# Root Locus Properties of Adaptive Beamforming and Capon Estimation for Uniform Linear Arrays

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**Abstract** In this paper we explore properties of the zeroes of the transfer function (*Z* transform) of the weight vector arising in adaptive beamforming and direction of arrival estimation (Capon) using sample matrix inversion. Our analysis sheds insights on properties of diagonal loading, as well as high-resolution properties of Capon's estimate. The analysis also provides hints at how to extend these properties to nonuniform array manifolds. Specifically we prove the following theorem.

Root locus theorem for ULAs: Let w be the clairvoyant weight vector of dimension N for a length N uniform linear array (ULA), given by $w = R^{-1}v$, where v is the steering vector to the target, and R is the (ensemble) covariance matrix. Then all *N–1* zeroes of the *Z* transform of w lie on the unit circle. (Note, since the sample matrix yields an unbiased estimator, the root locus for the adaptive beamformer has mean root loci on the unit circle as well.)

We then discuss three applications of this theorem:

(I) Diagonal loading: We show that the roots of the weight vector follow a trajectory (root locus) from the quiescent pattern to the interference angles as the interference-to-noise ratio grows. Diagonal loading can then be viewed as a regularization process that relaxes the root loci along this trajectory.

(II) Capon: The spectrum dynamic range is maximized when the zeroes are all on the unit circle; therefore, our result provides an alternative insight into the high-resolution properties of Capon estimation.

(III) Non-ULA extensions: We find in our proof that the root locus behavior results from symmetry properties of the MVDR objective function. This suggests guidelines for successful approaches to generalizing Capon estimation and diagonal loading to non-ULA settings.

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# Root Locus Properties of Adaptive Beam Forming and Capon Estimation for Uniform Linear Arrays

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# Problem and result

**Let $\vec{V}$ be a length N Vandermond steering vector,**

$$\vec{v}_{\omega_t} = [1, \exp(j\omega_t), .., \exp(j\omega_t(N-1))]^T$$

**where $\omega_t$ is the target arrival angle in normalized coordinates**

**It is well known that this vector has exactly N-1 nulls, i.e., its Z transform has all unit circle roots:**

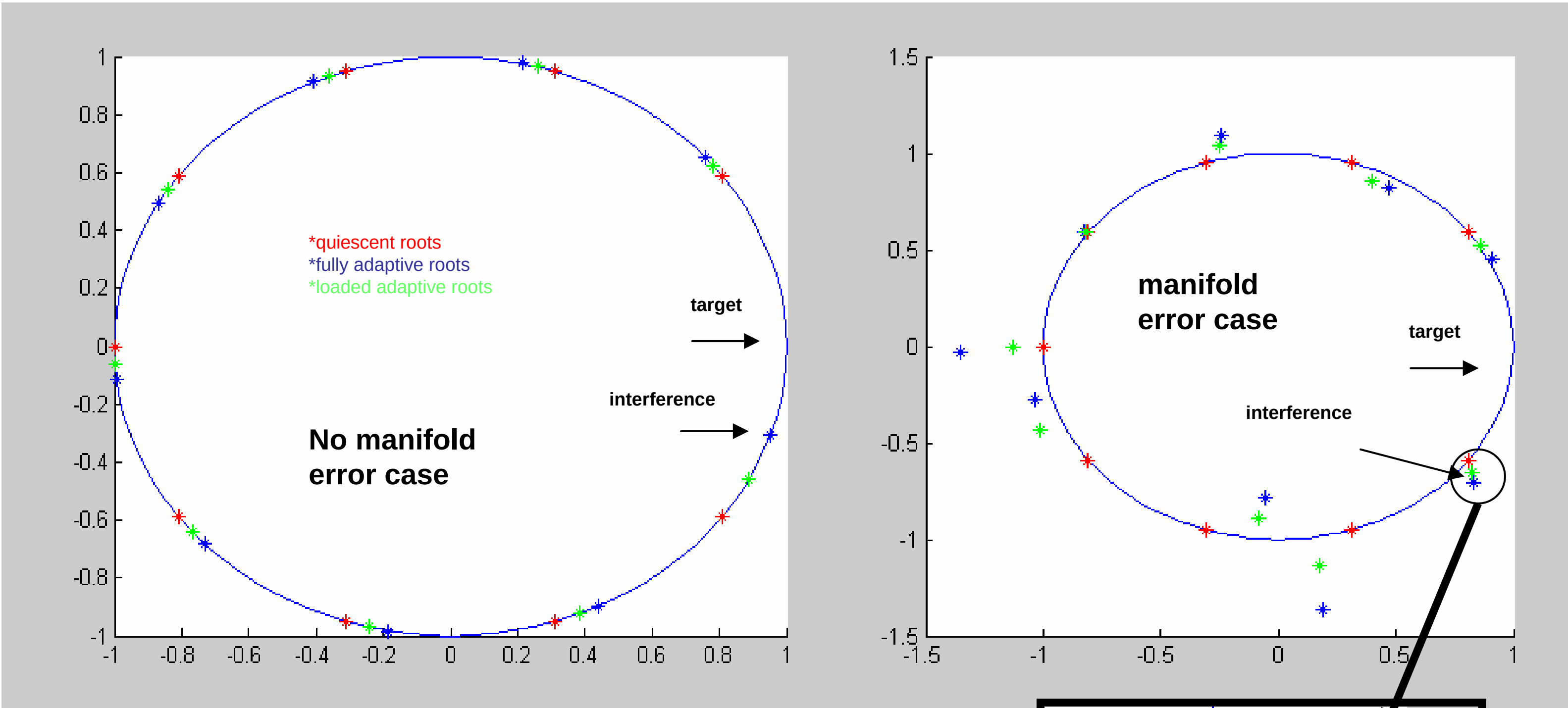
$$V(z) = \sum_{i=1}^{N} Exp(j\omega i)z^{-i} = \frac{1-(Exp(j\omega)z^{-1})^{N-1}}{1-(Exp(j\omega)z^{-1})}$$

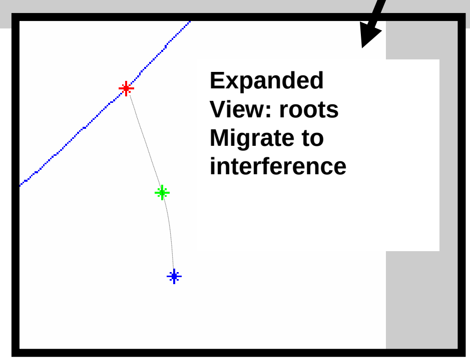
**Let R be a Toeplitz matrix (sample matrix for interference), and let us form the SMI MVDR weight vector:**

$$\vec{w} = \frac{R^{-1}\vec{v}}{\vec{v}^H R^{-1}\vec{v}}$$

Theorem: the weight vector **w** has all its roots on the unit circle

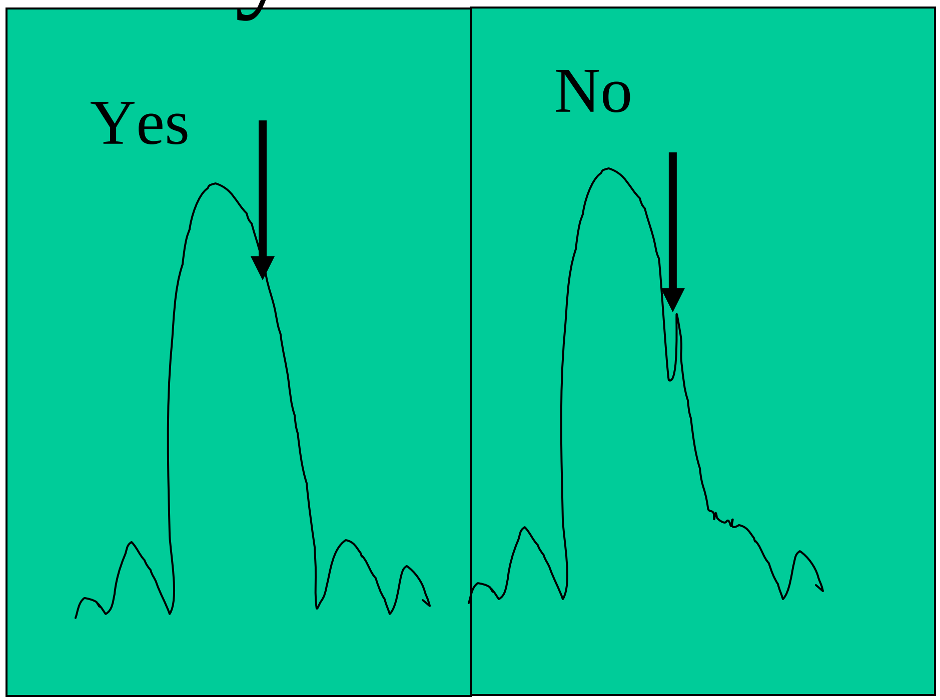
# Matlab examples




Matlab “proves” the theorem.
Matlab shows that nulls drift if array isnt linear: multipath or manifold error predictor

# Why we care




Even mainbeam nulling never leads to finite nulls!

# Proof

- Lemma: This is a surprising result!
  - Proof of Lemma: Dr Guerci and Dr Zatman think so!
  - ALL NULLS ALWAYS INFINITELY DEEP!!!!!!
- Proof of theorem: MVDR solves $\min_{\vec{v}^H \vec{w}=1} w^H Rw \equiv \mathrm{f}$

Wiener Khintchine, objective f= $\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\left|\sum_{i=1}^{N} w_i e^{-j\omega i}\right|^2 d\omega \quad S(\omega)>0\,\forall\omega$

Or f= $\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)W(z)W^*(z^{-1})d\omega, Z=\exp(j\omega)$ with $W(1)=1$

Let J be anti-identity. The JRJ=R, JRJw=v=Jv, so w=Jw
Hence roots appear as reciprocals. Are they unit modulus?

$$W(Z)=\prod_{i=1}^{n}(1-z_i z^{-1})/(1-z_i),W(1)=1$$

# Reformulation of MVDR cost function:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\left|\prod_{i-1}^{n}(1-z_i z^{-1})/(1-z_i)\right|^2 d\omega$$

If I replace a root by its inverse, constraint is preserved, and if root is NOT on the unit circle I have a different weight vector. But weight vector is unique by convexity. Hence we invoke reductio ad absurdum

QED