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14. ABSTRACT
Array processing techniques are primarily designed for point sources, i.e. spatially discrete sources of acoustic or electromagnetic energy. However, in many cases the transmitter is better modeled as a distributed, rather than a discrete source. The principal mechanism for making the source appear to be distributed in space is diffuse and specular multipath caused by scattering of the propagating waves. Another is transmitter motion. The goal of this project was to develop array processing techniques for distributed sources. The vector of the signals received by an array from a distributed source resides in a Grassmanian manifold. This is the natural extension of the array manifold used in the case of point sources, which can be considered to be a special (rank one) case of a subspace manifold. In other words, whereas the response of an array to discrete sources is characterized by what is commonly called the array manifold, its response to a distributed source is characterized by a subspace manifold.

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1 Introduction

This final report summarizes work done on the project "Array Processing for Discrete and Distributed Sources," under contract # N00014-01-1-0075 for the Office of Naval Research, during the period 10/01/00 - 9/30/04.

Array processing techniques are primarily designed for point sources, i.e. spatially discrete sources of acoustic or electromagnetic energy. However, in many cases the transmitter is better modeled as a distributed, rather than a discrete source. The principal mechanism for making the source appear to be distributed in space is diffuse (unresolvable) and specular (resolvable) multipath caused by scattering of the propagating waves. Another is transmitter motion.

The goal of this project was to develop array processing techniques for distributed sources. The vector of the signals received by an array from a distributed source resides in a Grassmanian manifold (or a subspace manifold). This is the natural extension of the array manifold used in the case of point sources, which can be considered to be a special (rank one) case of a subspace manifold. In other words, whereas the response of an array to discrete sources is characterized by what is commonly called the array manifold, its response to a distributed source is characterized by a subspace manifold.

The main emphasis of the project was on solving signal estimation and detection problems in an environment containing both discrete and distributed sources, where some of the sources are the signals of interest and others act as interference. We are especially interested in applications involving passive sonar systems which attempt to detect weak targets in the presence of strong possibly rapidly moving sources of interference.

The main objective of this research effort was to develop a more complete understanding of array processing for distributed sources, and to apply what we learn to various problems of interest in sonar and communication systems. More specifically, the objectives were to develop the theoretical framework of the subspace manifold within which array processing for distributed sources can be properly addressed, to develop algorithms for optimal signal estimation and detection for distributed sources, and to evaluate the performance of the candidate algorithms in the context of specific acoustic surveillance and communication scenarios.

2 Technical Approach

Our technical approach is based on representing the distributed sources by the subspace in which they exist. The signal $\mathbf{S}$ received by the array from a single source is assumed to be complex Gaussian with zero mean and covariance $\mathbf{R}_s$. Let $\mathbf{R}_s = \mathbf{U}\mathbf{D}\mathbf{U}^H$ be the singular value decomposition of the signal covariance matrix and let $\mathbf{U}_s$ be a matrix consisting of the first r singular vectors (i.e. the first r columns of $\mathbf{U}$), where r is chosen so that the sum of the first r singular values is very close to the sum of all

the singular values. In other words, $\mathbf{U}_s$ contains most of the signal energy. We will refer to $\mathbf{U}_s$ as the signal subspace, and to r as its effective rank. When the angular spread equals zero (or is very small), $r = 1$, and we have the conventional case of a point (discrete) source. For larger angular spreads we have $r > 1$ and this is the distributed source case.

Next consider the problem of detecting a signal $\mathbf{S}$ in the presence of zero-mean Gaussian interference-plus-noise with covariance matrix $\mathbf{R}$. Let $\mathbf{X}$ denote the vector of data measured at the array output. The detector needs to decide whether $\mathbf{X}$ is a zero mean Gaussian with covariance $\mathbf{R}$ or whether it is a zero mean Gaussian with covariance $\mathbf{R}_s + \mathbf{R}$. This basic detection problem can be restated in many variations differing by what is and is not known to the detector. In practice the interference-plus-noise covariance $\mathbf{R}$ is unknown and needs to be estimated from training data. The signal covariance is only partially known. We may, for example, assume that the rank of the subspace is known, but the direction of the signal and its energy are unknown. These different versions of the problem are of varying degrees of difficulty and require careful analysis and interpretation.

3 Summary of Work

Using the subspace model for distributed sources that was introduced in the previous section, we can formulate and attempt to solve a variety of detection and estimation problems that have been previously addressed for the case of discrete sources.

We have studied the basic problem of detecting a single distributed source in the presence of white Gaussian noise, focussing on the case of a linear uniformly spaced arrays. The structure of the "optimal" detector can be interpreted as a bank of beamformers whose output energies are combined. The beamformer coefficient vectors span a low rank subspace, whose rank depends on the angular spread of the signal. We developed the receiver operating characteristics for this subspace detector under various assumptions on which parameters are known or unknown. In particular, we made detailed comparisons of the performance of the subspace detector to the conventional beamformer. This work was summarized in project publications [1] and [2]. Reference [1] is incorporated as an appendix to this report, to provide a more detailed description of this work.

Next we considered the case of detecting a distributed source in the presence of interference. This problem is more involved, partly because it has several variations depending on the type of information available to the detector. For example, we may or may not have training data for estimating the interference-plus-noise covariance matrix. We have studied in some detail the case where training data is available, and were able to show that the optimum processor is a bank of Minimum Variance Distortionless Response (MVDR) beamformers, whose output energies are added up with appropriate weights. More specifically, the weight vector of the MVDR for a

point source is the product of the inverse interference plus noise covariance matrix and the steering vector in the target direction. For a distributed source we have multiple MVDRs, each with the same covariance matrix, but different steering vectors. The steering vectors are selected so as to span the signal subspace Us. The MVDR outputs are weighted according the the amount of signal energy contained in different parts of the signal subspace, and then combined. This work is summarized in project publications [3].

In [4], [5] we studied the problem of detecting distributed sources using sub-arrays as a means of improving performance when only a limited amount of training data is available. We developed two types of sub-array detectors, both having a sub-array level GMVDR as their first processing step. The first detector non-coherently combines the outputs of the GMVDRs. The second detector optimally combines these outputs using another GMVDR. We have analyzed the performance of the two sub-array detectors and were able to show that they offer significantly improved performance as the number of available data snapshots falls below the number of elements in the full-size array. As expected, in the case where the covariance matrix R is known the performance of the sub-array detectors is inferior to that of the full-size GMVDR, but the difference is quite small. Reference [4] is incorporated as an appendix to this report, to provide a more detailed description of this work.

In [6], [8] we studied the detection problem for the case where multiple observations of the signal are available. The detector is derived from the likelihood ratio detector designed for the case where the covariance R is known. After the test statistic is derived, the maximum likelihood estimate of the covariance matrix based on the secondary data is inserted in place of the known covariance. The resulting test statistic is a CFAR detector. By comparing this detector to a related detector derived for the case where the signal obeys a First Order Gaussian (FOG) model (where the signal is assumed to be a fixed but unknown vector), we observe that the derived CFAR detectors for both models have the same form for either single or multiple observations. The detector for the FOG model using a single observation has been shown to be a GLRT detector. This observation suggests that the detector we derived and the detector for the FOG model may be true GLRT detectors. Reference [8] is incorporated as an appendix to this report, to provide a more detailed description of this work.

4 Impact and Applications

The subspace array manifold provides a natural framework for extending array processing from point sources to distributed sources. It has both theoretical and practical significance. From a theoretical standpoint it provides an approach for extending much of the work which has been done on signal detection and estimation, interference cancellation and so on, to the underwater acoustic and communication environments.

This work has direct applications to sonar, radar, and wireless communication problems. A recent development in radar is the so-called MIMO radar which utilizes diversity techniques to enhance detection performance of radars equipped with multiple transmit/receive antennas. The problem formulation and results carried out in this project fit very well this problem. It may be possible to extend the MIMO radar concept to active sonar arrays. The application to wireless communications was explored in [9], [10], [11], [12], which uses a signal model and detectors based on the work described here.

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Adaptive Array Processing for Distributed Sources *

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Abstract

We consider the problem of detecting a spatially distributed source by an array of sensors. Source parameters such as signal power, direction and angular spread, as well as the noise power, are unknown to the detector. Using the generalized likelihood ratio (GLR) approach we derive the structure of the detector under different assumptions on which parameters are known or unknown. The performance of the detetors is evaluated and the effect of angular spread is investigated. It is shown that as angular spread increases, the GLR detector provides a significant advantages over the conventional beamformer based detector.

1 Introduction

The vast majority of modern array processing techniques are designed for point sources, i.e. spatially discrete sources of acoustic or electromagnetic energy. In many practical situations the transmitter is best modeled as a distributed, rather than a discrete source. The principal mechanism for making the source appear to be distributed in space, is diffuse (unresolvable) and specular (resolvable) multipath caused by scattering of the propagating waves. A secondary, but equally important mechanism, is transmitter motion. If the source moves significantly during the observation interval (or coherent integration time) it will appear to be distributed rather than discrete. In this paper we develop detectors for distributed sources, and study their performance as a function of the angular spread of the signal.

In section 2 we describe the statistical detection problem. In section 3 we present several GLR detectors which make different assumptions on what signal and noise parameters are known or unknown. Section 4 presents a brief performance analysis, and section 5 summarizes selected numerical examples which provide insights into the effects of angular spread on the performance of the conventional beamformer and the GLR detector.

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2 Problem Statement

Assume we have an array with P sensors having an array response vector $\mathbf{a}(\phi)$, where ϕ denotes azimuth. The array and all the sources are assumed to be in the same plane. We assume a narrowband model for all the signals, and all the signals are defined in baseband. Extensions to the broadband case will be described elsewhere.

The signal received by the array from a single source is modeled as

$$\mathbf{X}_t = \mathbf{S}_t + \mathbf{N}_t, \quad t = 1, \cdots, T \tag{1}$$

where $\mathbf{X}_t$ is the array output at sample time t, $\mathbf{S}_t$ is the signal received at the array elements, assumed to be complex Gaussian with zero mean and covariance $\mathbf{R}_s$, and $\mathbf{N}_t$ is complex Gaussian noise with zero mean and covariance $\mathbf{R}_n$. The signal and noise are assumed to be independent from sample to sample.

The array is characterized by the *array manifold* $\mathbf{a}(\phi)$, where ϕ is the source azimuth. The signal covariance matrix $\mathbf{R}_s$ is related to the array manifold by

$$\mathbf{R}_s(\phi, \beta) = E_s \int_{\phi-\beta/2}^{\phi+\beta/2} P(\theta; \phi)\mathbf{a}(\theta)\mathbf{a}(\theta)^H d\theta \tag{2}$$

where $P(\theta; \phi)$ is the spatial energy distribution of the source at azimuth ϕ. More specifically we may assume that $P(\theta; \phi) = P(\theta - \phi)$. The parameter β is the angular spread of the source, $-\pi \leq \beta \leq \pi$. The signal is assumed to have energy E_s, and therefore

$$\int_{\phi-\beta/2}^{\phi-\beta/2} P(\theta; \phi) d\theta = 1 \tag{3}$$

The signal covariance matrix $\mathbf{R}_s(\phi,\beta)$ depends on signal direction ϕ and angular spread β. The case where $\beta = 0°$ corresponds to a point source. Without loss of generality, we consider in the following an uniformly distributed signal model, i.e. $P(\theta;\phi) = 1/\beta$.

The problem considered in this paper is how to detect the signal, in the presence of noise and interference. Let us assume for the moment that the interference has known characteristics and is absorbed into the noise vector $\mathbf{N}_t$. The more realistic case of unknown interference is more complicated and will be presented elsewhere.

The detection problem is to decide between the null hypothesis $H_0 : \mathbf{X}_t = \mathbf{N}_t$ and $H_1 : \mathbf{X}_t = \mathbf{S}_t + \mathbf{N}_t$. The covariance matrix of the array output under H_0 is $\mathbf{R}_0 = \mathbf{R}_n$ and under H_1 it is $\mathbf{R}_1 = \mathbf{R}_s(\phi,\beta) + \mathbf{R}_n$. Without loss of generality we assume that $\mathbf{R}_n = \sigma^2\mathbf{I}$, where σ^2 is the noise variance.

The detector depends on a set of parameters $p = (E_s, \phi, \beta, \sigma^2)$. When all the parameters are known, this is a standard detection problem whose optimal solution is the likelihood ratio detector. When the parameters are unknown, we will use the generalized likelihood ratio (GLR) approach, involving the replacement of unknown parameters by their maximum likelihood estimates under each hypothesis. In the following we will consider the following types of detectors:

- Type I: All parameters are known.
- Type II: Unknown noise power σ^2.
- Type III: Unknown direction ϕ.
- Type IV: Unknown angular spread β.

3 The Detector

Given the assumption that the received signals are Gaussian with zero mean and covariance $\mathbf{R}_0$ and $\mathbf{R}_1$ under hypotheses H_0 and H_1 respectively, the probability density functions (pdf) for the observations are given by

$$f(\mathbf{X}_t;H_0) = [(2\pi)^{P/2}|\mathbf{R}_0|^{1/2}]^{-1} exp\{-\frac{1}{2}\mathbf{X}_t^H\mathbf{R}_0^{-1}\mathbf{X}_t\} \quad (4)$$

$$f(\mathbf{X}_t;H_1) = [(2\pi)^{P/2}|\mathbf{R}_1|^{1/2}]^{-1} exp\{-\frac{1}{2}\mathbf{X}_t^H\mathbf{R}_1^{-1}\mathbf{X}_t\} \quad (5)$$

When all the parameters are known the optimal detection statistic is given by the logarithm of the ratio $f(\mathbf{X}_t;H_1)/f(\mathbf{X}_t;H_0)$ of the likelihood functions. This log-likelihood ratio takes the form

$$L(\mathbf{X}_t) = \mathbf{X}_t^H\left(\mathbf{R}_0^{-1} - \mathbf{R}_1^{-1}\right)\mathbf{X}_t + \ln\frac{|\mathbf{R}_0|}{|\mathbf{R}_1|} \quad (6)$$

Next consider the case where some parameters are unknown. Denote by p_0 and p_1 the unknown parameters under H_0 and H_1 respectively. In this case the detection statistic is given by the generalized likelihood ratio

$$l(\mathbf{X}_t) = \frac{\max_{p_1} f(\mathbf{X}_t, H_1)}{\max_{p_0} f(\mathbf{X}_t, H_0)} \quad (7)$$

or its logarithm

$$\begin{aligned} L(\mathbf{X}_t,\hat{p}_0,\hat{p}_1) &= \mathbf{X}_t^H\left(\mathbf{R}_0^{-1}(\hat{p}_0) - \mathbf{R}_1^{-1}(\hat{p}_1)\right)\mathbf{X}_t \\ &\quad + \ln\frac{|\mathbf{R}_0(\hat{p}_0)|}{|\mathbf{R}_1(\hat{p}_1)|} \end{aligned} \quad (8)$$

where $\hat{p}_1$ and $\hat{p}_0$ are the parameter values which maximize the likelihood function under H_1 and H_0 respectively. In the following we will find it convenient to write this equation as

$$L(\mathbf{X}_t,\hat{p}_0,\hat{p}_1) = \mathbf{X}_t^H\mathbf{W}\mathbf{W}^H\mathbf{X}_t + \alpha_t(\hat{p}_0,\hat{p}_1) \quad (9)$$

where

$$\alpha(\hat{p}_0,\hat{p}_1) = \ln\frac{|\mathbf{R}_0(\hat{p}_0)|}{|\mathbf{R}_1(\hat{p}_1)|} \quad (10)$$

and

$$\mathbf{W}\mathbf{W}^H = \mathbf{R}_0^{-1}(\hat{p}_0) - \mathbf{R}_1^{-1}(\hat{p}_1) \quad (11)$$

where $\mathbf{W}$ is a $P \times P$ matrix. This leads to an interpretation of the GLR as a bank of beamformers. Let $\mathbf{y}_t$ as an output vector of the beamformers, or

$$\mathbf{y}_t = \mathbf{W}^H\mathbf{X}_t \quad (12)$$

We will refer to this as a "subspace beamformer", to distinguish it from the conventional beamformer where $\mathbf{W}$ is a $P \times 1$ vector. The detection statistic is given by

$$L(\mathbf{X}_t,\hat{p}_0,\hat{p}_1) = \|\mathbf{y}_t\|^2 + \alpha_t(\hat{p}_0,\hat{p}_1) \quad (13)$$

where $\|\mathbf{y}_t\|^2$ is the sum of squared magnitudes of the subspace beamformer outputs.

Next we conside the case of multiple measurements. Let $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_T]$ be the measured data. Assume that the measurements are mutually independent, in which case

$$L(\mathbf{X}, \hat{p}_0, \hat{p}_1) = \sum_{t=1}^{T} L(\mathbf{X}_t, \hat{p}_0, \hat{p}_1) \quad (14)$$
$$= \text{trace}\{\mathbf{W}^H \mathbf{X}\mathbf{X}^H \mathbf{W}\} + \alpha(\hat{p}_0, \hat{p}_1)$$

where $\alpha(\hat{p}_0, \hat{p}_1) = \sum_{t=1}^{T} \alpha_t(\hat{p}_0, \hat{p}_1)$.

Next we consider in more detail the structure of the different detectors introduced earlier. In all these cases the GLR statistics are given by equations (9) – (11).

Type I: In this case all the parameters are known and the subspace beamformer weight matrix is

$$\mathbf{W} = (\sigma^{-2}\mathbf{I} - (\sigma^2\mathbf{I} + \mathbf{R}_s(\phi, \beta))^{-1})^{1/2} \quad (15)$$

The term $\alpha(p_0, p_1)$ is constant (data independent) and can be ignored. In this case the detection statistic $L(\mathbf{X}_t)$ is a quadratic Gaussian form, and has a weighted Chi-squared distribution. See section 4.2 for details.

Type II: In this case noise variance σ^2 is unknown, and is replaced by $\hat{\sigma}_0^2$ under H_0 and $\hat{\sigma}_1^2$ under H_1 respectively. It is straightforward to show that

$$\hat{\sigma_0^2} = \frac{1}{P}\|\mathbf{X}_t\|^2 \quad (16)$$

The estimate $\hat{\sigma}_1^2$ can be computed by numerical maximization of the likelihood function. In the case where $\mathbf{R}_s$ has a low rank approximation.

$$\mathbf{R}_s \approx \mathbf{U}_r \mathbf{U}_r^H \quad (17)$$

where $\mathbf{U}_r$ is a $P \times r$ matrix, it can be shown that ([4])

$$\hat{\sigma_1^2} \approx \frac{1}{P-r}\|\mathbf{P}_{\mathbf{U}_r}^{\perp}\mathbf{X}_t\|^2 \quad (18)$$

The subspace beamformer weight matrix is given by

$$\mathbf{W} = (\frac{1}{\hat{\sigma_0^2}}\mathbf{I} - (\hat{\sigma_1^2}\mathbf{I} + \mathbf{R}_s)^{-1})^{1/2} \quad (19)$$

and

$$\alpha_t(\hat{\sigma_0^2}, \hat{\sigma_1^2}) = P \ln \hat{\sigma_0^2} - \ln|\mathbf{R}_1(\hat{\sigma}_0^2, \hat{\sigma}_1^2)| \quad (20)$$

Type III: In this case the signal direction ϕ is not known and needs to estimated. Note that the likelihood function under H_0 is independent of ϕ and thus it needs to be estimated only under H_1. The estimation is done by numerical maximization over the range $-180° \leq \phi \leq 180°$ (or $-90° \leq \phi \leq 90°$ for a linear array).

The subspace beamformer weight matrix is given by

$$\mathbf{W} = (\frac{1}{\sigma^2}\mathbf{I} - (\sigma^2\mathbf{I} + R_s(\hat{\phi}_1))^{-1})^{1/2} \quad (21)$$

and

$$\alpha_t(\hat{\phi}_1) = \ln|\mathbf{R}_1(\hat{\phi}_1)| \quad (22)$$

Type IV: In this case the angular spread β is unknown. The estimate $\hat{\beta}_1$ is computed by numerical maximization over range $0° \leq \beta \leq 360°$.

The subspace beamformer weight matrix is given by

$$\mathbf{W} = (\frac{1}{\sigma^2}\mathbf{I} - (\sigma^2\mathbf{I} + \mathbf{R}_s(\hat{\beta}_1))^{-1})^{1/2} \quad (23)$$

and

$$\alpha_t(\hat{\beta}_1) = \ln|\mathbf{R}_1(\hat{\beta}_1)| \quad (24)$$

4 Performance Analysis

4.1 Output SNR vs. angular spread

The subspace beamformer output is

$$\mathbf{y}_t = \mathbf{W}^H\mathbf{X}_t = \mathbf{W}^H\mathbf{S}_t + \mathbf{W}^H\mathbf{N}_t \quad (25)$$

Let us define the output SNR (SNRO) as the average signal power divided by the average noise power

$$\text{SNRO} = \frac{\text{tr}\{\mathbf{W}^H E\{\mathbf{S}_t\mathbf{S}_t^H\}\mathbf{W}\}}{\text{tr}\{\mathbf{W}^H E\{\mathbf{N}_t\mathbf{N}_t^H\}\mathbf{W}\}} \quad (26)$$

The input SNR is $\text{SNR} = E_s/\sigma^2$. The effective array gain AG will be defined as $\text{AG} = \text{SNRO}/\text{SNR}$.

We are interested in studying how the array gain varies with angular spread. Note that

$$\begin{aligned} \text{SNRO} &= \frac{\text{tr}\{\mathbf{R}_1(\beta)(\mathbf{R}_0^{-1} - \mathbf{R}_1^{-1}(\beta))\}}{\text{tr}\{\mathbf{R}_0(\mathbf{R}_0^{-1} - \mathbf{R}_1^{-1}(\beta))\}} - 1 \\ &= \frac{\text{tr}\{1/\sigma^2\mathbf{R}_1(\beta) - \mathbf{I}\}}{\text{tr}\{\mathbf{I} - \sigma^2\mathbf{R}_1^{-1}(\beta))\}} - 1 \\ &= \frac{1/\sigma^2\text{tr}\{E_s\mathbf{R}_s + \sigma^2\mathbf{I}\} - P}{P - \sigma^2\text{tr}\{(E_s\mathbf{R}_s + \sigma^2\mathbf{I})^{-1}\}} - 1 \\ &= \frac{E_s P/\sigma^2}{P - \sigma^2\text{tr}\{(E_s\mathbf{R}_s + \sigma^2\mathbf{I})^{-1}\}} - 1 \end{aligned} \quad (27)$$

The following can be concluded from this equation:

1. For the point source case $\mathbf{R}_s = E_s\mathbf{a}\mathbf{a}^H$ with $\mathbf{a}^H\mathbf{a} = 1$. Inserting this into the equation above we get $\text{AG} = P$.

2. For the case of completely uncorrelated fading $\mathbf{R}_s = E_s\mathbf{I}$. Inserting this into the equation above we get $\text{AG} = 1$.

3. If $\mathbf{R}_s$ has rank r, and all of its nonzero eigenvalues are equal, then $\text{AG} = P/r$ (see [4] for details).

In other words, the array gain is a monotonically decreasing function of angular spread, going from a maximum value of P to a minimum value of unity.

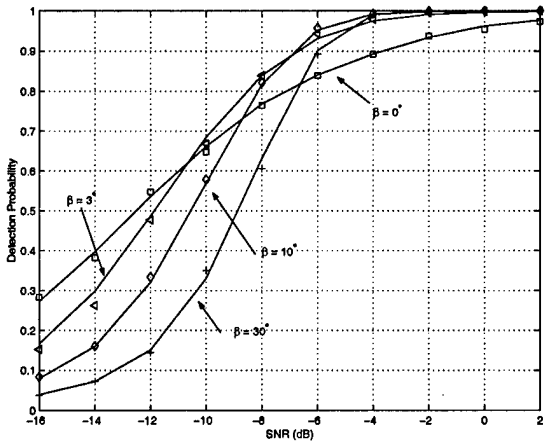

Figure 1. Probability of detection vs. SNR for Type I detector, for $\beta = 0°, 3°, 10°, 30°$. The lines depict the analytical results, while the markers show Monte-Carlo results.

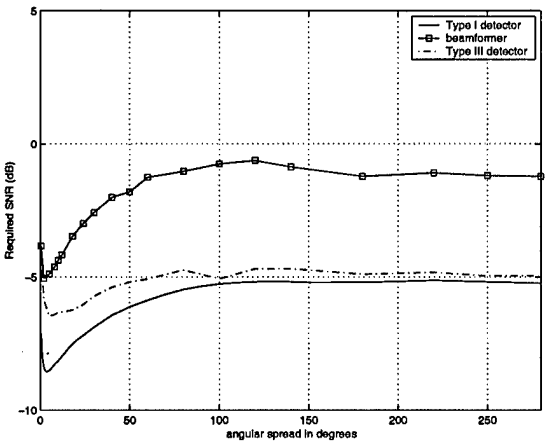

Figure 2. Required SNR vs. angular spread for Type I and Type III detectors and the beamformer, for $P_D = 0.8$.

4.2 Receiver Operating Characteristics

Let $p_M(z)$ be the pdf of a central Chi-squared distribution with M degrees of freedom. The log-likelihood ratio for a type I detector $L(\mathbf{X}_t, H_i) = \mathbf{X}_t^H \mathbf{W}\mathbf{W}^H \mathbf{X}_t$ is a quadratic form in complex Gaussian random variables. This can be rewritten as

$$L(\mathbf{X}_t, H_i) = \sum_{j=1}^{P} \gamma_{ji} |x_{ji}|^2 \tag{28}$$

where x_{ji} are independent zero mean unit variance Gaussian random variables, and γ_{ji} are the P real nonzero eigenvaules of the matrix $\mathbf{R}_i \mathbf{W}\mathbf{W}^H, i = 0, 1$. More precisely

$$\gamma_{j0} = \frac{\lambda_j}{\lambda_j + \sigma^2}, \quad \gamma_{j1} = \frac{\lambda_j}{\sigma^2}, \tag{29}$$

where λ_i are the eigenvalues of $\mathbf{R}_s$.

The pdf of the quadratic form $L(\mathbf{X}_t, H_i)$ is approximately a scaled Chi-squared random variable. It was shown in [3] that the scaling factor is $\tau_i = \sum \gamma_{ji}^2 / \sum \gamma_{ji}$, and the degrees of freedom $M_i = 2(\sum \gamma_{ji})^2 / \sum \gamma_{ji}^2$. In other words $L(\mathbf{X}_t, H_i) \sim \tau_i \chi^2_{M_i}$, and its pdf is $(1/\tau_i) p_{M_i}(z)$.,

The probability of false alarm (P_{FA}) and probability detection (P_D) are given by

$$P_{FA} = \int_{\eta}^{\infty} p_{M_0}(z) dz \tag{30}$$

and

$$P_D = \int_{\frac{\tau_0}{\tau_1}\eta}^{\infty} p_{M_1}(z) dz \tag{31}$$

The pdf of the log-likelihood function of detector types II - IV do not have closed analytical forms, but can be evaluated by Monte Carlo simulations.

5 Numerical Examples

In these examples we consider a linear array with $P = 100$ sensors uniformly spaced every one-half a wavelength. The source is at $\phi = 30°$ with a uniform energy distribution over the angular spread. The probability of false alarm is set to $P_{FA} = 0.01$.

5.1 Type I detector – Receiver Operating Characteristics

Figure 1 depicts the probability of detection of a type I detector, for different angular spreads. The figure shows both the analytical results using the formulas presented earlier, and the results of a Monte-Carlo simulation. As can be seen, there is a very good match between the two. Note that for high values of P_D the performance improves as angular spread increases up to a point, but then it starts decreasing. To see this more clearly it is convenient to pick just one point on the P_D vs. SNR curve. We define by SNRD the SNR

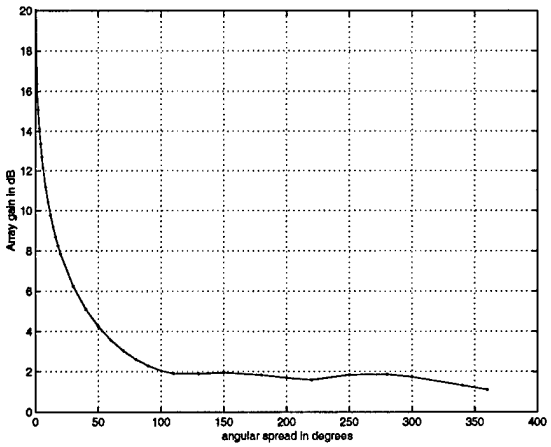

Figure 3. Array Gain vs. angular spread for the Type I detector.

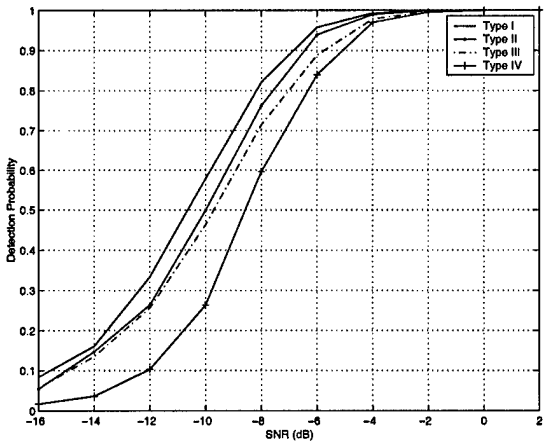

Figure 4. Probability of detection vs. SNR for detectors of types I, II, III, and IV, for $\beta = 10°$.

required to produce $P_D = 0.8$. In other words, SNRD is the SNR value of the points at which the horizontal line at $P_D = 0.8$ intersects with the curves.

5.2 Required SNR vs. angular spread

Figure 2 depicts the required SNR vs. angular spread, for detectors of type I and III, and the beamformer. It is assumed that the beamfomer searches over all possible directions and detection is done based on its largest output. As angular spread increases the required SNR initially decreases and then increases. Detector performance is affected by array gain and the pdf of the detection statistic. Array gain decreases monotonically with angular spread, while the change in the pdf improves performance monotonically with angular spread. The sum of these two effects has the form depicted in the figure.

5.3 Array Gain vs. angular spread

Figure 3 depicts the array gain AG vs. angular spread, for the Type I detector. As mentioned earlier, the type I detector has an array gain of P (20dB in this case) for $\beta = 0$, and a gain of unity (0 dB) for the uncorrelated case.

5.4 Comparison of different detectors

Figure 4 depicts the probability of detection curve for detector types I - IV, for an angular spread of 10°. Note that performance is most sensitive to lack of knowledge of the direction ϕ, and least sensitive to not knowing the noise power σ^2. However, over all the performance loss due to the unknown parameters is fairly small.

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Adaptive Subarray Processing For Distributed Sources

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Abstract

We introduce a framework for exploring array detection problems in a reduced dimensional space. This involves calculating a structured subarray transformation matrix for the detection of a distributed signal using large aperture linear arrays for short data records. We study the performance of the adaptive subarray detector and evaluate its potential improvement in detection performance compared with the full array detector with finite samples. One would expect that processing on subarrays may result in performance loss in that smaller number of degrees of freedom is utilized, yet lead to a better estimation accuracy for the interference and noise covariance matrix with finite data samples, which will yield some gain in performance. By studying the subarray detector for general linear arrays, we identify this gain under various scenarios. We show that when the number of samples is small, the subarray detector has a significant performance gain over the full array detector. We validate our results by computer simulations.

1 Introduction, Background and Motivation

The problem of detecting underwater acoustic sources using measurements by an array of sensors has been studied extensively in literature. For a large aperture acoustic array, a narrow beam can be formed so as to distinguish two closely spaced emitters. However, the acoustic energy source may be fairly close to the array and may move through several beams during the sonar system's temporal integration time. The effects of source motion on detector systems have been studied by several authors [2]. One may model the moving transmitter during an integration time as a source with energy scattering in space, or called a distributed source. The distributed source can be modelled by a subspace array manifold[5].

One of the enduring problems associated with the adaptive minimum variance distortionless beamformer (MVDR) lies in the classic dilemma of wanting long observation times for stable covariance matrix estimates yet needing short observation times to track dynamic field behavior. Reduced rank processing is one of the well known data processing methods [4] to deal with this issue. In this case, the data is mapped into a lower dimensional subspace via a transformation matrix prior to detection. In this paper, we study the problem of detecting distributed sources using subarrays, i.e., a partial collection of sensors of a full array, from the standpoint of the general reduced rank detection theory. In this case, the transformation matrix is a structured block diagonal matrix.

The motivation for this study lies in the following two observations. Firstly, a subarray with a smaller aperture gives rise to a wider beam which is able to cover the angular spread of the distributed sources if the subarray size is chosen properly. Secondly, the subarray processing offers a tradeoff: a better statistical accuracy of the estimate at the cost of reduced number of degrees of freedom. Hence substantial performance improvements are possible using the subarray detector in limited-data situations.

It should be noted that the idea of subarray processing has been proposed before, and has been studied by several authors, for instance, Cox [3], Morgan [6], and Owsley. Our work aims to derive the optimal subarray detector and its variations for distributed sources with short data record. In addition, our work is close in spirit to the well studied partially adaptive beamforming (see e.g. Van Veen [7] and the reference therein), where the number of adaptive degrees of freedom may be considerably fewer than the number of sensors, while still providing useful performance. Reducing the number of adaptive degrees of freedom degrades the interference cancellation performance. Thus minimizing the detection performance degradation is an important consideration in designing the optimal subarray detector for detecting signal sources with energy scattering.

2 Problem Formulation

2.1 Array signal model

We consider a general linear array composed of P sensors. Let $\{y_m, m = 1, \cdots, P\}$ be the coordinates of the m-th sensor measured in half wave-length units and $\mathbf{a}(\phi)$ be the steering vector of the array in the direction ϕ:

$$\mathbf{a}(\phi) = [e^{j\pi y_1 \sin\phi}, \cdots, e^{j\pi y_P \sin\phi}]^T \qquad (1)$$

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Let p narrow-band plane waves impinge on the array from distinct direction $\{\phi_1, \phi_2, \cdots, \phi_p\}$. For simplicity we assume that the sources and the sensors are coplanar. We denote the samples of the receiver outputs by x_{ti}, where i is the sensor number and t is the sample index (different samples at different t are assumed to be independent). Hence the signal received by the array is modelled as follows,

$$\mathbf{x}(t) = \bar{\mathbf{a}}_1(t)s_1(t) + \sum_{n=2}^{p} \bar{\mathbf{a}}_n(t)s_n(t) + \mathbf{n}(t), \quad t = 1, \cdots, K \tag{2}$$

where $\mathbf{x}(t)$ is array output at sample time t, $\mathbf{x}(t) = [x_{t1}, \cdots, x_{tP}]^T$. $s_n(t)$ is the n-th complex waveforms constituting the signal with total signal power $E\{|s_n(t)^2|\}$. We assume that the instantaneous array response $\bar{\mathbf{a}}_1(t)$ is a complex Gaussian vector with zero mean and covariance matrix $\mathbf{R}_s$. This covariance matrix is related to the subspace array manifold $\mathbf{A}$ via an eigenvalue decomposition, i.e., $\mathbf{R}_s \approx \mathbf{U}_r\Sigma_r\mathbf{U}_r^H = \mathbf{A}\mathbf{A}^H$ (see [5] for details), where r is the rank of the signal subspace. $\mathbf{n}(t)$ is the complex Gaussian noise with zero mean and covariance $\sigma^2\mathbf{I}$, and is uncorrelated with the signal sources. We assume that the first signal $s_1(t)$ is the desired signal with $E_s = E\{|s_1(t)|^2\}$, others are the interfering signals.

2.2 A reduced rank detection problem

A commonly used detection scheme is the binary hypothesis testing, that is, letting the null hypothesis be that the data is signal free and the alternative hypothesis be that the data contains a signal. Hence the detection problem on the basis of the *full array* data vector $\mathbf{x}$ (we drop t for simplicity purpose) is given as follows,

$$\begin{array}{ll} H_0: & \mathbf{x} \sim \mathcal{CN}(0, \mathbf{R}) \\ H_1: & \mathbf{x} \sim \mathcal{CN}(0, E_s\mathbf{R}_s + \mathbf{R}) \end{array} \tag{3}$$

The optimal full array detector $\mathbf{W}_f$ for the above detection problem is given as [5]

$$\mathbf{W}_f = \mathbf{R}^{-1}\mathbf{V} \tag{4}$$

where $\mathbf{V} = \sqrt{E_s}\mathbf{A}(E_s\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A} + \mathbf{I}_r)^{-1/2}$ is the weighted signal subspace matrix, while the matrix $(E_s\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}+\mathbf{I}_r)^{-1/2}$ is a diagonal dominant matrix and represents how the columns (or beams) of signal subspace $\mathbf{A}$ are weighted and combined. We call this matrix beamformer $\mathbf{W}_f$ a *Generalized MVDR* (GMVDR) beamformer in the sense that it extends the standard rank one MVDR beamformer $\mathbf{w}_f = \alpha\mathbf{R}^{-1}\mathbf{a}(\phi)$ to a multi-rank case.

The implementation of the full array detector requires *a priori* knowledge of $\mathbf{R}$, which is often estimated from finite training samples. The requirement for large number of data samples can be difficult or even impossible to meet in rapidly changing environments, especially for large aperture arrays. Thus we formulate a subarray detection problem within the framework of the general reduced rank detection theory described as follows. Let matrix $\mathbf{T} \in \mathbb{C}^{P\times L}$

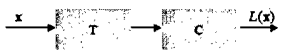

Figure 1. The reduced rank processing. T is a transformation matrix, C is a detector

be a linear transformation matrix, and matrix $\mathbf{C} \in \mathbb{C}^{L\times r}$ be a detector based upon transformed data

$$\mathbf{z} = \mathbf{T}^H\mathbf{x} \tag{5}$$

The detection problem is then described by the following binary hypothesis test

$$\begin{array}{ll} H_0: & \mathbf{z} \sim \mathcal{CN}(0, \mathbf{T}^H\mathbf{R}\mathbf{T}) \\ H_1: & \mathbf{z} \sim \mathcal{CN}(0, E_s\mathbf{T}^H\mathbf{R}_s\mathbf{T} + \mathbf{T}^H\mathbf{R}\mathbf{T}) \end{array} \tag{6}$$

The optimal detector for the above detection problem (6) is given as follows,

$$\mathbf{C} = (\mathbf{T}^H\mathbf{R}\mathbf{T})^{-1}\mathbf{T}^H\mathbf{A}(\frac{1}{E_s}\mathbf{I}+\mathbf{A}^H\mathbf{T}(\mathbf{T}^H\mathbf{R}\mathbf{T})^{-1}\mathbf{T}^H\mathbf{A})^{-\frac{1}{2}} \tag{7}$$

Hence, the overall detector for this reduced rank processing architecture shown in Fig. 1 is given as

$$\mathbf{W} = \mathbf{T}\mathbf{C} = \mathbf{T}(\mathbf{T}^H\mathbf{R}\mathbf{T})^{-1}\mathbf{V}_s \tag{8}$$

where the reduced rank signal subspace matrix $\mathbf{V}_s = \sqrt{E_s}\mathbf{T}^H\mathbf{A}(\mathbf{I} + E_s\mathbf{A}^H\mathbf{T}(\mathbf{T}^H\mathbf{R}\mathbf{T})^{-1}\mathbf{T}^H\mathbf{A})^{-1/2}$, which leads to the following test statistics

$$L(\mathbf{x}) = \mathbf{x}^H\mathbf{W}\mathbf{W}^H\mathbf{x} = \|\mathbf{W}^H\mathbf{x}\|^2 \tag{9}$$

The performance of the detector $\mathbf{W}$ depends on the choice of $\mathbf{T}$. This matrix $\mathbf{T}$ compresses P-dimensional data into a L-dimensional subspace prior to constructing a test statistic. This transformation reduces the nuisance parameter into $\mathbf{T}^H\mathbf{R}_\mathbf{x}\mathbf{T}$. This reduction in the number of nuisance parameters tends to improve the accuracy of the estimate $\mathbf{T}^H\mathbf{R}_\mathbf{x}\mathbf{T}$. Intuitively, $\mathbf{T}$ removes the dimensions that contain least "signal-to-interference-plus-noise" components. A desirable $\mathbf{T}$ should suppress strong interference components while match to the signal. Next, we derive a linear transformation matrix $\mathbf{T}$ where $\mathbf{T}$ has a block-diagonal structure.

3 The Subarray Detector For A General Linear Array

When we say subarray processing we mean that we divide the full array into many smaller arrays, or called sub-

array, and process the received data from each subarray individually. Consider a general linear array of P sensors, a common scheme is dividing the total array into a number of non-overlapping subarrays with equal size. Each subarray has M sensors. Without loss of generality, let us assume that $P = NM$ where N is the number of subarrays with sensors $\{1, \cdots, M\}$ forming the first subarray, sensors $\{M+1, \cdots, 2M\}$ forming the second subarray, etc. The full P-element input vector is given by equation (2). The M-element input data vector for the n-th subarray, which shall be denoted by $\mathbf{x}_n$, is expressed as follows,

$$\mathbf{x}_n = [x_{(n-1)M+1}, x_{(n-1)M+2}, \cdots, x_{nM}]^T, \quad n = 1, \cdots, N \tag{10}$$

Given the above subarray configuration, we constrain our processing matrix $\mathbf{T}$ to be a block diagonal matrix, i.e.,

$$\mathbf{T} = \text{diag}[\mathbf{T}_1, \cdots, \mathbf{T}_N] \tag{11}$$

where $M \times r_n$ matrix $\mathbf{T}_n$ is the optimal subarray detector of n-th subarray, and $L = \sum_{n=1}^{N} r_n$.

3.1 The $n-$th subarray beamformer $\mathbf{T}_n$

It is straightforward to show that, the optimal subarray beamformer for the n-th subarray is the GMVDR beamformer based on subarray data vector $\mathbf{x}_n$. Hence, the $M \times r_n$ matrix $\mathbf{T}_n$ is given as follows,

$$\mathbf{T}_n = \mathbf{R}_n^{-1}\mathbf{A}_n(\mathbf{A}_n^H\mathbf{R}_n^{-1}\mathbf{A}_n + \frac{1}{E_s}\mathbf{I})^{-\frac{1}{2}} = \mathbf{R}_n^{-1}\mathbf{V}_n \tag{12}$$

where $\mathbf{V}_n = \mathbf{A}_n(\mathbf{A}_n^H\mathbf{R}_n^{-1}\mathbf{A}_n + \frac{1}{E_s}\mathbf{I})^{-\frac{1}{2}}$. $\mathbf{R}_n$ is the n-th diagonal block of $\mathbf{R}$, and $\mathbf{A}_n \in \mathbb{C}^{M \times r_n}$ is the n-th subarray manifold.

3.2 The coherent subarray detector

Plugging equation (11) and (12) into equation (8), we obtain the coherent subarray detector as follows

$$\mathbf{W}_{coh} = \mathbf{TC} \tag{13}$$

3.3 The noncoherent subarray detector

For the noncoherent detector, the outputs of each subarray after the pre-processing matrix $\mathbf{T}$ are squared and summed regardless of the coherence of the signal along each subarray. This detector is given by

$$\mathbf{W}_{non} = \mathbf{T} \tag{14}$$

4 Performance Analysis

In this section, we quantify the performance of the subarray detector analytically. It is clear that the derived coherent subarray detector is a cascade of GMVDR beamformers, i.e., the beamformer $\mathbf{T}_n$ at each subarray, and a combiner $\mathbf{C}$. There are three basic issues that need to be understood. One is the interference cancellation through two stages of subarray processing. Smaller number of degrees of freedom is used at the subarray level to cancel out the interference, which causes performance loss, the second stage will gain back some of the loss by combining the outputs from the first stage. The second issue is the potential gain of the subarray processing compared with the full array processing with finite sample size due to a better statistical stability of the interference estimation. The third issue is the effect of signal source angular spread. We use a uniform linear array as an example.

4.1 Interference cancellation of coherent and noncoherent subarray detector

The $r_n \times r_n$ matrix $\mathbf{T}_n^H\mathbf{R}_n\mathbf{T}_n$ represents the covariance matrix of the interference plus noise at the output of n-th subarray detector $\mathbf{T}_n$.

$$\begin{aligned} \mathbf{T}_n^H\mathbf{R}_n\mathbf{T}_n &= \mathbf{I} - (E_s\mathbf{A}_n^H\mathbf{R}_n^{-1}\mathbf{A}_n + \mathbf{I})^{-1} \\ &= \mathbf{U}_n\text{diag}[\tfrac{\gamma_1}{1+\gamma_1}, \cdots, \tfrac{\gamma_{r_n}}{1+\gamma_{r_n}}]\mathbf{U}_n^H \end{aligned} \tag{15}$$

where $E_s\mathbf{A}_n^H\mathbf{R}_n^{-1}\mathbf{A}_n = \mathbf{U}_n\Gamma_n\mathbf{U}_n^H$ via an eigen-decomposition and Γ_n is the eigen-value matrix which represents the signal to noise ratio distributed in the signal subspace, and $\frac{\gamma_i}{1+\gamma_i}$ represents the residue interference plus noise appears at i-th beam at the output of $\mathbf{T}_n$. Furthermore, the general covariance matrix at the output of first stage processing matrix $\mathbf{T}$ is given by a block matrix $\mathbf{T}^H\mathbf{RT}$ with $\mathbf{T}_m^H\mathbf{R}_{m,n}\mathbf{T}_n$ being its (m,n)-th block, which indicates the cross-correlation of the interference plus noise outputs of beamformer $\mathbf{T}_m$ and $\mathbf{T}_n$.

It is often said that a beamformer requires one adaptive degree of freedom per point interferer to achieve interference cancellation. We extend the point sources to the distributed sources, and study the interference cancellation of the coherent and noncoherent subarray detectors for the following two cases: (a). $M \geq q$ and (b). $M < q$, where M is the subarray size and q is the rank of interference subspace.

If $M \geq q$, i.e., the available number of degrees of freedom at the subarray level is greater than the interference subspace rank, the equation (15) becomes

$$\mathbf{T}_n^H\mathbf{R}_n\mathbf{T}_n \approx \text{diag}[\varrho_{1,1}, \cdots, \varrho_{r_n,r_n}] \tag{16}$$

The above equation indicates that the beam noise outputs of $\mathbf{T}_n$ are uncorrelated. The same argument goes to the $\mathbf{T}^H\mathbf{RT}$ where

$$\mathbf{T}^H\mathbf{RT} \approx \text{diag}[\varrho_{1,1}, \cdots, \varrho_{L,L}] \tag{17}$$

To quantify the interference cancellation through different stages of subarray processing, we calculate the deflection, i.e., change in mean divided by standard deviation:

$$\text{DEFL}_{\mathbf{W}} = E_s\frac{\text{tr}[\mathbf{W}^H\mathbf{R}_s\mathbf{W}]}{\sqrt{\text{tr}[(\mathbf{W}^H\mathbf{RW})^2]}} \tag{18}$$

Therefore utilizing (16) and (17), we obtain

$$\frac{\mathrm{DEFL}_{\mathbf{T}}}{\mathrm{DEFL}_{\mathbf{T}_n}} \approx \sqrt{N} \tag{19}$$

The gain for the coherent subarray detector is rather complicated. Instead, we calculate the bounds of the gain. The gain function is given as follows (see [5] for details),

$$\sqrt{N} \leq \frac{\mathrm{DEFL}_{\mathbf{W}_{\mathrm{coh}}}}{\mathrm{DEFL}_{\mathbf{T}_n}} \leq N \tag{20}$$

The maximal gain of the coherent subarray detector is obtained when the signal source is a point source. In this case, the detector $\mathbf{C}$ is essentially a conventional beamformer which combines N beam outputs coherently and yields a gain of N (see also [3]). The lower bound of the gain function is certainly due to the fact that the coherent subarray detector has a better gain than the noncoherent subarray detector. Notice that the relative loss $\frac{\mathrm{DEFL}_{\mathbf{W}_f}}{\mathrm{DEFL}_{\mathbf{T}_n}}$ is lower bounded by $\sqrt{N}$. Equation (19) and (20) indicate that the second stage processing yields a constant gain that compensates the loss occurred at the first stage of processing. Consequently, we will also see by computer simulations that the *overall* performance loss of the subarray detector is insignificant compared with that of the full array detector.

When $M < q$, the number of adaptive degrees of freedom is smaller than the rank of the interference subspace, the performance loss due to incomplete cancellation of the interference may be significant. Generally the matrix $\mathbf{T}^H\mathbf{R}\mathbf{T}$ is a diagonal *dominant* matrix due to weak cross-correlation between different $\mathbf{T}_n$ beam outputs of residue interference. Hence, Eqn. (19) and (20) still hold. However, when M becomes extremely small, $\mathrm{DEFL}_{\mathbf{T}_n}$ tends to the average element deflection, which indicates a severe loss of interference cancellation capability.

4.2 Effect of signal angular spread

In this subsection, we use the output SINR, which is given as $\mathrm{SINR} = \frac{E_s \mathrm{tr}[\mathbf{W}^H\mathbf{R}_s\mathbf{W}]}{\mathrm{tr}[\mathbf{W}^H\mathbf{R}\mathbf{W}]}$, as our performance measure. It is assumed that the subarray size is chosen to be $M \geq q$. We notice that the SINR gain of $\mathbf{T}_n$, defined as output SINR vs. average element SINR, is given as

$$\mathrm{SINRG} \approx \frac{M}{r_n} \tag{21}$$

where r_n is the effective rank of subarray signal subspace, or equivalently, the number of main beams. This result is consistent with the SINR gain for the full array detector reported in [5]. Eqn. (21) suggests that the SINR gain is the array gain of the subarray normalized by the number of main beams. It shows that, SINR-wise, using subarrays ($\mathbf{T}_n$) with appropriate size so that a single wide beam ($r_n = 1$) is generated to obtain its array gain has little difference than using a full array with multiple narrow beams ($r > 1$). However, the advantage of using subarrays is evident when finite sample size is used because of the better estimation accuracy.

4.3 SINR gain with finite samples K

In this subsection we will examine the effect of the reduced rank processing with finite samples by comparing the performance between the coherent subarray detector and the full array detector. We consider the case where the training data include the signal component. For simplicity, we assume that the signal source is a point source, Hence, for the full array detector, by citing the results from Mati, we have

$$\mathrm{SINR}_{full} \approx \frac{E_s}{[\mathbf{a}^H\mathbf{R}^{-1}\mathbf{a}]^{-1} + \frac{P-1}{K}E_s} \tag{22}$$

Without too much difficulty and assuming that $\mathbf{T}$ is a relatively constant matrix independent of particular set of data, we have the following SINR for the coherent subarray detector

$$\mathrm{SINR}_{sub} \approx \frac{E_s}{[\mathbf{a}^H\mathbf{T}(\mathbf{T}^H\mathbf{R}\mathbf{T})^{-1}\mathbf{T}^H\mathbf{a}]^{-1} + \frac{L-1}{K}E_s} \tag{23}$$

Thus the SINR gain factor ρ_K in this case is defined as

$$\begin{aligned}
\rho_K &= \frac{\mathrm{SINR}_{sub}(K)}{\mathrm{SINR}_{full}(K)} \\
&\approx \frac{[\mathbf{a}^H\mathbf{R}^{-1}\mathbf{a}]^{-1} + \frac{P-1}{K}E_s}{[\mathbf{a}^H\mathbf{T}(\mathbf{T}^H\mathbf{R}\mathbf{T})^{-1}\mathbf{T}^H\mathbf{a}]^{-1} + \frac{L-1}{K}E_s} \\
&\approx \frac{\mathbf{a}^H\mathbf{T}(\mathbf{T}^H\mathbf{R}\mathbf{T})^{-1}\mathbf{T}^H\mathbf{a}}{\mathbf{a}^H\mathbf{R}^{-1}\mathbf{a}} \frac{1 + \mathbf{a}^H\mathbf{R}^{-1}\mathbf{a}\frac{E_s(P-1)}{K}}{1 + \mathbf{a}^H\mathbf{T}(\mathbf{T}^H\mathbf{R}\mathbf{T})^{-1}\mathbf{T}^H\mathbf{a}E_s\frac{L-1}{K}}
\end{aligned} \tag{24}$$

We notice that the left term of the last line of Eqn. (24) represents the asymptotic overall performance loss of coherent subarray processing relative to the full array processing, while the right terms stands for the performance gain of coherent subarray processing due to finite sample size effect. The two opposite factors decide the overall performance.

5 Monte Carlo Simulations

In this section we evaluate the performance of subarray detectors described above, using Monte Carlo simulations. The simulation is based upon a general linear array with $P = 40$ sensors. Fig. 2 depicts the SINR and deflection along two stages of subarray processing with different subarray size. When $M > 8$, the detector $\mathbf{T}$ shows little performance loss in deflection. When $M < 8$, the drop-off gets larger due to partial cancellation of interference, even the second stage processing will not gain back the loss. Fig. 3 shows that the SINR of the coherent subarray detector is very close to that of the full array detector as $M > q$. Also the deflection gain of coherent subarray detector relative to $\mathbf{T}_n$ is bounded by 7 dB ($N = 5$) and 3.5 dB. While

the deflection gain of the noncoherent subarray detector remains at 3.5 dB. The comparison of performance with finite samples is also depicted in Figure 4 and 5. Examination of these figures shows a significant detection improvement with short data record via subarray processing.

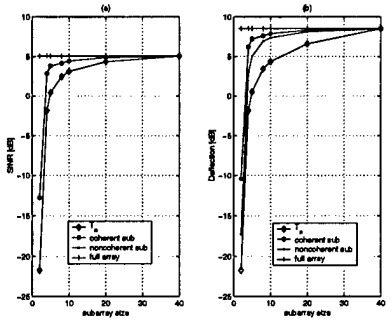

Figure 2. SINR and deflection of subarray and full array processing vs. subarray size. Signal angular spread $\beta_{sig} = 4$ BW, $\beta_{int} = 8$ BW. The array is a ULA, the SNR $= -4$ dB, SIR $= -30$ dB. The interference rank is $q = 8$.

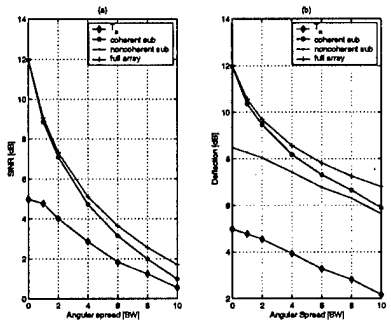

Figure 3. Comparison of SINR and deflection of subarray and full array processing as the angular spread changes. The array is a ULA, the SNR $= -4$ dB, SIR $= -30$ dB. $M = 8$.

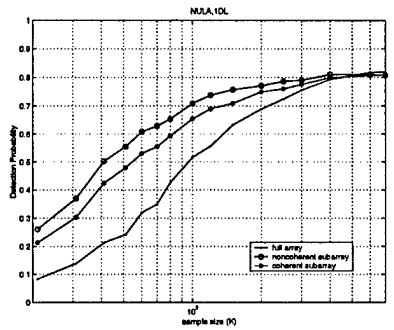

Figure 4. Comparison of detection performance for subarray detectors and full array detector as training sample size K changes. The array is a NULA, the SNR $= 0$ dB, SIR $= -30$ dB, loading level is noise level, training sample data include signal components

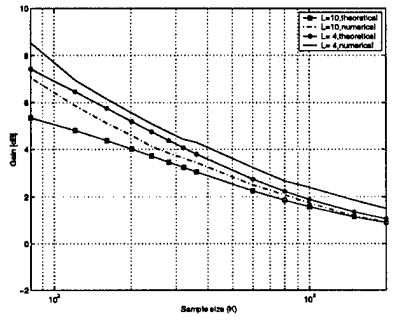

Figure 5. The SINR gain of the subarray processing relative to the full array processing as sample size K changes. The training data include signal components. The source is a point source in this case.

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CFAR Adaptive Detection Of Distributed Signals

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Abstract

We consider the problem of detecting distributed signals described by the second order Gaussian models in the presence of noise whose covariance structure and level are both unknown. Such a detection problem is often called the "Gauss-Gauss" problem in that both the signal and the noise are assumed to have Gaussian distributions. We derive an adaptive detector for the Second Order Gaussian (SOG) model signals based on multiple observations. The detector is derived in a manner similar to that of the generalized likelihood ratio test (GLRT), but the unknown covariance structure is replaced by sample covariance matrix based on training data. The proposed detector is a constant false alarm rate (CFAR) detector. We give an approximate closed form of the probability of detection and false alarm, and compute performance curves.

1 Introduction

Detecting the signals, either deterministic or random, in the presence of interference or noise is a common problem in multi-dimensional signal processing. In array processing, the problem under study concerns the extraction of information from measurements using an array of sensors. Given the observations of the sensor outputs, the objective is to estimate the unknown parameters associated with the wavefronts and to decide whether the measurements consist of noise only (the null hypothesis) or of a number of sources corrupted by noise (the alternative hypothesis). For a large class of detection problems, the signal of interest is modelled as Second Order Gaussian (SOG) model, that is, each observation of the signal waveform $\mathbf{s}$ can be modelled as some linear combination of p basis vectors or "modes" where p is the subspace rank. The signal $\mathbf{s}$ obeys the linear subspace model $\mathbf{s} = \mathbf{Ab}$, and $\mathbf{b}$ has a Gaussian distribution. Such a problem is usually called the "Gauss-Gauss" problem in that both the signal and the noise are assumed to have Gaussian distributions. One example of this type of model is the passive sonar application in which the signal of interest is the acoustic source generated by, for instance, ships or submarines. The acoustic signal is random in nature and is spread in space [4].

We study in this paper the problem of adaptive detecting a distributed signal based upon multiple observations. The detector is derived based upon the GLRT assuming the covariance is known. After the test statistic is derived, the maximum likelihood estimate of the covariance matrix based on the secondary data is inserted in place of the known covariance. The resulting test statistic is a CFAR detector.

Detection of signals on the basis of multiple observations is of interest in many applications, such as adaptive radar [2]. In general, the detector based on multiple observations is not a straightforward extension of that based on a single observation. The proposed CFAR adaptive detectors for the SOG model signals based upon multiple observations, to best of our knowledge, appear to be new.

One important piece of work in this paper is that, for the SOG model, we are able to derive a closed form expression of the *approximate* distribution function of the detection statistic for the false alarm rate (P_{FA}) and the detection probability (P_D). The form of approximation, verified by Monte Carlo simulation, appears to have quite good accuracy. This approximate closed form simplifies the computation of the detection threshold. In addition, the closed form reveals that the test statistic derived for the second order Gaussian has a central $F-$ distribution.

By comparing the detectors for the SOG model and the First Order Gaussian (FOG) model of which the $\mathbf{b}$ is a deterministic unknown vector, we observe that the derived CFAR detectors for both models have the *same form* under either single or multiple observations. In fact, the detection problem for the FOG model on the basis of a single observation is proven to be a GLRT detector (see Kraut [6]). This observation suggests that the derived adaptive detectors for SOG model and FOG model may be the true GLRT detector although have not been proven mathematically. Our results extend the results on non-adaptive matched subspace detectors (see e.g. [8]) to adaptive subspace detectors and expand the range of applications of adaptive subspace detectors for the FOG model to that for the SOG model.

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2 Problem Formulation

Let $\{\mathbf{x}_k\} \in \mathcal{C}^P$ be a sequence of statistically independent, stationary, complex Gaussian distributed sensor data vectors for $k \in \{1, \cdots, K\}$. In an array processing application, $\mathbf{x}_k$ typically represents a snapshot (sample of the sensor outputs at time k) collected from an array of P sensors, and it is assumed that $\{\mathbf{x}_k\}$ is the superposition of the signal of interest and the noise. We define the random data matrix $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_K] \in \mathcal{C}^{P \times K}$ as a concatenation of all available data. We consider the data $\{\mathbf{x}_k\}$ as the primary data for detection.

The general signal model we consider in this paper is

$$\mathbf{x}_k = \mathbf{A}\mathbf{b}_k + \mathbf{d}_k \in \mathcal{C}^P \tag{1}$$

where $\mathbf{A} \in \mathcal{C}^{P \times p}$ is the known signal subspace depending on source direction ϕ. $\mathbf{b}_k \in \mathcal{C}^p$ is distributed as $\mathcal{N}(\mathbf{0}, \mathbf{R}_{\mathbf{bb}})$. $\mathbf{R}_{\mathbf{bb}}$ is positive definite matrix of dimension $p \times p$. In other words, the signal covariance matrix is given as

$$\mathbf{R}_s = \mathbf{A}\mathbf{R}_{\mathbf{bb}}\mathbf{A}^H \tag{2}$$

For the distributed signals, $\mathbf{R}_s$ is described by signal angular spread β, nominal direction of arrival ϕ and the spatial energy distribution $P(\phi, \beta)$ (see [3]),

$$\mathbf{R}_s(\phi, \beta) = \int_{-\pi}^{\pi} P(\theta; \phi, \beta)\mathbf{a}(\theta)\mathbf{a}(\theta)^H d\theta \tag{3}$$

$\mathbf{d}_k$ is the noise data vector and is distributed as $\mathbf{d}_k \sim \mathcal{N}(\mathbf{0}, \alpha\mathbf{R})$, where α denotes the noise level.

The CFAR requirement of a candidate algorithm ensures that the false alarm rate may be prescribed at a given value independent of the correlation properties between the various noise components. In the adaptive detection literature (see, e.g.,[10]), "CFAR" is with respect to the noise covariance matrix $\mathbf{R}$, assumed to be uniform over test and training data. However, if we allow the noise level to vary between training and test data with covariance matrix $\alpha\mathbf{R}$ and $\mathbf{R}$, respectively, we then mean "CFAR" with respect to both the shared noise covariance matrix structure $\mathbf{R}$ and independent scaling α of the noise in the test data. This generalizes the meaning of "CFAR" in both the nonadaptive and adaptive detection literature, where "CFAR" is respect to a shared covariance matrix or presumed gain factor between test data and training data, respectively [7].

Our goal is to determine the existence of a signal in the received data matrix $\mathbf{X}$. Posing the problem as a hypothesis test, we let the null hypothesis be that the data is signal free and the alternative hypothesis be that the data contains a signal. We also assume that one has access to secondary data, or the "signal free" data $\{\mathbf{y}_m, m = 1, \cdots, M | \mathbf{y}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{R})\}$. Hence, the SOG detection problem we consider in the paper can be formulated as

$$H_0 : \quad \begin{cases} \mathbf{x}_k \sim \mathcal{N}(0, \alpha\mathbf{R}), & k = 1, \cdots, K \\ \mathbf{y}_m \sim \mathcal{N}(0, \mathbf{R}), & m = 1, \cdots, M \end{cases} \tag{4}$$

and

$$H_1 : \quad \begin{cases} \mathbf{x}_k \sim \mathcal{N}(0, \mathbf{R}_s + \alpha\mathbf{R}), & k = 1, \cdots, K \\ \mathbf{y}_m \sim \mathcal{N}(0, \mathbf{R}), & m = 1, \cdots, M \end{cases} \tag{5}$$

Notice that α is the scale factor accounting for the power mismatch between the primary and secondary data. The arbitrary scaling between the test data and training data is of significant in practical situation. $\alpha = 1$ can be considered as an idealized condition. In fact, we hope that the false alarm rate is insensitive to α when it deviates from unity [6]. The number of primary data vectors K could be arbitrary, we are interested in the case of single observation detection ($K = 1$) and multiple observation detection ($K > 1$).

3 Adaptive Subspace Detector for SOG model with K observations

According to the Neyman-Pearson criterion, the optimum solution to the above hypothesis testing problem (4) and (5) is the likelihood ratio test. However, for the case under consideration, it cannot be employed since the total ignorance of the parameters

$$\mathbf{R}, \mathbf{R}_{\mathbf{bb}}, \alpha \tag{6}$$

is assumed. A possible way to cope with the aforementioned *a priori* uncertainty is to resort to the GLRT, which is tantamount to replacing the unknown parameters by their maximum likelihood estimates under each hypothesis. In other words, the GLRT is to be derived from

$$L(\mathbf{X}, \mathbf{Y}) = \frac{\max_{(\mathbf{R}, \mathbf{R}_{\mathbf{bb}}, \alpha)} f_1(\mathbf{X}, \mathbf{Y})}{\max_{(\mathbf{R}, \alpha)} f_0(\mathbf{X}, \mathbf{Y})} \tag{7}$$

where $f_{0,1}(\mathbf{X}, \mathbf{Y})$ is the joint densities under H_0 and H_1.

Unfortunately, it has been well known that when both the interference and signal covariance are unknown, the GLRT detector is intractable (see e.g.[1],[9]). In order to circumvent this drawback we resort to an ad hoc two-step design procedure: first we derive the GLRT detector assuming the covariance is known. After the test statistic is derived, the maximum likelihood estimate of the covariance matrix based upon the secondary data is inserted in place of the known covariance matrix. The resulting detector has the desirable CFAR property.

3.1 Derivation of CFAR Adaptive Detector

Before we proceed, we define a pre-whitening filter $\mathbf{R}^{-\frac{1}{2}}$, where $\mathbf{R}^{1/2}\mathbf{R}^{1/2}$ is the Cholesky factorization of $\mathbf{R}$. Then the pre-whitened measurement $\mathbf{z}$ is given as follows,

$$\mathbf{z} = \mathbf{R}^{-\frac{1}{2}}\mathbf{x} \tag{8}$$

where the disturbance becomes white noise $\mathbf{R}^{-\frac{1}{2}}\mathbf{d} \sim \mathcal{N}(\mathbf{0}, \alpha\mathbf{I})$. In reality, we will replace this filter by its estimate from the training data, by means of either maximum likelihood estimate or some forms of reduced rank

processing. Let us define several notations before we begin the derivation of the GLRT detector. We use the notation $\tilde{\mathbf{R}}_s = \mathbf{R}^{-1/2}\mathbf{R}_s\mathbf{R}^{-1/2}$ to represent the whitened signal covariance matrix. Thus the detection problem can be formulated as the following simple hypothesis test,

$$\begin{aligned} H_0 &: \quad \mathbf{z}_k \sim \mathcal{N}(0, \alpha\mathbf{I}) \\ H_1 &: \quad \mathbf{z}_k \sim \mathcal{N}(0, \tilde{\mathbf{R}}_s + \alpha\mathbf{I}) \end{aligned} \tag{9}$$

The proposed CFAR subspace detectors are summarized in Table 1. With $K = 1$ single data snapshot, the GLRT is given as (see Appendix I.B in [4] for details)

$$L(\mathbf{z}) = \frac{\mathbf{z}^H\mathbf{P}_\mathbf{H}\mathbf{z}}{\mathbf{z}^H\mathbf{P}_\mathbf{H}^\perp\mathbf{z}} \tag{10}$$

where $\mathbf{H} = \mathbf{R}^{-\frac{1}{2}}\mathbf{A}$, $\mathbf{P}_\mathbf{H}$ is the projection operator on the subspace $\mathbf{H}$ while $\mathbf{P}_\mathbf{H}^\perp = \mathbf{I} - \mathbf{P}_\mathbf{H}$ is its nulling projector. This detector is also called *generalized energy detector* in that the detection statistic is basically the ratio of signal power projected onto the signal subspace to the noise power projected onto its null space. Taking into account the pre-whitening filter, we obtain the following test

$$L(\mathbf{x}) = \frac{\mathbf{x}^H\mathbf{R}^{-\frac{1}{2}}\mathbf{P}_{\mathbf{R}^{-\frac{1}{2}}\mathbf{A}}\mathbf{R}^{-\frac{1}{2}}\mathbf{x}}{\mathbf{x}^H\mathbf{R}^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P}_{\mathbf{R}^{-\frac{1}{2}}\mathbf{A}})\mathbf{R}^{-\frac{1}{2}}\mathbf{x}} \tag{11}$$

With $K > 1$ data snapshots, the proposed CFAR detector is given as below (see Appendix I.C in [4]),

$$L(\mathbf{Z}) = \frac{\sum \mathbf{z}_k^H\mathbf{P}_\mathbf{H}\mathbf{z}_k}{\sum \mathbf{z}_k^H\mathbf{P}_\mathbf{H}^\perp\mathbf{z}_k} \tag{12}$$

Or taking into account the pre-whitening filter

$$L(\mathbf{X}) = \frac{\sum \mathbf{x}_k^H\mathbf{R}^{-\frac{1}{2}}\mathbf{P}_{\mathbf{R}^{-\frac{1}{2}}\mathbf{A}}\mathbf{R}^{-\frac{1}{2}}\mathbf{x}_k}{\sum \mathbf{x}_k^H\mathbf{R}^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P}_{\mathbf{R}^{-\frac{1}{2}}\mathbf{A}})\mathbf{R}^{-\frac{1}{2}}\mathbf{x}_k} \tag{13}$$

It should be noted that this is *not* a GLRT detector in a strict sense. In fact, the GLRT when $K > p$ is given by (see Appendix I.A in [4])

$$L(\mathbf{Z}) = \frac{(\mathrm{tr}[\mathbf{R}_z])^P}{(\mathrm{tr}[\mathbf{P}_\mathbf{H}^\perp\mathbf{R}_z])^{P-p}}|\tilde{\mathbf{R}}_z|^{-1} \tag{14}$$

where $\mathbf{R}_z = \sum_{k=1}^K \frac{\mathbf{z}_k\mathbf{z}_k^H}{K}$, $\tilde{\mathbf{R}}_z = (\mathbf{H}^H\mathbf{H})^{-\frac{1}{2}}\mathbf{H}^H\mathbf{R}_z\mathbf{H}(\mathbf{H}^H\mathbf{H})^{-\frac{1}{2}}$. When $p = 1$, it is easy to see that the detector (12) and (14) are equivalent. However, the proposed CFAR detector (12) generally outperforms the GLRT (14). In fact, it can be shown that the proposed CFAR detector can be obtained through a maximization based on a loosened condition (see Appendix I.C in [4]). We also see that for the SOG signal of rank p, the GLRT detectors take different forms with different K. This is because the detection statistic shall depend on signal power distribution along each dimension for the SOG model. When there are $K > p$ data snapshots, the total signal power can be resolved onto each dimension of the signal subspace, and the signal-to-noise powers along each dimension of the signal subspace are accounted for. With only $K = 1$ data snapshot and without a priori knowledge of $\mathbf{R}_{\mathbf{bb}}$, resolving signal onto each dimension of the signal subspace is intractable, hence only the total signal power projected onto the whole signal subspace is accounted for.

Table 1. CFAR Subspace Detectors for SOG model

	$\mathbf{R}$ is known	$\mathbf{R}$ is unknown
$K = 1$	$L(\mathbf{z}) = \frac{\mathbf{z}^H\mathbf{P}_\mathbf{H}\mathbf{z}}{\mathbf{z}^H\mathbf{P}_\mathbf{H}^\perp\mathbf{z}}$ Eq.(10), (GLRT)	$L(\mathbf{z}) = \frac{\mathbf{z}^H\mathbf{P}_{\hat{\mathbf{H}}}\mathbf{z}}{\mathbf{z}^H\mathbf{P}_{\hat{\mathbf{H}}}^\perp\mathbf{z}}$ (GLRT)
$K > 1$	$L(\mathbf{Z}) = \frac{\sum \mathbf{z}_k^H\mathbf{P}_\mathbf{H}\mathbf{z}_k}{\sum \mathbf{z}_k^H\mathbf{P}_\mathbf{H}^\perp\mathbf{z}_k}$ Eq. (12)	$L(\mathbf{Z}) = \frac{\sum \mathbf{z}_k^H\mathbf{P}_{\hat{\mathbf{H}}}\mathbf{z}}{\sum \mathbf{z}_k^H\mathbf{P}_{\hat{\mathbf{H}}}^\perp\mathbf{z}_k}$
$K > p$	$L(\mathbf{Z}) = \frac{(\mathrm{tr}[\mathbf{R}_z])^P\lvert\tilde{\mathbf{R}}_z\rvert^{-1}}{(\mathrm{tr}[\mathbf{P}_\mathbf{H}^\perp\mathbf{R}_z])^{P-p}}$ Eq. (14), (GLRT)	$L(\mathbf{Z}) = \frac{(\mathrm{tr}[\mathbf{R}_z])^P\lvert\widehat{\tilde{\mathbf{R}}_z}\rvert^{-1}}{(\mathrm{tr}[\mathbf{P}_{\hat{\mathbf{H}}}^\perp\mathbf{R}_z])^{P-p}}$

If the noise covariance matrix were known, then we would use the detector described by (11) and (13). In general, the covariance matrix is unknown and must be estimated by using adaptive techniques. In this paper, we use an *ad hoc* procedure by substituting the unknown covariance matrix with its maximum likelihood estimate based on the secondary data. The resulting detector when $K = 1$ is given as follows,

$$L(\mathbf{x}, \mathbf{Y}) = \frac{\mathbf{x}^H\mathbf{S}^{-\frac{1}{2}}\mathbf{P}_{\mathbf{S}^{-\frac{1}{2}}\mathbf{A}}\mathbf{S}^{-\frac{1}{2}}\mathbf{x}}{\mathbf{x}^H\mathbf{S}^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P}_{\mathbf{S}^{-\frac{1}{2}}\mathbf{A}})\mathbf{S}^{-\frac{1}{2}}\mathbf{x}} \tag{15}$$

where $\mathbf{S}$ is the maximum likelihood estimate of the noise covariance matrix from the secondary data, i.e.,

$$\mathbf{S} = \frac{1}{M}\sum_{m=1}^M \mathbf{y}_m\mathbf{y}_m^H \tag{16}$$

and $\hat{\mathbf{H}} = \mathbf{S}^{-\frac{1}{2}}\mathbf{A}$, $\hat{\tilde{\mathbf{R}}}_z = (\hat{\mathbf{H}}^H\hat{\mathbf{H}})^{-\frac{1}{2}}\hat{\mathbf{H}}^H\mathbf{R}_z\hat{\mathbf{H}}(\hat{\mathbf{H}}^H\hat{\mathbf{H}})^{-\frac{1}{2}}$. Similarly, when $K > 1$, the corresponding detector is given as follows,

$$L(\mathbf{X}, \mathbf{Y}) = \frac{\sum \mathbf{x}_k^H\mathbf{S}^{-\frac{1}{2}}\mathbf{P}_{\mathbf{S}^{-\frac{1}{2}}\mathbf{A}}\mathbf{S}^{-\frac{1}{2}}\mathbf{x}_k}{\sum \mathbf{x}_k^H\mathbf{S}^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P}_{\mathbf{S}^{-\frac{1}{2}}\mathbf{A}})\mathbf{S}^{-\frac{1}{2}}\mathbf{x}_k} \tag{17}$$

So far, we have not specified the structure of the covariance matrix $\mathbf{R}$. Certainly, if $\mathbf{R}$ is a structured covariance matrix, for instance, $\mathbf{R} = \mathbf{R}_i + \sigma^2\mathbf{I}$, where $\mathbf{R}_i$ is a low rank

matrix of rank q, we then would use ML estimate which incorporates the structure of $\mathbf{R}$. (see e.g [5]). However, the estimation of $\mathbf{R}$ when $\mathbf{R}$ has a particular structure is out of the scope of this paper. Equivalently, the CFAR detector (15) can also be written as

$$L'(\mathbf{X},\mathbf{Y}) = \frac{\sum_k \mathbf{x}_k^H \mathbf{S}^{-1/2}\mathbf{P}_{\mathbf{S}^{-1/2}\mathbf{A}}\mathbf{S}^{-1/2}\mathbf{x}_k}{\sum_k \mathbf{x}_k^H \mathbf{S}^{-1}\mathbf{x}_k} \quad (18)$$

due to the fact that $L'(\mathbf{X},\mathbf{Y})$ is a monotonic function of equation (15). or

$$L'(\mathbf{X},\mathbf{Y}) = \frac{\sum_k \mathbf{x}_k^H \mathbf{S}^{-1}\mathbf{A}(\mathbf{A}^H\mathbf{S}^{-1}\mathbf{A})^{-1}\mathbf{A}^H\mathbf{S}^{-1}\mathbf{x}_k}{\sum_k \mathbf{x}_k^H \mathbf{S}^{-1}\mathbf{x}_k} \quad (19)$$

As a special case where a single observation is used, we have

$$L(\mathbf{x},\mathbf{Y}) = \frac{\mathbf{x}^H \mathbf{S}^{-1}\mathbf{A}(\mathbf{A}^H\mathbf{S}^{-1}\mathbf{A})^{-1}\mathbf{A}^H\mathbf{S}^{-1}\mathbf{x}}{\mathbf{x}^H \mathbf{S}^{-1}\mathbf{x}} \quad (20)$$

We recognize immediately that this detector has the same form as the GLRT CFAR adaptive subspace detector for the FOG model[6].

3.2 Decision Thresholds

A closed form of the test's probability of false alarm and detection is usually difficult to obtain. However, we find that when the signal subspace $\mathbf{A}$ is known, a *approximate* closed form of the P_{FA} and P_D is achievable (see [4] for details). This *approximate* closed form reveals an insight as how the designed detector differs from the one designed for the first order Gaussian model although the two bear the same structure, that is, the test statistic for the second order Gaussian model has a central $F-$ distribution while the test statistic for the first order Gaussian model when $K = 1$ has a non-central $F-$ distribution [11]. We write $F_{(M_1,M_2)}$ to denote the central F-distribution with degrees of freedom of (M_1, M_2), the detection statistics under H_0 and H_1 are given as

$$L(\mathbf{X}; H_0) \sim \frac{p}{P-p} \times F_{(2Kp,2K(P-p))} \quad (21)$$

$$L(\mathbf{X}; H_1) \sim \frac{g_1 h_1}{g_2 h_2} \times F_{(h_1,h_2)} \quad (22)$$

where h_1, g_1, h_2, g_2 are dependent of eigenvalues of $\mathbf{R}$ and $\mathbf{A}$ (see [4] for details). Equation (21) is a function of the signal subspace rank (p), the total dimension (P) and the number of observations (K). It is independent of noise structure ($\mathbf{R}$) and level (α). This clearly indicates that the proposed detector (10) and (12) are constant false alarm rate (CFAR) detectors.

3.3 Performance Results

In this section, the computer simulation is conducted to verified the analytical result presented in the previous subsection. The detectors to be studied are given in Eqn.(10) and Eqn. (12). We choose a uniform linear array with $P = 30$ sensors. We consider the detector performance when the number of training samples is sufficient. The choice $M = 2P$ is made since this condition provides a reasonable accuracy for estimation of the covariance matrix of noise $\mathbf{R}$. We will study the performance of the detector with $K = 1$ and $K = 20$ snapshots respectively, which represents the cases of single snapshot and multiple snapshots detection. The rank of signal subspace depends on the angular spread implicitly. Its effective rank can be calculated by counting the dominant eigenvalues of the signal covariance matrix $\mathbf{R}_s$.

In Fig. 1 - 3, the detection performance vs. the SNR value for different dimension of signal subspace is depicted. It is assumed that the signal angular spread or the signal subspace is known completely. We calculate the detection probability under $K = 1$ and $K = 20$ snapshots for false alarm rate of $P_{FA} = 10^{-2}, 10^{-3}$. In these figures, the symbols denote the Monte Carlo trial results while the lines denote the theoretical results. The three trial cases show that the theoretical results match with the Monte Carlo results quite well. However, cautions should be taken when using this approximation if a precise detection probability is required. Fig. 1 and 2 show that for a single snapshot detection, there is a cross-over point on the receiver operating characteristic (ROC) curves. It demonstrates that within a certain SNR range, the increase of the dimension of subspace improves the detection performance due to a change of the shape of probability density function. However, when multiple snapshots are used for detection as is shown in figure 3, the increase of the rank of subspace reduces the detection performance. The results are consistent with the results reported in [3] and [9].

4 Conclusion

We have proposed the adaptive subspace detectors for the Second Order and the First Order Gaussian models based upon multiple observations. The proposed tests are CFAR tests. They are invariant to arbitrary scaling of the training data and the test data. With multiple observations, the GLRT test is not optimal in the Neyman-Pearson sense. The proposed CFAR test has a higher probability of detection than the GLRT for the test scenarios.

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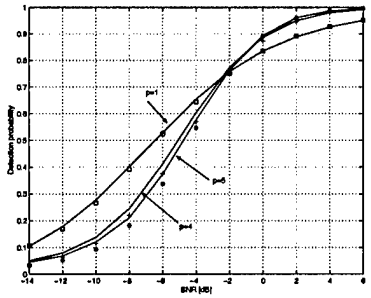

Figure 1. P_D vs. SNR for Gaussian random signals confined in subspace A with $p = [1, 4, 5]$ dimension. $P_{FA} = 0.01, K = 1$. The symbols denote the Monte Carlo trial results while the lines denote the analytical results.

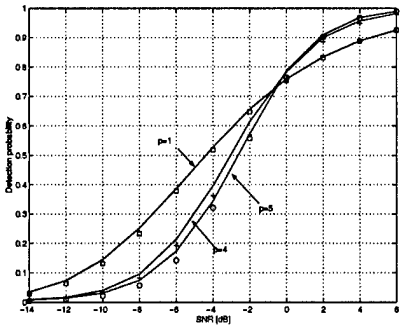

Figure 2. P_D vs. SNR for Gaussian random signals confined in subspace A with $p = [1, 4, 5]$ dimension. $P_{FA} = 0.001, K = 1$. The symbols and the lines denote the Monte Carlo trial results and analytical results respectively.

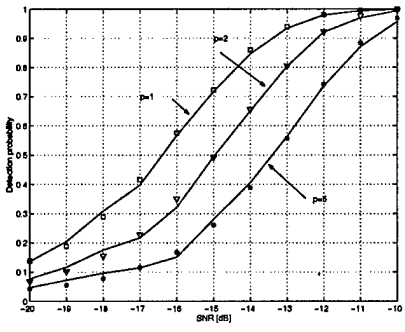

Figure 3. P_D vs. SNR for Gaussian random signals confined in subspace A with $p = [1, 2, 5]$ dimension. $P_{FA} = 0.01, K = 20$. The symbols and the lines denote the Monte Carlo trial results and the analytical results respectively.

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