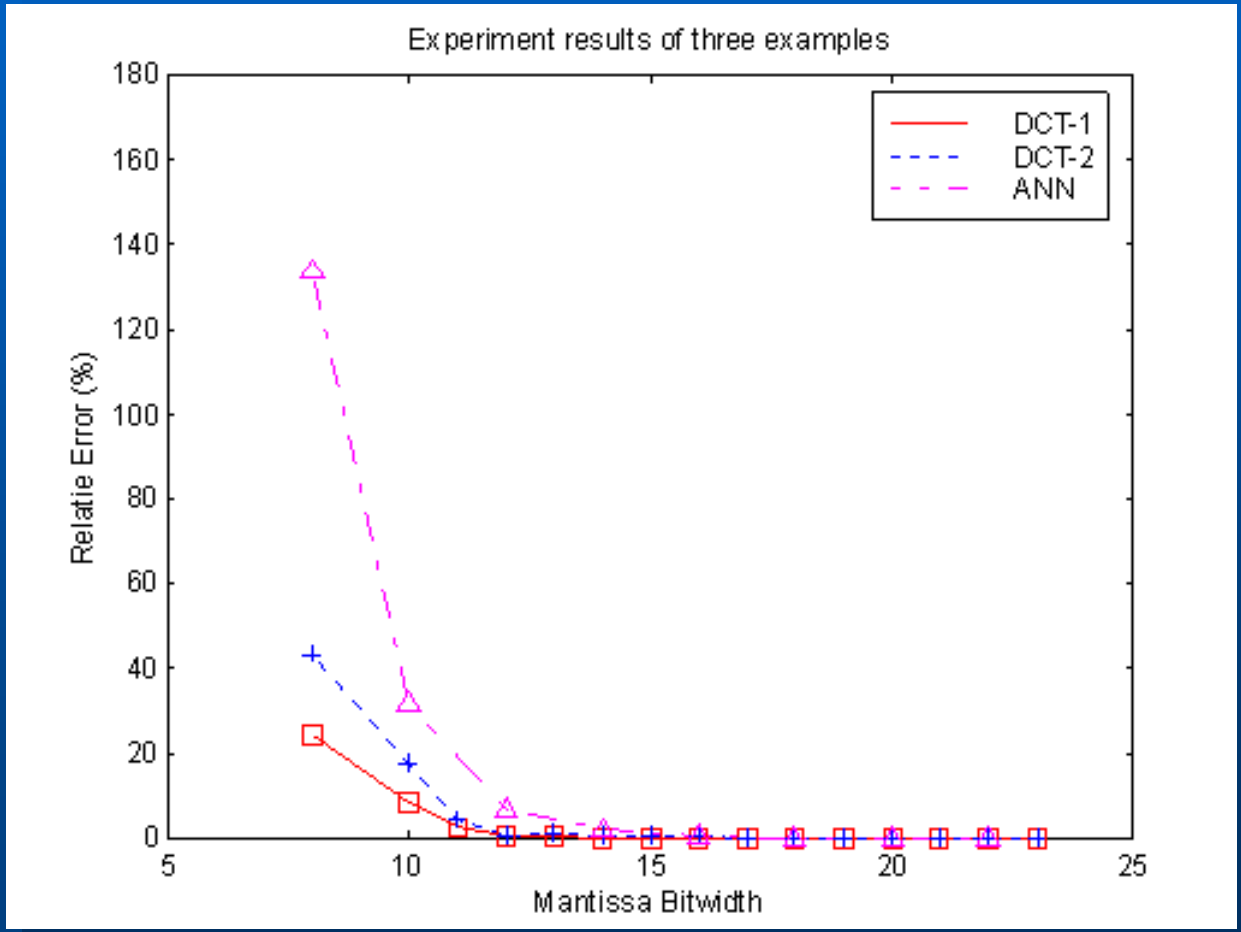
Variable Bitwith Computing

- Why Variable Bitwidth Computing ?
- Obtaining Optimized Bitwidths
  - Other's approach : simulation-based bitwidth searching
  - Our approach: model-based bitwidth optimization
  - System flowchart of the model-based bitwidth optimization

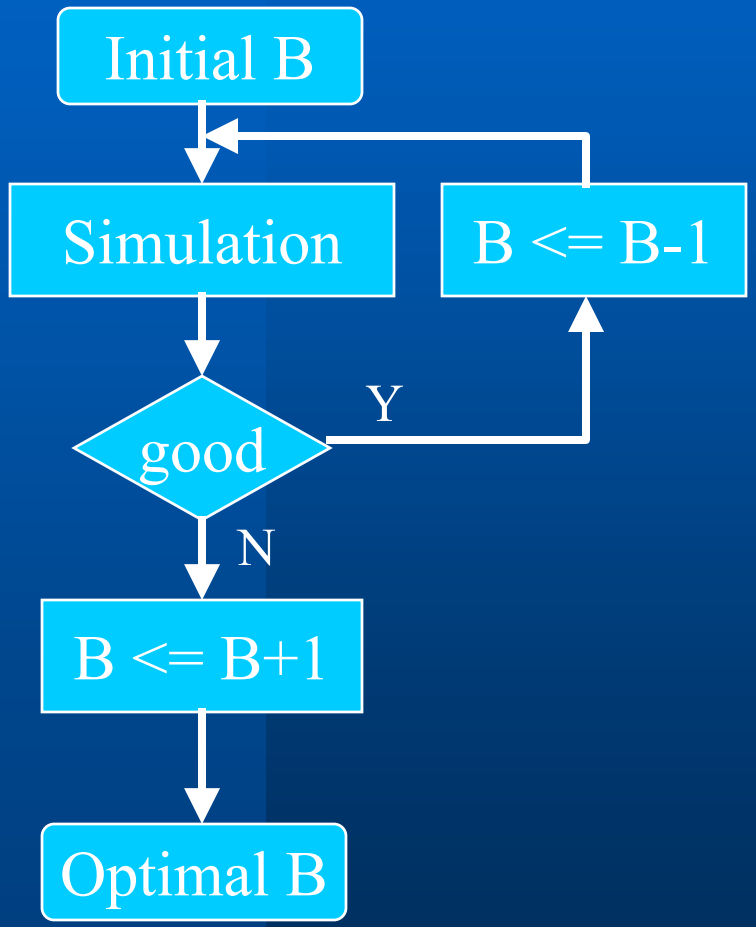
Why Variable Bitwith Computing ?

- Standard FP representation (Single Precision): 32 bits
  - sign: 1 bit
  - exponent: 8 bits
  - mantissa: 23 bits
- Implementation of standard FP operations in custom circuits is expensive
- Mantissa bitwidth can be reduced without compromising precision requirements

Relative Error VS Bitwidth

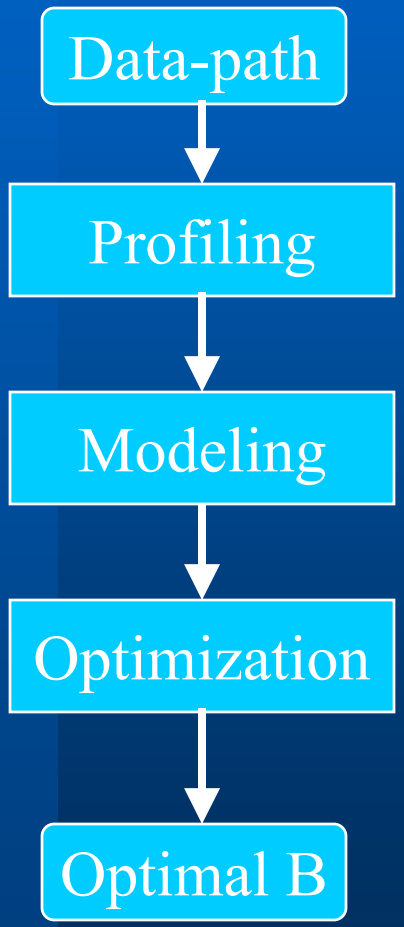


Simulation-based Searching



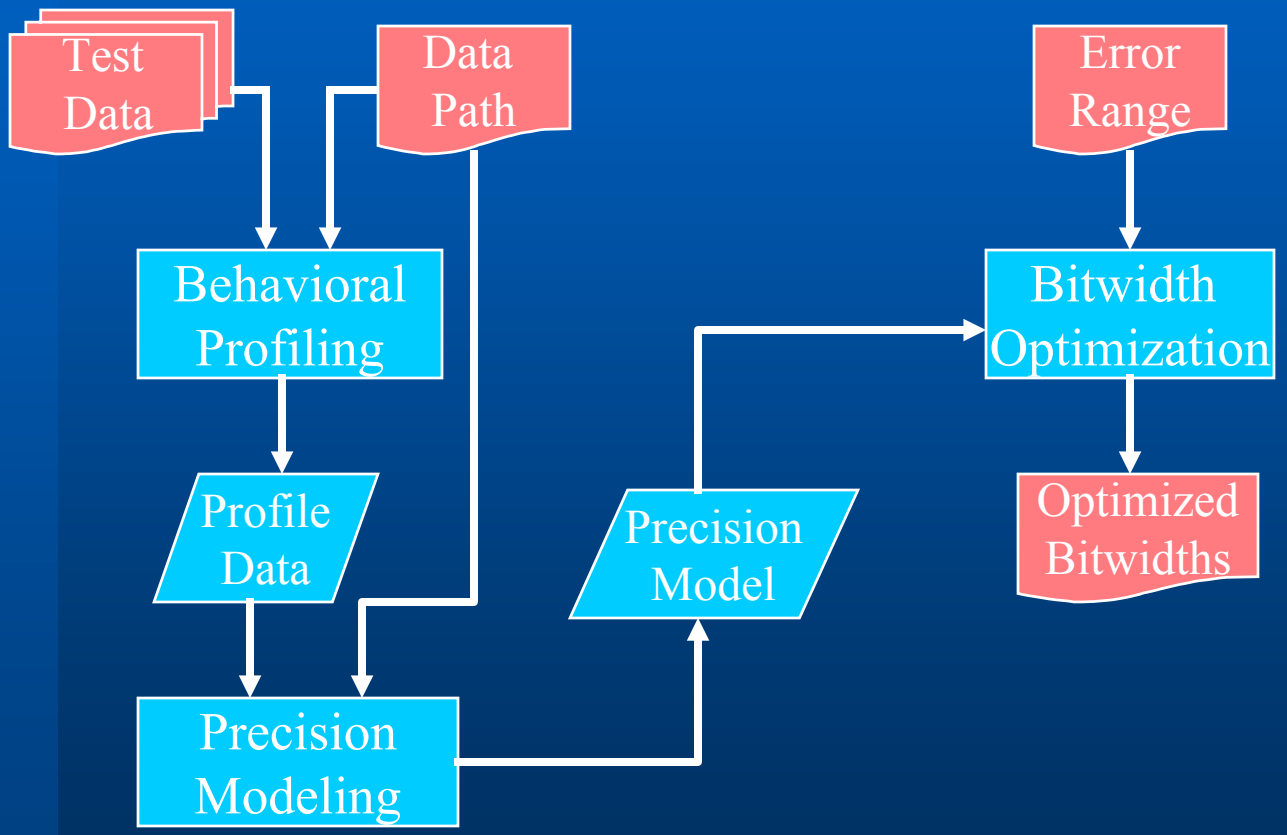
- Iterative simulation
- No explicit arithmetic precision model generated or needed
- Time consuming

Model-based Optimization



- No iterative simulation
- Application-specific arithmetic precision model generated
- One bitwidth value for each operation node

System Flowchart



Precision Modeling Methodology

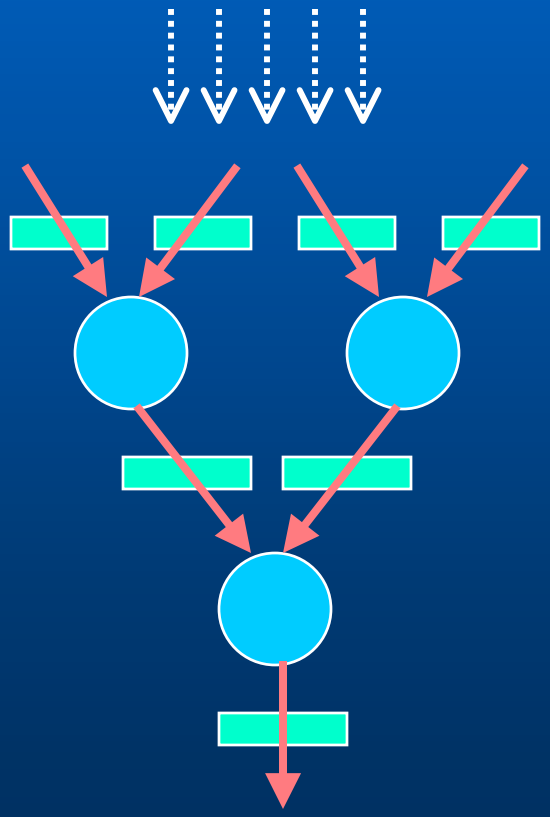
- Behavioral Profiling
- Error Models of FP Operations
  - Rounding Error
  - Propagation Error
- Constructing Precision Model

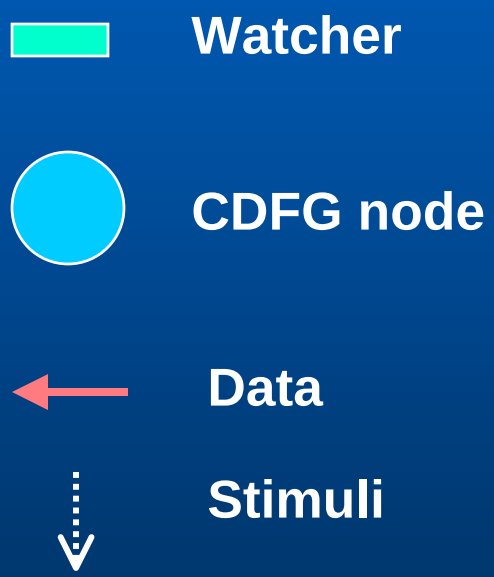
Profile-Driven Modeling

- Behavioral profiling gathers profile data through one-time simulation
  - bit probability
  - statistical values of variables in data-path
- Profiling is performed on a graphical representation (usually CDFG) of the application
- Profile data is used in precision modeling
- Selecting stimuli is important

Behavioral Profiling

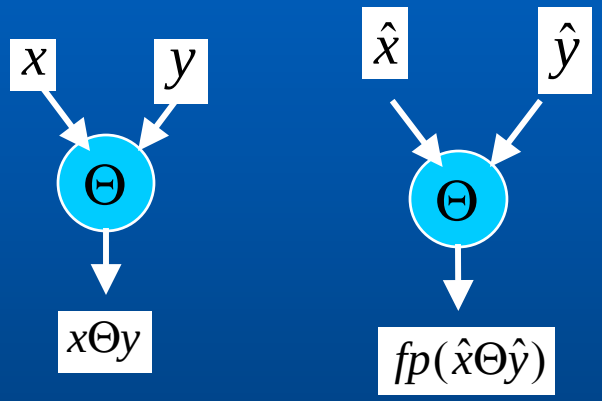
- Watcher insertion & simulation





Error in Floating-Point Model

- The overall error of a FP operation:



$$(x\Theta y) - fp(\hat{x}\Theta\hat{y}) = (x\Theta y - \hat{x}\Theta\hat{y}) + (\hat{x}\Theta\hat{y} - fp(\hat{x}\Theta\hat{y}))$$

- Overall error at the result of an operation :
  Propagation Error(PE) + Rounding Error(RE)

Propagation Error (1)

- Calculation of PE

$$f(x,y) \approx f(\hat{x},\hat{y}) + \frac{\partial f(\hat{x},\hat{y})}{\partial x}(x-\hat{x}) + \frac{\partial f(\hat{x},\hat{y})}{\partial y}(y-\hat{y})$$

$$\varepsilon_f = \frac{|f(x,y) - f(\hat{x},\hat{y})|}{f(x,y)} \approx \frac{f'(\hat{x},\hat{y})\hat{x}}{f(\hat{x},\hat{y})}\varepsilon_x + \frac{f'(\hat{x},\hat{y})\hat{y}}{f(\hat{x},\hat{y})}\varepsilon_y = k_x\varepsilon_x + k_y\varepsilon_y$$

- ***K*** **is the amplification factor, determined based on the operation's type and data**

Propagation Error (2)

Equation:

$$\varepsilon_{z\_p} = \frac{|f(x,y) - f(\hat{x},\hat{y})|}{f(x,y)} \approx \frac{f'(\hat{x},\hat{y})\hat{x}}{f(\hat{x},\hat{y})}\varepsilon_x + \frac{f'(\hat{x},\hat{y})}{f(\hat{x},\hat{y})}\varepsilon_y = k_x\varepsilon_x + k_y\varepsilon_y$$

MULT:

$$k_x = \frac{f'(\hat{x},\hat{y})\hat{x}}{f(\hat{x},\hat{y})} = \frac{\hat{x}\hat{y}}{\hat{x}\hat{y}} = 1.0$$

$$k_y = \frac{f'(\hat{x},\hat{y})\hat{y}}{f(\hat{x},\hat{y})} = \frac{\hat{x}\hat{y}}{\hat{x}\hat{y}} = 1.0$$

ADD:

$$k_x = \frac{f'(\hat{x},\hat{y})\hat{x}}{f(\hat{x},\hat{y})} = \frac{\hat{y}}{\hat{x}+\hat{y}}$$

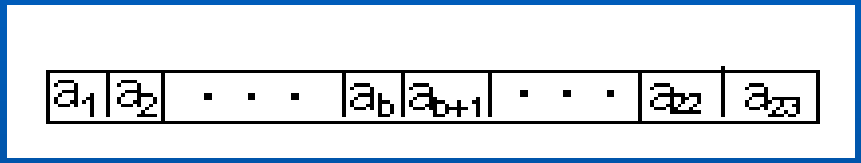
$$k_y = \frac{f'(\hat{x},\hat{y})\hat{y}}{f(\hat{x},\hat{y})} = \frac{\hat{x}}{\hat{x}+\hat{y}}$$

SQRT:

$$k_x = \frac{f'(\hat{x})\hat{x}}{f(\hat{x})} = 0.5$$

Rounding Error

Mantissa:



Precise value:

$$z = (1.0 + a_1 2^{-1} + a_2 2^{-2} + ☹ + a_b 2^{-b} + a_{b+1} 2^{-b-1} + ☹ + a_{22} 2^{-22} + a_{23} 2^{-23}) \times 2^e$$

FP value:

$$\hat{z} = (1.0 + a_1 2^{-1} + a_2 2^{-2} + ☹ + a_b 2^{-b}) \times 2^e$$

Error:

$$\varepsilon_{z\_r} = \frac{z - \hat{z}}{z} = \frac{(a_{b+1} 2^{-b-1} + ☹ + a_{22} 2^{-22} + a_{23} 2^{-23})}{(1.0 + a_1 2^{-1} + a_2 2^{-2} + ☹ + a_{23} 2^{-23})} \approx \frac{p_{b+1} 2^{-b-1} + ☹ + p_{22} 2^{-22} + p_{23} 2^{-23}}{1.0 + p_1 2^{-1} + p_2 2^{-2} + ☹ + p_{23} 2^{-23}}$$

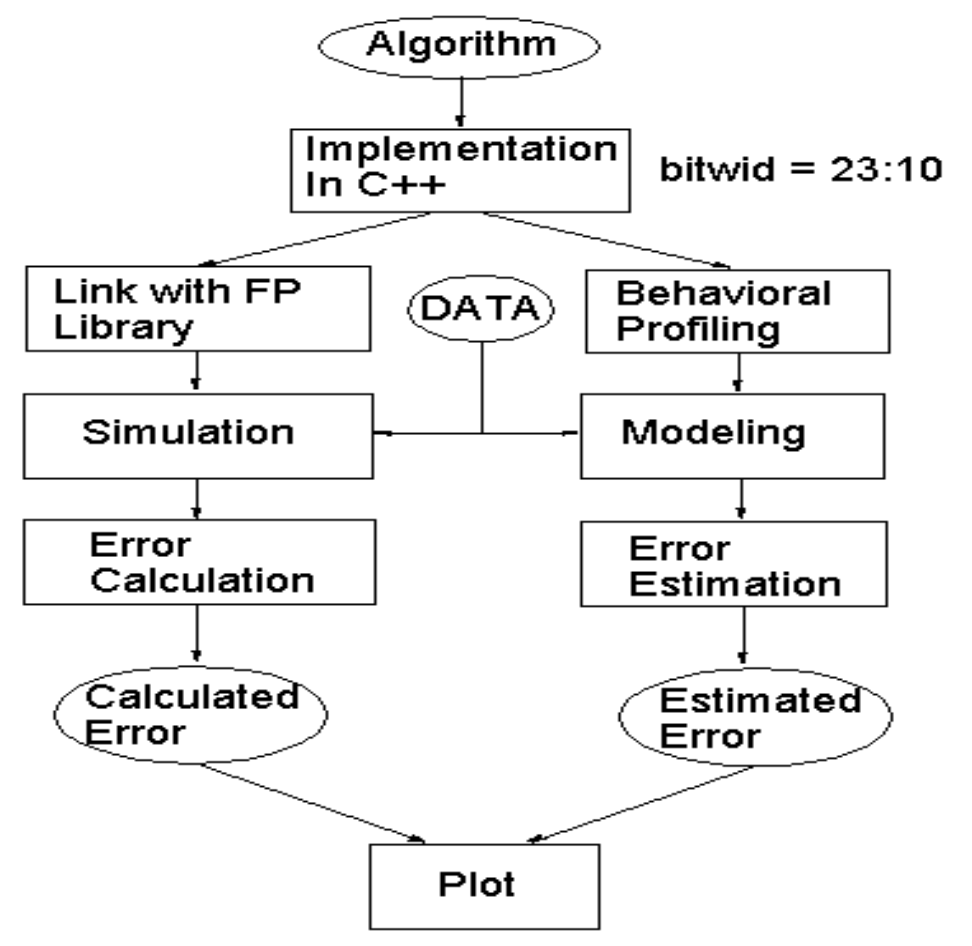
Constructing Precision Model

- Establish error model for each node in data-path using the RE+PE model
- Construct precision model for the application based-on data-path structure
- The precision model is a function of output error of the application in terms of bit-widths in data-path and input error of the application
- The precision model can be used to predict output error and optimize bitwidths

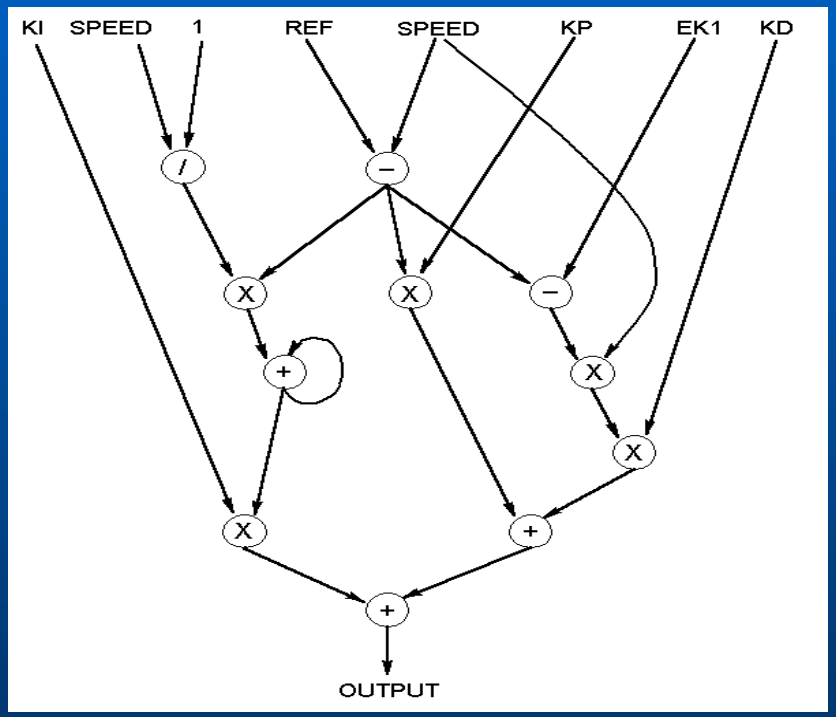
Experimental Results

- Experimental Procedure
- Example Data-paths (CDFGs)
- Comparison of Predicted Error Range and Actual Errors

Experimental Procedure

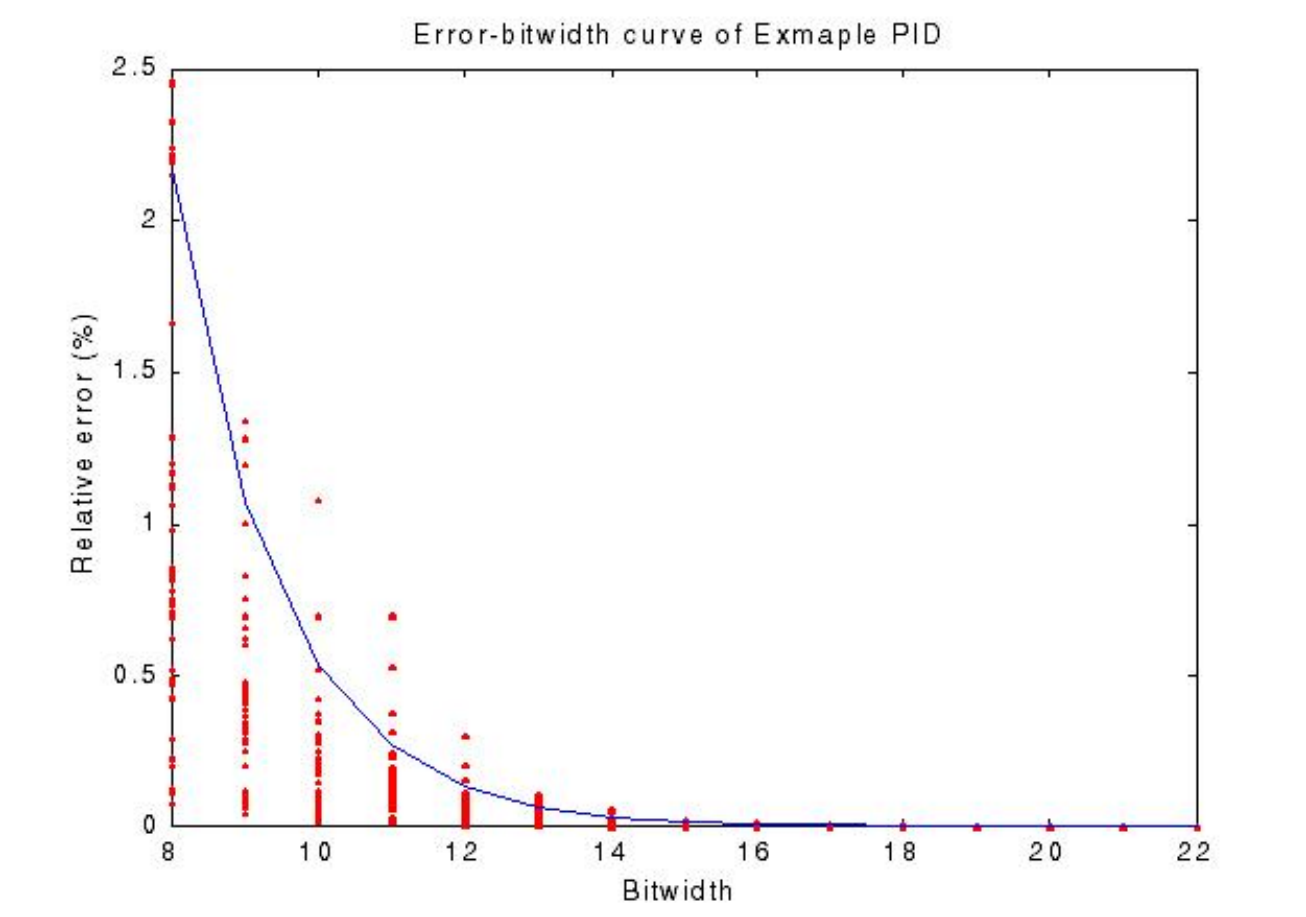


CDFG: PID Controller



$$Irefk <= (Kp * Ek) + Ki * \sum (Ek * \delta T) + Kd * \Delta(Ek) / \delta T$$

Result: PID Controller



Bitwidth Optimization

- Problem Formulation
- Optimization Methods
  - Grid Steepest Descent (GDS)
  - Accelerated Grid Steepest Descent (GSD-A)
- Optimization Results

Problem Formulation

- **Precision model based on CDFG and profile data:**

$$F = f(b_1, b_2, ..., b_N)$$

N: number of nodes in CDFG

- **Objective function:**

$$\min \sum_{1}^{N} b_i$$

b_i: bit-width of node *i*

- **Constraints:**

$$\begin{cases} F \geq P \\ b_i \in Z \end{cases}$$

P: accuracy requirements

$$Z = [1, 2, ..., 23]$$

Z: range for bit-width selection

Optimization Algorithms (1)

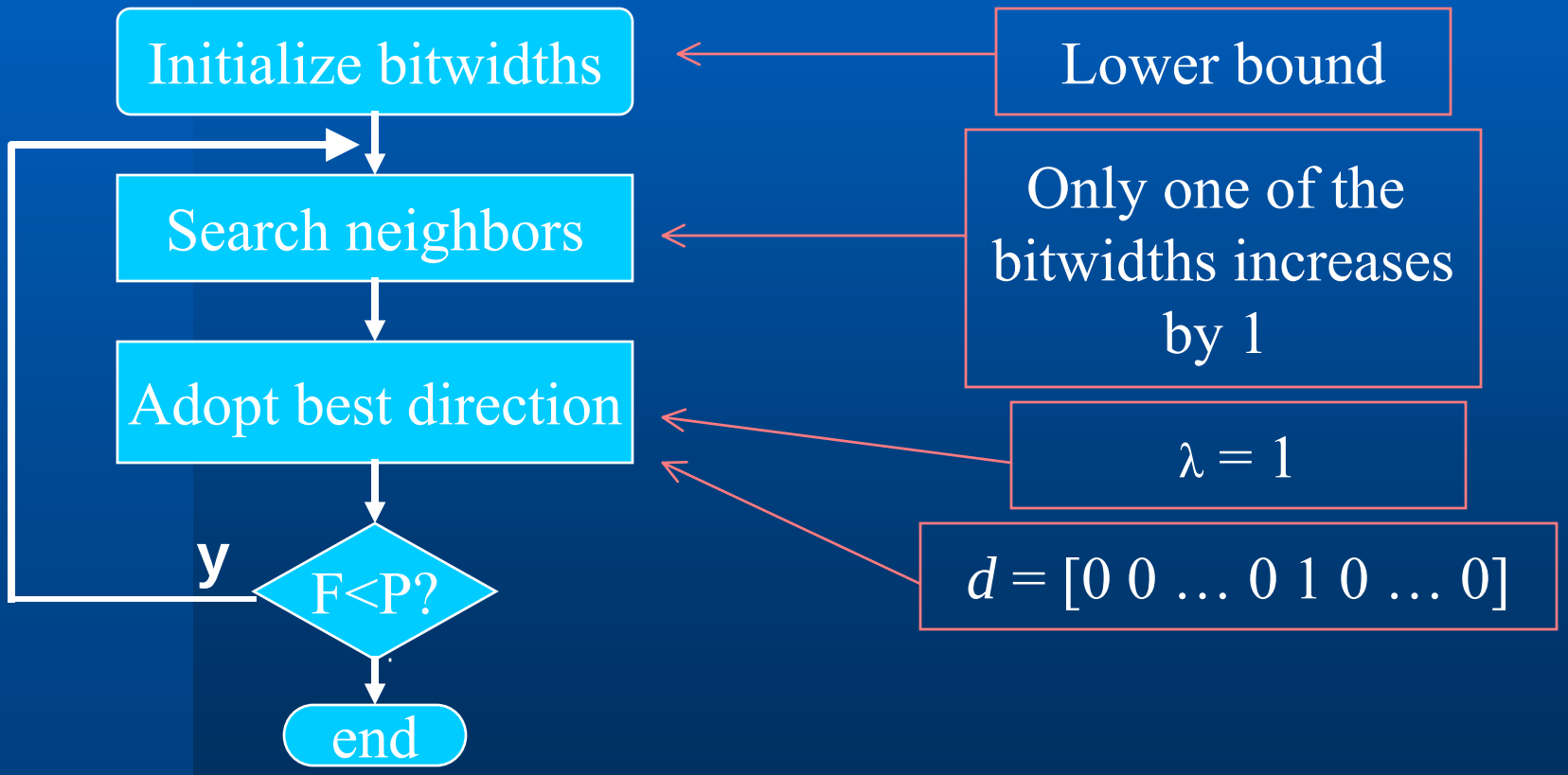
$$\begin{cases} \min\ f(x) \\ x_{k+1} = x_k + \lambda_k d_k \\ \lambda_k = -\nabla f(x_k) \end{cases}$$

x: vector of bitwidths
k: search step

- **Regular Steepest Decent**
  - **In each step, direction is calculated, step length is determined by searching in the direction**
- **Grid Steepest Decent (GSD)**
  - **In each step, step length is fixed (λ=1), direction is determined by searching**

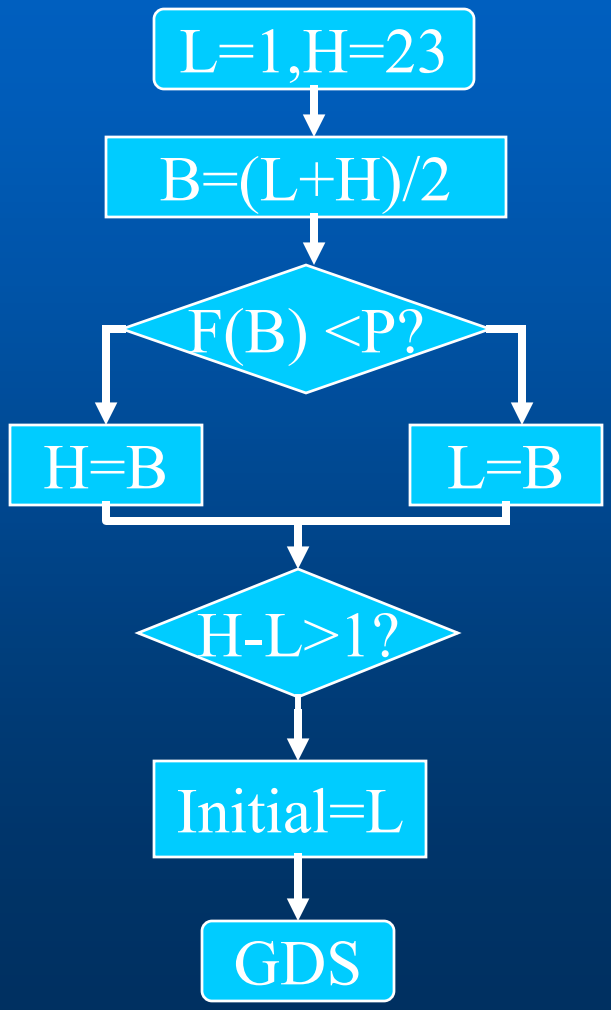
Optimization Algorithms (2)

- **Grid Steepest Decent (GSD)**



Optimization Algorithms (3)

- Accelerated GSD (GSD-A)
  - "Smart" Initial Point
  - Binary search to locate initial point
  - Total search time is reduced to a fraction
  - Initial Point: All bitwidths have the same initial value
  - GSD: Each bitwidth is calculated individually



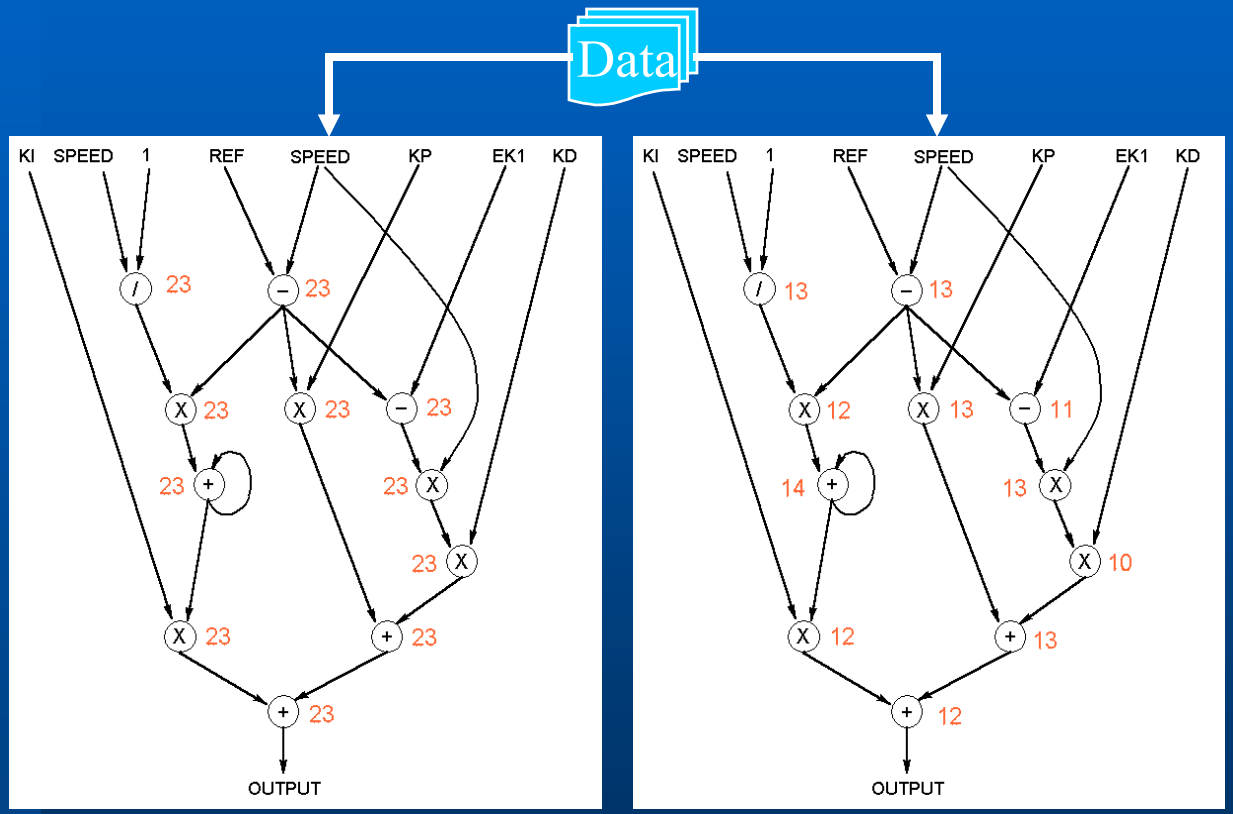
Optimization Results (1)

Result Comparison (P = 5%)

EXAMPLE	IEEE BITWIDTH	OPTIMIZED BITWIDTH	RGSD STEPS	RGSD-A STEPS
DIFFEQ	230	132	98	9
PID	253	136	87	8
Three MULT	96	37	59	2

Optimization Result (2)

- IEEE Format
- Total Bitwidth **253**
- Output Error **0**



- Variable Bitwidth
- Total Bitwidth **136**
- Output Error **5%**

Future Work

- Validate the optimized bitwidths
  - *C++ floating-point library* supporting variable bitwidth
  - *VHDL Variable precision floating-point component library*, developed by Rapid Prototyping Lab at Northeastern University, available under GPL at http://www.ece.neu.edu/groups/rpl/projects/floatingpoint/
- Improve error models
  - Propagation error models and rounding error models
  - Singularity issues
- Integrate in high level synthesis flow
  - IEEE 1076.3 working group: variable bitwidth floating-point for synthesis

Conclusion (1)

- Variable bitwidth FP computing is viable
- Model-based bitwidth optimization has advantages over simulation-based searching
- A methodology of FP precision modeling has been developed
- The precision model predicts output error and can be used for bit-width optimization

Conclusion (2)

- A customized optimization algorithm, Grid Steepest Descent (GSD), has been developed
- Search acceleration techniques have been applied to GSD
- Optimized bitwidths for a given precision target can be found quickly
- Sum of the optimized bitwidths is significantly smaller than that of standard IEEE format