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and control theory
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Numerical techniques for approximating the solution of matrix ODE on the general linear group

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Outline

- ❖ The matrix ODE we deal with
- ❖ Theoretical results
- ❖ Examples
- ❖ Numerical tools:
 - Substituting approach
 - Solution via Riccati equation
 - SVD approach
- ❖ Rectangular Case
- ❖ Numerical examples

The differential system

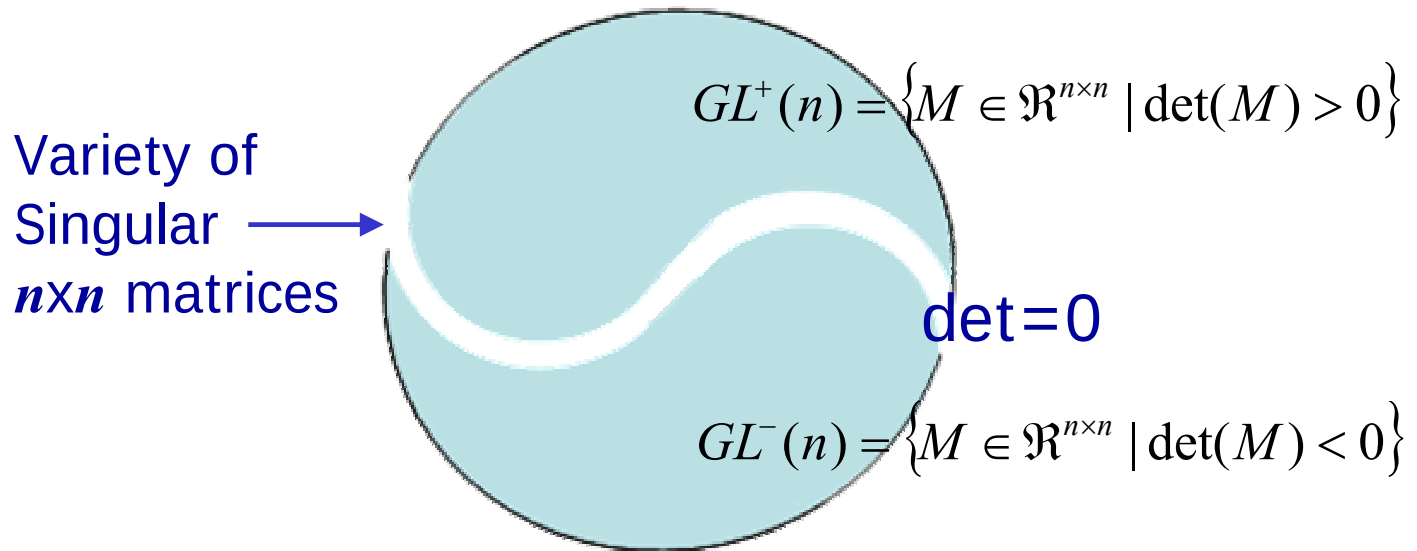
- ❖ Consider the matrix differential equation

$$\begin{aligned}\dot{Y}(t) &= Y(t)^{-T} F(Y(t), Y(t)^{-T}) \\ Y(0) &= Y_0 \in GL(n)\end{aligned}$$

- ❖ F is a continuous matrix function, globally Lipschitz on a subdomain of $GL(n)$
- ❖ the solution $Y(t)$ exists and is unique in a neighborhood $]-\tau, \tau[$ of the origin 0

The structure of $GL(n)$

- ❖ Two maximal connected and disjoint open subsets comprising $GL(n)$



Theoretical results

- ❖ The existence of the solution $Y(t)$ for all t is not guaranteed *a priori* and the presence of a finite escape time behavior is not precluded.
- ❖ The value of the escape point depends on the function F
 - If the escape point τ is finite then $Y(t)$ approaches a singular matrix as $t \rightarrow \tau$
 - if $\tau < \infty$ then $Y(t)$ exists for all $t > 0$

Theoretical results

❖ Example: ***F*** constant function with ***trace(F) = 0***

$$\dot{Y} = Y^{-T} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad Y(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

solution

$$Y(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+t} & -\sqrt{1+t} \\ \sqrt{1-t} & \sqrt{1-t} \end{bmatrix}$$

Existence interval
(-1,1)
Escape point 1

Theoretical results

- ❖ Relationship between the singular values of the solution $Y(t)$, the initial condition $Y(0)$ and the symmetric matrix function:

$$E(t) = \int_0^t [F^T(Y(s), Y^{-T}(s)) + F(Y(s), Y^{-T}(s))] ds$$



$$\sigma_{\min}(t) \geq \sigma_{\min}^0 + \lambda_{\min}(E(t))$$

Smallest Singular
Value of $Y(t)$

Smallest Singular
Value of $Y(0)$

Smallest
Eigenvalue of $E(t)$

Systems with structure

- ❖ If the matrix function F maps all matrices into the Lie algebra of skew-symmetric matrices

 $Y(t)$ belongs to the orthogonal manifold
(whenever $Y(0)$ is orthogonal)

- ❖ If $\text{diag}(F) = 0$ for all nonsingular matrices

 $\text{diag}(Y(t)^T Y(t)) = \text{diag}(Y(0)^T Y(0))$

Examples

❖ Control Theory

- **Optimal system assignment via Output Feedback Control**
- **Balanced Matrix Factorizations**
- **Balanced realizations (Isodynamical flows)**

❖ Multivariate Data Analysis

- **Weighted Oblique Procrustes problem**

❖ Inverse Eigenvalue Problem

- **Pole placement or eigenvalue assignment problem via output feedback**
- **Prescribed Entries Inverse Eigenvalue Problem**

Examples in Control Theory

❖ Output Feedback Control of linear system

- Consider the linear dynamical system defined by the triple $(A,B,C) \in \mathbb{P}^{n \times n} \times \mathbb{P}^{n \times m} \times \mathbb{P}^{p \times n}$

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

- The process of “*feeding back*” the output or the state variables in a dynamical system configuration through the input channels
- **Output Feedback:** $u(t)$ is replaced by $u(t) = Ky(t) + v(t)$

$K \in \mathbb{P}^{m \times p}$ feedback gain matrix

Examples in Control Theory

❖ Output Feedback Control of linear system

- The feedback system is

$$\begin{aligned}\dot{x}(t) &= (A + BKC)x(t) + Bv(t) \\ y(t) &= Cx(t)\end{aligned}$$

❖ Optimal system assignment

- Given a **target system** described by the triple $(F, G, H) \in \mathbb{P}^{n \times n} \times \mathbb{P}^{n \times m} \times \mathbb{P}^{p \times n}$ find an **optimal feedback transformation** of (A, B, C) which results the best approximation of (F, G, H) .

Examples in Control Theory

- ❖ The set $GL(n) \times P^{m \times p}$ of feedback transformation is a Lie group under the operation

$$(T_1, K_1) \circ (T_2, K_2) = (T_1 T_2, K_1 + K_2)$$

- ❖ We can consider action on the output feedback group and orbits, particularly:

$$\Phi(A, B, C) = \{(T(A + BKC)T^{-1}, TB, CT^{-1} \mid (T, K) \in GL(n) \times P^{m \times p}\}$$

- ❖ The distance function

$$\Phi = \|T(A + BKC)T^{-1} - F\|^2 + \|TB - G\|^2 + \|CT^{-1} - H\|^2$$

Examples in Control Theory

- ❖ The gradient flow of this distance function with respect to a specific Riemannian metric on $\Phi(A,B,C)$ can be written as:

$$\dot{T} = T^{-T} f(T, T^{-T}, K)$$

$$\dot{K} = -B^T T^T (T(A + BKC)T^{-1} - F)T^{-T} C^T$$

Examples in Control Theory

❖ **Balanced matrix factorizations**

➤ General matrix factorization problem:

Given a matrix $H \in \mathbb{P}^{k \times l}$ find two $X \in \mathbb{P}^{k \times n}$ and $Y \in \mathbb{P}^{n \times l}$ such that $H = XY$

➤ balanced factorization $X^T X = Y Y^T$

➤ diagonal balanced factorization $X^T X = Y Y^T = D$

❖ Balanced and diagonal balanced factorization can be characterized as critical points of cost functions defined on the orbit

$$\mathcal{O}(X, Y) = \{(X T^{-1}, T Y) \in \mathbb{P}^{k \times n} \times \mathbb{P}^{n \times l} \mid T \in GL(n)\}$$

Examples in Control Theory

❖ The cost functions are respectively:

$$\begin{aligned}\Phi : \mathcal{O}(X, Y) &\rightarrow \mathbb{P} & \Phi(XT^{-1}, TY) &= \|XT^{-1}\|^2 + \|TY\|^2 \\ \Phi_N : \mathcal{O}(X, Y) &\rightarrow \mathbb{P} & \Phi_N(XT^{-1}, TY) &= \text{tr}(NT^{-T}X^T XT^{-1} + NTYY^T T^T)\end{aligned}$$

❖ Applying a gradient flow techniques differential systems on $GL(n)$ can be constructed:

balanced

$$\dot{T} = T^{-T} (X^T X (T^T T)^{-1} - T^T T Y Y^T) \quad T(0) = T_0$$

$$\dot{T} = T^{-T} (X^T X T^{-1} N T^{-T} - T^T N T Y Y^T) \quad T(0) = T_0$$

diagonal
balanced

Examples in Control Theory

❖ Balanced realizations in linear system theory

- Consider the linear dynamical system defined by the triple $(A, B, C) \in \mathbb{P}^{n \times n} \times \mathbb{P}^{n \times m} \times \mathbb{P}^{p \times n}$

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

- Gramians: $W_C = \int_0^\infty e^{At} B B^T e^{A^T t} dt \quad W_O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$
- (A, B, C) is a **balanced realization** if $W_C = W_O$
- (A, B, C) is a **diagonal balanced realization** if $W_C = W_O = D$

Examples in Control Theory

❖ Any $T \in GL(n)$ changes a realization by

$$(A, B, C) \rightarrow (TAT^{-1}, TB, CT^{-1})$$

❖ and the Gramians via

$$W_C \rightarrow T W_C T^{-1} \quad W_0 \rightarrow T^{-T} W_0 T^{-1}$$

❖ Balanced and diagonal balanced realizations have been proved to be critical points of costs functions defined on the orbit

$$\mathcal{O}(A, B, C) = \{(TAT^{-1}, TB, CT^{-1}) \in \mathbb{P}^{n \times n} \times \mathbb{P}^{n \times m} \times \mathbb{P}^{k \times n} \mid T \in GL(n)\}$$

Examples in Control Theory

❖ The cost functions are respectively:

$$\begin{aligned}\Phi : \mathcal{O}(A, B, C) &\rightarrow \mathcal{P} & \Phi(T) &= \text{tr}(TW_C T^{-1} + T^{-T} W_O T^{-1}) \\ \Phi_N : \mathcal{O}(A, B, C) &\rightarrow \mathcal{P} & \Phi_N(T) &= \text{tr}(NTW_C T^{-1} + NT^{-T} W_O T^{-1})\end{aligned}$$

❖ All balancing transformation $T \in GL(n)$ for a given asymptotically stable system (A, B, C) can be obtained solving the gradient flow

balanced

$$\dot{T} = T^{-T} (W_O (T^T T)^{-1} - T^T T W_C) \quad T(0) = T_0$$

$$\dot{T} = T^{-T} (W_O T^{-1} N T^{-T} - T^T N T W_C) \quad T(0) = T_0$$

diagonal
balanced

Examples in Multivariate Data Analysis

❖ Weighted oblique Procrustes problem (WObPP)

➤ Manifold of the oblique rotation matrices

$$OB(n) = \{X \in \mathbb{R}^{n \times n} \mid \det(X) \neq 0, \text{diag}(X^T X) = I\}$$

❖ Given A, B, C fixed matrices with conformal dimensions

➤ Minimize $\|AXC - B\|$ subject to $X \in OB(n)$

➤ Problem in factor analysis known as a “rotation to *factor-structure matrix*”

➤ Minimize $\|AX^T C - B\|$ subject to $X \in OB(n)$

➤ Problem of finding an approximation to a “*factor-pattern*” matrix

Examples in Multivariate Data Analysis

- ❖ The solution of the WObPP problem can be obtained solving a **descent matrix ODE**:

$$\frac{dX}{dt} = -\pi_{OB(n)}(\nabla) = -X^{-T} \text{off}(X^T \nabla)$$

- ❖ being ∇ the gradient of the function to be minimize with respect to the chosen metric

(N. Trendafilov FGCS 2003)

Examples in Inverse Eigenvalue Problem and control theory

❖ Pole placement or eigenvalue assignment via output feedback:

- Given a linear system described by the triple (A, B, C) and a self-conjugate set of complex points $\{\lambda_1 \lambda_2 \dots \lambda_n, \}$
- find a feedback gain matrix K such that $A+BKC$ has eigenvalues λ_i

❖ Denoted by Λ a fixed matrix with eigenvalues λ_i the pole placement task is equivalent to find a matrix $T \in GL(n)$ and $K \in \mathbb{R}^{m \times p}$ minimizing the distance $\mu_1 \quad \| \Lambda - T(A + BKC)T^{-1} \|$

Examples in Inverse Eigenvalue Problem and control theory

❖ Using a gradient flow techniques the solution can be obtained solving

$$\begin{aligned}\dot{T} &= T^{-T} [(A + BKC)^T, T^T (\Lambda - (A + BKC)) T^{-T}] \\ \dot{K} &= -B^T T^T (T(A + BKC)T^{-1} - F) T^{-T} C^T\end{aligned}$$

Examples in Inverse Eigenvalue Problem

- ❖ **Matrix completion with prescribed eigenvalues**
- ❖ PEIEP (prescribed entries inverse eigenvalue problem) :

Given

➤ $\Lambda = \{(i_v, j_v) \mid v = 1, \dots, m\}$ m pairs of integers $1 \leq i_v < j_v \leq n$

➤ $\mathbf{a} = \{a_1, \dots, a_m\} \subset \mathbb{P}$

➤ $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{X}$ closed under conjugation

Find a matrix $X \in \mathbb{P}^{n \times n}$ such that $\sigma(X) = \{\lambda_1, \dots, \lambda_n\}$

and $x_{i_v j_v} = a_v \quad v = 1, \dots, m$

Examples in Inverse Eigenvalue Problem

❖ Let A a matrix with eigenvalues λ_i and denoting

$$M(A) = \{ VAV^{-1} \mid V \in GL(n) \}$$

the orbit of matrices isospectral to A under the action group of $GL(n)$ and

$$\Sigma(\Lambda, \mathbf{a}) = \{ X = [x_{ij}] \in \mathbb{P}^{n \times n} \mid x_{i_v j_v} = a_v \quad v = 1, \dots, m \}$$

❖ Solving the PEIEP is to find intersection of the two geometric entities $M(A)$ and $\Sigma(\Lambda, \mathbf{a})$

Examples in Inverse Eigenvalue Problem

- ❖ Minimize for each given $X \in M(\Lambda)$ the distance between X and $\Sigma(\Lambda, \mathbf{a})$

$$\min_{V \in M(\Lambda)} \frac{1}{2} \langle V\Lambda V^{-1} - P(V\Lambda V^{-1}), V\Lambda V^{-1} - P(V\Lambda V^{-1}) \rangle$$

↑
Projection on $\Sigma(\Lambda, \mathbf{a})$

- ❖ Using a descent flow approach we get

$$\frac{dV}{dt} = \kappa(V\Lambda V^{-1})V^{-T} \quad \text{with} \quad \kappa(X) = [X^T, X - P(X)]$$

(M.T. Chu et al. FGCS 2003)

Numerical Approximation: substituting approach

❖ Consider our system:

$$\begin{aligned}\dot{Y}(t) &= Y(t)^{-T} F(Y(t), Y(t)^{-T}) \\ Y(0) &= Y_0 \in GL(n)\end{aligned}$$

❖ Setting $Z=Y^{-T}$ from $Y^T Z=I$ we get

$$\dot{Y}^T Z + Y^T \dot{Z} = 0 \Leftrightarrow \dot{Z} = -Y^{-T} \dot{Y}^T Z$$

$$\begin{cases} \dot{Y} = ZF(Y, Z) = H(Y, Z), & Y(0) = Y_0 \\ \dot{Z} = -ZF^T(Y, Z)Z^T Z = -ZH^T(Y, Z)Z, & Z(0) = Y_0^{-T} \end{cases}$$

Substituting Approach

❖ Advantages:

- No direct use of the inverse of $Y(t)$ (computational advantages)

❖ Drawbacks:

- Solution of a new matrix ODE with **double dimension** with respect to the original system;
- High stiffness (when $Y(t)$ tends to a singular matrix or the Lipschitz constant of H is large);
- The presence of an additional structure of the solution matrix $Y(t)$ is not considered ➡ need of *ad hoc* numerical scheme

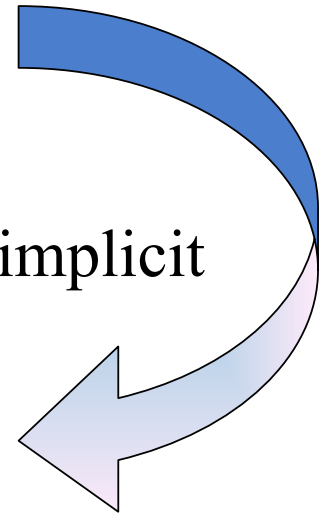
Solution via Riccati equation

- ❖ When the matrix function F does not depend explicitly on Y^{-T} , i.e.:

$$\begin{aligned}\dot{Y}(t) &= Y(t)^{-T} F(Y(t)) \\ Y(0) &= Y_0 \in GL(n)\end{aligned}$$

- ❖ It could **be convenient** work with the implicit equation

$$\begin{aligned}Y(t) \dot{Y}(t) &= F(Y(t)) \\ Y(0) &= Y_0 \in GL(n)\end{aligned}$$



Solution via Riccati equation

- ❖ Applying the second order Gauss Legendre method, we get:

$$Y_{n+1}^T Y_{n+1} + Y_n^T Y_{n+1} - Y_{n+1}^T Y_n - Y_n^T Y_n - 2hF\left(\frac{Y_n + Y_{n+1}}{2}\right) = 0$$

- ❖ The previous equation can be iteratively solved starting from an initial approximation $Y_{n+1}^{(0)}$ (avoiding the nonlinearity of F)

$$Y_{n+1}^T Y_{n+1} + Y_n^T Y_{n+1} - Y_{n+1}^T Y_n - Y_n^T Y_n - 2hF\left(\frac{Y_n + Y_{n+1}^{(0)}}{2}\right) = 0$$

Solution via Riccati Equation

- ❖ The latter equation is the prototype of an Algebraic Riccati equation, in fact setting

$$A = Y_n \quad \text{and} \quad C = Y_n^T Y_n + 2hF \left(\frac{Y_n + Y_{n+1}^{(0)}}{2} \right)$$

- ❖ we get

$$R(X) = X^T X + A^T X - X^T A + C = 0$$

Solution via Algebraic Riccati equation

❖ Numerical methods to solve **Algebraic Riccati equation** are based on fixed point or Newton iteration:

➤ **Picard iteration:**

$$A^T X_{k+1} - X_{k+1}^T A = -C - X_k^T X_k$$

➤ **Newton method:**

➤ $R : \mathbb{P}^{n \times n} \rightarrow \mathbb{P}^{n \times n}$

➤ its Frechét derivative is: $R'_X(H) = H^T(X - A) + (X + A)^T H$

➤ the Newton iteration starts from X_0 and solves $R(X)=0$ via $X_{k+1} = X_k + D_k$ being D_k the solution of Sylvester equation

$$R'_X(D_k) = -R(X_k) \Leftrightarrow (X_k + A)^T D_k + D_k^T (X_k - A) = -R(X_k)$$

Solution via Riccati equation

- ❖ Solving Riccati equation implies the numerical treatment of the Sylvester equation

$$AX + X^T B = X$$

with A , B , X given $n \times n$ matrices

Existence: there exists a solution X of the Sylvester equation iff

$$\begin{bmatrix} X & A \\ B & O \end{bmatrix} \text{ and } \begin{bmatrix} O & A \\ B & O \end{bmatrix}$$

are equivalent

Solution via Riccati equation

- ❖ To obtain conditions for uniqueness of solution and for constructing it, we reformulate the Sylvester equation as a $n^2 \times n^2$ linear system:

$$(I \otimes A) \text{vec}(X) + (B^T \otimes I) \text{vec}(X^T) = \text{vec}(X)$$

$$\text{vec}(X^T) = P(n, n) \text{vec}(X)$$

$$P(n, n) = \sum_{i=1}^n \sum_{j=1}^n E_{ij} \otimes E_{ij}^T$$

$$\underbrace{\left[(I \otimes A) + (B^T \otimes I) P(n, n) \right]}_M \text{vec}(X) = \text{vec}(X)$$

Solution via Riccati equation

$$M = \begin{bmatrix} A + e_1 b_1^T & e_2 b_1^T & \cdots & e_n b_1^T \\ e_1 b_2^T & A + e_2 b_2^T & \cdots & e_n b_2^T \\ \vdots & \vdots & \ddots & \vdots \\ e_1 b_n^T & e_2 b_n^T & \cdots & A + e_n b_n^T \end{bmatrix}$$

being b_i the columns of the matrix B

Uniqueness: there exists a unique solution X of the Sylvester equation $AX + X^T B = X$ if the matrix M is non-singular ($\text{rank}(M) = n^2$)

Solution via Riccati equation

- ❖ Considering the linear equation derived from:
 - **Picard iteration:** $A=A^T$ and $B=A \Rightarrow M$ is singular
 - **Newton iteration:** $A=X_k+A^T$ and $B=X_k-A \Rightarrow M$ is non-singular $\Rightarrow$ **unique solution !**
- ❖ Newton method converges in a reasonable number of iterations
- ❖ Numerical solution of Sylvester equation :
 - Direct methods (QR, Gaussian Elimination);
 - Iterative algorithms;
 - Generalize Conjugate Residual method.

Singular Value Decomposition

- ❖ To avoid the inverse matrix computations and to control the singularities of the matrix solution $Y(t)$ we can adopt a continuous Singular Value Decomposition approach
- ❖ The continuous SVD of $Y(t)$ is a continuous factorization

$$Y(t) = U(t) \Sigma(t) V^T(t)$$

- $U(t), V(t)$ orthogonal matrices ($U^T U = I_n$ and $V^T V = I_n$)
- $\Sigma(t)$ diagonal matrix with diagonal elements the singular values $\sigma_i(t)$ of $Y(t)$
- ❖ The motion of $Y(t)$ is now described by the variables $U(t)$, $\Sigma(t)$, $V(t)$ giving more information on the flow

Singular Value Decomposition

❖ Suppose that the solution $Y(t)$ possesses distinct and nonzero singular values $\sigma_i(t)$, for $i=1, \dots, n$ and t in $[0, \tau)$ then there exists a continuous SVD of $Y(t)$ and the factors $U(t)$, $\Sigma(t)$, $V(t)$ of such a decomposition satisfy the following ODEs:

$$\dot{\Sigma} = \Sigma^{-1} V^T F(Y, Y^{-T}) V - H \Sigma + \Sigma K, \quad \Sigma(0) = \Sigma_0$$

$$\dot{U} = UH, \quad U(0) = U_0$$

$$\dot{V} = VK, \quad V(0) = V_0$$

Singular Value Decomposition

- ❖ The differential equations for the singular values are

$$\dot{\sigma}_i = \frac{1}{\sigma_i} \left(V^T F(Y, Y^{-T}) V \right)_{ii}, \quad i = 1, \dots, n$$

- ❖ The elements of the skew-symmetric matrices H, K are

$$H_{ij} = \frac{1}{\sigma_i \sigma_j (\sigma_j^2 - \sigma_i^2)} \left[\sigma_j^2 (V^T F V)_{ij} + \sigma_i^2 (V^T F V)_{ji} \right]$$
$$K_{ij} = \frac{1}{(\sigma_j^2 - \sigma_i^2)} \left[(V^T F V)_{ij} - (V^T F V)_{ji} \right]$$

Singular Value Decomposition

❖ Numerical solution of:

- a diagonal system in σ_i (information on the conditioning of the matrix solution $Y(t)$)
- two linear systems in H_{ij} K_{ij}
- two orthogonal systems in U and V
 - **our aim** is to preserve the **non-singular behavior** of the numerical solution → **explicit integration** of the systems in U and V (orthogonality preserved up to the order of the method)

❖ Drawback **distinct singular values**

- **Block Continuous SVD**

Rectangular case

- ❖ Some of the previous results can be extended to differential problems on the manifold

$$GL(m, n) = \{Y \in P^{m \times n} \mid \text{rank}(Y) = n\}, \quad n \leq m$$

- ❖ Differential systems on $GL(m, n)$ have the following form:

$$\dot{Y} = G(Y), \quad Y(0) = Y_0 \in GL(n, p)$$

- ❖ with G belonging to the tangent space of $GL(m, n)$:

$$G(Y) = Y \underbrace{(Y^T Y)^{-1} F_1(Y)}_{n \times n} + \left[I_n - Y \underbrace{(Y^T Y)^{-1} Y^T}_{m \times n} \right] \underbrace{F_2(Y)}_{m \times n}$$

Rectangular Case: numerical treatment

❖ Continuous SVD (economy)

$$Y(t) = U_1(t) \Sigma_1(t) V^T(t)$$

$m \times n$ matrix
 $U_1^T U_1 = I_n$

$diag(\sigma_1, \dots, \sigma_n)$

$n \times n$ matrix
 $V^T V = V V^T = I_n$

❖ Differentiating we obtain the differential systems satisfied by the three factors:

Rectangular Case: numerical treatment

$$\dot{\sigma}_i = \frac{1}{\sigma_i} \left(V^T F_1(Y) V \right)_{ij} \quad i = 1, \dots, n$$

$$\dot{V} = VK, \quad V(0) = V_0$$

$$\dot{U}_1 = U_1 H + (I_n - U_1 U_1^T T) F_2(Y) \Sigma_1^{-1}, \quad U(0) = U_0$$

Differential System on the Stiefel manifold

$$H_{ij} = \frac{1}{\sigma_i \sigma_j (\sigma_j^2 - \sigma_i^2)} \left[\sigma_j^2 \left(V^T F_1(Y) V \right)_{ij} + \sigma_i^2 \left(V^T F_1(Y) V \right)_{ji} \right]$$

$$K_{ij} = \frac{1}{(\sigma_j^2 - \sigma_i^2)} \left[\left(V^T F_1(Y) V \right)_{ij} + \left(V^T F_1(Y) V \right)_{ji} \right]$$

Rectangular Case: numerical treatment

❖ Substituting approach:

$$\dot{Y} = Y(Y^T Y)^{-1} F_1(Y) + \left[I_n - Y(Y^T Y)^{-1} Y^T \right] F_2(Y)$$

❖ Setting $Z = (Y^T Y)^{-1}$ we obtain

$$\begin{aligned}\dot{Y} &= YZF_1(Y) + \left[I - YZY^T \right] F_2(Y) \\ \dot{Z} &= -Z \left[F_1(Y) + F_1^T(Y) \right] Z\end{aligned}$$

Numerical Illustrations

❖ First example:

$$\dot{Y} = Y^{-T} \begin{bmatrix} 0 & -\frac{\delta}{2} \\ -\frac{\delta}{2} & 0 \end{bmatrix} \quad Y(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

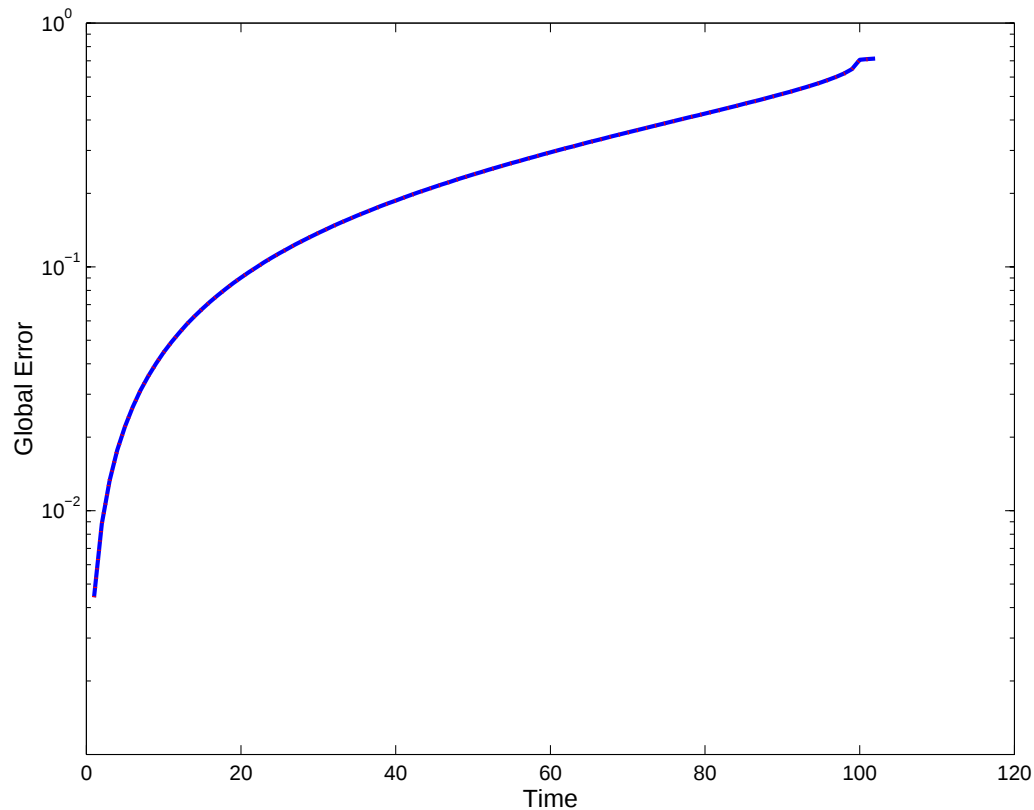
❖ With solution existing in $(-1/\delta, 1/\delta)$

$$Y(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+\delta t} & -\sqrt{1+\delta t} \\ \sqrt{1-\delta t} & \sqrt{1-\delta t} \end{bmatrix}$$

❖ We solve the problem with $\delta = 1/2$

Numerical Illustrations

❖ Behaviour of the global error on $[0, 2)$



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Numerical Illustrations

❖ Second example

$$\dot{Y} = Y^{-T} \begin{bmatrix} -\sin(t) \cos(t) & \cos(t) \\ -t \sin(t) & t \end{bmatrix} \quad Y(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

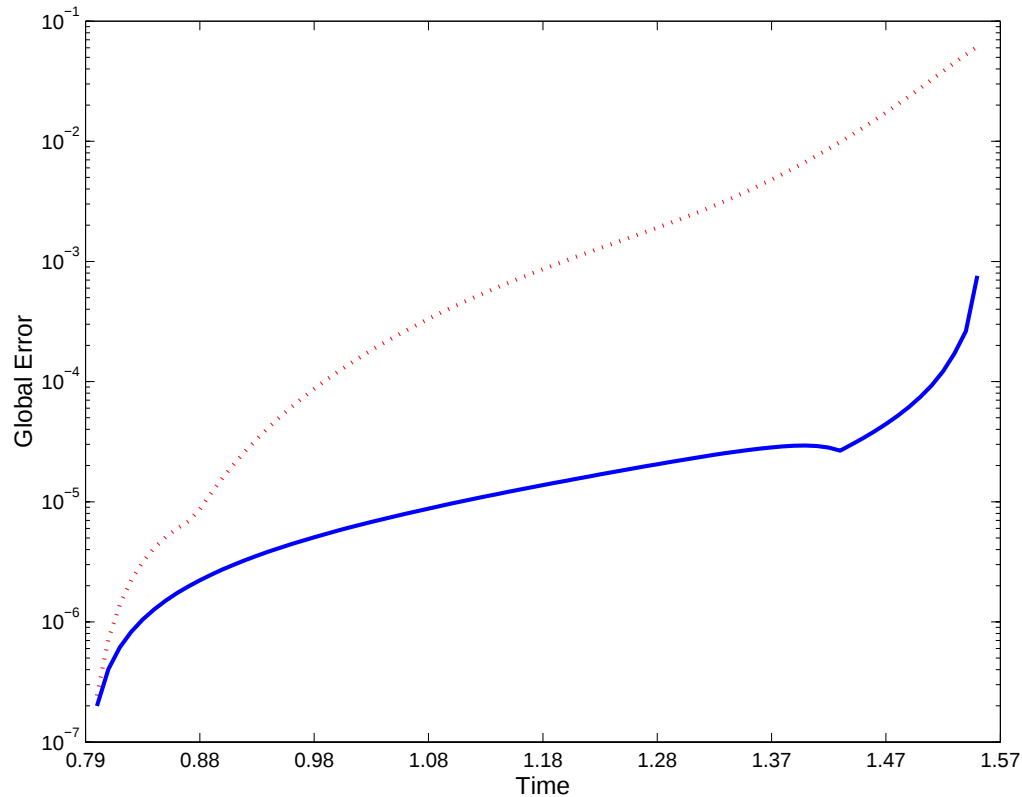
❖ with solution

$$Y(t) = \begin{bmatrix} \cos(t) & t \\ 0 & 1 \end{bmatrix}$$

❖ periodically singular (for each $\tau_k = k \pi/2$)

Numerical Illustrations

❖ Semilog plot of the global error on $(\pi/4, \pi/2)$



Conclusions

- ❖ We have considered a particular ODEs on $GL(n)$ often occurring in applications
- ❖ Several problems modeled by such ODEs
- ❖ Different numerical approaches avoiding the direct use of matrix inversion and detection of singular behavior
- ❖ Future works:
 - Improving the validation of the proposed approaches by tackling numerical tests on real examples