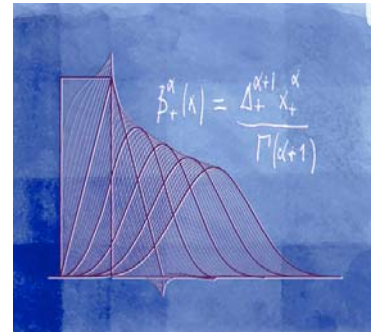




On fractals, fractional splines and wavelets

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FRACTALS AND PHYSIOLOGY

■ Fractal characteristics:

- Complex, patterned
- Statistical self-similarity
- Scale-invariant structure
- Generated by simple iterative rules
- $1/\omega^{2H+d}$ spectral decay

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■ Growth processes, biofractals

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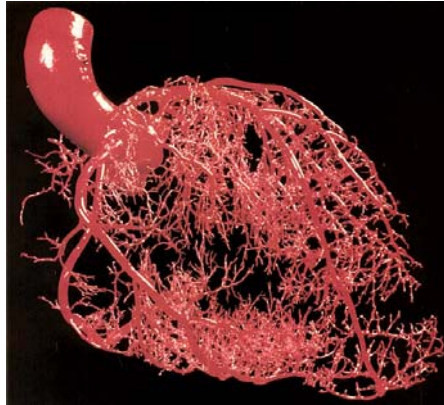
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Cardiovascular system

- Heart
 - Arterial tree
 - Dendritic anatomy



- Lung

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Fractal bones

- Trabecular bone



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CT of a vertebra

μCT

Courtesy F. Peyrin ESRF

Mammograms

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DDSM: University of Florida

(Digital Database for Screening Mammography)

(Arnéodo et al., 2001)

Brain as a biofractal



(Bullmore, 1994)

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1mm

Courtesy R. Mueller ETHZ

OUTLINE

- Fractals in physiology 
- Wavelets and fractals
 - Motivation for using wavelets
 - Fractal processing: order is the key
 - What about fractional differentiation
- Fractional splines
- Fractional wavelets
- Wavelets in medical imaging
 - Survey of applications
 - Analysis of functional images

Motivation for using wavelets

- Wavelets provide basis functions that are self-similar [Mallat, 1989]

$$\forall f(x) \in L_2, \quad f(x) = \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_{i,k} \rangle \psi_{i,k}(x)$$

$$\psi_{i,k} = 2^{-i/2} \psi\left(\frac{x - 2^i k}{2^i}\right)$$



- Wavelets approximately decorrelate statistically self-similar processes [Flandrin, 1992; Wornell, 1993]
- Unlike Fourier exponentials, wavelets are jointly localized in space and frequency
- The basis functions themselves are fractals [Blu-Unser, 2002]



Wavelets are prime candidates for processing fractal-like signals and images

On the fractal nature of wavelets

■ Harmonic spline decomposition of wavelets

Theorem: Any valid compactly supported scaling function $\varphi(x)$ (or wavelet $\psi(x)$) can be expressed either as

- (1) a weighed sum of the integer shifts of a self-similar function (fractal) ;
- (2) a linear combination of harmonic splines with complex exponents.

[*Blu-Unser*, 2002]

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D4 as a sum of harmonic splines

- Sum of spline components

$$\varphi_N(x) = \sum_{n=-N/2}^{+N/2} \gamma_n s_n(x)$$

where

$$s_n(x) = \sum_{k \in \mathbb{Z}^+} p_k(x - k)_{+}^{\frac{\log \lambda}{\log 2} + j \frac{2\pi n}{\log 2}}$$

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Fractal processing: order is the key !

■ Vanishing moments

Classical N th order transform $\Leftrightarrow$ analysis wavelet $\tilde{\psi}(x)$ has N vanishing moments

$$\int_{x \in \mathbb{R}} x^n \tilde{\psi}(x) dx = 0, \quad n = 0, \dots, N-1$$

$\Rightarrow$ $\tilde{\psi}$ kills all polynomials of degree $n < N$

■ Multi-scale differentiation property

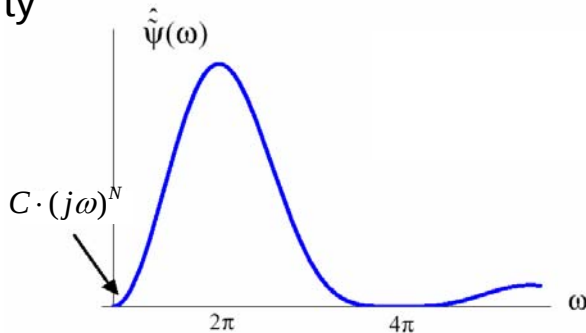
Property

An analysis wavelet of order N acts like a N th order differentiator:

$$\hat{\tilde{\psi}}(\omega) = O(\omega^N)$$

$\Rightarrow$ $\langle f(x), \tilde{\psi}(x-u) \rangle = \frac{d^N}{du^N} \{ \phi * f \}(u)$

Smoothing kernel: $\hat{\phi}(\omega) = \hat{\tilde{\psi}}^*(\omega) / (j\omega)^N$

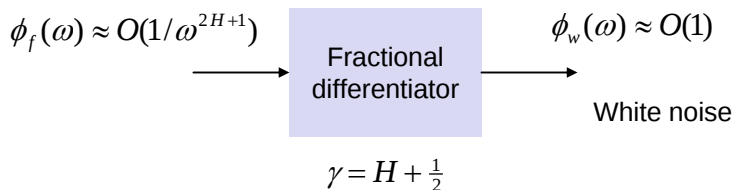


What about fractional differentiation ?

- Fractional differentiation operator

$$\mathcal{D}^\gamma f(x) \quad \xleftrightarrow{\text{F}} \quad (j\omega)^\gamma \hat{f}(\omega) \quad \gamma \in \mathbb{R}^+$$

- Motivation: whitening of fBM-like processes



- QUESTION

Are there wavelets that act like fractional differentiators ?

- ANSWER

Not within the context of standard wavelet theory where the order is constrained to be an *integer*, but ...

SPLINES

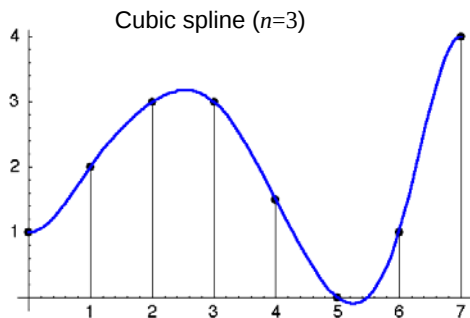
- Polynomial splines
- Fractional B-splines
- Properties
 - Fractional differentiation
 - Fractional order of approximation

Polynomial splines (Schoenberg, 1946)

Definition:

$s(x)$ is *cardinal* polynomial spline of degree n iff

- Piecewise polynomial:
 $s(x)$ is a polynomial of degree n in each interval $[k, k+1)$;
- Higher-order continuity:
 $s(x), s^{(1)}(x), \dots, s^{(n-1)}(x)$ are continuous at the knots k .

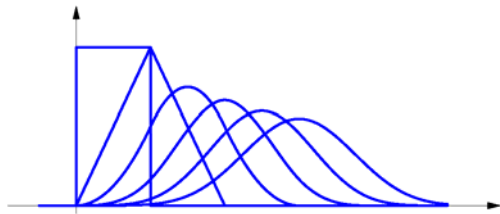


B-spline representation

- B-splines of degree n

$$\beta_+^n(x) = \underbrace{\beta_+^0 * \dots * \beta_+^0}_{(n+1) \text{ times}}(x)$$

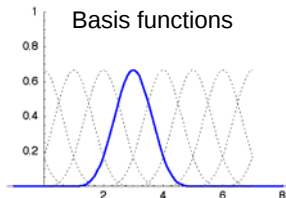
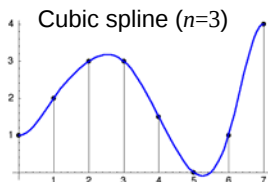
$$\text{Explicit formula : } \beta_+^n(x) = \frac{\Delta_+^{n+1} x_+^n}{n!}$$



Theorem [Schoenberg, 1946]

A cardinal spline of degree n has a stable, unique representation as a linear combination of shifted B-splines

$$s(x) = \sum_{k \in \mathbb{Z}} c(k) \beta_+^n(x - k)$$



Can we fractionalize splines ?

- Schoenberg's formula

$$\beta_+^n(x) = \frac{\Delta_+^{n+1} x_+^n}{n!}$$



$$\beta_+^\alpha(x) = \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)}$$

Basic tools for fractionalization

- Generalized factorials—Euler's Gamma function

$$n! = \Gamma(n+1) \qquad \Gamma(u) = \int_0^{+\infty} x^{u-1} e^{-x} dx$$

- Generalized binomial

$$(1+z)^\gamma = \sum_{k=0}^{+\infty} \binom{\gamma}{k} z^k \qquad \binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}$$

- Fractional derivative [Liouville, 1855]

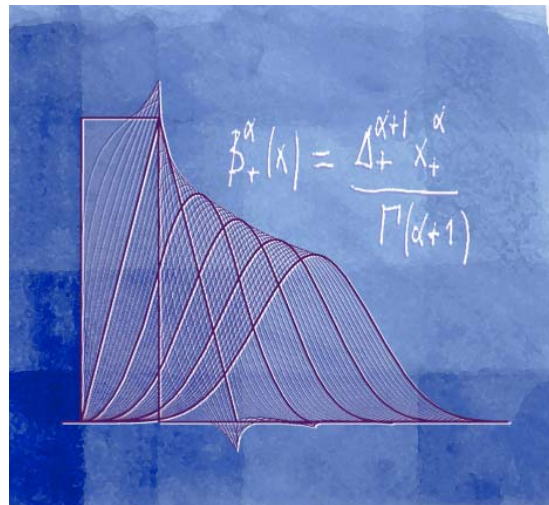
$$\mathcal{D} \xleftrightarrow{\text{Fourier}} (j\omega)^s$$

- Fractional finite differences

$$\Delta_+^s \xleftrightarrow{\text{Fourier}} (1 - e^{-j\omega})^s \qquad \Rightarrow \qquad \Delta_+^s f(x) = \sum_{k=0}^{+\infty} (-1)^k \binom{s}{k} f(x-k)$$

Fractional B-splines

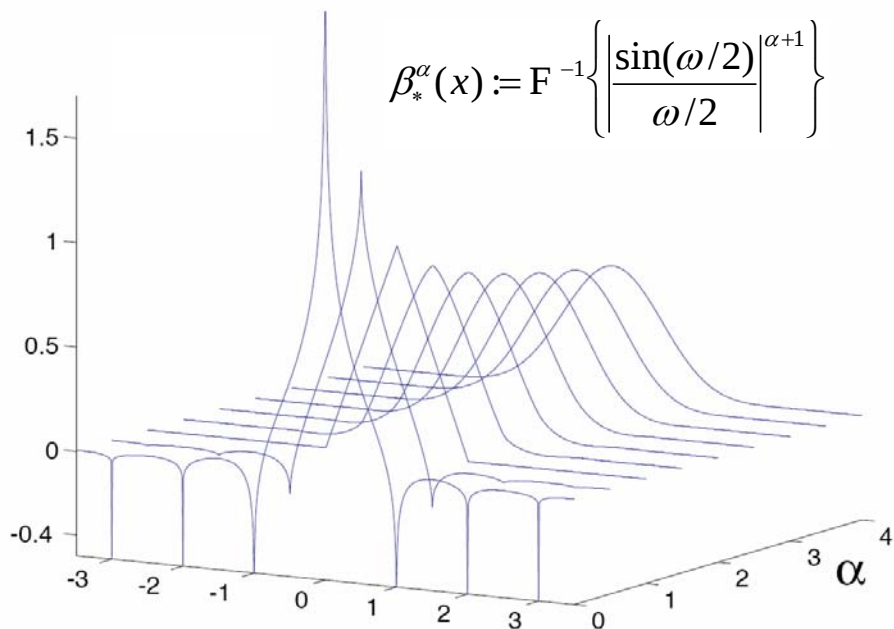
$$\begin{aligned}
 \beta_+^0(x) &:= x_+^0 - (x-1)_+^0 && \xleftrightarrow{\text{Fourier}} && \left(\frac{1-e^{-j\omega}}{j\omega} \right) \\
 &\vdots \\
 \beta_+^\alpha(x) &:= \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)} && \xleftrightarrow{\text{Fourier}} && \left(\frac{1-e^{-j\omega}}{j\omega} \right)^{\alpha+1}
 \end{aligned}$$



One-sided power functions: $x_+^\alpha = \begin{cases} x^\alpha & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Symmetric B-splines

- Symmetrization in Fourier domain: $|\hat{\beta}_+^\alpha(x)| = \left| \left(\frac{1 - e^{-j\omega}}{j\omega} \right)^{\alpha+1} \right| = \left| \frac{\sin(\omega/2)}{\omega/2} \right|^{\alpha+1}$



Properties

Generic notation : β^α for either β_+^α (causal) or β_*^α (symmetric)

■ Equivalence with classical B-splines

$$\beta_+^\alpha(x) \text{ with } \alpha = n \quad (\text{integers})$$

$$\beta_*^\alpha(x) \text{ with } \alpha = 2n+1 \quad (\text{odd integers})$$

$\Rightarrow$ Compact support !

■ Decay

Theorem : For $\alpha > -1$, there exists a constant C such that $|\beta^\alpha(x)| \leq \frac{C}{|x|^{\alpha+2}}$.

(U. & Blu, *SIAM Rev*, 2000)

■ Convolution property

$$\beta^{\alpha_1} * \beta^{\alpha_2} = \beta^{\alpha_1 + \alpha_2 + 1}$$

$$\langle \beta^\alpha(\cdot), \beta^\alpha(\cdot - x) \rangle = \beta_*^{2\alpha+1}(x)$$

Riesz basis

$\{\beta^\alpha(x-k)\}_{k \in \mathbb{Z}}$ is a Riesz basis for the cardinal fractional splines

- Generic B-spline representation of a fractional spline

$$s(x) = \sum_{k \in \mathbb{Z}} c[k] \beta^\alpha(x-k)$$

*Continuous-time function
(fractional spline)*



$$\{c[k]\}_{k \in \mathbb{Z}}$$

*Discrete representation
(digital signal)*

- Stable, one-to-one representation

For $\alpha > -\frac{1}{2}$, there exist two constants $A_\alpha > 0$ and $B_\alpha < +\infty$ such that

$$\forall c \in l_2, \quad A_\alpha \cdot \|c\|_{l_2} \leq \left\| \sum_{k \in \mathbb{Z}} c[k] \beta^\alpha(x-k) \right\|_{L_2} \leq B_\alpha \cdot \|c\|_{l_2}$$

Explicit fractional differentiation formula

- Fractional derivative operators

$$\mathcal{D} \xleftrightarrow{\text{Fourier}} (j\omega)^s$$

$$\mathcal{D} \beta_+^\alpha(x) = \Delta_+^s \beta_+^{\alpha-s}(x)$$

Fractional finite difference operator:

$$\Delta_+^s \xleftrightarrow{\text{Fourier}} (1 - e^{-j\omega})^s$$

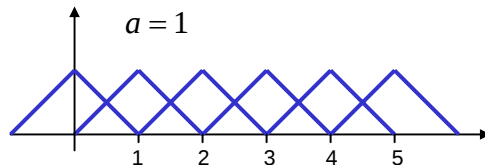
Sketch of proof:

$$\mathcal{D} \beta_+^\alpha(x) \longleftrightarrow (j\omega)^s \cdot \left(\frac{1 - e^{-j\omega}}{j\omega} \right)^{\alpha+1} = (1 - e^{-j\omega})^s \cdot \left(\frac{1 - e^{-j\omega}}{j\omega} \right)^{\alpha+1-s}$$

Order of approximation

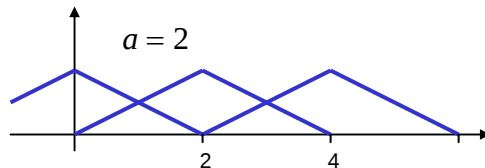
- Approximation space at scale a

$$V_a = \left\{ s_a(x) = \sum_{k \in \mathbb{Z}} c(k) \phi\left(\frac{x}{a} - k\right) : c(k) \in l_2 \right\}$$



- Projection operator

$$\forall f \in L_2, \quad P_a f = \arg \min_{s_a \in V_a} \|f - s_a\|_{L_2} \in V_a$$



- Order of approximation

DEFINITION

A scaling function ϕ has order of approximation γ iff

$$\forall f \in W_2^\gamma, \quad \|f - P_a f\| \leq C \cdot a^\gamma \|f^{(\gamma)}\| = O(a^\gamma)$$

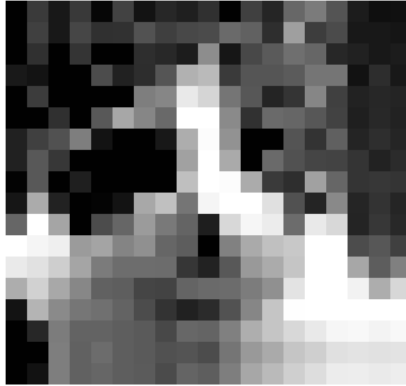


B-splines of degree α have order of approximation $\gamma = \alpha + 1$

Spline reconstruction of a CAT-scan

Piecewise constant

$$\gamma = 1$$



Cubic spline

$$\gamma = 4$$



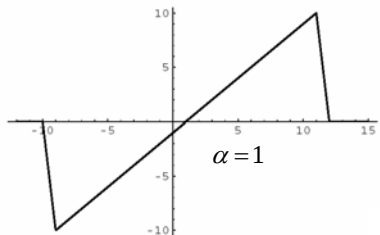
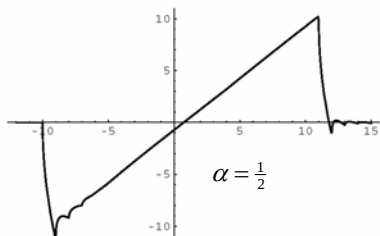
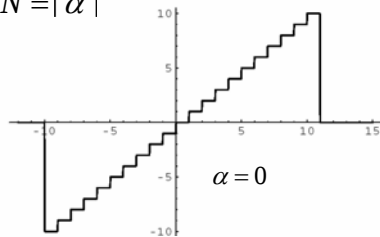
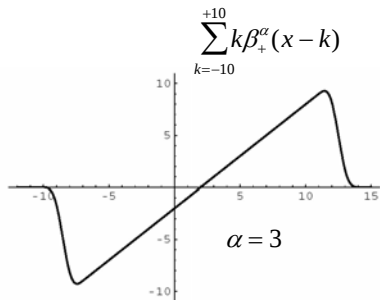
Reproduction of polynomials

- B-splines reproduce polynomials of degree $N = |\alpha|$

$$\sum_{k \in \mathbb{Z}} \beta_+^\alpha(x-k) = 1$$

$$\vdots$$

$$\sum_{k \in \mathbb{Z}} k^n \beta_+^\alpha(x-k) = x^n + a_1 x^{n-1} + \dots + a_n$$



More fractals...

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

QuickTime™ and a
TIFF (LZW) decompressor
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Pollock

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Dali

Mandelbrot meets Mondrian

FRACTIONAL WAVELETS

- Basic ingredients
- Constructing fractional wavelets
- Fractional B-spline wavelets
- Multi-scale fractional differentiation
- Adjustable wavelet properties

Scaling function

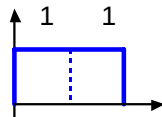
DEFINITION: $\varphi(x)$ is an admissible scaling function of L_2 iff:

- Riesz basis condition

$$\forall c \in l_2, \quad A \cdot \|c\|^2 \leq \left\| \sum_k c(k) \varphi(x - k) \right\|_{L_2}^2 \leq B \cdot \|c\|^2$$

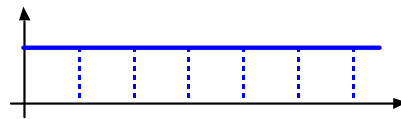
- Two-scale relation

$$\varphi(x/2) = 2 \sum_{k \in \mathbb{Z}} h(k) \varphi(x - k)$$



- Partition of unity

$$\sum_{k \in \mathbb{Z}} \varphi(x - k) = 1$$



From scaling functions to wavelets

- Wavelet bases of L_2 (Mallat-Meyer, 1989)

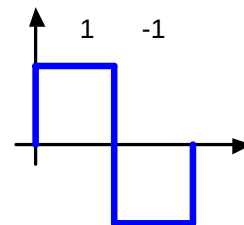
For any given admissible scaling function of L_2 , $\varphi(x)$, there exists a wavelet

$$\psi(x/2) = 2 \sum_{k \in \mathbb{Z}} g(k) \varphi(x - k)$$

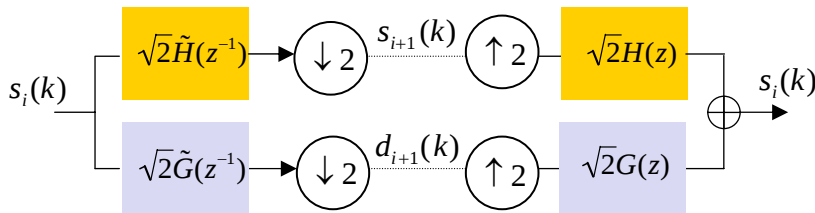
such that the family of functions

$$\left\{ \frac{1}{\sqrt{2^i}} \psi \left(\frac{x - 2^i k}{2^i} \right) \right\}_{i \in \mathbb{Z}, k \in \mathbb{Z}}$$

forms of Riesz basis of L_2 .



- Constructive approach: perfect reconstruction filterbank



Constructing fractional wavelets

Theorem : Let $\varphi(x)$ be the L_2 -stable solution (scaling function) of the two-scale relation

$$\varphi(x/2) = 2 \sum_{k \in \mathbb{Z}} h(k) \varphi(x - k)$$

Then $\varphi(x)$ is of order γ (**fractional**) if and only if

$$H(z) = \underbrace{\left(\frac{1+z^{-1}}{2} \right)^\gamma}_{\text{spline part}} \cdot \underbrace{Q(z)}_{\text{distributional part}} \quad \text{with} \quad |Q(e^{j\omega})| < \infty$$



Multi-scale differentiator

$$\hat{\psi}(\omega) \propto (-j\omega)^\gamma, \quad \omega \rightarrow 0$$



B-spline factorization:

$$\varphi = \beta_+^{\gamma-1} * \varphi_0$$



Approximation order:

$$\|f - P_a f\|_{L_2} = O(a^\gamma)$$



Vanishing moments:

$$\int x^n \tilde{\psi}(x) dx = 0, \quad n = 0, \dots, \lceil \gamma - 1 \rceil$$

Binomial refinement filter

- Two-scale relation

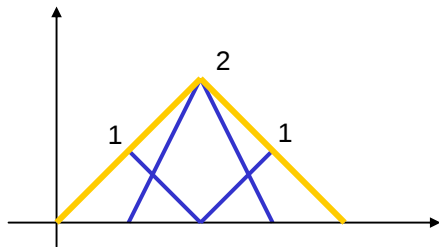
$$\beta_+^\alpha(x/2) = 2 \sum_{k \in \mathbb{Z}} h_+^\alpha(k) \beta_+^\alpha(x - k)$$

- Generalized binomial filter

$$h_+^\alpha(k) = \frac{1}{2^{\alpha+1}} \binom{\alpha+1}{k} \longleftrightarrow H^\alpha(z) = \left(\frac{1+z^{-1}}{2} \right)^{\alpha+1}$$

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}$$

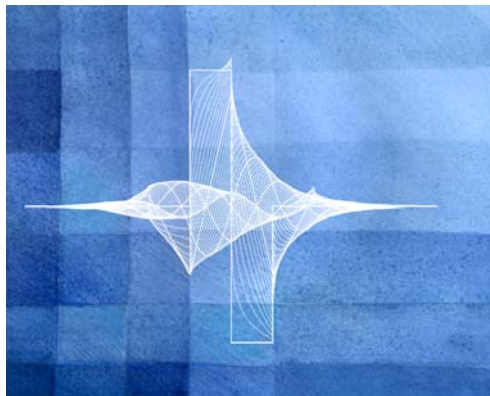
- Example of linear splines: $\alpha=1$



Fractional B-spline wavelets

$$\psi_+^\alpha(x/2) = \sum_{k \in \mathbb{Z}} \underbrace{\frac{(-1)^k}{2^\alpha} \sum_n \binom{\alpha+1}{n} \beta_*^{2\alpha+1}(n+k-1)}_{g(k)} \beta_+^\alpha(x-k)$$

QuickTime™ and a
Video decompressor
are needed to see this picture.

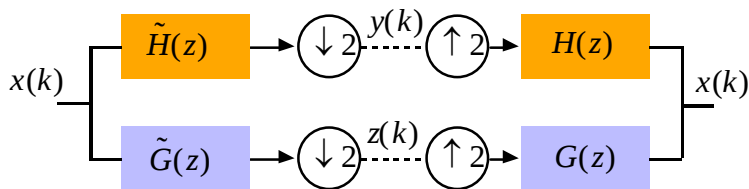


■ Remarkable property

Each of these wavelets generates a *semi-orthogonal* Riesz basis of L_2

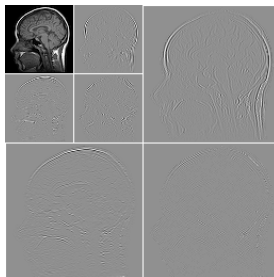
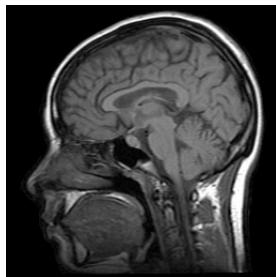
FFT-based wavelet algorithm

- Filterbank algorithm



$$\varphi(x/2) = \sqrt{2} \sum_{k \in \mathbb{Z}} h(k) \varphi(x - k)$$

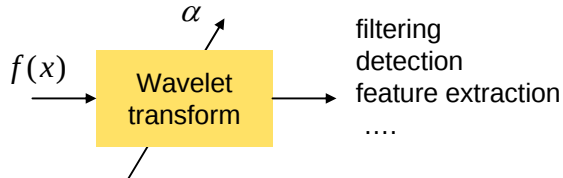
$$\psi(x/2) = \sqrt{2} \sum_{k \in \mathbb{Z}} g(k) \varphi(x - k)$$



Click for demo

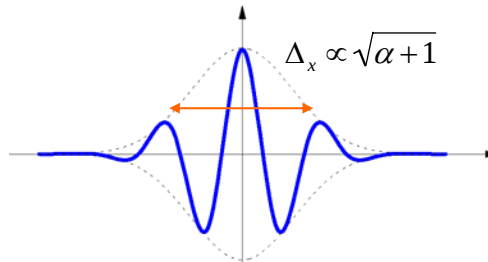
(Blu & Unser, *ICASSP'2000*)

Adjustable wavelet properties

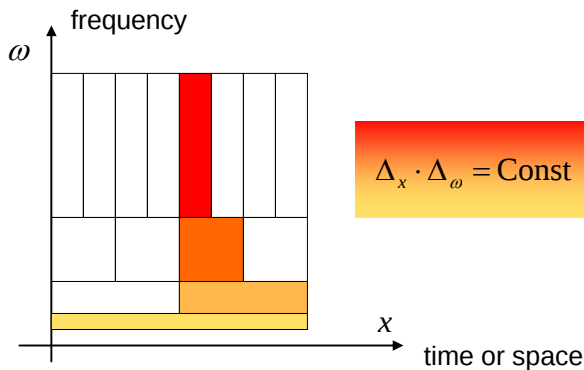


■ Transform is tunable in a continuous fashion !

- Order of differentiation: $\gamma = \alpha + 1$
 - Whitening of fBMs, fractals
- Regularity
 - Hölder continuity: α
 - Sobolev: $s_{\max} = \alpha + 1/2$
- Localization:



Wavelets and the uncertainty principle



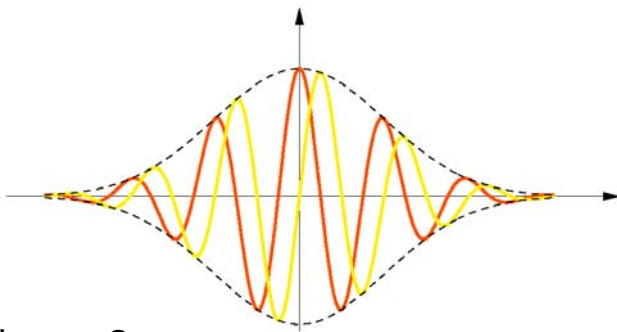
$$\Delta_x = \min_{x_0} \frac{\|(x - x_0)\psi(x)\|_{L_2}}{\|\psi\|_{L_2}}$$

$$\Delta_\omega = \min_{\omega_0} \frac{\|(\omega - \omega_0)\hat{\psi}(\omega)\|_{L_2}}{\|\hat{\psi}\|_{L_2}}$$

- Heisenberg's uncertainty relation

$$\Delta_x \cdot \Delta_\omega \geq \frac{1}{2}$$

with equality iff $\psi(x) = a \cdot e^{-b(x-x_0)^2 + j\omega_0 x}$



Question: are there such wavelet bases ?

Localization of the B-spline wavelets

Theorem

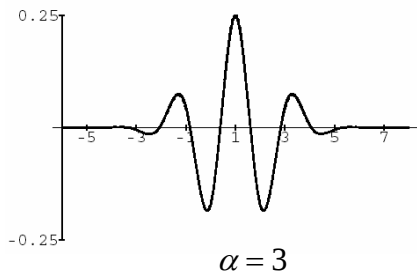
The B-spline wavelets converge (in L_p -norm) to modulated Gaussians as the degree goes to infinity :

$$\lim_{\alpha \rightarrow \infty} \{\beta_+^\alpha(x)\} = C \cdot e^{-(x-x'_\alpha)^2 / 2\sigma_\alpha'^2}$$

$$\sigma_\alpha = \sqrt{\frac{\alpha+1}{12}}$$

$$\lim_{\alpha \rightarrow \infty} \{\psi_+^\alpha(x)\} = \underbrace{C' \cdot e^{-(x-x'_\alpha)^2 / 2\sigma_\alpha'^2}}_{\text{Gaussian}} \times \underbrace{\cos(\omega_0 x + \theta_\alpha)}_{\text{sinusoid}}$$

$$\sigma_\alpha' = B \cdot \sigma_\alpha \quad \text{with} \quad B \cong 2.59$$

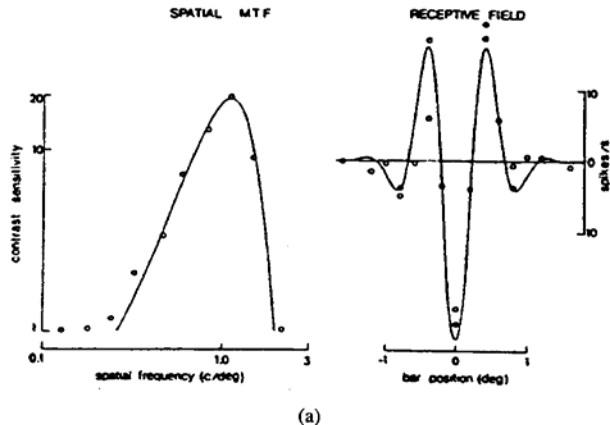


QuickTime™ and a Video decompressor are needed to see this picture.

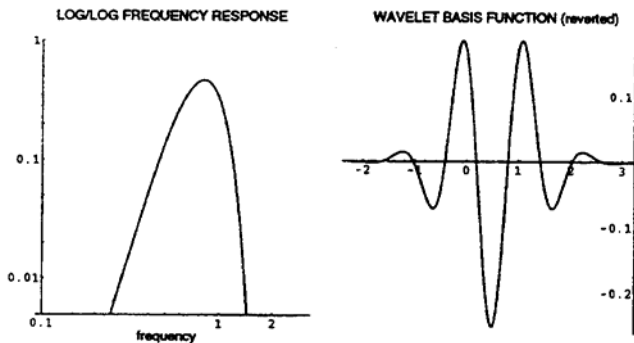
Cubic B-spline wavelets:
within 2% of the uncertainty limit !

(Unser et al., *IEEE-IT*, 1992)

Are there wavelets in my brain ?



(a)

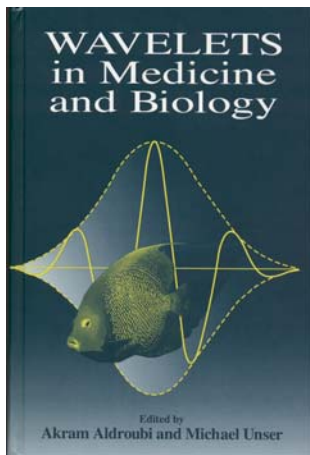


(b)

Fig. 2. Similarity between the receptive field of simple cortical cells and a wavelet basis function. (a) Response of a simple X cell from a monkey visual cortex and its fitted Gabor elementary signal [26], [67, Fig. 3]. (b) Semi-orthogonal cubic B-spline wavelet and its log-log frequency response [100].

WAVELETS IN MEDICAL IMAGING

- Survey of applications
- Analysis of functional imaging data (fMRI)



Wavelets in medical imaging: Survey 1991-1999

References

- Unser and Aldroubi, *Proc IEEE*, 1996
- Laine, *Annual Rev Biomed Eng*, 2000
- Special issue, *IEEE Trans Med Im*, 2003

Image processing task	Application / modality	Principal Authors
Image compression	• MRI • Mammograms • CT • Angiograms, etc...	Angéles 94; DeVore 95; Manduca 95; Wang 96; etc ...
Filtering	<i>Image enhancement</i> • Digital radiograms • MRI • Mammograms • Lung X-rays, CT	Laine 94, 95; Lu, 94; Qian 95; Guang 97; etc ...
	<i>Densifying</i> • MRI • Ultrasound (speckle) • SPECT	Weaver 91; Xu 94; Coifman 95; Abdel-Malek 97; Laine 98; Novak 98, 99
Feature extraction	<i>Detection of micro-calcifications</i> • Mammograms	Qian 95; Yoshida 94; Strickland 96; Dhawan 96; Baoyu 96; Heine 97; Wang 98
	<i>Texture analysis and classification</i> • Ultrasound • CT, MRI • Mammograms	Barman 93; Laine 94; Unser 95; Wei 95; Yung 95; Bush 97; Mojsilovic 97
	<i>Snakes and active contours</i> • Ultrasound	Chuang Kuo 96
Wavelet encoding	• Magnetic resonance imaging	Weaver-Healy 92; Parych 94, 96; Geman 96; Shimizu 96; Jian 97
Image reconstruction	• Computer tomography • Limited angle data • Optical tomography • PET, SPECT	Olson 93, 94; Peyrin 94; Walnut 93; Diney 95; Sahiner 96; Zhu 97; Kdaczyn 94; Rahja 99
Statistical data analysis	<i>Functional imaging</i> • PET • fMRI	Ruttman 93, 94, 98; Unser 95; Feiher 99; Raz 99
Multi-scale Registration	<i>Motion correction</i> • fMRI, angiography <i>Multi-modality imaging</i> • CT, PET, MRI	Unser 93; Thévenaz 95, 98; Kybic 99
3D visualization	• CT, MRI	Gross 95, 97; Muraki 95; Kamath 98; Holbet 99

QuickTime™ and a TIFF (Uncompressed) decompressor are needed to see this picture.

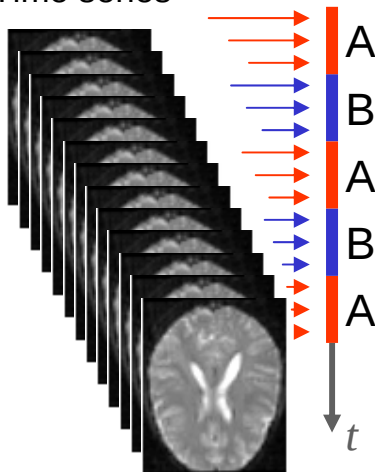


Functional brain imaging by fMRI

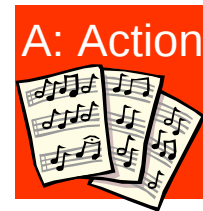
BOLD (Blood Oxygenation Level Dependence)

Basic principle: deoxygenated blood is more paramagnetic than oxygenated blood

Time series



B: Rest



A: Action

EPI acquisition

Matrix size: 128 x 128 x 30 Pixels x 68 measurements

Resolution: 1.56 x 1.56 x 4 mm x 6 seconds

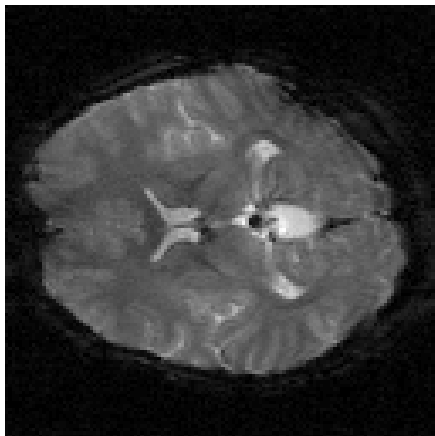
Functional brain imaging by fMRI (Cont'd)

Where?

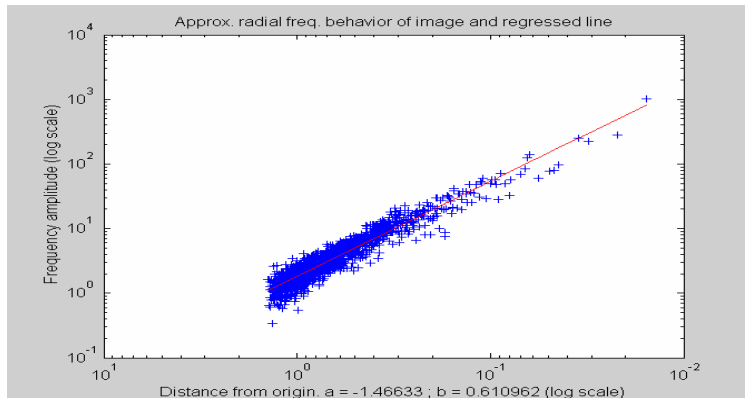
QuickTime™ and a decompressor are needed to see this picture.

- **Main problems:**
 - Small signal changes (1-5%)
 - Very noisy data — averaging
- **Standard solution**
 - Spatial Gaussian smoothing (SPM)

On the fractal nature of fMRI data



Brain: courtesy Jan Kybic

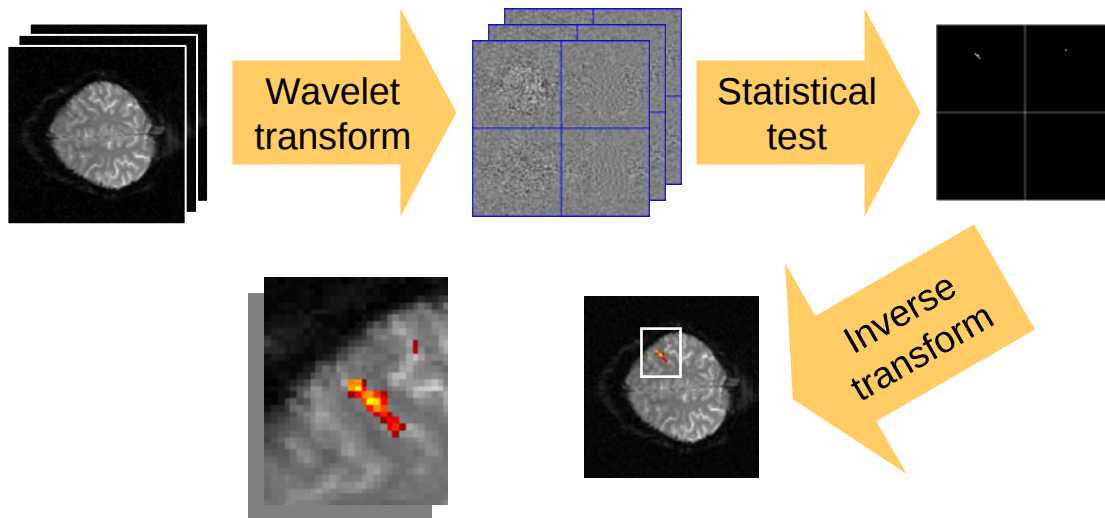


Log-Log plot of spectral density

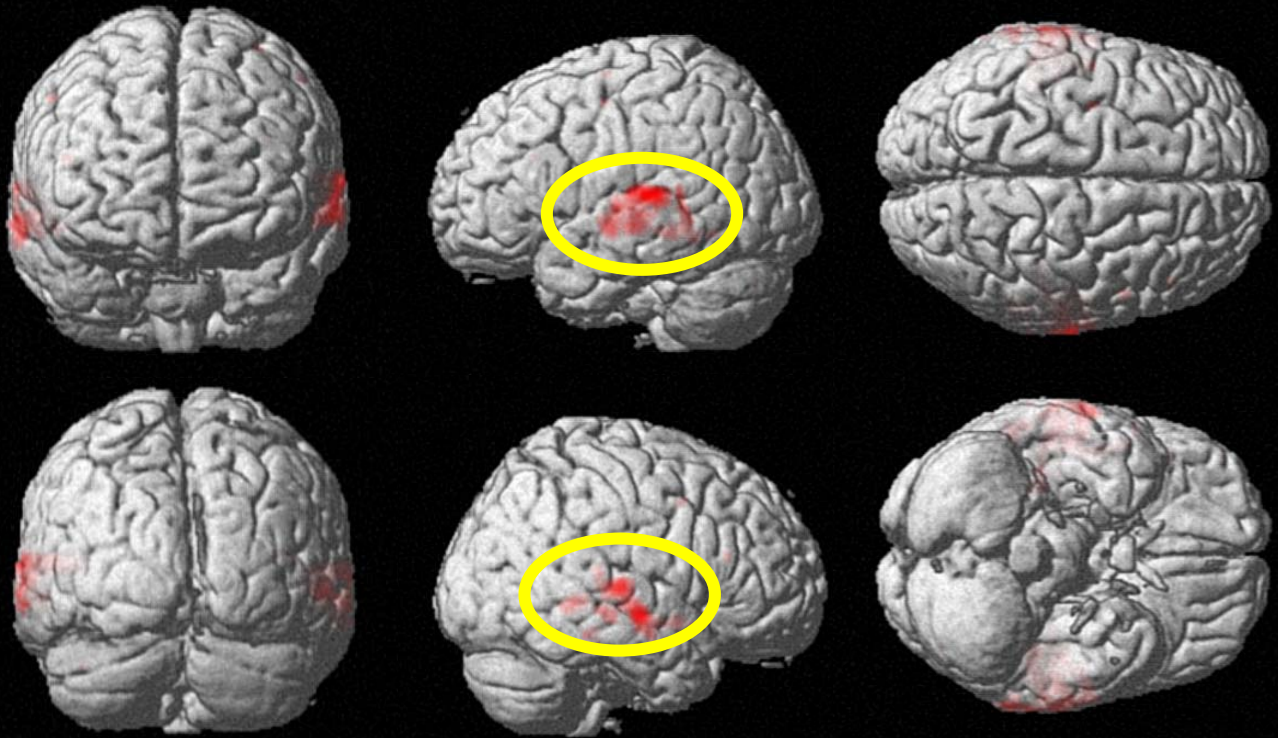
Fractal dimension: $D = 1 + d - H = 2.534$ with $d = 2$ (topological dimension)

Wavelet analysis of fMRI

- Advantages of the wavelet transform (Ruttiman et al., *IEEE-TMI*, 1998)
 - Orthogonal transformation : white noise $\rightarrow$ white noise
 - Decorrelates/whitens fMRI signal
 - Data compression
 - Increased signal-to-noise ratio (averaging effect)
 - Preserves space localization



An example: auditory stimulation



Conclusion

- Fractional splines
 - Natural extension of Schoenberg's polynomial splines
 - Stable, convenient B-spline representation
 - Most polynomial B-spline properties are retained
 - Intimate link with fractional calculus
 - Elementary building blocks: Green functions of fractional derivative operators
 - Efficient digital-filter-based solutions
- New fractional wavelets
 - Multiresolution bases of L_2
 - Fast algorithm
 - Tunable
 - Regularity
 - Localization
 - Order of differentiation
 - Optimal for the processing of fractal-like processes (pre-whitening)
- Application in signal and image processing
 - Processing of fractal-like signals
 - Wavelet-based processing and feature extraction

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+ many other researchers,
and graduate students

- Software and demos at:

<http://bigwww.epfl.ch>



Extensions (on-going work)

- Richer family: alpha-tau splines

$$\mathcal{O}_\tau \xleftrightarrow{\text{Fourier}} (j\omega)^{\frac{\gamma}{2}+\tau} (-j\omega)^{\frac{\gamma}{2}-\tau}$$

[Blu et al., ICASSP'03]

- Multi-dimensional: fractional polyharmonic splines

- Polyharmonic smoothing splines

[Tirosh et al., ICASSP'04]

- Polyharmonic wavelets

[Van de Ville et al., under review]

$$\Delta^{\gamma/2} \xleftrightarrow{\text{Fourier}} \|\omega\|^\gamma$$