

From Cascades to Multifractal Processes

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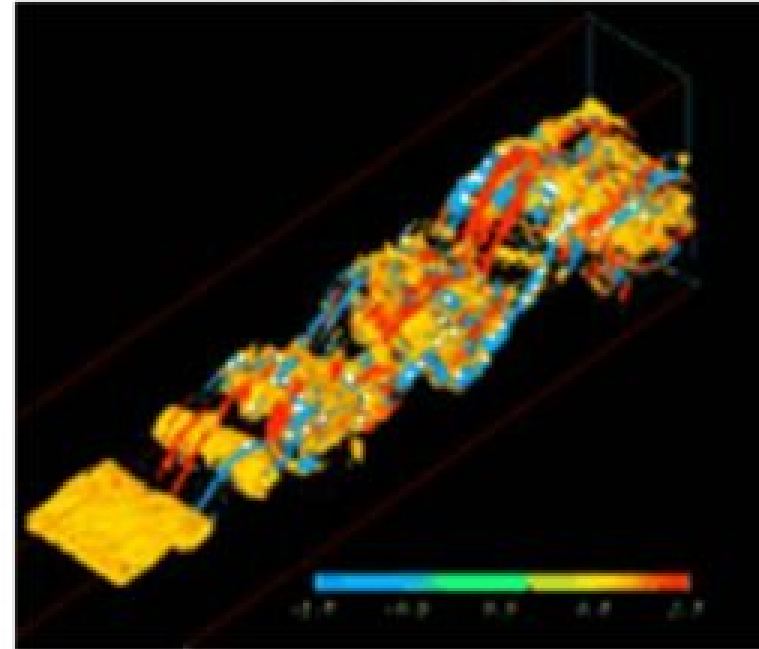
Reading on this talk

- www.stat.rice.edu/~riedi
- This [talk](#)
- Intro for the “untouched mind”
 - Explicit [computations on Binomial](#)
- Monograph on “Multifractal processes”
 - [Multifractal formalism](#) (proofs, [references](#))
 - Multifractal subordination ([warping](#))
- Papers, links

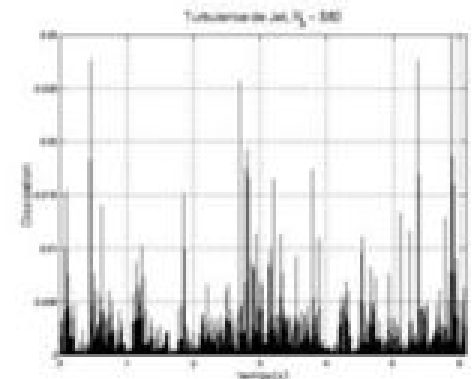
Why Cascades

Turbulence: models wanted

- Kolmogorov 1941 :
 $\langle [v(x+r)-v(x)]^q \rangle \sim r^{q/3}$
- Kolmogorov 1962 :
 $\langle [v(x+r)-v(x)]^q \rangle \sim r^{H(q)}$
- ...and beyond

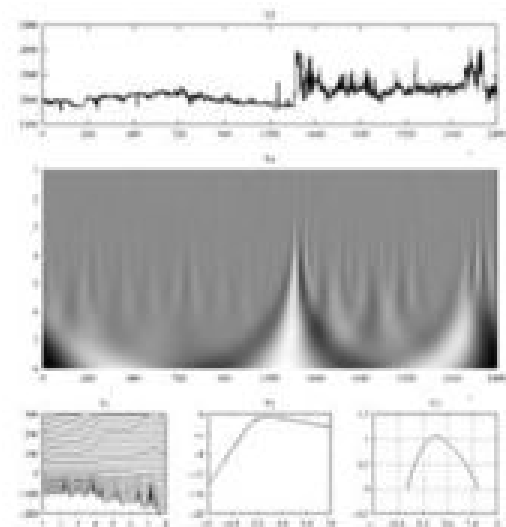
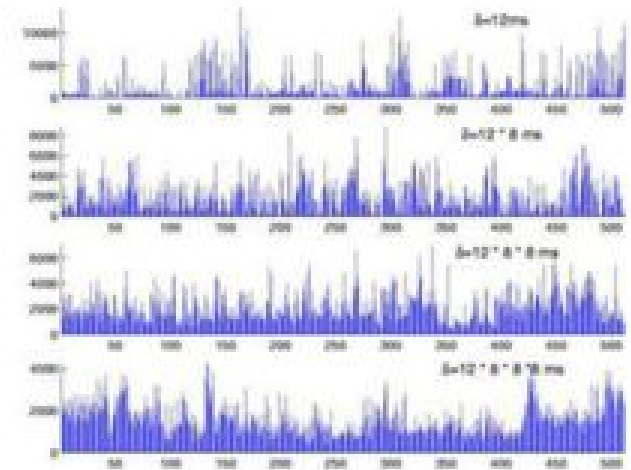
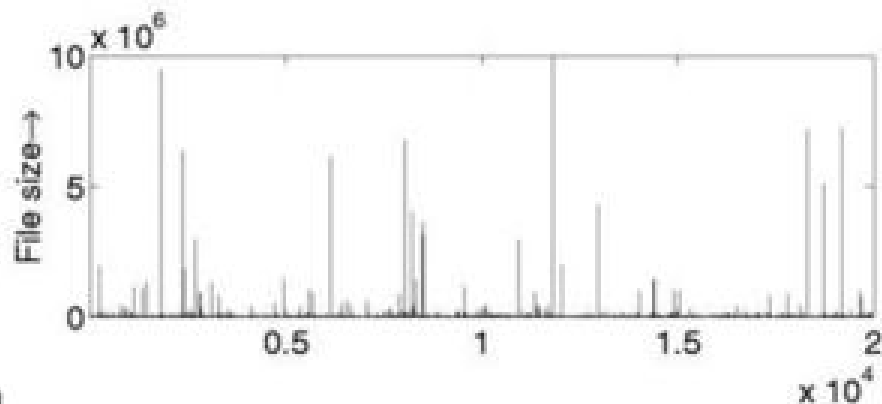


Courtesy P. Chainais



Measured Data

- Networks
- Geophysics
- WWW
- Stock Markets





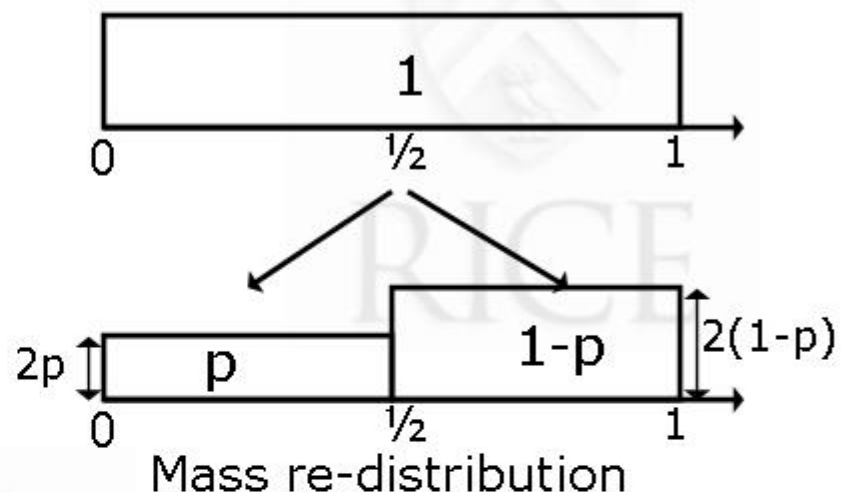
Multifractal Analysis

Toy Example



The Toy: Binomial Cascade

- Start with unit mass
- Redistribute uniformly
portion $p < 1/2$ to the left
portion $1-p$ to the right

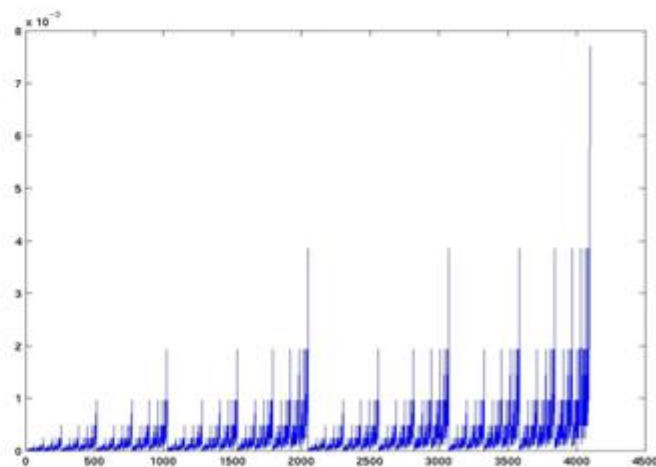


- Iterate
- Converges to measure μ

$$t = \sum_{k=1}^{\infty} \epsilon_k / 2^k \quad \text{with } \epsilon_k = 0, 1$$

$$I(\epsilon_1 \dots \epsilon_n) := \left[\sum_{k=1}^n \epsilon_k / 2^k, \sum_{k=1}^n \epsilon_k / 2^k + 1/2^n \right)$$

$$l_n(t) := \#\{k \leq n : \epsilon_k = 1\} = \sum_{k=1}^n \epsilon_k$$



$$\mu(I(\epsilon_1 \dots \epsilon_n)) = p^{n-l_n(t)} (1-p)^{l_n(t)}$$

Multifractal Spectrum

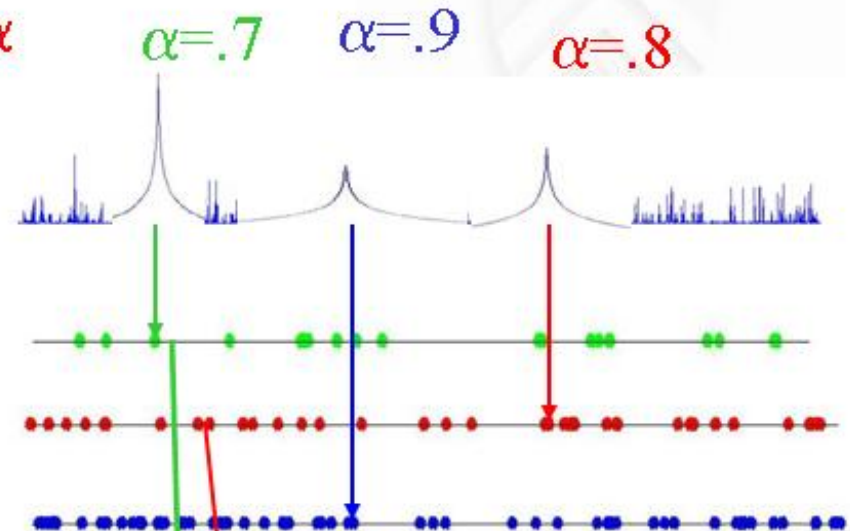
- Oscillate $\sim |t|^\alpha \rightarrow$ **local** strength α

$$\alpha(t) := \liminf_n \alpha_n(t)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$I_n(t)$: dyadic interval containing t

$\Delta I_n(t)$: oscillation indicator
total increment over I_n ,
max increment in I_n ,
wavelet coefficients,...

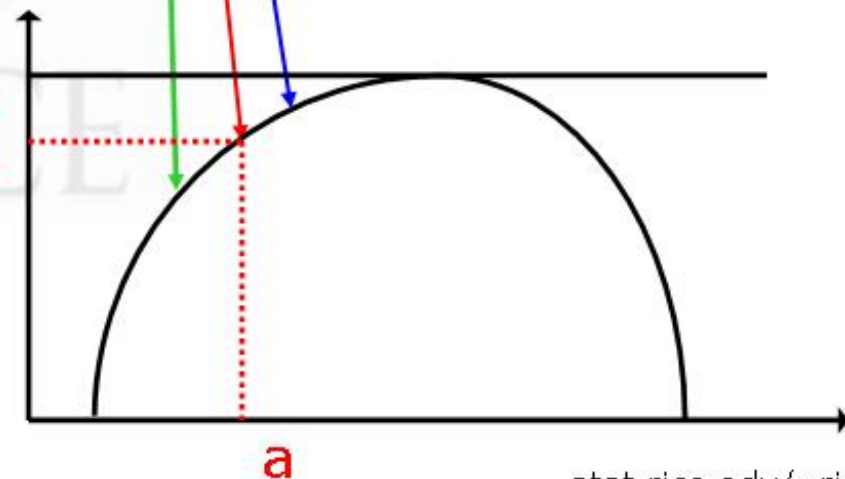


- Collect points t with same α :

$$E_a := \{t : \alpha(t) = a\}$$

$\text{Dim}(E_a)$

- $\text{Dim}(E_a)$** : Spectrum
 $\rightarrow$ prevalance of α



Binomial

We take dyadic partition:

$$I_n(t) = I(\epsilon_1 \dots \epsilon_n) := \left[\sum_{k=1}^n \epsilon_k / 2^k, \sum_{k=1}^n \epsilon_k / 2^k + 1/2^n \right)$$

$$\begin{aligned} \Delta I_n(t) &= \mu(I_n(t)) \\ &= p^{l_n(t)} (1-p)^{n-l_n(t)} \end{aligned}$$

$$\alpha_n(t) = -\frac{n - l_n(t)}{n} \log_2(p) - \frac{l_n(t)}{n} \log_2(1-p)$$

Range of exponents:

$$t = 0: l_n = 0, \alpha_n \rightarrow a_\infty := -\log_2(p) < 1$$

$$t = 1: l_n = n, \alpha_n \rightarrow a_{-\infty} := -\log_2(1-p) > 1$$

Recall

$$\alpha(t) := \liminf_n \alpha_n(t)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$l_n(t) = \#\{k \leq n : \epsilon_k = 1\}$$

"Typical" exponents

$t=0, t=1$ seem "atypical".

Intuition: for a "typical" t :

$$l_n(t) \simeq n/2$$

Rigorously: **Law of Large Numbers**

- Binary digits ϵ_k are independent, $P[\epsilon_k=0] = P[\epsilon_k=1] = 1/2$:
- t is uniformly distributed (i.e., with Lebesgue measure $\mathcal{L}$)

$$\frac{l_n(t)}{n} = \frac{1}{n} \sum_{k=1}^n \epsilon_k \rightarrow \mathbb{E}_{\mathcal{L}}[\epsilon] = 1/2$$

- "Typical" exponent:

$$\begin{aligned} \alpha_n(t) &= -\frac{n - l_n(t)}{n} \log_2(p) - \frac{l_n(t)}{n} \log_2(1 - p) \\ &\rightarrow a_0 := -\frac{1}{2} \log_2(p) - \frac{1}{2} \log_2(1 - p) > 1 \end{aligned}$$

Recall

$$\alpha(t) := \liminf_n \alpha_n(t)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$l_n(t) = \#\{k \leq n : \epsilon_k = 1\}$$

A first point on the Spectrum

Conclusion:

- $\mathcal{L}(E_{a_0}) > 0$
- Mass Distribution Principle
(Lebesgue measure $\mathcal{L}$ is 1-dim Hausdorff measure)

$$\dim E_{a_0} = 1$$

“Where” and “how many” are the other exponents?

- Choose digits “unfairly”, e.g., prefer 1 over 0.

Other exponents

Recall

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$l_n(t) = \#\{k \leq n : \epsilon_k = 1\}$$

The measure μ prefers 1 over 0 (ratio 1-p to p).

Intuitive:

$$l_n(t) \simeq n(1 - p)$$

Rigorously: **Law of Large Numbers using μ**

- Binary digits ϵ are independent, $P[\epsilon_k=0]=p$, $P[\epsilon_k=1]=1-p$:
- t is distributed according to μ
-

$$\frac{l_n(t)}{n} = \frac{1}{n} \sum_{k=1}^n \epsilon_k \rightarrow \mathbb{E}_{\mu}[\epsilon] = 1 - p$$

- μ -typical exponent

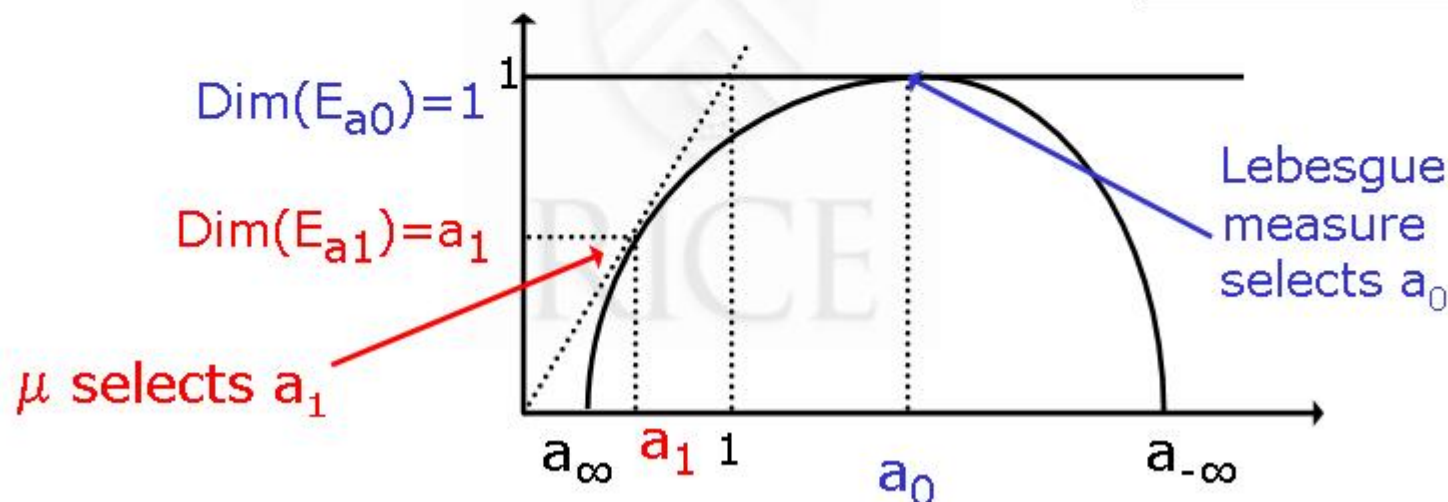
$$\begin{aligned} \alpha_n(t) &= -\frac{n - l_n(t)}{n} \log_2(p) - \frac{l_n(t)}{n} \log_2(1 - p) \\ &\rightarrow a_1 := -p \log_2(p) - (1 - p) \log_2(1 - p) < 1 \end{aligned}$$

A second point on the Spectrum

Conclusion:

- $\mu(E_{a_1}) > 0$
- Mass Distribution Principle $\rightarrow \dim E_{a_1} \geq a_1$
(Hausdorff dimension of μ ? It is $a_1 < 1$!)

$$\alpha_n(t) = \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \rightarrow a_1$$



- All exponents: Inspiration from Large Deviation Theory



Large Deviations and the Multifractal Formalism

Box Spectrum

Recall

$$I_n(t) = I(\epsilon_1 \dots \epsilon_n)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$E_a := \{t : \alpha(t) = a\}$$

• Notation:

$$N_n(a, \delta) := \#\{(\epsilon_1 \dots \epsilon_n) : a - \delta \leq \alpha_n(\epsilon_1 \dots \epsilon_n) < a + \delta\}.$$

$$f(a) := \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 N_n(a, \delta)$$

• Thm: we always have

$$\dim E_a \leq f(a)$$

Proof

Fix a . To prove the first part of the lemma consider an arbitrary $\gamma > f(a)$, and choose $\eta > 0$ such that $\gamma > f(a) + 2\eta$. Then, there is $\varepsilon > 0$ and integer m_0 such that

$$N_n(a, \varepsilon) \leq 2^{n(f(a) + \eta)}$$

for all $n > m_0$. Let us define $J(m) := \cup \{k_n : n \geq m \text{ and } a - \varepsilon \leq \alpha_n^k \leq a + \varepsilon\}$. Then, for any m the intervals I_n^k with $k_n \in J(m)$ form a cover of E_a . These intervals are of length less than 2^{-m} . Moreover, for any $m > m_0$ we have

$$\begin{aligned} \sum_{k_n \in J(m)} |I_n^k|^\gamma &= \sum_{n \geq m} N_n(a, \varepsilon) \cdot 2^{-n\gamma} \\ &\leq \sum_{n \geq m} 2^{-n(\gamma - f(a) - \eta)} \leq \sum_{n \geq m} 2^{-n\eta} \end{aligned}$$

tends to zero with $m \rightarrow \infty$. We conclude that the γ -dimensional Hausdorff measure of E_a is zero, hence, $\dim E_a \leq \gamma$. Letting $\gamma \rightarrow f(a)$ completes the proof.

[www.stat.rice.edu/~riedi]

• Beware the **folklore**: $f(a)$ is NOT the box-dim of E_a

Legendre spectrum

Recall

$$I_n(t) = I(\epsilon_1 \dots \epsilon_n)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$E_a := \{t : \alpha(t) = a\}$$

- Notation: **partition sum and function**

$$S_n(q) := \sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q = \sum_{\epsilon_1 \dots \epsilon_n} |2^n|^{q\alpha_n(\epsilon_1 \dots \epsilon_n)}.$$

$$\tau(q) := \liminf_{n \rightarrow \infty} -\frac{1}{n} \log_2 S_n(q)$$

- Thm: we always have

$$f(a) \leq \tau^*(a) := \inf_q (qa - \tau(q))$$

Proof

$$\begin{aligned} \sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q &\geq \sum_{\alpha_n(\epsilon_1 \dots \epsilon_n) \in [a-\delta, a+\delta]} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q \geq N_n(a, \delta) 2^{-n(qa + |q|\delta)} \\ &\geq 2^{-n(qa - f(a) + \delta' + |q|\delta)} \end{aligned}$$

Legendre spectrum

- Thm: provided $\alpha_n(t)$ are bounded we have

$$f(a) = \tau^*(a) \quad \text{for } a = \tau'(q).$$

- Proof idea: steepest ascent (**large deviations**)

$$\begin{aligned} \sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q &\leq \sum_{l=1}^m \sum_{\alpha_n(\epsilon_1 \dots \epsilon_n) \in [l\delta - \delta, l\delta + \delta]} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q \\ &\leq \sum_{l=1}^m N_n(l\delta, \delta) 2^{-n(ql\delta - |q|\delta)} \\ &\leq \sum_{l=1}^m 2^{-n(ql\delta - f(l\delta) - \delta' - |q|\delta)} \leq m 2^{-n(\inf_a (qa - f(a)) - \delta' - |q|\delta)} \end{aligned}$$

- **Thus:** $\tau(q) = f^*(q) = \inf_a (qa - f(a))$ for all q .
- τ is concave, non-decreasing, differentiable with exceptions
- Recover $f = f^{**}$ at $a = \tau'(q)$ using lower semi-continuity

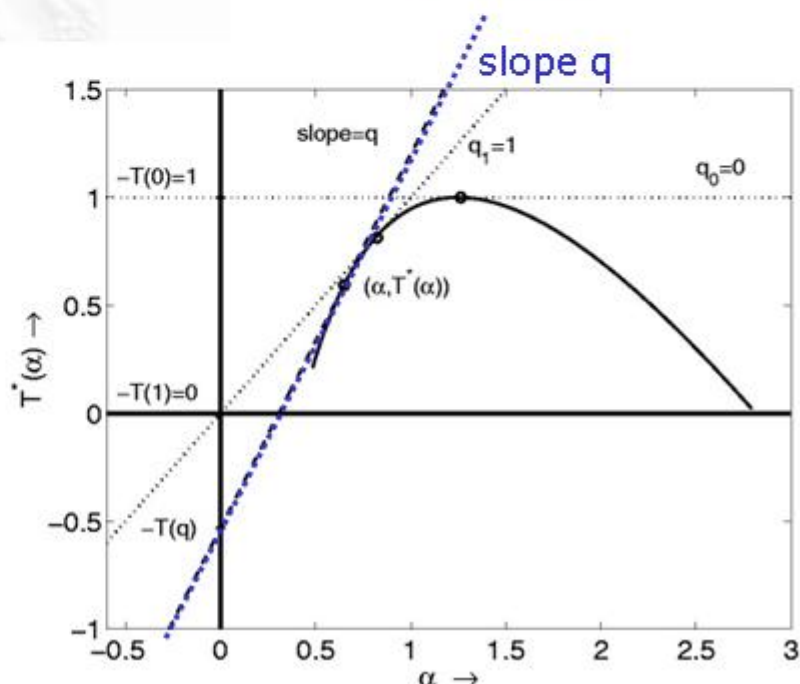
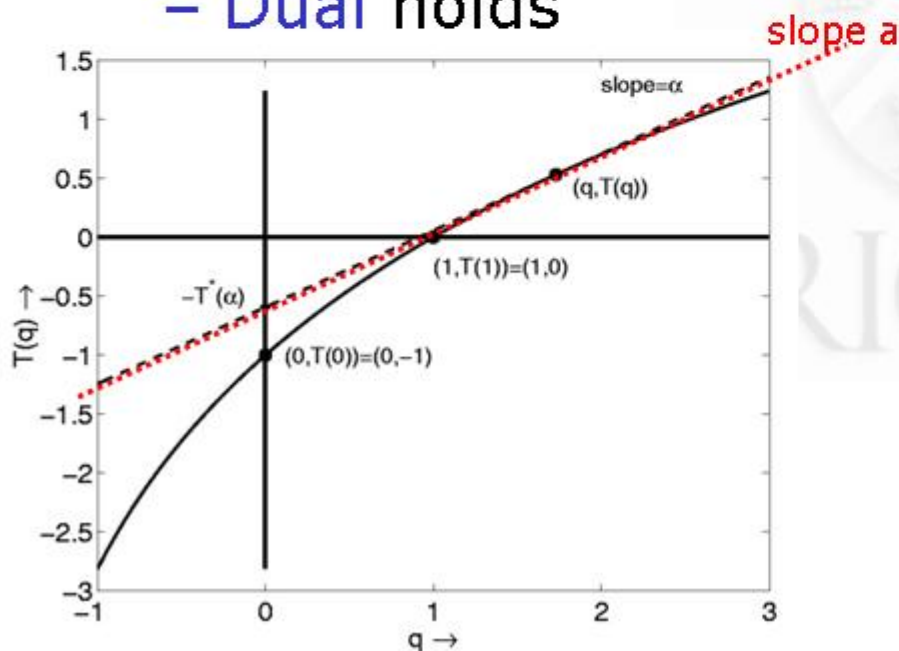
Legendre transform 101

- Elementary calculus:

$$\tau^*(a) := \inf_q (qa - \tau(q)) = \bar{q}a - \tau(\bar{q})$$

where $\bar{q}$ is defined by $a = \tau'(\bar{q})$

- Tangent of **slope a** to $\tau(q)$
- Intersection with ordinate yields **$-\tau^*(a)$**
- **Dual** holds





Binomial Spectrum

continued

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RICE

Partition function of the Binomial

$$\begin{aligned} S_n(q) &= \sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q \\ &= \sum_{\epsilon_1 \dots \epsilon_n} [p^{n-l_n(\epsilon_1 \dots \epsilon_n)} (1-p)^{l_n(\epsilon_1 \dots \epsilon_n)}]^q \\ &= \sum_{l=0}^n \binom{n}{l} [p^{n-l} (1-p)^l]^q \\ &= [p^q + (1-p)^q]^n. \end{aligned}$$

- (Upper) envelop of $\dim(E_a)$:

$$\begin{aligned} \tau(q) &= \liminf_{n \rightarrow \infty} -\frac{1}{n} \log_2 S_n(q) \\ &= -\log_2 [p^q + (1-p)^q] \end{aligned}$$

Insight from Large Deviations

- From steepest ascent:

$$\begin{aligned} S_n(q) &= \sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q \simeq 2^{-n(\inf_a (qa - f(a)))} \\ &= 2^{-n(q\bar{a} - f(\bar{a}))} \simeq \sum_{\alpha_n(\epsilon_1 \dots \epsilon_n) \simeq \bar{a}} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q \end{aligned}$$

- Dominant** terms in $S_n(q)$, for fixed q , are the ones with

$$\alpha_n(\epsilon_1 \dots \epsilon_n) = \frac{\log \Delta I_n}{\log |I_n|} \simeq \bar{a} = \tau'(q)$$

- ...and vice versa: these terms contribute such that

$$S_n(q) \simeq 2^{-n\tau(q)} = (p^q + (1-p)^q)^n$$

For the Binomial these correspond
to mass re-distribution in **ratio p^q to $(1-p)^q$**

Locating the exponents

Fix q .

Consider the measure μ_q defined as μ but with mass ratio p^q to $(1-p)^q$. Intuitively, we have then:

$$l_n(t) = \#\{k \leq n : \epsilon_k = 1\} \simeq n \frac{(1-p)^q}{p^q + (1-p)^q} = n(1-p)^q 2^{\tau(q)}$$

Rigorously: **Law of Large Numbers using μ_q**

- Binary digits ϵ : indep, $P[\epsilon_k=0]=p^q 2^{\tau(q)}$, $P[\epsilon_k=1]=(1-p)^q 2^{\tau(q)}$
- t is distributed according to μ_q

- $$\frac{l_n(t)}{n} = \frac{1}{n} \sum_{k=1}^n \epsilon_k \rightarrow \mathbb{E}_{\mu_q}[\epsilon] = (1-p)^q 2^{\tau(q)}$$

- μ -typical exponent

$$\begin{aligned} \alpha_n(t) &= -\frac{n - l_n(t)}{n} \log_2(p) - \frac{l_n(t)}{n} \log_2(1-p) \\ &\rightarrow a_q := -p^q 2^{\tau(q)} \log_2(p) - (1-p)^q 2^{\tau(q)} \log_2(1-p) \end{aligned}$$

Completing the Spectrum

Conclusion:

- $\mu_q(E_{a_q}) > 0$
- $a_q = \tau'(q)$

Recall

$$\tau(q) = -\log_2[p^q + (1-p)^q]$$

$$a_q = -p^q 2^{\tau(q)} \log_2(p) - (1-p)^q 2^{\tau(q)} \log_2(1-p)$$

- Hausdorff dimension of μ_q :

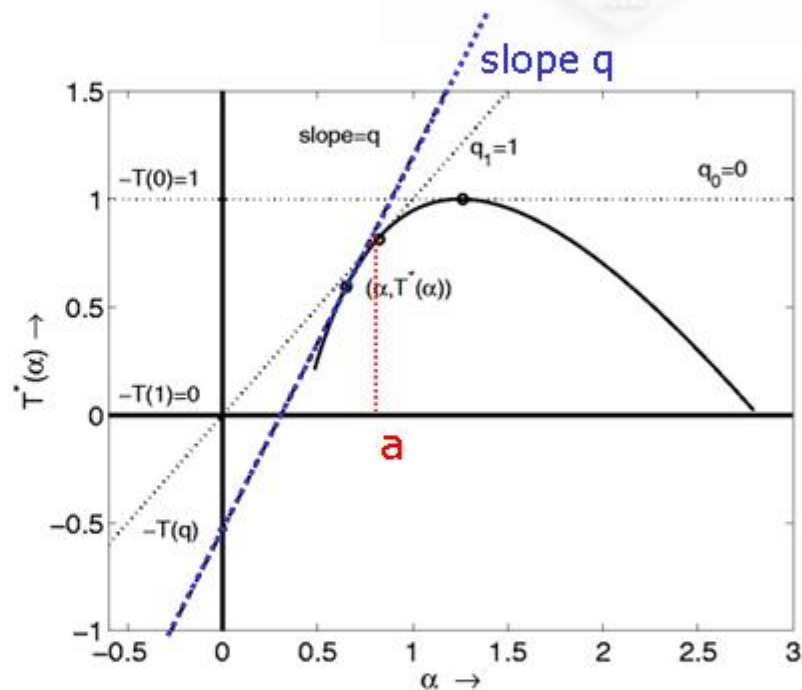
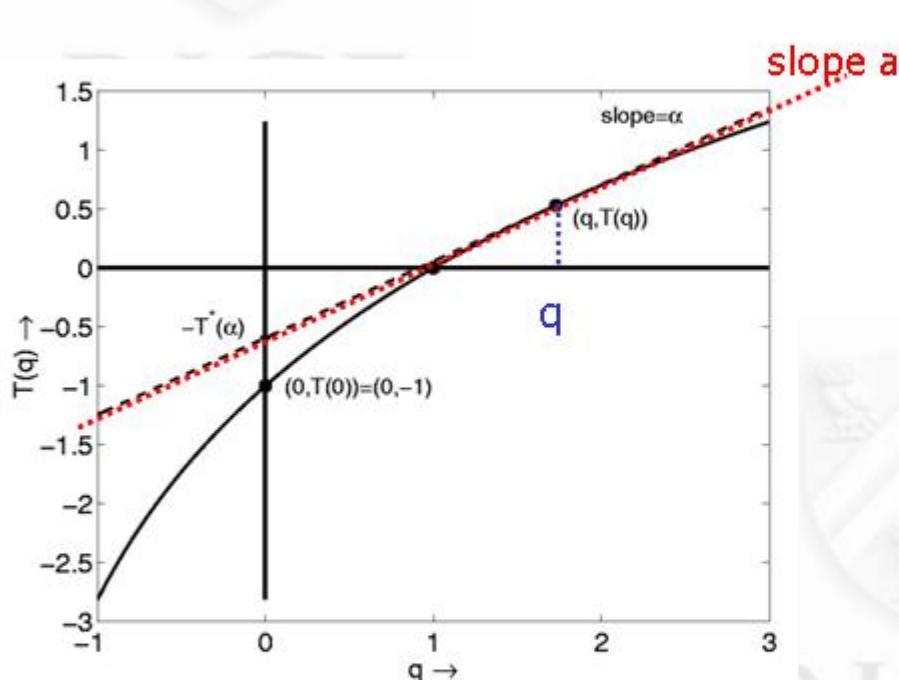
$$\begin{aligned} \frac{\log \mu_q(I_n(t))}{\log |I_n(t)|} &= -\frac{n - l_n(t)}{n} \log_2[p^q 2^{\tau(q)}] - \frac{l_n(t)}{n} \log_2[(1-p)^q 2^{\tau(q)}] \\ &= -\tau(q) + q\alpha_n(t) \\ &\rightarrow qa_q - \tau(q) = \tau^*(a_q) \end{aligned}$$

- Mass Distribution Principle

$$\dim E_{a_q} \geq \tau^*(a_q)$$

Lessons

Binomial cascade: $\dim E_a = f(a) = \tau^*(a)$



- Points with exponent $\log_\mu(I(t))/\log|I(t)| \sim a = \tau'(q)$
 - Are concentrated on the support of μ_q
 - Dominate the partition sum $S_n(q)$
- Partition function allows to bound/estimate $\dim(E_a)$



Random Cascades

A further multifractal envelop
Convergence and Degeneracy



Multifractal Spectra and Randomness

- $\Delta I_n(t)$: oscillation indicator for process or measure

- Pathwise

$$\dim E_a \leq f(a) \leq \tau^*(a)$$

Recall

$$E_a = \{t : \alpha(t) = a\}$$

$$N_n(a, \varepsilon) \simeq 2^{nf(a)}$$

$$S_n(q) \simeq 2^{-n\tau(q)}$$

- $S_n(q)$ is q -th moment estimator.
- Replace by true moment:

$$T(q) := \liminf_{n \rightarrow \infty} -\frac{1}{n} \log_2 \mathbb{E} S_n(q)$$

Recall

$$S_n(q) = \sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q$$

- ...analytically easier to handle and often sufficient
- $T(q)$ is **concave** like $\tau(q)$, but NOT always increasing

Pathwise and deterministic envelop

- Lemma: With probability one for all q with $T(q) < \infty$.

$$\tau(q, \omega) \geq T(q)$$

[Proof: www.stat.rice.edu/~riedi]

- Cor: $\tau^*(a, \omega) \leq T^*(a)$

$$\mathbb{E}[\log(X)] \leq \log \mathbb{E}[X]$$

- Weaker result from Chebichev inequality:

$$\mathbb{E}[\tau(q, \omega)] \geq T(q)$$

Quenched Average

Annealed Average

- Material science: free energy is “self-averaging” iff quenched and annealed averages are equal.

Multifractal Envelops

- Almost surely, for all a :

Recall at $a = \tau'(q)$
 $f(a) = \tau^*(a)$

$$\dim E_a \leq f(a) \leq \tau^*(a) \leq T^*(a)$$

- Holds **always** provided use **same** ΔI_n in all spectra
- Choice of scales I_n
 - I_n is here **dyadic**, could be any sub-exponential
 - This could affect/change f , τ and/or T due to boundary effects
 - Robust: ΔI_n = oscillation in I_n and its neighbor intervals
- Choice of **oscillation indicator** ΔI_n
 - For true Hoelder regularity ΔI_n = **max increment** “around” I_n
 - ΔI_n = Wavelet coefficient: only a **proxy** to Hoelder regularity!
 - For **measures** supported on $[0,1]$: $\Delta I_n = \mu(I_n)$ gives Hoelder!

Multifractal Envelopes

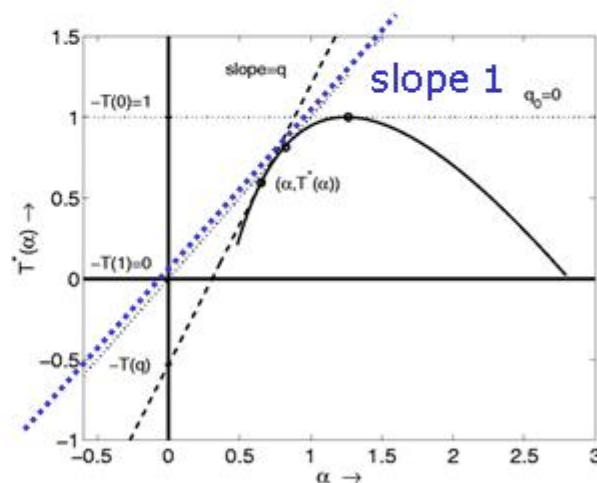
Recall at $a = \tau'(q)$
 $f(a) = \tau^*(a)$

- Almost surely, for all a :

$$\dim E_a \leq f(a) \leq \tau^*(a) \leq T^*(a)$$

- Special feature:

- If a property of
 “bounded total variation”
 holds then the spectrum f
 touches the bi-sector:



If $\sum_{\epsilon_1 \dots \epsilon_n} \Delta I_n(\epsilon_1 \dots \epsilon_n) \leq C$ for all n
 then $\tau(1) = 0$.

Multifractal Envelops

- Almost surely, for all a :

Recall at $a = \tau'(q)$
 $f(a) = \tau^*(a)$

$$\dim E_a \leq f(a) \leq \tau^*(a) \leq T^*(a)$$

- Terminology:

- Multifractal formalism “holds” if

- $\dim(E_a) = f(a) = \tau^*(a)$

with your preferred oscillation indicators ΔI_n ,
e.g., Holder exponent in E_a , wavelet decay in $f(a)$.

[First step: show T is same for Holder and wavelets.]

- Falconer: “A concise definition of a multifractal tends to be avoided.”
- Others: “An object is multifractal if the formalism holds for it.”
- Others: “An object is multifractal if it has more than one singularity exponent”. (not mono-fractal)



Multifractals and classical regularity



Besov spaces

- For oscillation indicator from wavelets:

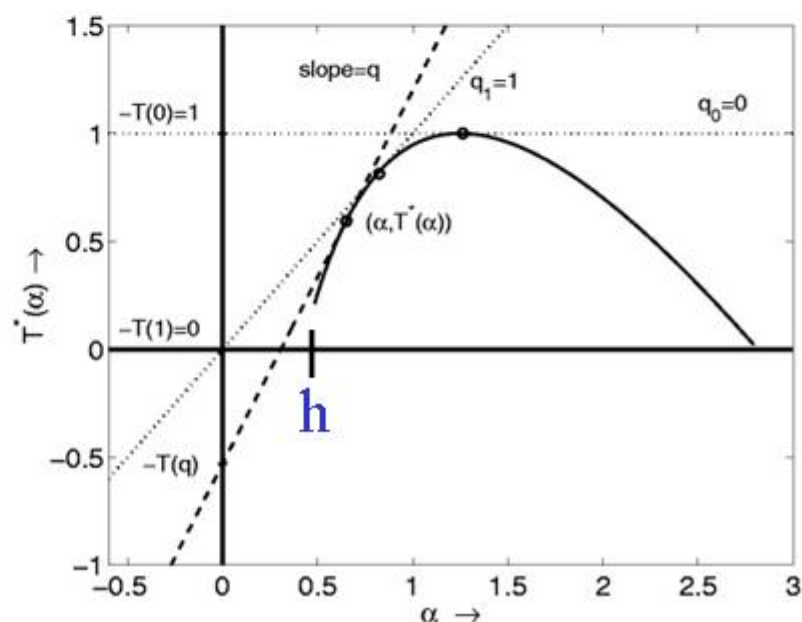
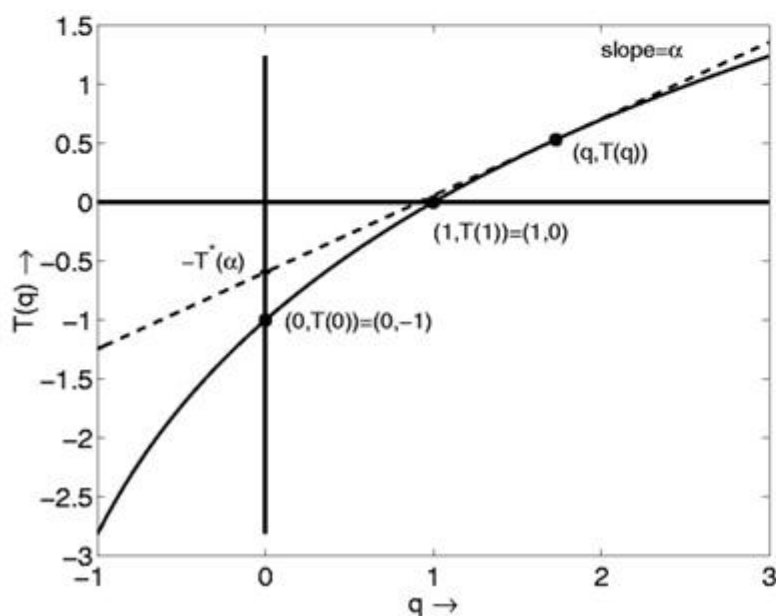
$$\sup\{s : Y \in B_v^s(L^u)\} = \frac{\tau(u) + 1}{u}$$

- Proof: use wavelet coefficients $C_{j,k} = \Delta I_j(k2^j)$ and equivalent Besov norm

$$\left(\sum_k |D_{0,0}|^v\right)^{1/v} + \left(\sum_{j \geq J_0} \left(\sum_k 2^{jsu} 2^{-j} |2^{j/2} C_{j,k}|^u\right)^{v/u}\right)^{1/v}.$$

Kolmogorov

- Thm [Kolmogorov]:
 - If $E[|A(s)-A(t)|^b] < C |s-t|^{1+d}$ then almost all paths of A are of (global) Holder-continuity for all $h < d/b$,
 - i.e., for all $h < T(q)/q$.
- The best such h is $\min(a : T^*(a) > 0)$.
 - $T(q)/q = \text{slope of tangent through the origin}$.





Binomial Spectrum

continued

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Binomial with Random Multipliers

- Random re-distribution
- **Multipliers Independent**
between scales

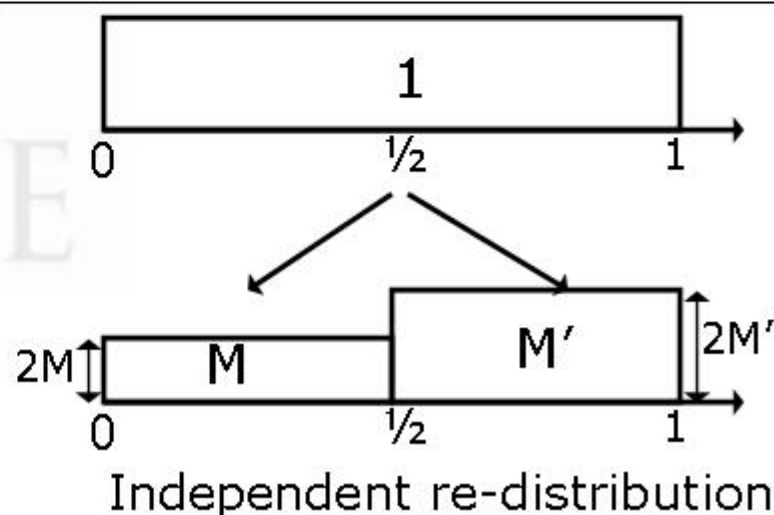
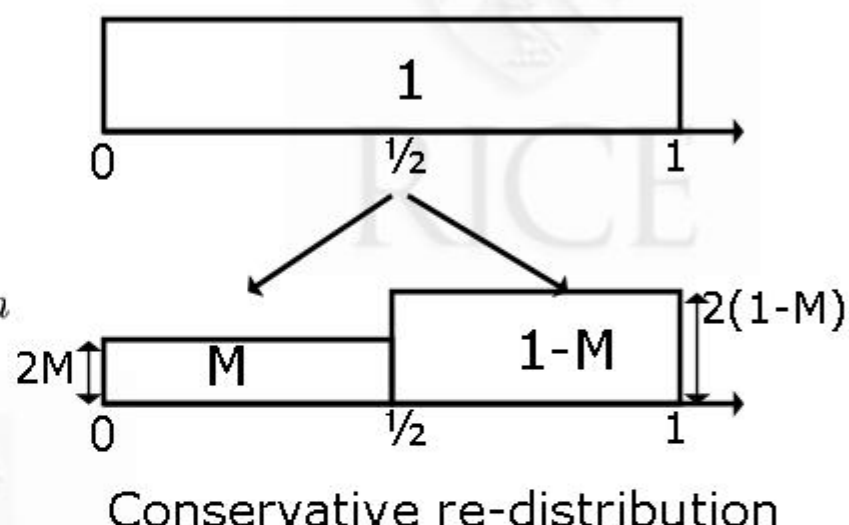
$$\mu_n(I(\epsilon_1 \dots \epsilon_n)) = M_{\epsilon_1} \dots M_{\epsilon_1 \dots \epsilon_n}$$

- Conservative:

$$M_{\epsilon_1 \dots \epsilon_n 0} + M_{\epsilon_1 \dots \epsilon_n 1} = 1$$

- Conservation is **too restrictive** for stationarity!
- “Martingale de Mandelbrot”:

$$\mathbb{E}[M_{\epsilon_1 \dots \epsilon_n 0} + M_{\epsilon_1 \dots \epsilon_n 1}] = 1$$



Convergence of Random Binomial

- Conservative:

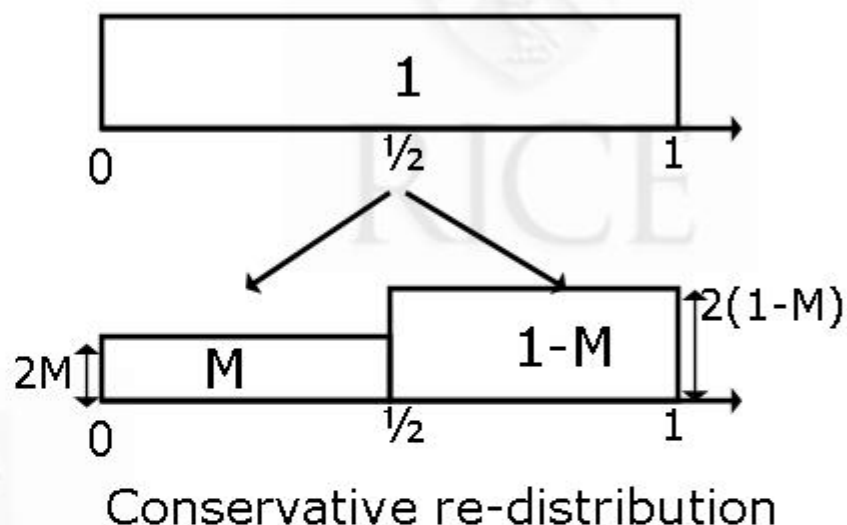
- $M_{\epsilon_1 \dots \epsilon_n 0} + M_{\epsilon_1 \dots \epsilon_n 1} = 1$

- For all $m > n$

$$\mu_m(I(\epsilon_1 \dots \epsilon_n)) = M_{\epsilon_1} \dots M_{\epsilon_1 \dots \epsilon_n}$$

- Thus converges to

$$\mu(I(\epsilon_1 \dots \epsilon_n)) = M_{\epsilon_1} \dots M_{\epsilon_1 \dots \epsilon_n}$$



Convergence of Random Binomial

- “Martingale de Mandelbrot”:
 - A **price** to pay towards stationarity

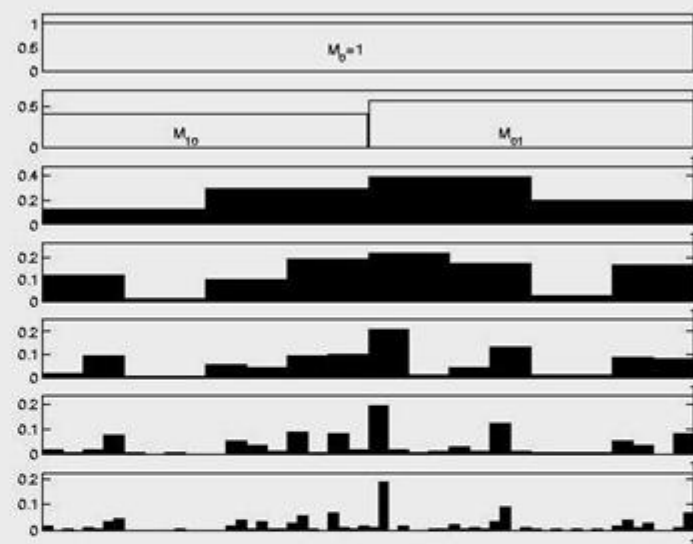
$$- \mathbb{E}[M_{\epsilon_1 \dots \epsilon_n 0} + M_{\epsilon_1 \dots \epsilon_n 1}] = 1$$

- Martingale: For all $m > n$

$$\mathbb{E}[\mu_m(I(\epsilon_1 \dots \epsilon_n)) | \mathcal{F}_n] = M_{\epsilon_1} \dots M_{\epsilon_1 \dots \epsilon_n} = \mu_n(I(\epsilon_1 \dots \epsilon_n))$$

- Thus converges almost surely (but may **degenerate**)
- We have

$$\mathbb{E}[\mu(I(\epsilon_1 \dots \epsilon_n)) | \mathcal{F}_n] = M_{\epsilon_1} \dots M_{\epsilon_1 \dots \epsilon_n}$$



Envelope for Random Binomial

- By independence of multipliers
 - Martingale of Mandelbrot:

$$\mathbb{E}[S_n(q)] = \sum_{\epsilon_1 \dots \epsilon_n} \mathbb{E}|\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q = \sum_{\epsilon_1 \dots \epsilon_n} \mathbb{E}|M_{\epsilon_1} \dots M_{\epsilon_1 \dots \epsilon_n}|^q = 2^n \mathbb{E}[M^q]^n.$$

$$T(q) = -1 - \log_2 \mathbb{E}[M^q]$$

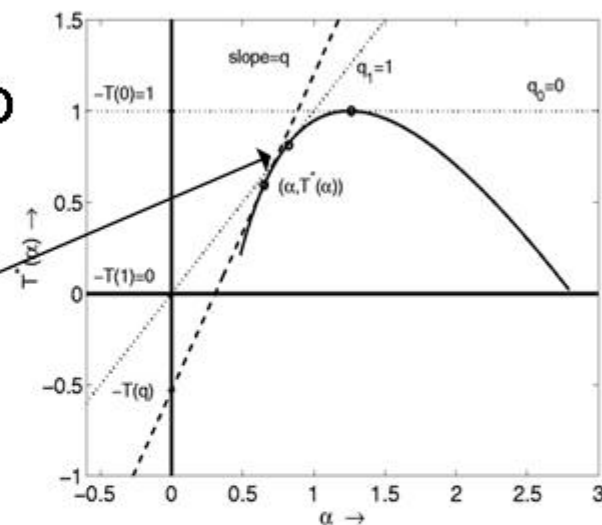
- Conservative: similar

$$\begin{aligned} \mathbb{E}[S_n(q)] &= \sum_{\epsilon_1 \dots \epsilon_n} \mathbb{E}[M^q]^{n-l_n(\epsilon_1 \dots \epsilon_n)} \mathbb{E}[(1-M)^q]^{l_n(\epsilon_1 \dots \epsilon_n)} \\ &= (\mathbb{E}[M^q] + \mathbb{E}[(1-M)^q])^n \\ &= (2\mathbb{E}[M^q])^n. \end{aligned}$$

Kahane-Peyriere theory for the Martingale of Mandelbrot

- Martingale “degenerates”

- iff $\mu([0,1])=0$ almost surely zero
- iff $E \mu([0,1])=0$
- iff $T'(1) \leq 0$



- Intuition:

- $T'(1) = a_1$ = dimension of the carrier of μ .
- If $T'(1) > 0$ then
 - $\exists q > 1$ with $T(q) > 0$
 - μ converges in L_q
 - $\mathbb{E}[\mu([0, 1])] = \lim_n \mathbb{E}[\mu_n([0, 1])] = 1$

Multifractal formalism holds

- Thm for random **binomial** [Barral, Arbeiter-Patschke, Falconer]:
 - Set $\Delta I_n = \mu(I_n)$.
 - Assume M has a **finite** moment of some **negative order**
 - Then, with probability 1: for all a such that $T^*(a) > 0$

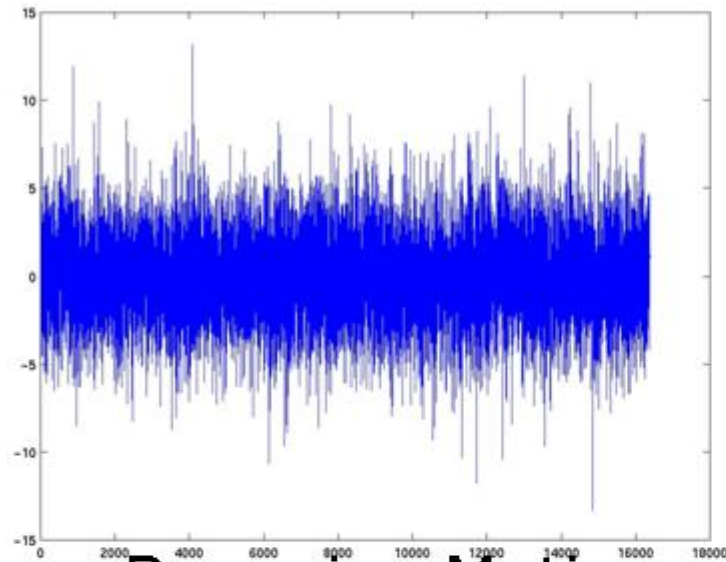
$$\dim E_a = f(a) = \tau^*(a) = T^*(a)$$

- Note:
 - $T^*(a) > 0$ means $a = T'(q)$ with q limited by tangents through the origin: $T'(q) = T(q)/q$.
 - Little known in general for other a ...or q ! Possible: $\tau(q) > T(q)$
 - Proofs: Use Mass distortion Principle with factors M^q

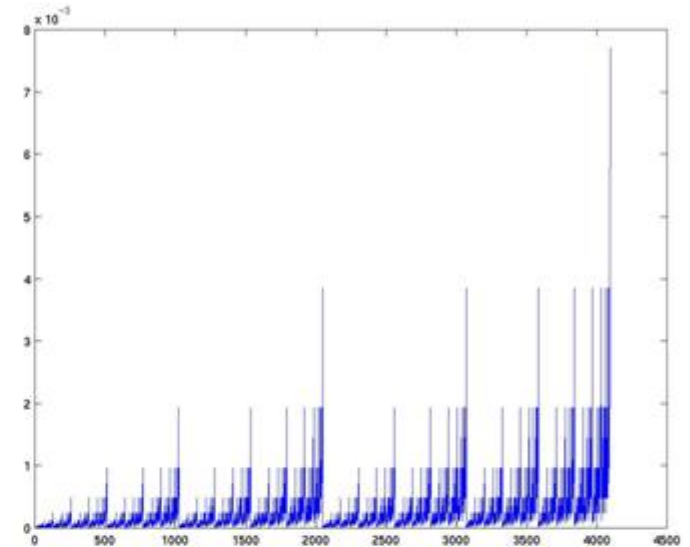
Wavelets for the Binomial

- Compactly supported wavelet
 - ΔI_n = wavelet coefficient corresponding to I_n
 - ΔI_n same rescaling property as measure itself
 - Same $T(q)$
 - Multifractal formalism holds

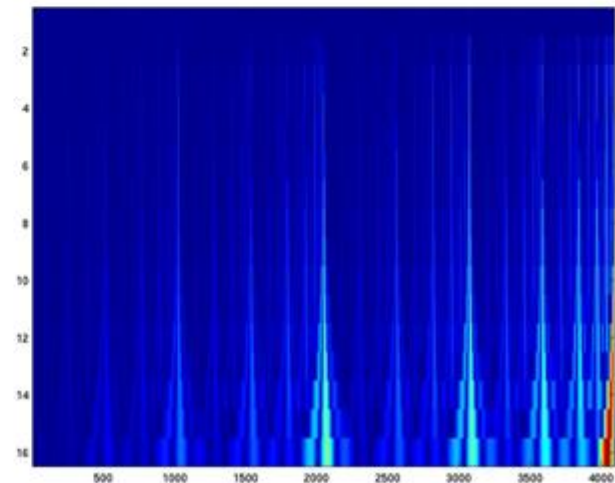
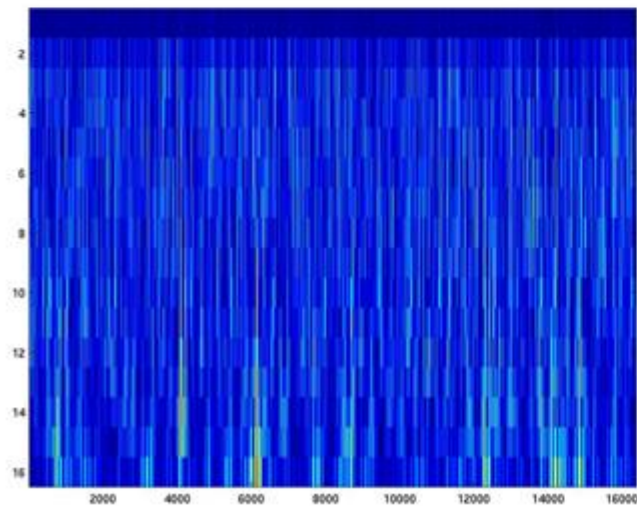
Toy examples



White noise



Cascade

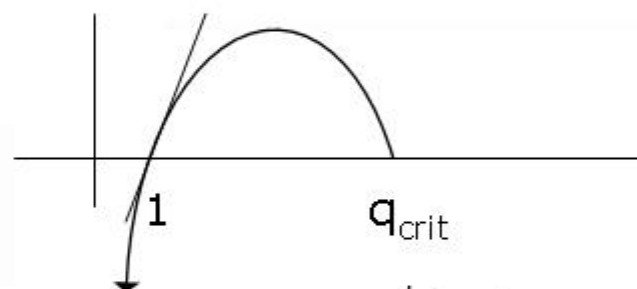


Log-Normal Binomial

- Deterministic envelope is a parabola: [Mandelbrot]

$$T(q) = (q-1) \left(1 - \frac{\sigma^2}{2 \ln(2)} q \right) \quad \text{for } q < q_{\text{crit}} := 2 \ln(2) / \sigma^2.$$

- Zeros: $q=1$, $q=q_{\text{crit}}$



- Non-Degeneracy: $T'(1) > 0 \Leftrightarrow q_{\text{crit}} > 1 \Leftrightarrow 2 \ln(2) > \sigma^2$

- Spectrum is parabola as well

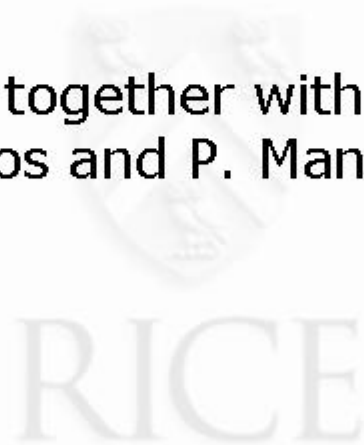
$$T^*(a) = 1 - \frac{\ln(2)}{2\sigma^2} \left(a - 1 - \frac{\sigma^2}{2 \ln(2)} \right)^2$$

- Partition function $\tau(q)$ is non-decreasing,
- thus $\tau(q) > T(q)$ (at least) for $q > (1+q_{\text{crit}})/2$



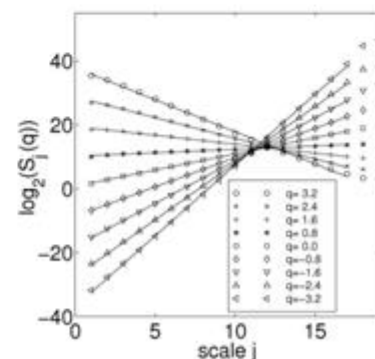
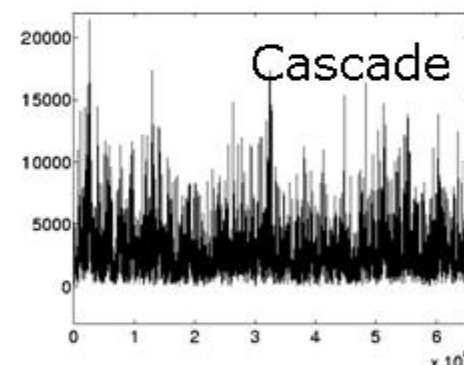
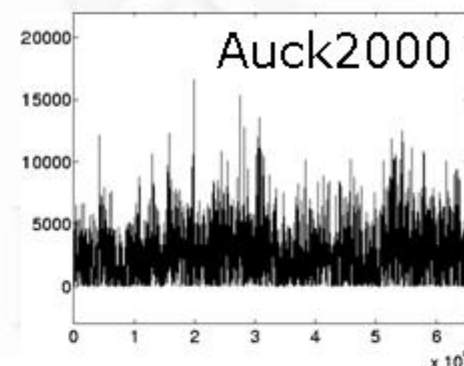
Multifractal Product of Pulses

together with
I. Norros and P. Mannersalo



Network Traffic is Multifractal

- Visually striking
- Scaling of impressive quality
(Levy Vehel & RR '96,
Norros & Mannersalo '97,
Willinger et al '98)
- Statistical models:
 - Binomial cascades with
scale dependent multipliers
(Crouse & RR '98, Willinger et al '98)
- Not stationary!
 - Cumbersome for statistics
 - and probability (Queueing)



Multifractal paradigm

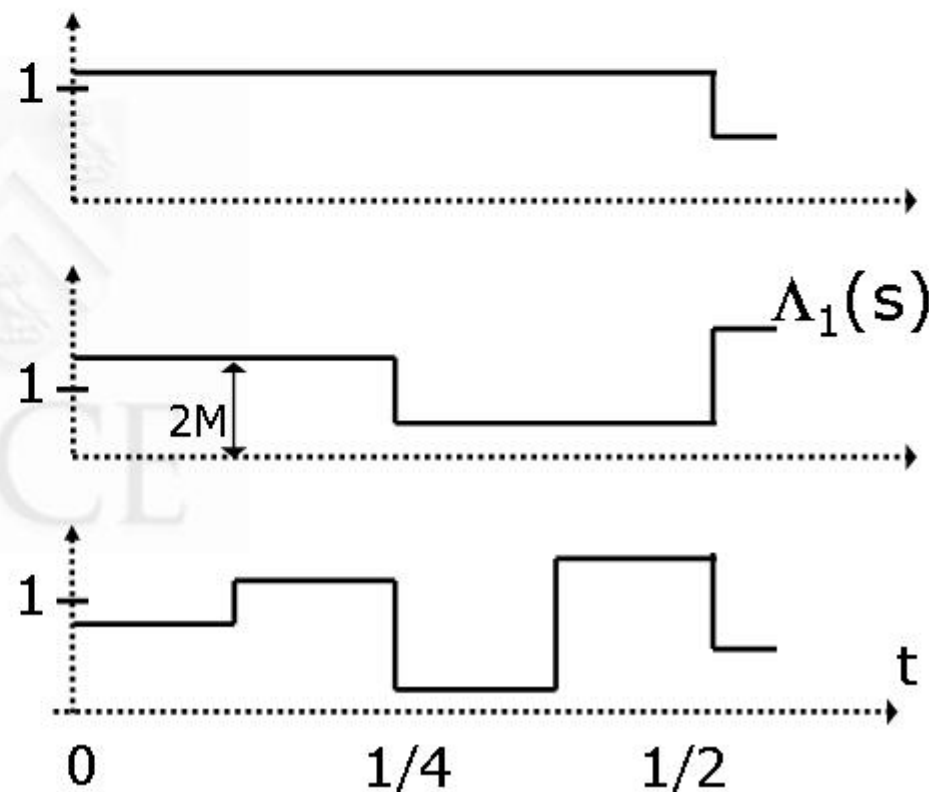
Multiplicative Processes:

- From **redistributing** mass to **multiplying pulses**

$$A(t) = \lim_{n \rightarrow \infty} \int_0^t \Lambda_0(s) \dots \Lambda_n(s) ds$$

Binomial Cascade

- $\Lambda_n(s)$ is constant on dyadic intervals
- Conservative:
 $\Lambda_n(2k/2^n) + \Lambda_n((2k+1)/2^n) = 2$
- Martingale de Mandelbrot:
 $E \Lambda_n(s) = 1$
- **Not stationary**



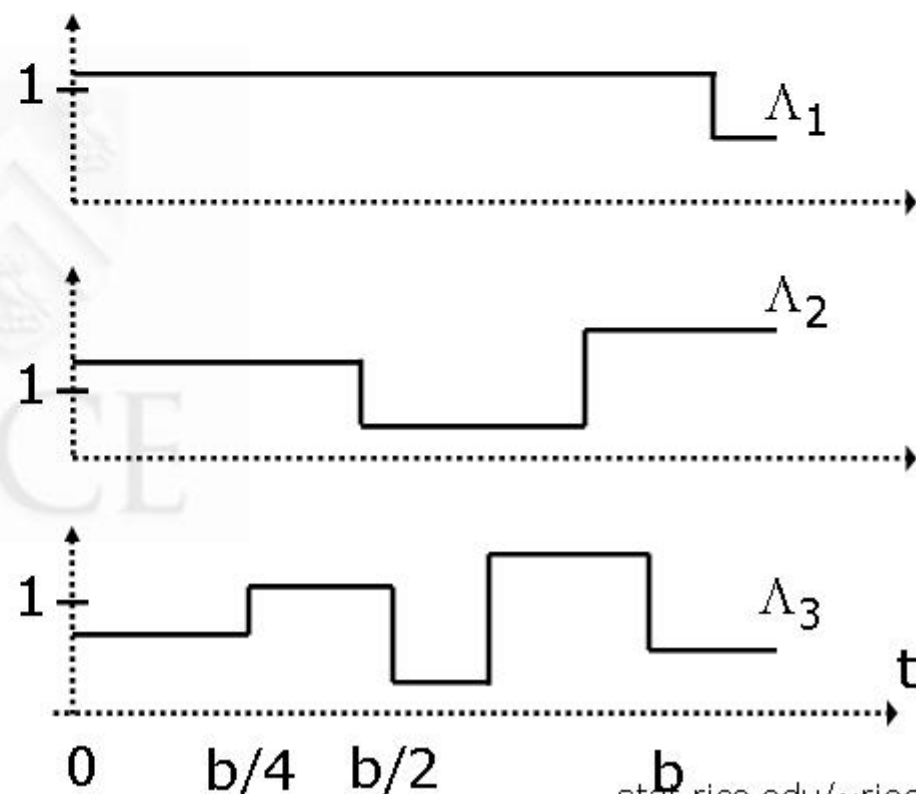
Multifractal paradigm

- Multiplicative Processes:

$$A(t) = \lim_{n \rightarrow \infty} \int_0^t \Lambda_0(s) \dots \Lambda_n(s) ds$$

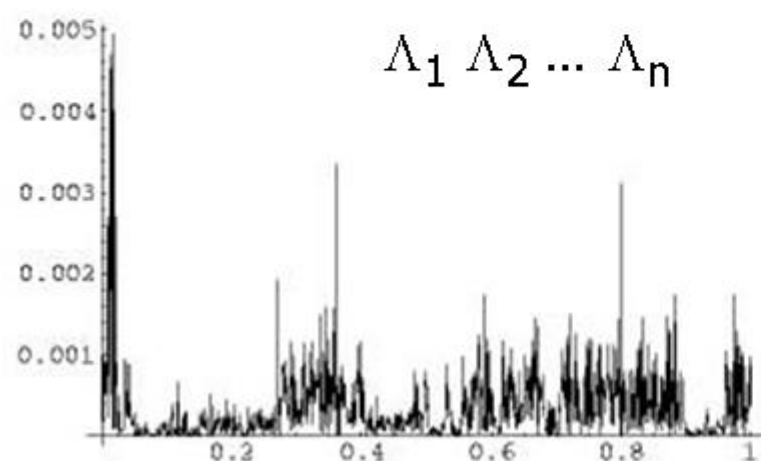
- Stationary Cascade

- $\Lambda_n(s)$ is stationary
- Conservation:
 $E\Lambda_n(t) = 1$
- “self-similarity”:
 $\Lambda_n(s) =_d \Lambda_1(sb^n)$



Parameters and Scaling

- Parameter estimation
 - $\Lambda_i(s)$: i.i.d. values with Poisson arrivals (λ_i):
 - $Z(s) = \log [\Lambda_1(s) \Lambda_2(s) \dots \Lambda_n(s)]$
 - $\text{Cov}(Z(t)Z(t+s)) = \sum_{i=1..n} \exp(-\lambda_i s) \text{Var } \Lambda_i(s)$
- Performance of predictors / simulations



- Multifractal Envelope
(with Norros and Mannersalo)
 $T(q) = q - 1 - \log_2 E[\Lambda^q]$



Interlude

Self-similar processes

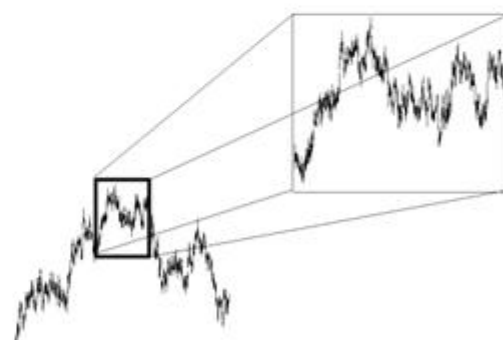


Statistical Self-similarity

- Self-similarity: canonical form
 - $B(at) \stackrel{\text{fdd}}{=} C(a) B(t)$ B : process, C : scale function
 - Iterate: $B(abt) \stackrel{\text{fdd}}{=} C(a)C(b) B(t)$
 - $C(a)C(b) = C(ab)$
 - $C(a) = a^H$: Powerlaw is **default**

- H-self-similar:

$$B(at) \stackrel{\text{fdd}}{=} a^H B(t)$$



- Examples

- Gaussian: **fractional Brownian motion** $B_H(t)$ is unique H-self-similar Gaussian process with stationary increments.
- Stable: not unique in general, $a=1/H$: Levy motion

Statistical Self-similarity

- How do self-similar processes occur?
 - X_k : stationary time series
 - $U(t) := X_1 + \dots + X_t$
 - If $U(nt)/f(n) \rightarrow_{f.d.d.} Z(t)$
 - then necessarily $H = \lim_{n \rightarrow \infty} \log f(n) / \log(n)$ exists and $Z(t)$ is H -self-similar.
 - If X_k are iid with finite variance, then $H=1/2$ and Z is **Brownian motion**
 - If X_k are LRD, then $H>1/2$ and Z is **fractional Brownian motion**
- Prediction and estimation windows

Self-similar Processes

- What do they model?

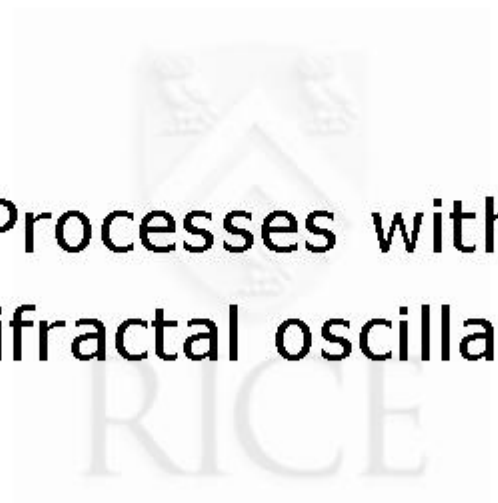


- Sustained excursions above/below the mean
- Different from (finite order) linear models
 - Auto-Regressive
 - ARMA
 - (G)ARCH
 - Exponential decay of correlations
- Corresponds to infinite order AR models
 - FARIMA
 - FIGARCH

$$\text{fBm}(t) = \int_{-\infty}^t K(t,s) dW(s)$$



Multifractal Subordination



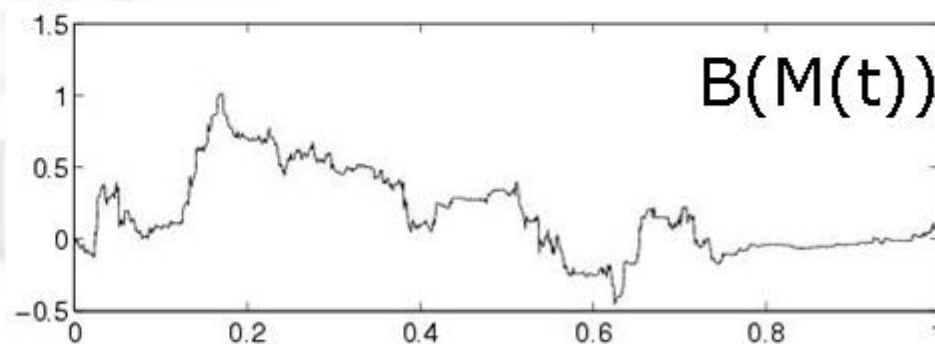
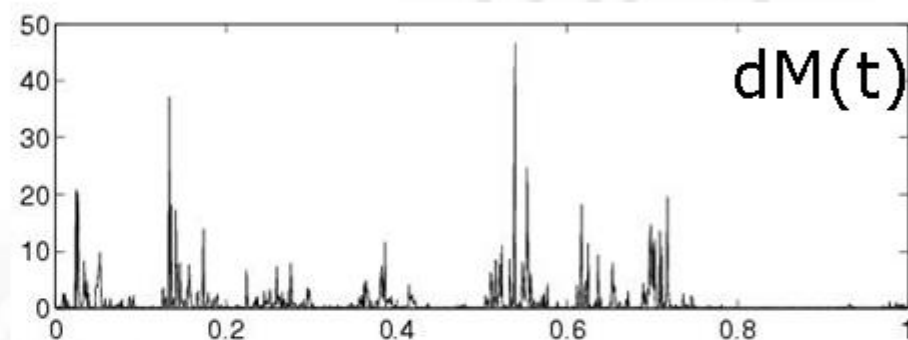
Processes with
multifractal oscillations

Multifractal time warp

$B_H(M(t))$: B_H fBm, dM independent measure

A versatile model

- $M(t)$: Multifractal
Time change
Trading time
- B : Brownian motion
Gaussian fluctuations



Hölder regularity

- Levy modulus of continuity:

- With probability one for all t

$$|B_H(t + \delta) - B_H(t)| \simeq |\delta|^H$$

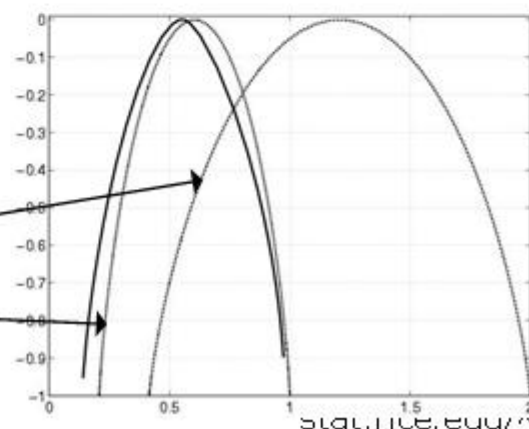


- Thus, exponent gets stretched:

$$|B_H(M(t+\delta)) - B_H(M(t))| \simeq |M(t+\delta) - M(t)|^H \simeq |\delta|^{H\alpha(t)}$$

- and spectrum gets squeezed:

$$\dim E_\alpha[B_H(M)] = \dim E_{\alpha/H}[M]$$



Multifractal formalism for $B_H(M(t))$

- Conditioning on M one finds:

$$\begin{aligned}\mathbb{E}|B_H(M(t+\delta)) - B_H(M(t))|^q &= \mathbb{E}|B_H(1)|^q \mathbb{E}|M(t+\delta) - M(t)|^{qH} \\ &\simeq |\delta|^{T_M(qH)}\end{aligned}$$

– thus
$$T_{B(M)}(q) = T_M(qH)$$

- which confirms the stretched exponent:

$$T'_{B(M)}(q) = HT'_M(qH)$$

- and matches with warp formula before:

$$T^*_{B(M)}(a) = T^*_M(a/H)$$

- If the formalism holds for M , then also for $B_H(M(t))$

Auto-Correlation

- **Conditioned** on knowing $M(t)$:
 - $E[B(M(t)) B(M(s)) \mid M]$
 $= (\sigma^2/2) [M^{2H}(t) + M^{2H}(s) - M^{2H}(t-s)]$
 - Non stationary **Gaussian** Process
 - Increments: $X(t)=B(M(t+1)) - B(M(t))$
 - $E[X(t) X(s) \mid M] = (\sigma^2 / 2) \times$
 $([M(t+1) - M(s)]^{2H} - [M(t) - M(s)]^{2H} - [M(t+1) - M(s+1)]^{2H} + [M(t) - M(s+1)]^{2H})$
- **Unconditioned**: For $H=1/2$ and $E[M(t)]=t$
 - $E[B(M(t)) B(M(s))] = \sigma^2 \min(s,t)$
 - $E[X(t+k) X(t)] = E[M(k+1) - 2M(k) + M(k-1)] = 0$
 - Uncorrelated increments, stationary, 2nd order
 - But **not Gaussian**
 - Dependence of higher order through $M(t)$

Estimation: Wavelets decorrelate

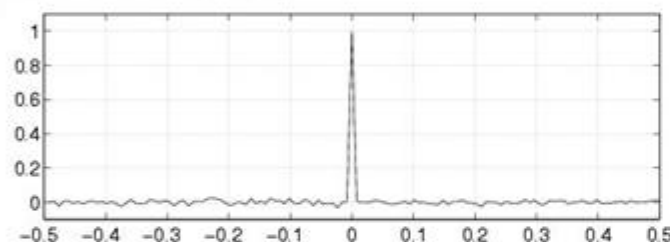
(with P. Goncalves)

- $W_{jk} = \int \psi_{jk}(t) B(M(t)) dt$
N: number of vanishing moments
- $E[W_{jk} W_{jm}]$
 $= \int \int \Psi_{jk}(t) \Psi_{jm}(s) E[B(M(t)) B(M(s))] dt ds$
 $= \int \int \Psi_{jk}(t) \Psi_{jm}(s) E[|M(t) - M(s)|^{2H}] dt ds$
 $\sim O(|k-m|^{T(2H)+1-2N}) \quad (|k-m| \rightarrow \infty)$

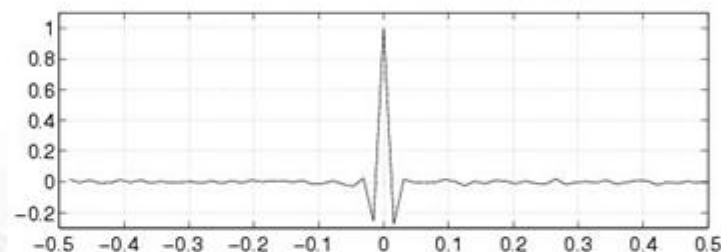
Multifractal Estimation for $B(M(t))$

- Weak Correlations of Wavelet-Coefficients:
(with P. Goncalves)

Haar



Daubechies2



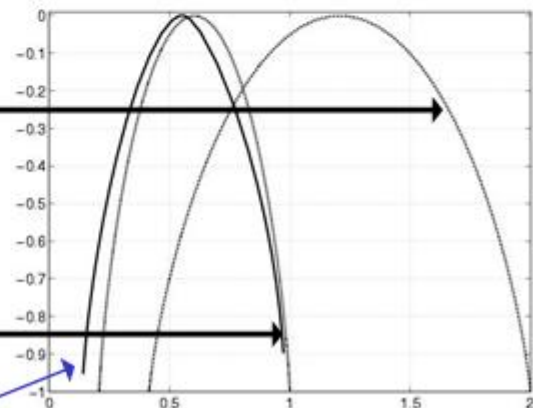
- Improved estimator due to weak correlations
- Multifractal Spectrum

$$M(t+s) - M(t) \sim s^{a(t)}$$

$$B(t+u) - B(t) \sim u^H \quad (\forall t)$$

→

$$B(M(t+s)) - B(M(t)) \sim s^{H^*a(t)}$$



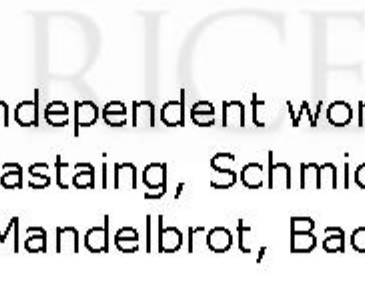
Estimation



From Multiplicative Cascades to Infinitely Divisible Cascades



with
P. Chainais and P. Abry



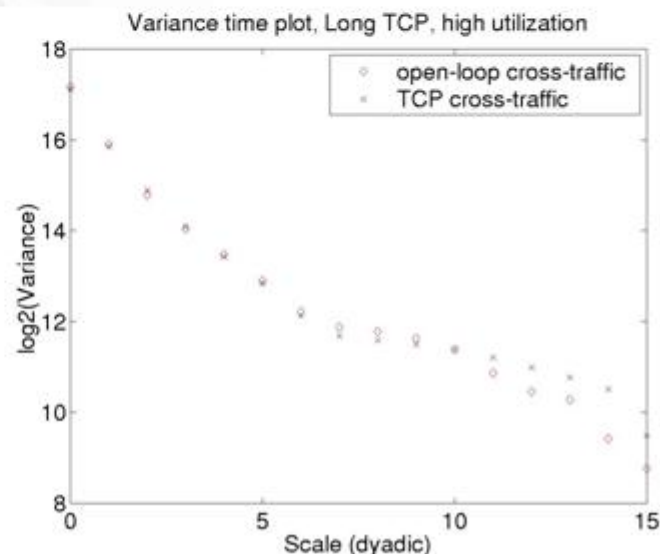
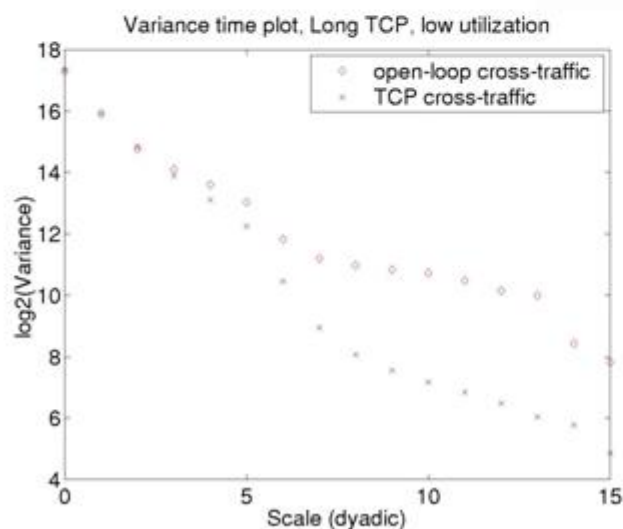
Independent work:
Castaing, Schmidt,
Barral-Mandelbrot, Bacry-Muzy

Adapting to the real world

Real world data

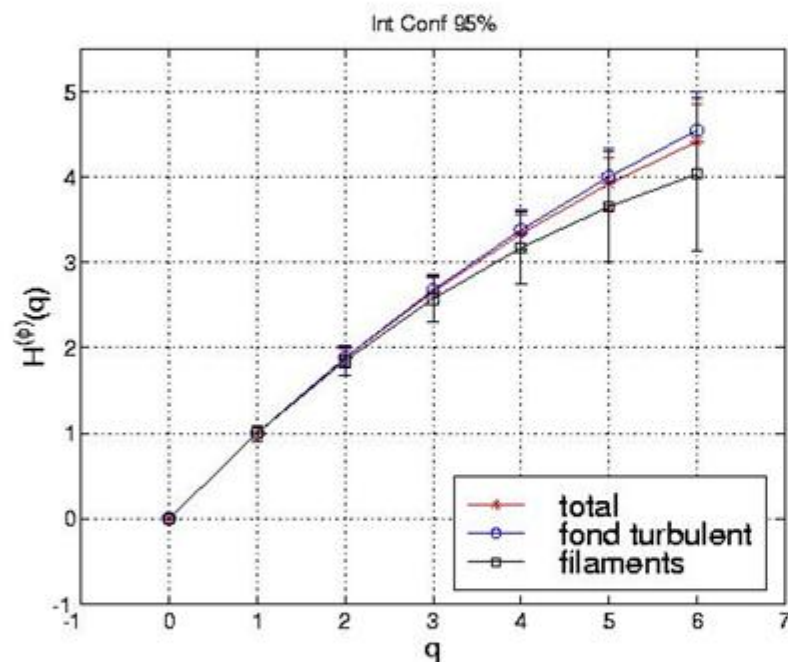
- can deviate from powerlaws: traffic
- has no preference for dyadic scales

Lukacs: if the data does not fit to the model then too bad for the data.

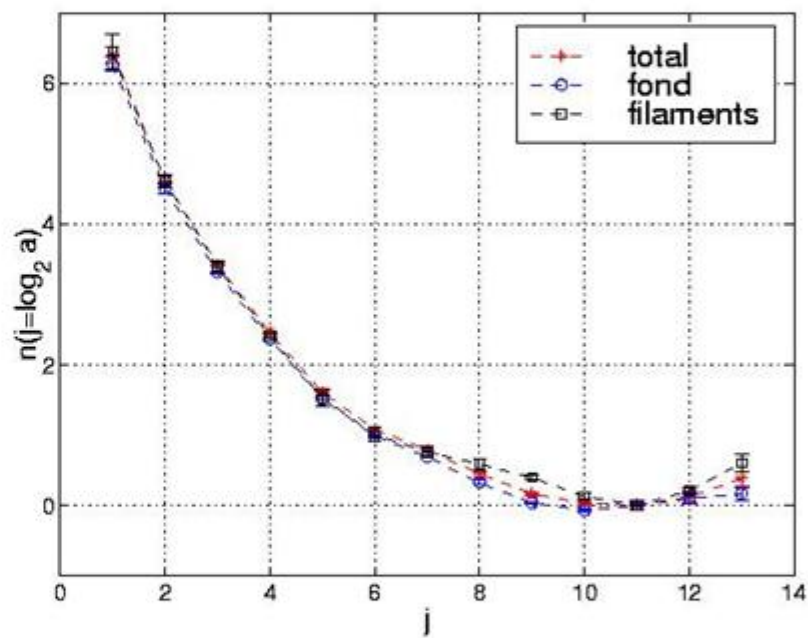


Experimental results

Courtesy P. Chainais



$H(q)$



$n(a)$: non-powerlaw

Beyond Self-similarity

- Self-similarity revisited:
 - $B(at) \stackrel{d}{=} C(a) B(t)$ B : process, C : scale function
 - $B(abt) \stackrel{d}{=} C(a)C(b) B(t)$
 - $C(a)C(b)=C(ab) \rightarrow C(a) = a^H$
 - $E[|B(a^n)|^q] = c(q) (a^{qH})^n$
 - linear in q (mono-fractal)

Beyond Self-similarity

- Self-similarity revisited:
 - $B(at) \stackrel{d}{=} C(a) B(t)$ B : process, C : scale function
 - $B(abt) \stackrel{d}{=} C(a)C(b) B(t)$
 - $C(a)C(b)=C(ab) \rightarrow C(a) = a^H$
 - $E[|B(a^n)|^q] = c(q) (a^{qH})^n$
 - linear in q (mono-fractal)
- More flexible rescaling “Ansatz”:
 - $C=C(a,t) ?$: non-stationary increments
 - C =independent r.v. for every re-scaling :
 - $X(a...at) = X(a^n t) = C_1(a)...C_n(a) X(t)$: **multiplicative**
 - $E[|X(a^n)|^q] = c(q) E[|C(a)|^q]^n$
 - non-linear in q ; **powerlaw**

Infinitely divisible scaling

Self-similarity: $\mathbb{E}[|B(t + \delta) - B(t)|^q] \simeq \delta^{qH}$

Multifractal scaling: $\mathbb{E}[|M(t + \delta) - M(t)|^q] \simeq \delta^{1+T(q)}$

IDC scaling: $\mathbb{E}[|X(t + \delta) - X(t)|^q] \simeq \exp[n(\delta)\zeta(q)]$

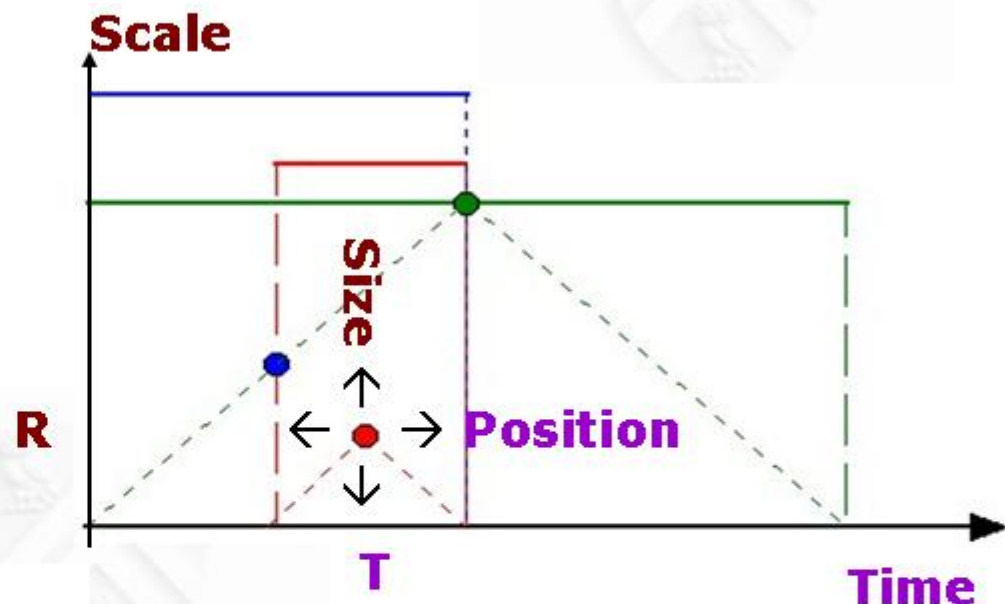
- Multifractal scaling reduces to self-similarity if T is linear in q . (sometimes called **mono-fractal**)
- IDC reduces to multifractal scaling if $n(\delta) = -\log(\delta)$
- In general **$n(\delta)$** gives the speed of the cascade

Geometry of Binomial Pulses

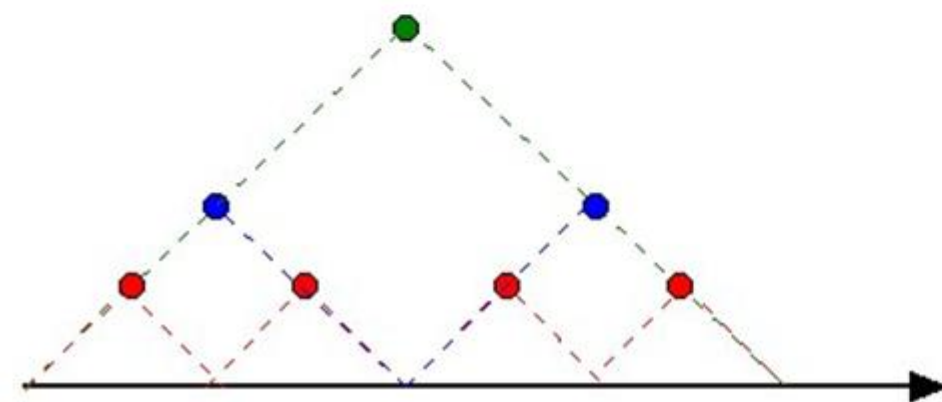
- Time-Scale plane: codes shape of pulses
 - Position** (T =center)
 - Size** (R =length)

Pulses:

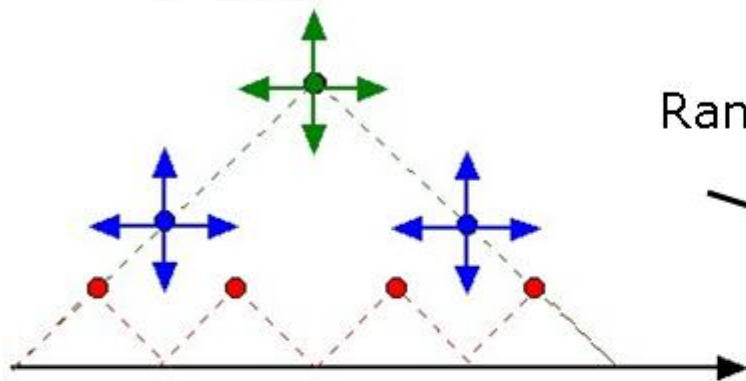
$$P_i(t) = \begin{cases} W_i & \text{if } |t-t_i| < r_i/2 \\ 1 & \text{else} \end{cases}$$



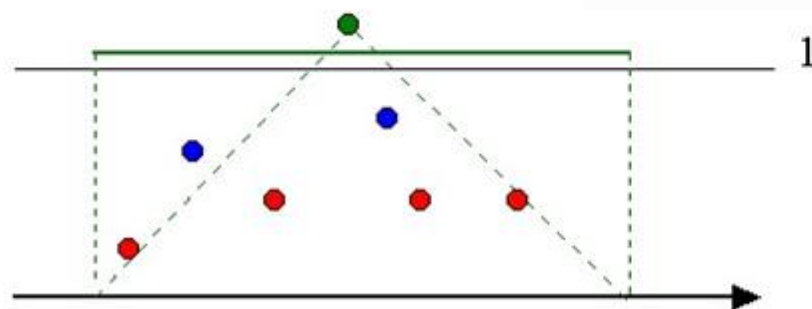
For Binomial:
Strict dyadic
geometry



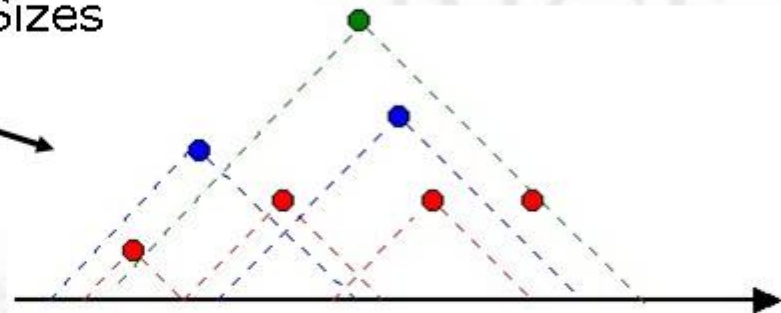
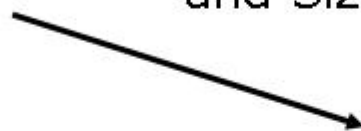
Stationary geometry



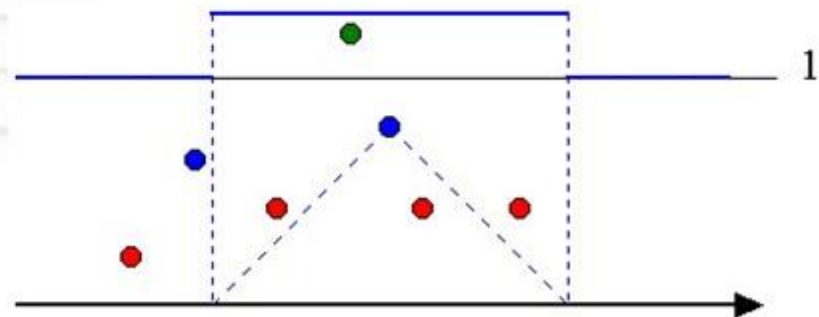
Large Scale Pulses



Randomize Positions
and Sizes



Medium Scale Pulses



Compound Poisson Cascade

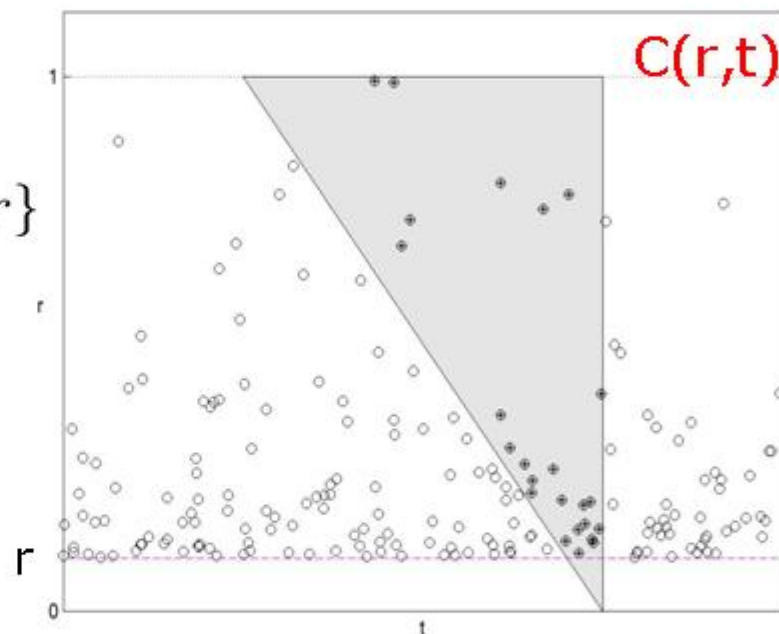
Poisson points (t_i, r_i) in time-scale plane with marks W_i

Cone of influence at t

$$C(r, t) = \{(t_i, r_i) : t - r_i < t_i < t, r_i > r\}$$

Cascade Process:

$$Q_r(t) = \prod_{(t_i, r_i) \in C(r, t)} W_i$$



- Poisson Cascades exhibit **scaling properties akin to IDC scaling**

$$m(C(r, t)) = m(C(r, 0)) = \mathbb{E}[\#\{(t_i, r_i) \in C(r, t)\}]$$

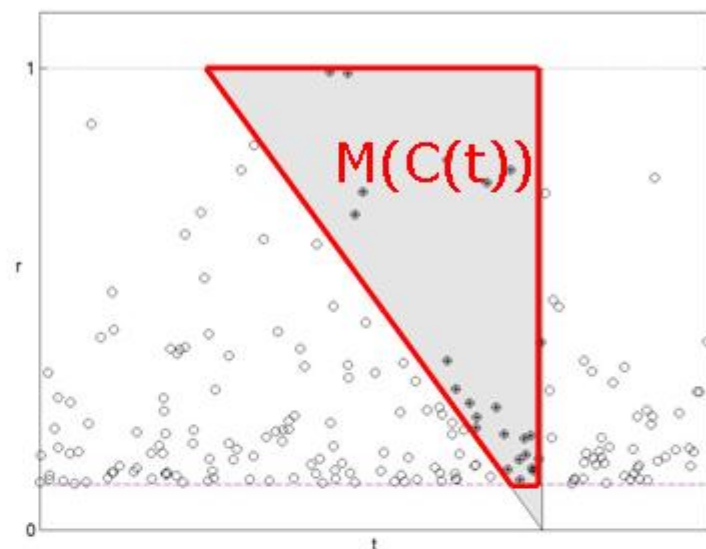
$$\mathbb{E}Q_r(t)^q = \exp[-\varphi(q)m(C(r, *))]$$

Cascade and AR processes

- Continuous version (IDC):

$$\begin{aligned} Q(t) &= \exp M(C(t)) \\ &= \exp \int k_C(t, s) dM(s) \end{aligned}$$

- M is an infinitely divisible measure



- Classic theory to be exploited:
 - AR-type processes

$$B_H(t) = \int \tilde{k}_H(s, t) dW(s)$$

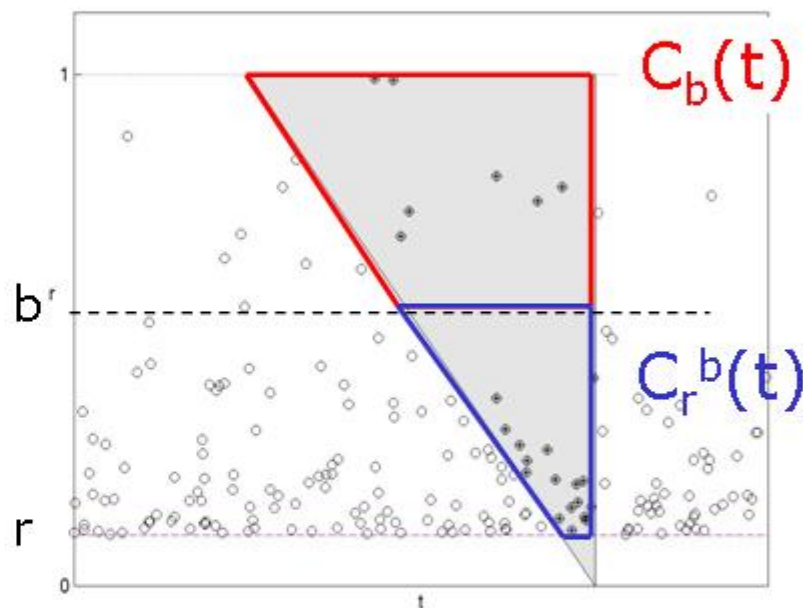
- kernel estimate of the random measure dM

Cascades: Invariance and scaling

Infinitely divisible nature
and scaling of the cascade:

$$\begin{aligned}
 Q_r(t) &= \prod_{C(r,t)} W_i = \underbrace{\prod_{C_b} W_i}_{C_b} \times \underbrace{\prod_{C_r^b} W_i}_{C_r^b} \\
 &= Q_b(t) \times \underbrace{\prod_{C_r^b} W_i}_{C_r^b}
 \end{aligned}$$

Rescaled version of $Q_{r/b}$
in the scale-invariant case only!



Poisson Cascade has **re-scaling properties**;
in scale invariant case: akin to Product of Processes

Multifractal scaling

- Multifractal formalism holds in self-similar case [Barral-Mandelbrot]
- Infinitely Divisible Scaling

Recall

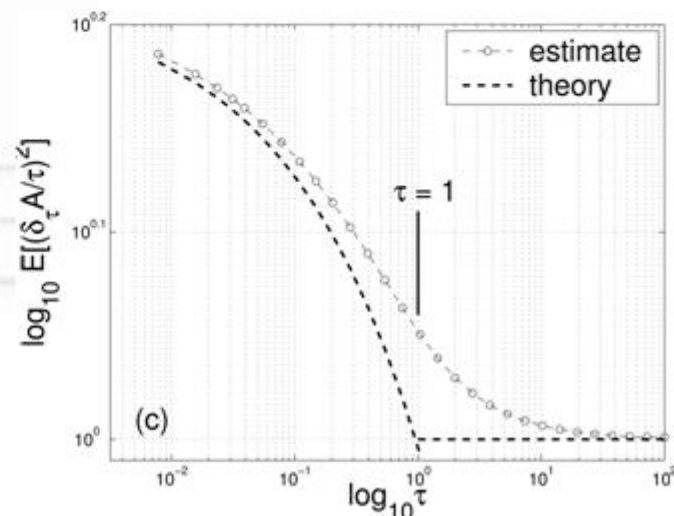
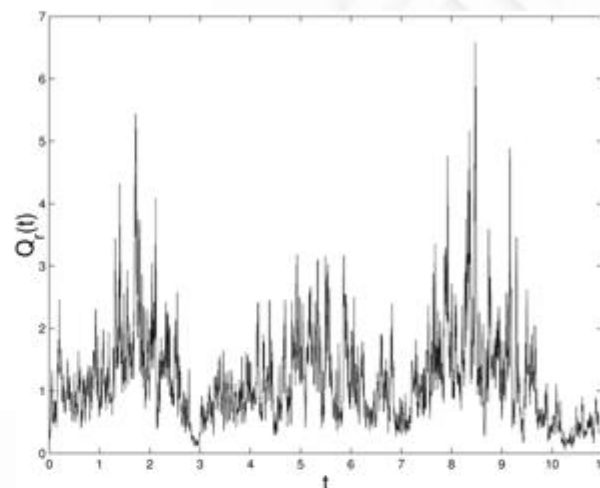
$$\mathbb{E}Q_r(t)^q = \exp[-\varphi(q)m(C(r, *))]$$

$$\mathbb{E}A(t)^q \simeq t^q \exp[-\varphi(q)m(C(t, *))]$$

- **powerlaw** only if $m(C(t, *)) = -\log(t)$
- for IDC in self-similar case [Bacry-Muzy, Barral]
- for CPC and log-normal IDC in certain **non-powerlaw** cases [Chainais-R-Abry]

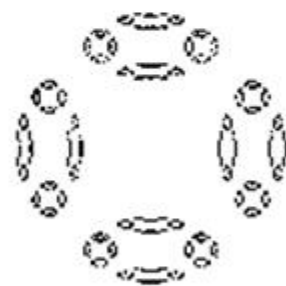
Simulations

- Stationary Cascade:
- Non-powerlaw scaling



“Never happy”: More flexibility

- Better control of scaling
- Wider range of known non-powerlaw scaling
- Higher dimensions: anisotropy
 - “As expected” in generic cases [Falconer, Olsen]
 - Formalism may break if directional preferences [McMullen, Bedford, Kingman, R]



Overall Lessons

- Multifractal spectrum $\leftrightarrow$ regularity
 - Besov spaces
 - Global Hoelder regularity
- Powerful modeling via multiplication through scales
 - Poisson product of Pulses
 - Multifractal warping
 - Degeneracy: price to pay for stationarity
- Estimation via wavelets
 - Multifractal envelopes
 - numerical $\tau(q)$,
 - Analytical $T(q)$
 - Choice of wavelet, of order q
 - Interpretation: what kind of spectrum did you estimate
 - Hoelder exponent
 - Wavelet decay

To take away

- Cascades matured to versatile multifractal models
- There remains much to do.

Reading on this talk

- www.stat.rice.edu/~riedi
- This talk
- Intro for the “untouched mind”
 - Explicit computations on Binomial
- Monograph on “Multifractal processes”
 - Multifractal formalism (proofs)
 - Multifractal subordination (warping)
- Papers, links