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Highly oscillatory quadrature and its applications

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Part I:

Quadrature of HiOsc integrals

Suppose that the **kernel**

$$K(x, \omega) \in L[[0, 1] \times \mathbb{R}_+]$$

oscillates rapidly for $\omega \gg 1$, e.g.

- The **Fourier oscillator** $e^{i\omega x}$;
- The **irregular oscillator** $e^{i\omega g(x)}$, where g is real;
- The **singular oscillator** $e^{i\omega|x-y|}/|x-y|^\alpha$, where $y \in [0, 1]$ and $\alpha < 1$;
- The **Bessel oscillator** $J_\nu(\omega x)$, $\nu \in \mathbb{R}$;
- The **Airy oscillator** $\text{Ai}(-\omega x)$.

We are interested in approximating

$$I[f] = \int_0^1 f(x) K(x, \omega) dx.$$

What's wrong with Gaussian quadrature?

Gauss–Christoffel quadrature:

$$I[f] \approx Q^{\text{GC}}[f] = \int_0^1 \phi(x) dx = \sum_{l=1}^{\nu} b_l f(c_l) K(c_l, \omega),$$

where ϕ is the $(\nu - 1)$ -degree polynomial that interpolates $f(x)K(x, \omega)$ at the **quadrature nodes**

$$c_1 < c_2 < \cdots < c_{\nu} \quad \text{in} \quad [0, 1].$$

In particular: $c_1, c_2, \dots, c_{\nu}$ zeros of $P_{\nu}(2x - 1) \Rightarrow$ the quadrature is of maximal order 2ν : this is the **Gauss–Legendre quadrature**.

Suppose for simplicity that $K(x, \omega) = e^{i\omega x}$.

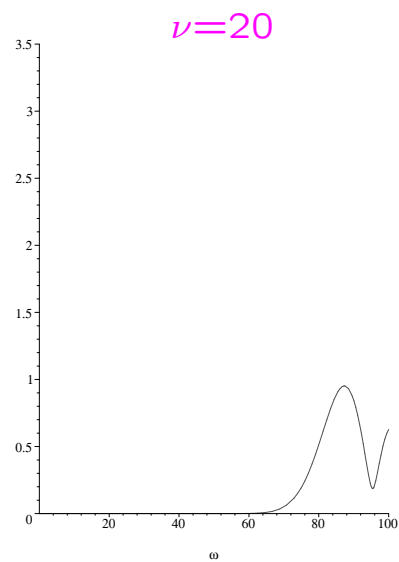
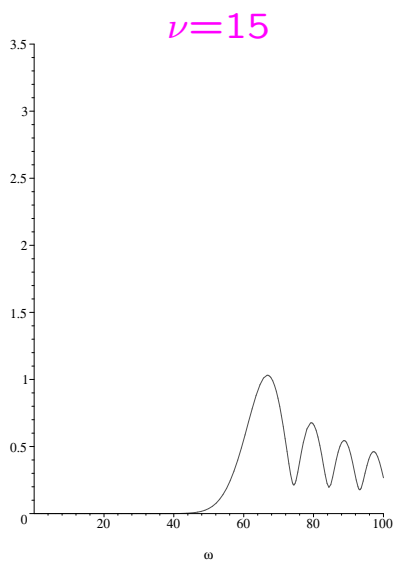
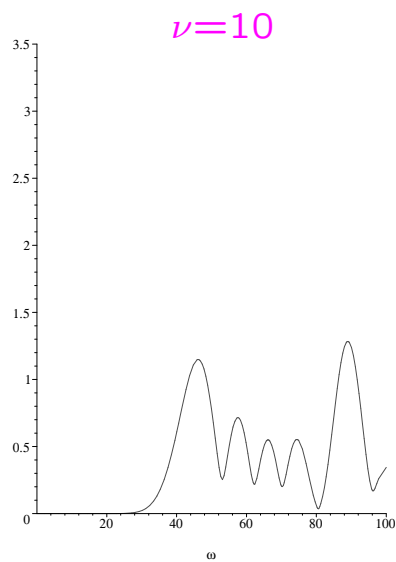
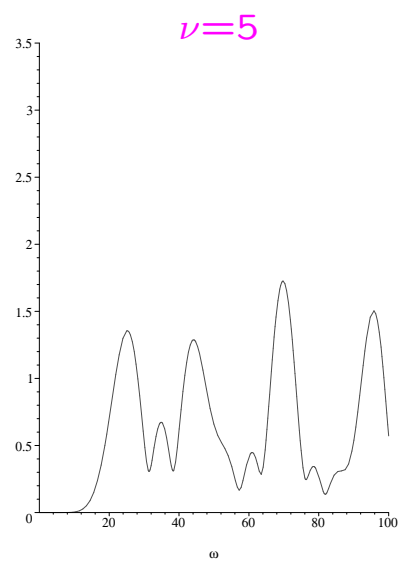
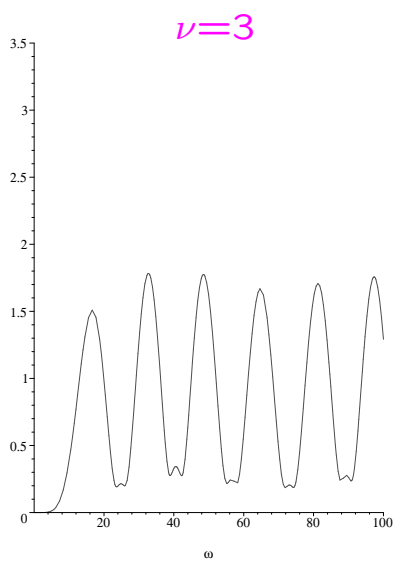
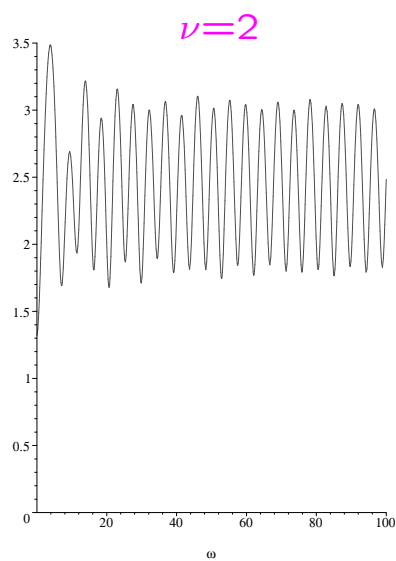
For $\omega \gg 1$ and fixed $x \neq 0$ the value $e^{i\omega x}$ is, to all intents and purposes, a **random number** on the complex unit circle $|z| = 1$. Therefore, for fixed ν

$$Q^{\text{GC}}[f] = \sum_{l=1}^{\nu} b_l f(c_l) e^{i\omega c_l} \sim \mathcal{O}(1), \quad \omega \rightarrow \infty.$$

On the other hand, **Riemann–Lebesgue** implies that

$$\lim_{\omega \rightarrow \infty} I[f] = 0, \quad f \in L_1[0, 1].$$

$$\int_0^1 e^{(1+i\omega)x} dx = \frac{e^{1+i\omega} - 1}{1 + i\omega}$$



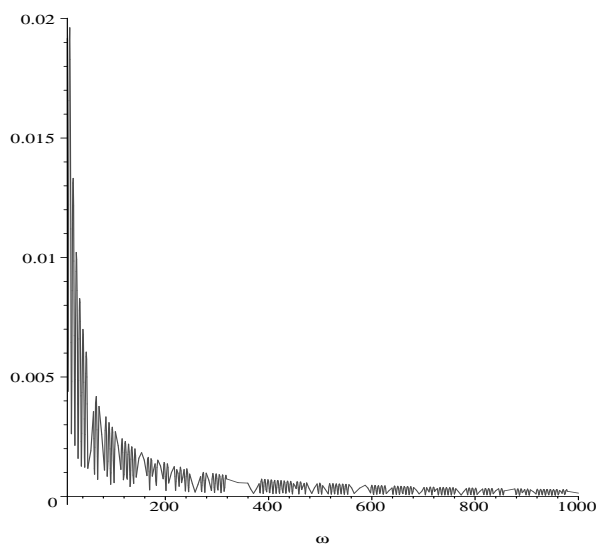
Generalising the Filon method

Instead of interpolating the integrand $f(x)K(x, \omega)$ at the quadrature nodes, we interpolate the values of $f(x)$ there by the polynomial $\tilde{\phi}$:

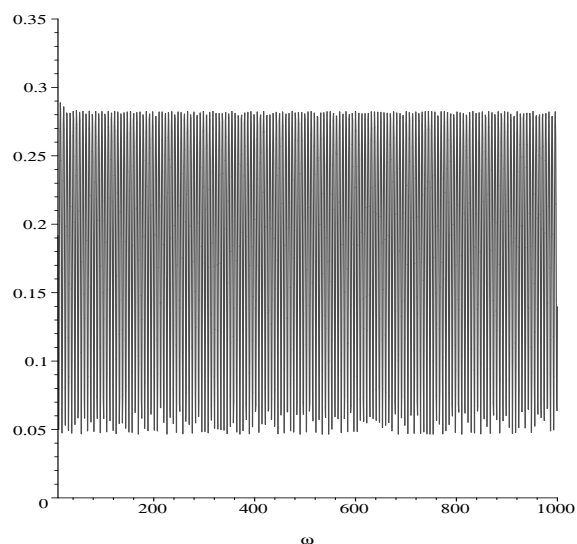
$$I[f] \approx Q^F[f] = \int_0^1 \tilde{\phi}(x) K(x, \omega) dx = \sum_{l=1}^{\nu} b_l(\omega) f(c_l).$$

Note that the weights depend on the frequency ω .

Filon–Legendre: $\nu = 2$, $c = [\frac{1}{2} - \frac{\sqrt{6}}{3}, \frac{1}{2} + \frac{\sqrt{6}}{3}]$.



$$|Q^F[e^x] - I[e^x]|$$

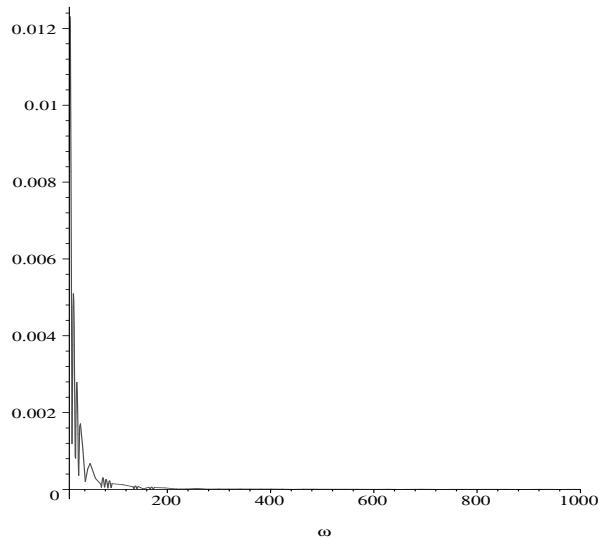


$$\omega |Q^F[e^x] - I[e^x]|$$

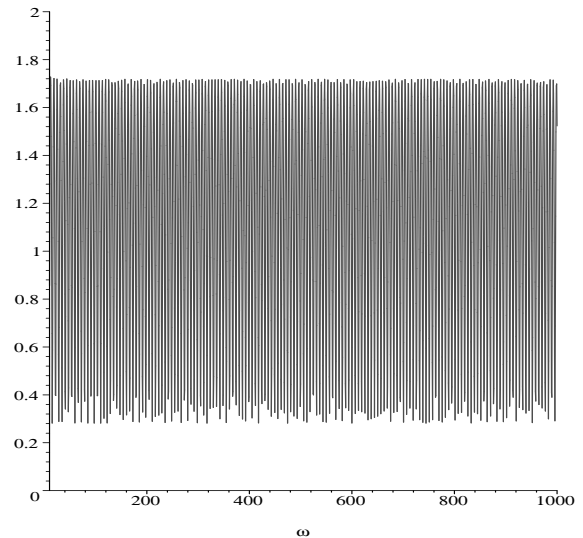
Just two quadrature points. . .

... but we can do even better!

Choose instead **Lobatto** points: $\nu = 2$, $c = [0, 1]$:



$$|Q^F[e^x] - I[e^x]|$$



$$\omega^2 |Q^F[e^x] - I[e^x]|$$

The wonder-method is

$$\int_0^1 f(x) e^{i\omega x} dx \approx b_1(\omega) f(0) + b_2(\omega) f(1),$$

where

$$b_1(\omega) = \frac{1}{-i\omega} + \frac{e^{i\omega} - 1}{(-i\omega)^2},$$
$$b_2(\omega) = -\frac{e^{i\omega}}{-i\omega} - \frac{e^{i\omega} - 1}{(-i\omega)^2}.$$

But why does it work so well?

Asymptotic expansion

Let $K(x, \omega) = e^{i\omega g(x)}$, where g is real and smooth.

In addition, we require that $g' \neq 0$ in $[0, 1]$.

Integrating by parts,

$$\begin{aligned} I[f] &= \frac{1}{i\omega} \int_0^1 \frac{f(x)}{g'(x)} \frac{de^{i\omega g(x)}}{dx} dx \\ &= \frac{1}{i\omega} \left[e^{i\omega g(1)} \frac{f(1)}{g'(1)} - e^{i\omega g(0)} \frac{f(0)}{g'(0)} \right] \\ &\quad - \frac{1}{i\omega} I[d(f/g')/dx]. \end{aligned}$$

We continue by induction. Let

$$\begin{aligned} \sigma_0(x) &= f(x), \\ \sigma_{m+1}(x) &= \frac{d}{dx} \frac{\sigma_m(x)}{g'(x)}, \quad m \in \mathbb{Z}_+. \end{aligned}$$

Then, in the limit,

$$I[f] \sim \sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[\frac{\sigma_m(0)}{g'(0)} e^{i\omega g(0)} - \frac{\sigma_m(1)}{g'(1)} e^{i\omega g(1)} \right].$$

We have

$$\begin{aligned}\sigma_0 &= f, \\ \sigma_1 &= -\frac{g''}{g'^2}f + \frac{1}{g'}f', \\ \sigma_2 &= \frac{3g''^2 - gg'''}{g'^4}f - 3\frac{g''}{g'^3}f' + \frac{1}{g'^2}f''\end{aligned}$$

and so on.

Asymptotic quadrature

Let

$$Q_s^A[f] = \sum_{m=0}^{s-1} \frac{1}{(-i\omega)^{m+1}} \left[\frac{\sigma_m(0)}{g'(0)} e^{i\omega g(0)} - \frac{\sigma_m(1)}{g'(1)} e^{i\omega g(1)} \right].$$

The method uses $s - 1$ derivatives of f and

$$Q_s^A[f] - I[f] \sim \mathcal{O}(\omega^{-s-1}), \quad \omega \rightarrow \infty.$$

For $g(x) = x$ we have

$$Q_s^A[f] = \sum_{m=0}^{s-1} \frac{1}{(-i\omega)^{m+1}} [f^{(m)}(0) - e^{i\omega} f^{(m)}(1)].$$

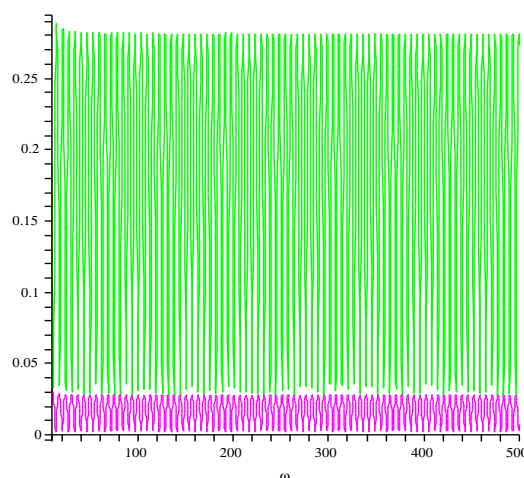
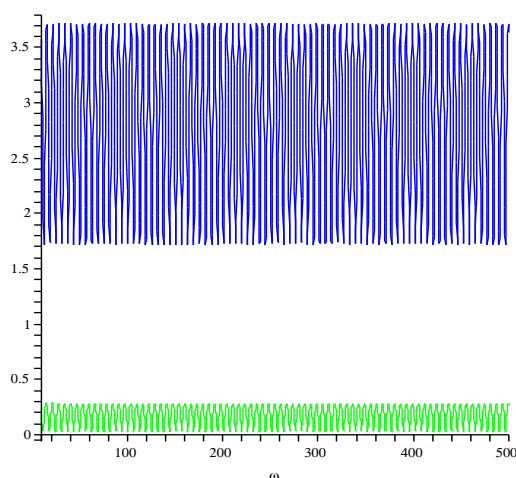
Filon-type methods

Given nodes $c_1, < \dots < c_\nu$ and $n_1, \dots, n_\nu \in \mathbb{N}$, we choose the unique polynomial $\tilde{\phi}$ of degree $\sum_l n_l - 1$ such that for all $l = 1, \dots, \nu$

$$\tilde{\phi}^{(j)}(c_l) = f^{(j)}(c_l), \quad j = 0, \dots, n_l - 1.$$

A **Filon-type method** is

$$Q^F[f] = I[\tilde{\phi}].$$



$$\omega^3 |Q^F[e^x] - I[e^x]|$$

BLUE: Q_2^A ;

GREEN: Q^F , $\nu = 2$, $n_1 = n_2 = 2$, $c_1 = 0$, $c_2 = 1$;

MAGENTA: Q^F , $\nu = 3$, $n_1 = n_3 = 2$, $n_2 = 1$,
 $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = 1$.

THEOREM If $c_1 = 0$, $c_\nu = 1$ and $n_1 = n_\nu = s$ then $Q^F[f] - I[f] \sim \mathcal{O}(\omega^{-s-1})$ when $\omega \rightarrow \infty$.

Proof Since $Q^F[f] - I[f] = I[\tilde{\phi} - f]$ and

$$\tilde{\phi}^{(j)}(0) = f^{(j)}(0), \quad \tilde{\phi}^{(j)}(1) = f^{(j)}(1),$$

for $j = 0, 1, \dots, s-1$, the proof follows from the asymptotic expansion of $I[\tilde{\phi} - f]$. □

Thus, Filon has the same asymptotic order as the asymptotic method. Typically it has a smaller error constant, which can be further decreased, by adding extra nodes in $(0, 1)$.

All this is true as long as there are no **stationary points** of the oscillator in $[0, 1]$, i.e. $g' \neq 0$ in the interval.

Stationary points

Suppose first that $g'(y) = 0$, $g''(y) \neq 0$, for some $y \in [0, 1]$ and that $g' \neq 0$ elsewhere.

Naive integration by parts breaks down, since division by g' introduces a polar singularity. An alternative is the **method of stationary phase** (Cauchy, Stokes, Kelvin), except that, while requiring nasty contour integration, it does not deliver all the information we need. Instead, let

$$\mu_0(\omega) = \int_0^1 e^{i\omega g(x)} dx$$

and

$$\begin{aligned} I[f] &= f(y)\mu_0(\omega) + \frac{1}{i\omega} \int_0^1 \frac{f(x) - f(y)}{g'(x)} \frac{d e^{i\omega g(x)}}{dx} dx \\ &= f(y)\mu_0(\omega) + \frac{1}{i\omega} \left[e^{i\omega g(1)} \frac{f(1) - f(y)}{g'(1)} \right. \\ &\quad \left. - e^{i\omega g(0)} \frac{f(0) - f(y)}{g'(0)} \right] \\ &\quad - \frac{1}{i\omega} \int_0^1 \left[\frac{d}{dx} \frac{f(x) - f(y)}{g'(x)} \right] e^{i\omega g(x)} dx. \end{aligned}$$

We continue by induction. Letting

$$\begin{aligned}\sigma_0(x) &= f(x), \\ \sigma_{m+1}(x) &= \frac{d}{dx} \frac{\sigma_m(x) - \sigma_m(y)}{g'(x)}, \quad m \in \mathbb{N},\end{aligned}$$

we have

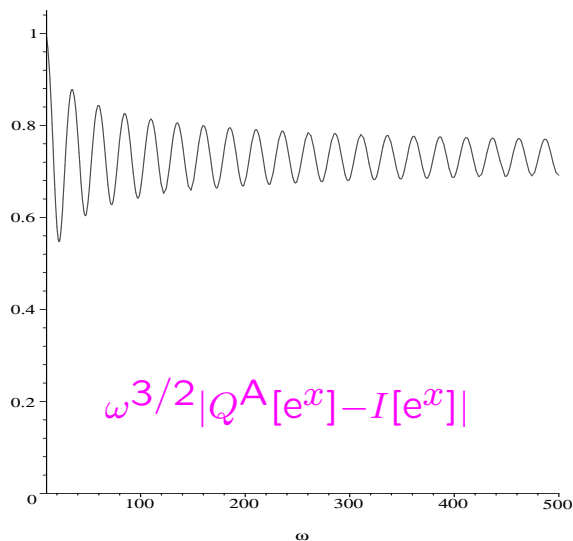
$$\begin{aligned}I[f] \sim \mu_0(\omega) \sum_{m=0}^{\infty} \frac{\sigma_m(y)}{(-i\omega)^m} \\ + \sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[e^{i\omega g(0)} \frac{\sigma_m(0) - \sigma_m(y)}{g'(0)} \right. \\ \left. - e^{i\omega g(1)} \frac{\sigma_m(1) - \sigma_m(y)}{g'(1)} \right].\end{aligned}$$

The **van der Corput lemma** $\Rightarrow \mu_0(\omega) = \mathcal{O}(\omega^{-1/2})$.
Therefore, using the first s derivatives at 0 , y and 1 gives an asymptotic method with an asymptotic error of $\mathcal{O}(\omega^{-s-\frac{3}{2}})$.

Easy generalisation to several stationary points and to $g'(y) = \dots = g^{(r)}(y) = 0$, $g^{(r+1)}(y) \neq 0$, $r \geq 1$.

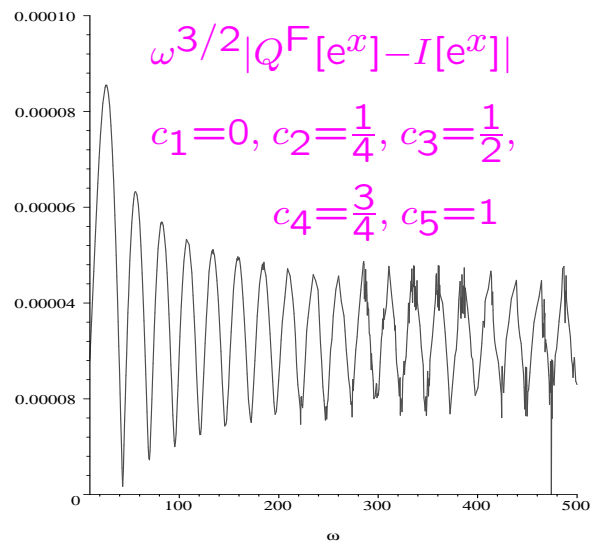
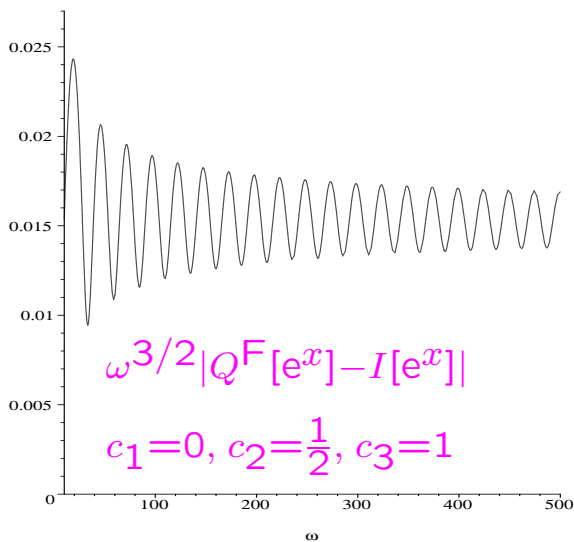
Filon again...

The Filon method can be generalised to cater for stationary points. Again, the idea is to interpolate to f and its derivatives at $\{0, y_1, y_2, \dots, y_n, 1\}$, where $y_1, \dots, y_n$ are the stationary points.



$$g(x) = x(1 - x)$$

$$y = \frac{1}{2}, \quad s = 0$$



‘Exotic’ oscillators

Most results can be extended to more ‘exotic’ oscillators, for example

$$J_\nu(\omega x) \quad \text{and} \quad \text{Ai}(-\omega x)$$

but, clearly, much remains to be done.

Computation of special functions

In the stage of tentative ideas: using Filon quadrature for fast computation of special functions (e.g. **hypergeometric** and **Bessel functions**) for large arguments, (hopefully) more precise than using standard asymptotic formulæ.

Singular integrals (Hermann Brunner, AI & SPN)

Similar techniques have been applied to the kernels

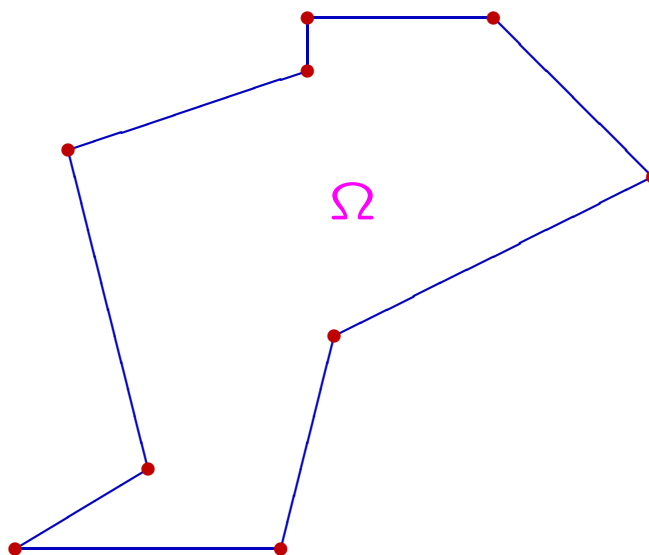
$$\frac{e^{i\omega|x-y|}}{|x-y|^\alpha} \quad \text{and} \quad e^{i\omega|x-y|} \log|x-y|,$$

where $\alpha < 1$ and $y \in [0, 1]$. The asymptotic expansion is more difficult, mainly since σ_m need not be smooth at y , but an important observation is that singularities play similar role to stationary points.

Multivariate integrals

This is perhaps the most fascinating chapter of our work!

RESULT 1 Let $\Omega \subset \mathbb{R}^d$ be a compact domain with piecewise-linear boundary,



Then, as long as $\nabla g(x) \neq 0$ in $\text{cl } \Omega$,

$$\int_{\Omega} f(x) e^{i\omega g(x)} dV \sim \sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+d}} \sum_k a_{m,k}[f],$$

where each functional $a_{m,k}$, a periodic function in ω , depends just on f and its first m derivatives at the k th vertex.

Consequently, we again have two options, both resulting in an $\mathcal{O}(\omega^{-s-d})$ quadrature: either truncate the asymptotic expansion at $m = s$ or replace f by Hermite interpolation to the function and its first s directional derivatives at the vertices.

RESULT 2 Let $\Omega \subset \mathbb{R}^d$ be a compact domain with piecewise-smooth boundary and without cusps. Suppose again that there are no critical points in $\text{cl } \Omega$, i.e. that $\nabla g \neq 0$ there. Then

$$\begin{aligned} & \int_{\Omega} f(x) e^{i\omega g(x)} dV \\ &= \frac{1}{i\omega} \int_{\partial\Omega} \frac{f(x)}{\|\nabla g(x)\|^2} n(x)^{\top} \nabla g(x) e^{i\omega g(x)} dS \\ & \quad - \frac{1}{i\omega} \int_{\Omega} \nabla^{\top} \frac{f(x)}{\|\nabla g(x)\|^2} \nabla g(x) e^{i\omega g(x)} dV, \end{aligned}$$

where n is the outward unit normal.

This can be converted into an asymptotic expansion, a **Stokes-type theorem**, “pushing” the integral from Ω to the boundary. All this can be extended to cater for **nondegenerate critical points** $x_0 \in \Omega$, where $\nabla g(x_0) = 0$, $\det \nabla \nabla^{\top} g(x_0) \neq 0$.

Part II:

Fredholm equations of the 2nd kind

(Hermann Brunner, AI & SPN)

Consider the problem

$$\mathcal{K}[\phi](y) = \lambda\phi(y) - g(y), \quad y \in [0, 1],$$

where g is given,

$$\mathcal{K}[\phi](y) = \int_0^1 \phi(x) e^{i\omega|x-y|} dx$$

and $\lambda \notin \sigma(\mathcal{K})$.

A naive approach Cover $[0, 1]$ with the grid

$$0 = y_0 < y_1 < \cdots < y_{N-1} < y_N = 1.$$

Let $\phi_k \approx \phi(y_k)$ and replace integrals with Filon. We obtain a linear system of the form

$$\sum_{l=0}^N b_{k,l}(\omega) \phi_l = \lambda\phi_k - g_k, \quad k = 0, 1, \dots, N.$$

This will not work, since the solution ϕ also oscillates with frequency ω , and this means that our asymptotics break down.

An alternative We seek complex numbers λ_m and complex-valued functions ϕ_m s.t.

$$\mathcal{K}[\phi_m] = \lambda_m \phi_m.$$

Since

$$\mathcal{K}[\phi](y) = \int_0^y \phi(x) e^{i\omega(y-x)} dx + \int_y^1 \phi(x) e^{i\omega(x-y)} dx,$$

we have

$$\begin{aligned} \frac{d\mathcal{K}[\phi](y)}{dy} &= i\omega \left[\int_0^y \phi(x) e^{i\omega(y-x)} dx \right. \\ &\quad \left. - \int_y^1 \phi(x) e^{i\omega(x-y)} dx \right], \\ \frac{d^2\mathcal{K}[\phi](y)}{dy^2} &= (i\omega)^2 \mathcal{K}[\phi](y) + 2i\omega\phi(y). \end{aligned}$$

But

$$\frac{d\mathcal{K}[\phi](y)}{dy} = \lambda\phi'(y), \quad \frac{d^2\mathcal{K}[\phi](y)}{dy^2} = \lambda\phi''(y).$$

Therefore

$$\lambda\phi'' = (i\omega)^2\lambda\phi + 2i\omega\phi.$$

Let

$$\theta(\omega) = \sqrt{\omega^2 - \frac{2i\omega}{\lambda}},$$

then

$$\phi'' + \theta^2\phi = 0.$$

Moreover,

$$\lambda\phi'(0) = \frac{d\mathcal{K}[\phi](0)}{dy} = -i\omega\lambda\phi(0),$$

$$\lambda\phi'(1) = \frac{d\mathcal{K}[\phi](1)}{dy} = i\omega\lambda\phi(1).$$

The condition at $y = 0$ results (up to normalization) in

$$\phi(x) = (\theta - \omega)e^{i\theta x} + (\theta + \omega)e^{-i\theta x}.$$

Note however that θ depends on the unknown eigenvalue λ .

Using the boundary condition at $y = 1$ we obtain

$$(\theta - \omega)^2 e^{i\theta} = (\theta + \omega)^2 e^{-i\theta}.$$

Therefore

$$(\theta - \omega) e^{\frac{1}{2}i\theta} = \pm (\theta + \omega) e^{-\frac{1}{2}i\theta},$$

Taking the **plus sign** we obtain the **transcendental equation**

$$i\theta \tan \frac{\theta}{2} = \omega,$$

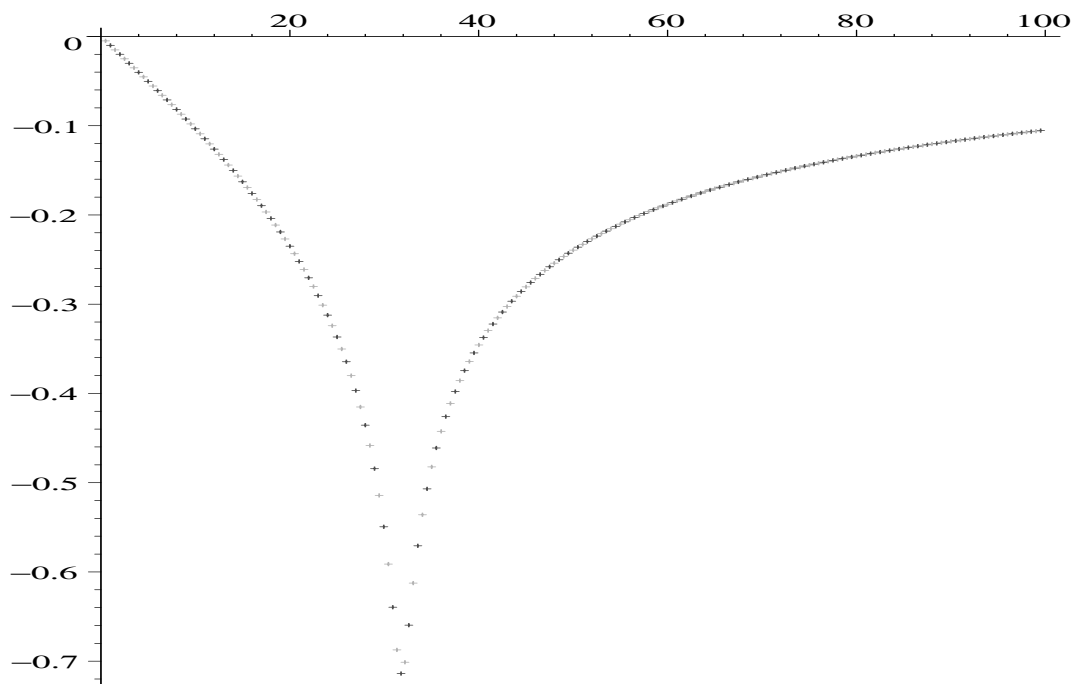
while the **minus sign** yields

$$i\theta \cot \frac{\theta}{2} = \omega.$$

The solutions of these equations **interlace**: the first has a solution for $\operatorname{Re} \theta \in (2m\pi, (2m+1)\pi)$ and the second in $\operatorname{Re} \theta \in ((2m+1)\pi, (2m+2)\pi)$. We observe that

The real part of θ behaves like $\mathcal{O}(m)$,

The imaginary part of θ is $\mathcal{O}(1)$ and small.



The values of $\theta_m/(2\pi)$ in the complex plane for $1 \leq m \leq 200$.

How is this going to help?

Let (with greater generality)

$$\mathcal{K}[f](y) = \int_0^1 f(x)K(x,y)dx, \quad y \in [0,1].$$

Then the **Hilbert–Schmidt theory** tells us that $\mathcal{K}$ has a countable number of distinct eigenvalues and eigenfunctions $\{\lambda_m, \phi_m\}$. Let

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

be the standard **real L_2 inner product**.

For $m \neq n$

$$\lambda_m \phi_m(y) = \int_0^1 \phi_m(y) K(x, y) dx$$
$$\Rightarrow \lambda_m \langle \phi_m, \phi_n \rangle = \int_0^1 \int_0^1 \phi_m(x) \phi_n(x) K(x, y) dx dy.$$

By symmetry, also

$$\lambda_n \langle \phi_m, \phi_n \rangle = \int_0^1 \int_0^1 \phi_m(x) \phi_n(x) K(x, y) dx dy$$

and we just deduce L_2 orthogonality of the eigenfunctions,

$$\langle \phi_m, \phi_n \rangle = 0, \quad m \neq n.$$

Note that $\langle \cdot, \cdot \rangle$ is **not** a **positive definite** inner product: it is complex-valued and it is entirely possible that

$$\langle f, f \rangle = 0, \quad f \neq 0.$$

However, $\langle \phi_m, \phi_m \rangle \neq 0$, and that's all we need.

A spectral method

We expand

$$f(y) = \sum_{m=0}^{\infty} f_m \phi_m(y).$$

Therefore,

$$f_m = \frac{g_m}{\lambda - \lambda_m}, \quad m \geq 0,$$

where

$$g_m = \frac{\langle g, \phi_m \rangle}{\langle \phi_m, \phi_m \rangle} = \frac{\langle g, \phi_m \rangle}{2(\lambda_m^2 - 2i\omega - \omega^2)}.$$

We thus need to compute $\langle g, \phi_m \rangle$ for a **large** number of m s. However, if

$$\theta_m = \alpha_m - i\beta_m$$

then

$$\begin{aligned} \langle g, \phi_m \rangle &= (\theta_m - \omega) \int_0^1 g(x) e^{(\beta_m + i\alpha_m)x} dx \\ &\quad + (\theta_m + \omega) \int_0^1 g(x) e^{-(\beta_m + i\alpha_m)x} dx. \end{aligned}$$

Recall: while $\alpha_m \approx 2\pi m$ is large, $|\beta_m|$ is small.
Moreover, g is **nonoscillatory**. Therefore

All the integrals can be computed very fast and accurately by either the asymptotic method or a Filon-type method.

An ongoing challenge is to generalize all this to other **Fredholm kernels**, e.g.

$$\mathcal{K}[f](y) = \int_0^1 f(x) x^\gamma e^{i\omega|x-y|} dx$$

for $\gamma > -1$ and

$$\mathcal{K}[f](y) = \int_0^1 f(x) \frac{e^{i\omega|x-y|}}{|x-y|^\gamma} dx$$

for $\gamma \in (0, 1)$.

Part III:

Solving HiOsc differential equations

We commence from the **linear ODE**

$$\mathbf{y}' = A(t)\mathbf{y}, \quad t \geq 0, \quad \mathbf{y}(0) = \mathbf{y}_0.$$

Suppose that its solution oscillates fast, e.g. that all the eigenvalues of A live in $\text{cl } \mathbb{C}_-$ and there are large eigenvalues on $i\mathbb{R}$.

Standard numerical methods perform very poorly, the reason being that the principal error term of a p th-order classical method is of the form

$$h^{p+1} \mathcal{D}_{p+1}(\mathbf{y}(t_N)),$$

where $\mathcal{D}_{p+1}$ is a linear combination of **elementary differentials** of order $p + 1$.

$\mathbf{y}(t)$ oscillates with **frequency** $\omega \Rightarrow$

$$\|\mathbf{y}^{(p+1)}(t)\| \sim \omega^{p+1} \|\mathbf{y}(t)\|,$$

hence $\|\mathcal{D}_{p+1}\|$ is **very** large!

An alternative: Change of variables

To time-step from t_N to $t_{N+1} = t_N + h$, set

$$\mathbf{y}(t) = e^{(t-t_N)\tilde{A}}\mathbf{x}(t-t_N), \quad t \geq t_N,$$

where $\tilde{A} = A(t_{N+\frac{1}{2}})$. Then

$$\mathbf{x}' = B(t)\mathbf{x}, \quad t \geq 0, \quad \mathbf{x}(0) = \mathbf{y}_N,$$

where

$$B(t) = e^{-t\tilde{A}}[A(t) - \tilde{A}]e^{t\tilde{A}}.$$

Since $e^{\pm t\tilde{A}}$ oscillates rapidly, so does $B(t)$.

We have already seen that high oscillation can be turned to our advantage. The main idea is to ‘invert’ the reason for the failure of classical methods:

Integrate, don’t differentiate!

Specifically, for an s -fold integral and $\mathcal{B}(\mathbf{x})$ a product of s terms from $\{B(x_1), \dots, B(x_s)\}$,

$$\left\| \int \cdots \int \mathcal{B}(\mathbf{x}) dx_s \cdots dx_1 \right\| \sim \mathcal{O}(\omega^{-s}).$$

The Magnus method Letting

$$x(t) = e^{\Omega(t)} x_0,$$

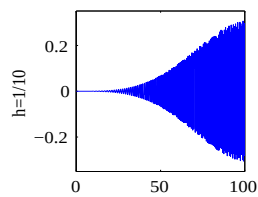
we have

$$\begin{aligned}\Omega(t) = & \int_0^t B(x) dx \\ & - \frac{1}{2} \int_0^t \int_0^{x_1} [B(x_2), B(x_1)] dx_2 dx_1 \\ & + \frac{1}{4} \int_0^t \int_0^{x_1} \int_0^{x_2} [[B(x_3), B(x_2)], B(x_1)] dx_3 dx_2 dx_1 \\ & + \frac{1}{12} \int_0^t \int_0^{x_1} \int_0^{x_1} [B(x_3), [B(x_2), B(x_1)]] dx_3 dx_2 dx_1 \\ & + \dots\end{aligned}$$

Thus, repeated integration. . . .

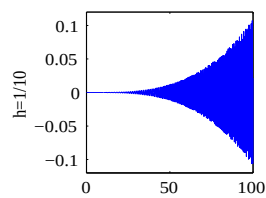
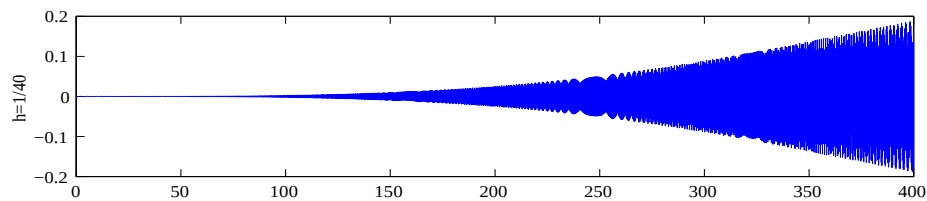
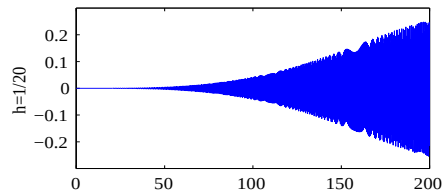
An advantage of Magnus: If $A(t)$ lives in a **Lie algebra** $\mathfrak{g}$ then $y(t)$ evolves on a homogeneous space $\mathcal{M}$, acted upon by the corresponding **Lie group** $\mathcal{G}$. Using Magnus (with or without change of variables) ensures $y_N \in \mathcal{M}$, $N \geq 0$.

An example: The Airy equation Let $y'' + ty = 0$.



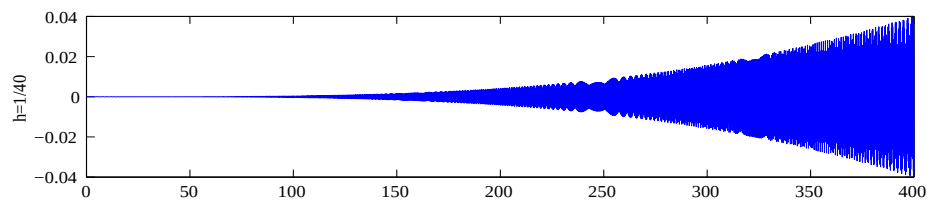
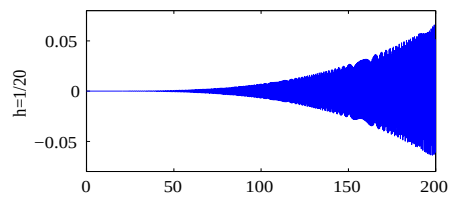
Runge-Kutta 4

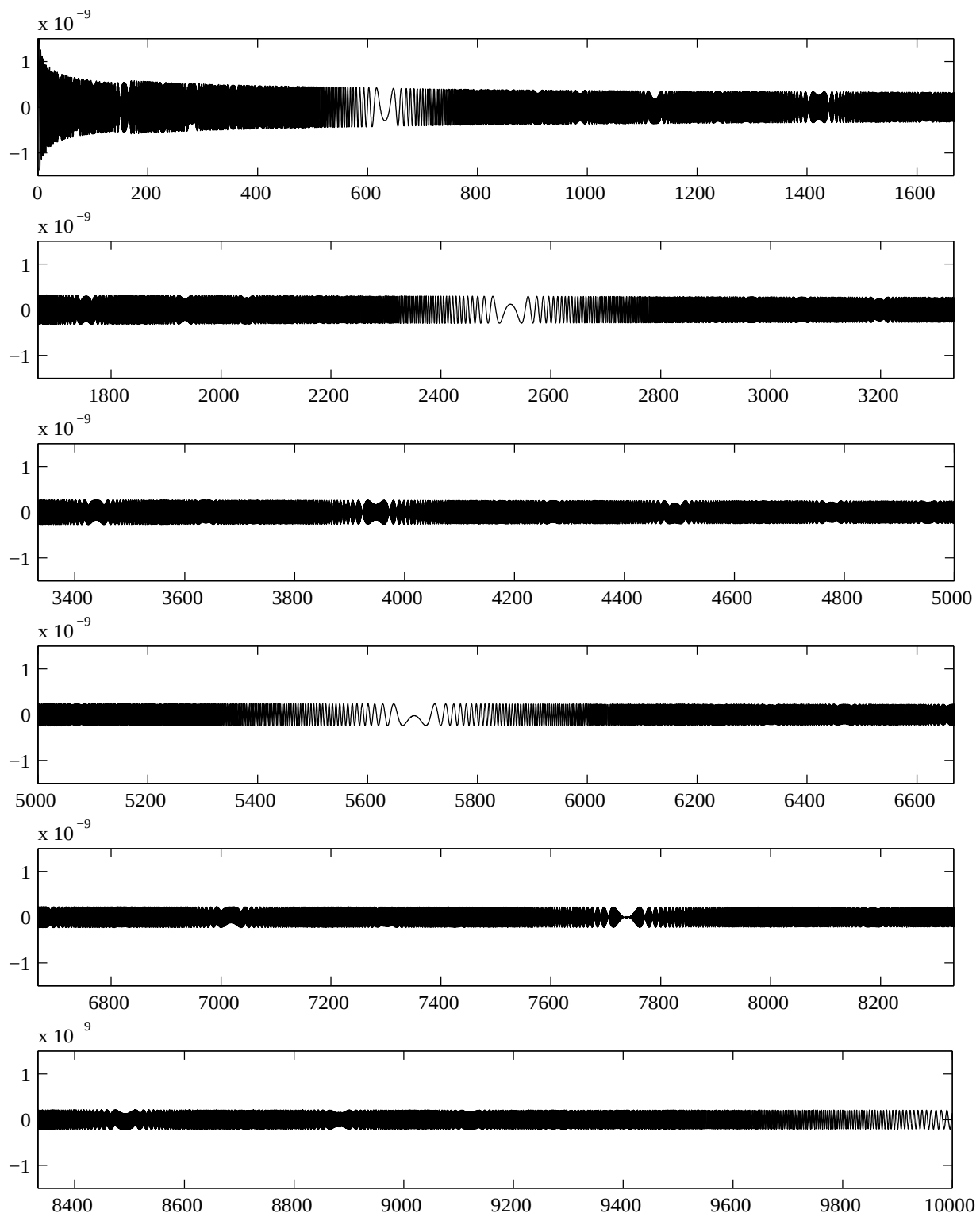
$$\|y_N - y(t_N)\| \sim \mathcal{O}(h^4 t^{13/4})$$



RK Gauss-Legendre 4

$$\|y_N - y(t_N)\| \sim \mathcal{O}(h^4 t^{13/4})$$





Modified Magnus, $h = \frac{1}{4}$, $\|y_N - y(t_N)\| \sim \mathcal{O}(h^3 t^{-1/4})$

Calculating integrals: We use Filon: all (multivariate) integrals can be calculated to high precision using just $B(0)$ and $B(h)$. As before, high oscillation helps computation!

A disadvantage of Magnus: We need to calculate **two** exponentials per step:

$$e^{\pm h\tilde{A}} \quad \text{and} \quad e^{\Omega(h)}.$$

This can be problematic when the dimension is large. **However**, while Ω is typically unstructured, this is not the case with $\tilde{A}$.

Suppose that the ODE originates in a **semidiscretized PDE**. Then often $\tilde{A}$ is **block Toeplitz** and $e^{\pm\tilde{A}}$ can be calculated very fast by **FFT**. The challenge is thus **to do away with the need for the calculation of $e^{\Omega(h)}$** .

The Neumann method To avoid the calculation of the second exponential, we abandon Magnus in favour of the **Neumann expansion**

$$\boldsymbol{x}(t) = \sum_{m=0}^{\infty} \mathcal{N}_m(t) \boldsymbol{y}_N,$$

where $\mathcal{N}_0(t) \equiv I$ and

$$\mathcal{N}_m(t) = \int_0^t \int_0^{x_1} \cdots \int_0^{x_{m-1}} B(x_1) \cdots B(x_m) dx_m \cdots dx_1.$$

Because of high oscillation, $\|\mathcal{N}_m(h)\| \sim \mathcal{O}((h/\omega)^m)$, hence **very** rapid convergence.

Multivariate integrals can be computed in a very small number of function evaluations, similarly to Magnus integrals. Again, high oscillation of B means that Filon methods are very precise.

Numerical results for Airy are virtually identical to Magnus, but the method comes into its own for HiOsc PDEs, e.g. the **Schrödinger equation**.

Nonlinear equations Suppose that

$$y' = A(t)y + g(y)$$

is highly oscillatory. Transforming as before, but letting $\tilde{A} = A(t_N)$, we have

$$x' = B(t)x + e^{-t\tilde{A}}g(t_N + t, e^{t\tilde{A}}x).$$

Let

$$\Phi' = B(t)\Phi, \quad t \geq 0, \quad \Phi(0) = I.$$

Note that we can evaluate Φ by either **Magnus** or **Neumann**. Then

$$\begin{aligned} x(t) &= \Phi(t)y_N \\ &\quad + \int_0^t \Phi(t - \xi)e^{\xi\tilde{A}}g(t_N + \xi, e^{\xi\tilde{A}}x(\xi))d\xi. \end{aligned}$$

This motivates the **waveform relaxation** approach,

$$\begin{aligned} x^{[0]}(t) &\equiv y_N, \\ x^{[m+1]}(t) &= \Phi(t)y_N \\ &\quad + \int_0^t \Phi(t - \xi)e^{\xi\tilde{A}}g(t_N + \xi, e^{\xi\tilde{A}}x^{[m]}(\xi))d\xi. \end{aligned}$$

Next steps...

- **Filon without derivatives:** Work in progress.
Letting nodes depend on ω , it is possible to obtain arbitrary degree of error attenuation without using derivatives;
- **Exotic oscillators:** For starters, how to compute $\int_0^1 f(x) \sin(\omega \sin \pi x) dx$?
- **Multivariate HiOsc integrals:** What are the implications of the Stokes-type theorem?
- **Volterra HiOsc equations:** The current approach doesn't scale up e.g. to singular kernels;
- **HiOsc PDEs:** Much further work required for specific PDEs, e.g. **Schrödinger** and **Hamilton–Jacobi**;
- **Stochastic DEs:** Perhaps...