# Wavelets and Approximation

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# Report Documentation Page

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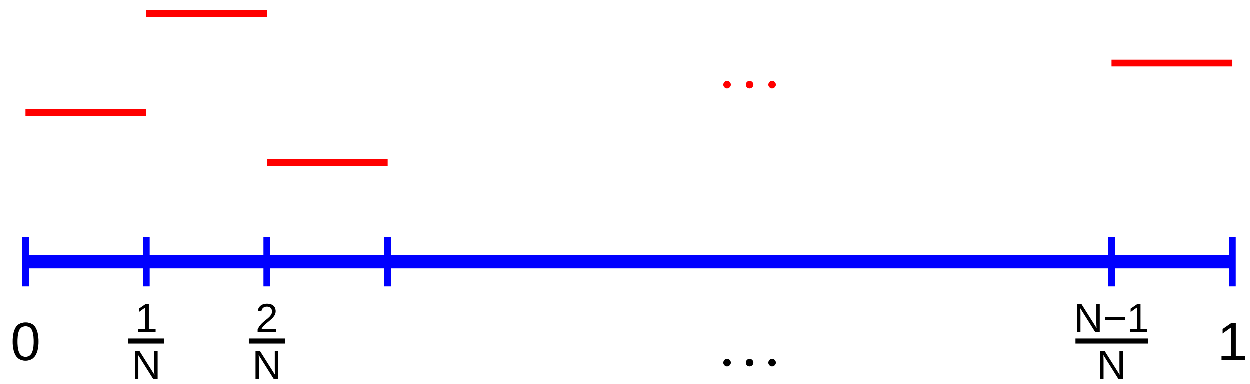
# Typical function in $\mathcal{S}_n$



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- Stop when $\mathcal{B}_\epsilon = \emptyset$, $\mathcal{P}_\epsilon := \mathcal{G}_\epsilon$, $N_\epsilon := \#(\mathcal{P}_\epsilon)$

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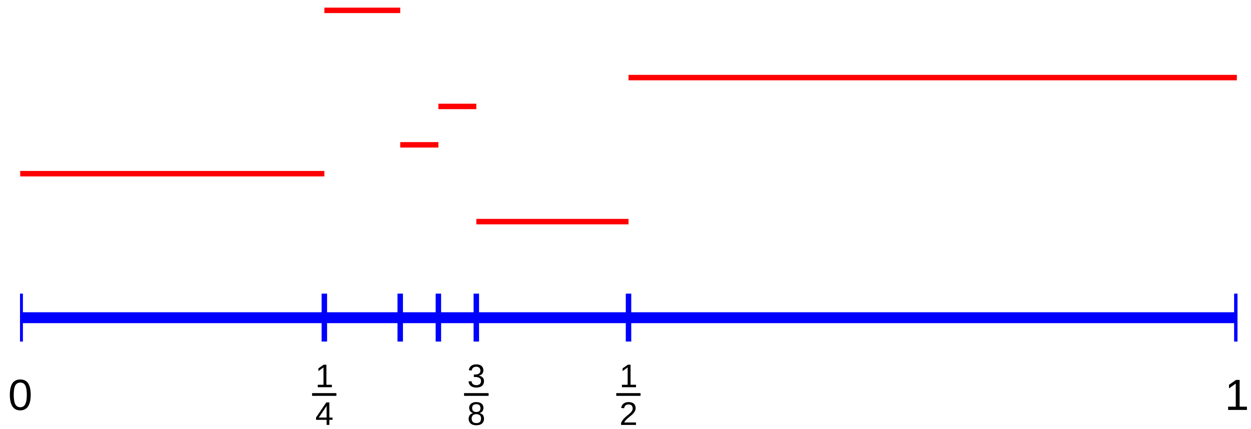
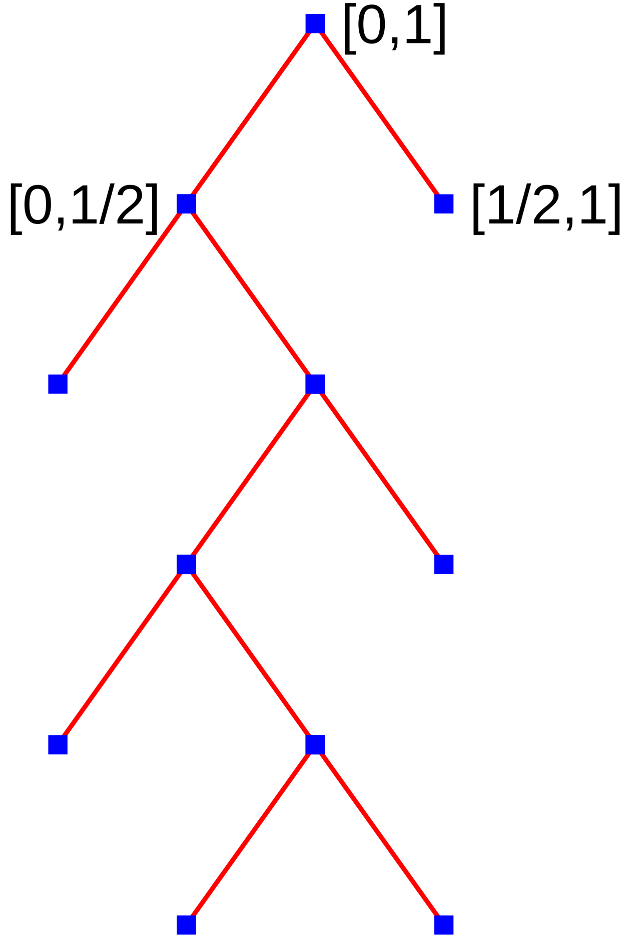
# Adaptively generated partition



# Tree associated to adaptive partition



# Comparison

- Approximation classes: $\alpha > 0$
  define $\mathcal{A}^\alpha(L_p,\ linear\ splines)$ as the set of all $f \in L_p[0,1]$ such that

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- Similarly define $\mathcal{A}^\alpha(L_p)$ for the other forms of approximation
- $\mathcal{A}^\alpha_q(L_p)$ finer scaling: same approximation order $\alpha$

$$|f|_{\mathcal{A}^\alpha_q(L_p)} := (\sum_{n=1}^{\infty}[n^\alpha E_n(f)_p]^q)^{1/q}$$

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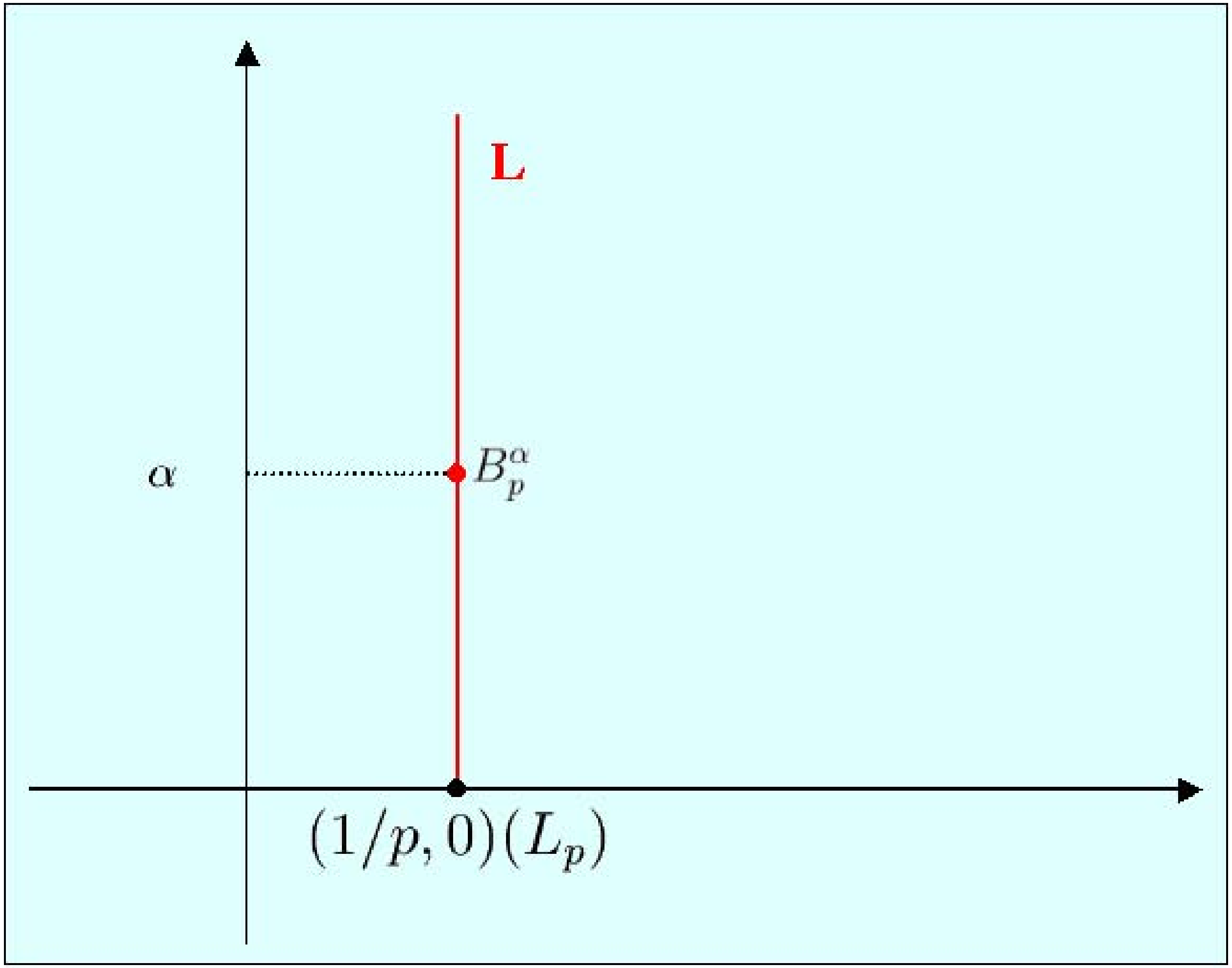

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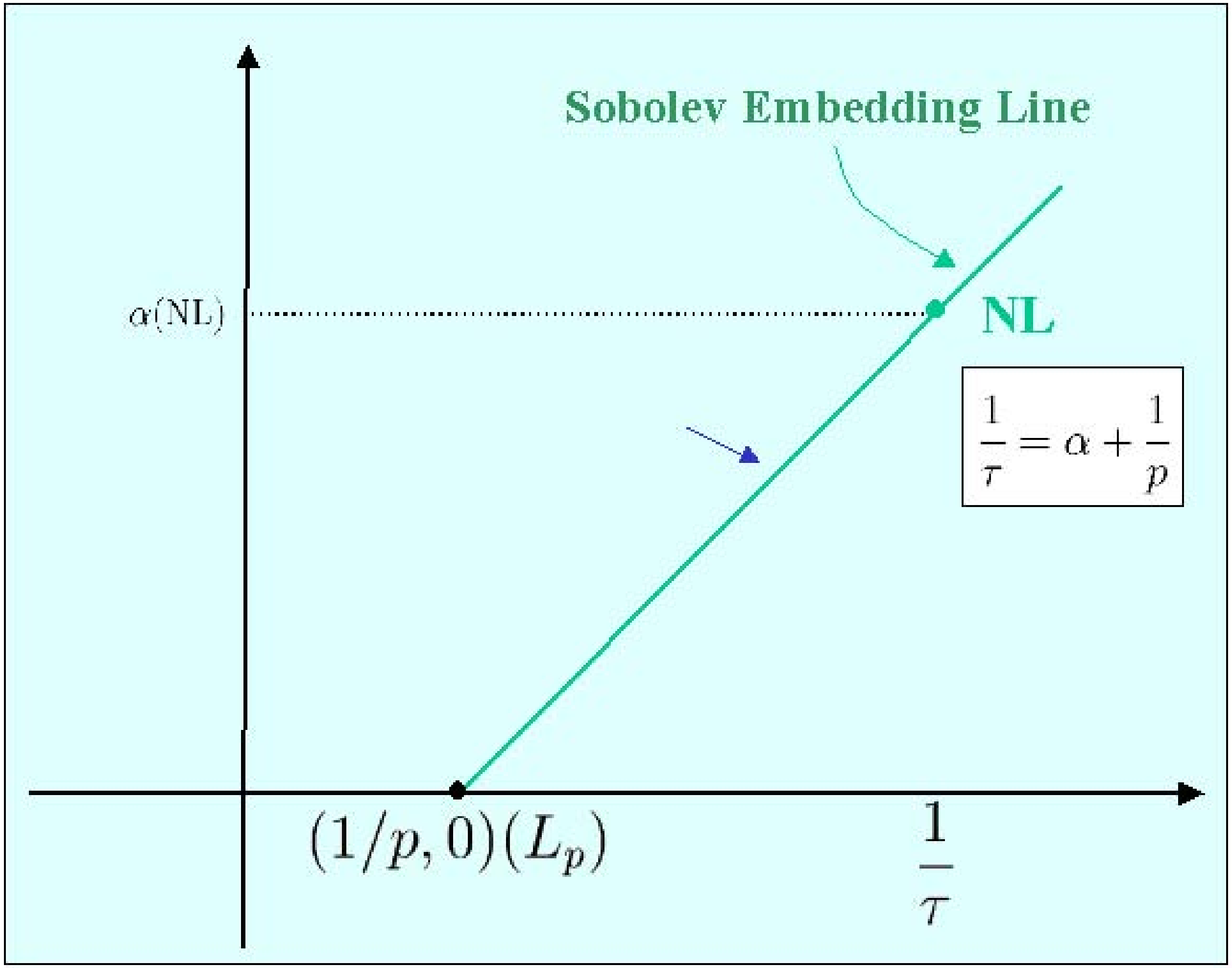
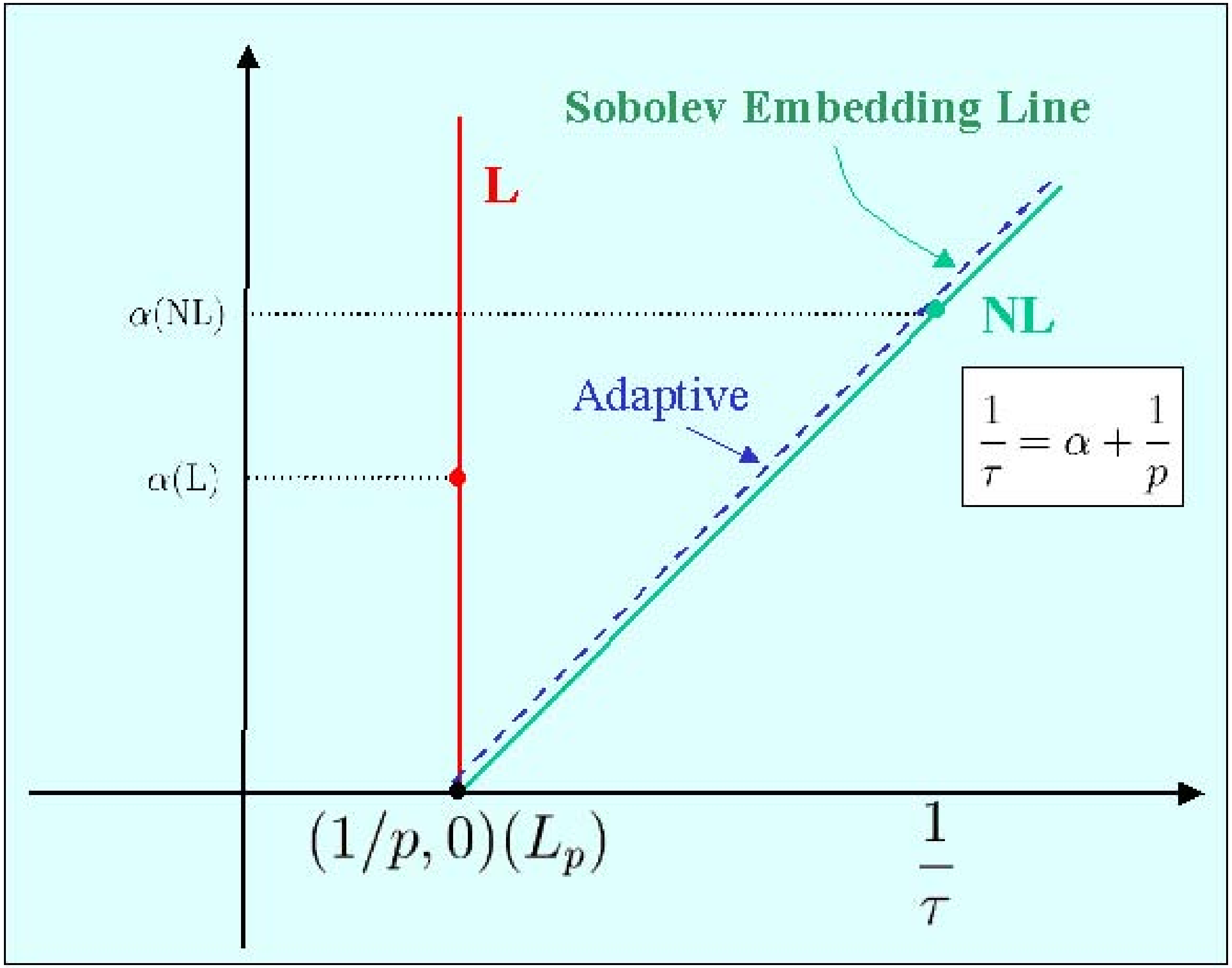
- Petrushev, DeVore-Popov (splines); DeVore-Jawerth-Popov (wavelets)

# Approximation class: free knot splines



# Adaptive approximation



# Example: Approximation in $L_\infty$

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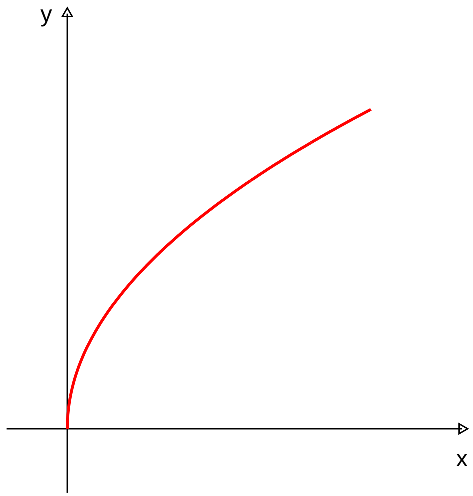
- Adaptive approximation $f' \in LlogL$: for example $f' \in L_p$ for some $p > 1$

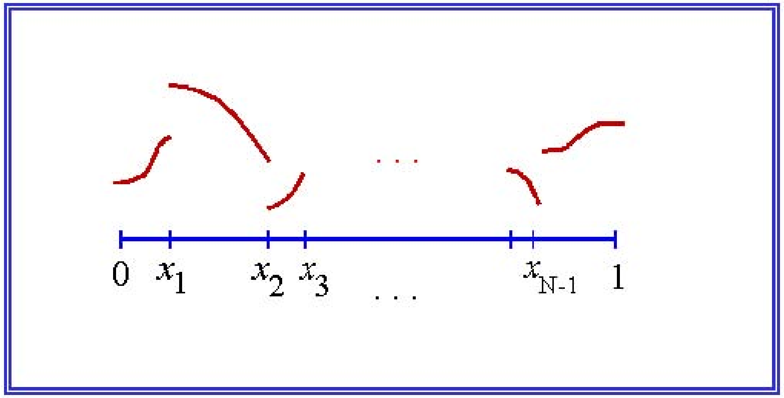
# **Example:** $f(x) = x^{\alpha}$, $0 < \alpha < 1 - 1/p$




$$E_n(f)_p \approx Cn^{-(\alpha+1/p)} \qquad \sigma_n(f)_p \le Cn^{-1}$$

Break points/ wavelets concentrate near singularity at $0$

# Example: piecewise smooth

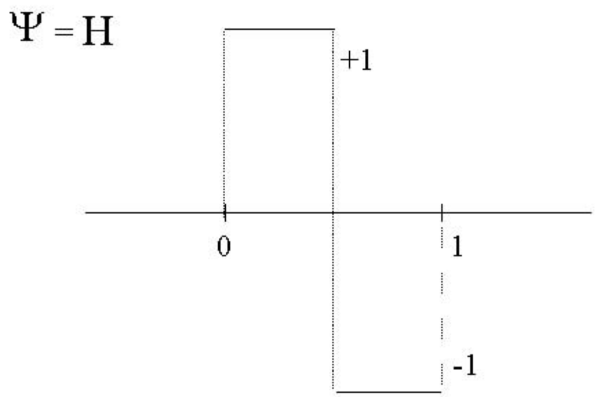



$$E_n(f)_p \geq Cn^{-1/p} \quad \sigma_n(f)_p \leq Cn^{-1}$$

Breakpoints/wavelets concentrate near singularities

# Wavelets: Haar Wavelet

$$H(x) := \begin{cases} -1, & x \in [0, 1/2) \\ +1, & x \in [1/2, 1] \end{cases},$$



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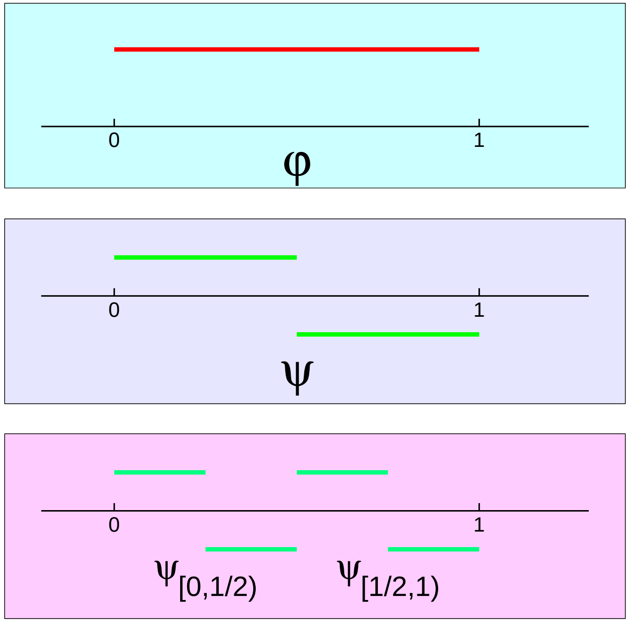
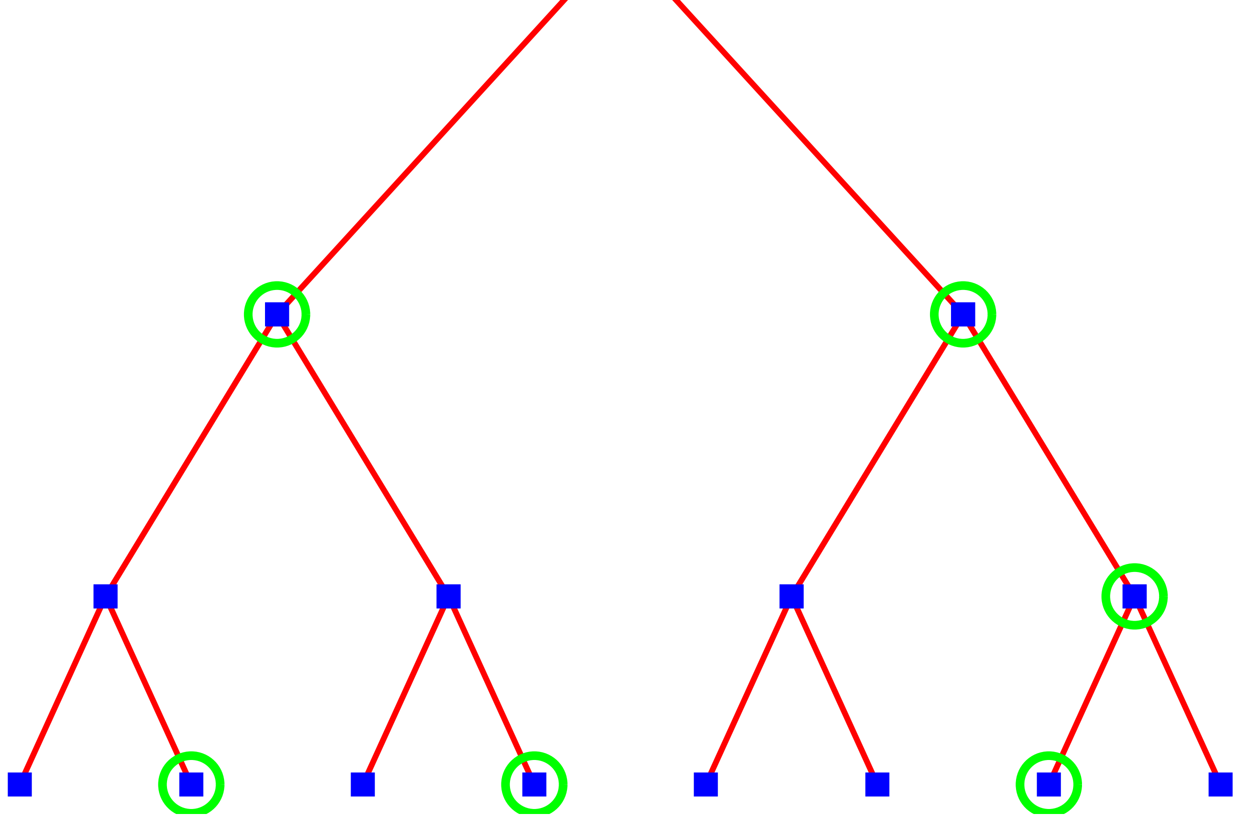
- $\{\chi_{[0,1]}\} \cup \{H_I\}_{I \in \mathcal{D}_+}$ is a complete orthonormal system in $L_2[0,1]$

# Haar Basis



# Wavelet tree



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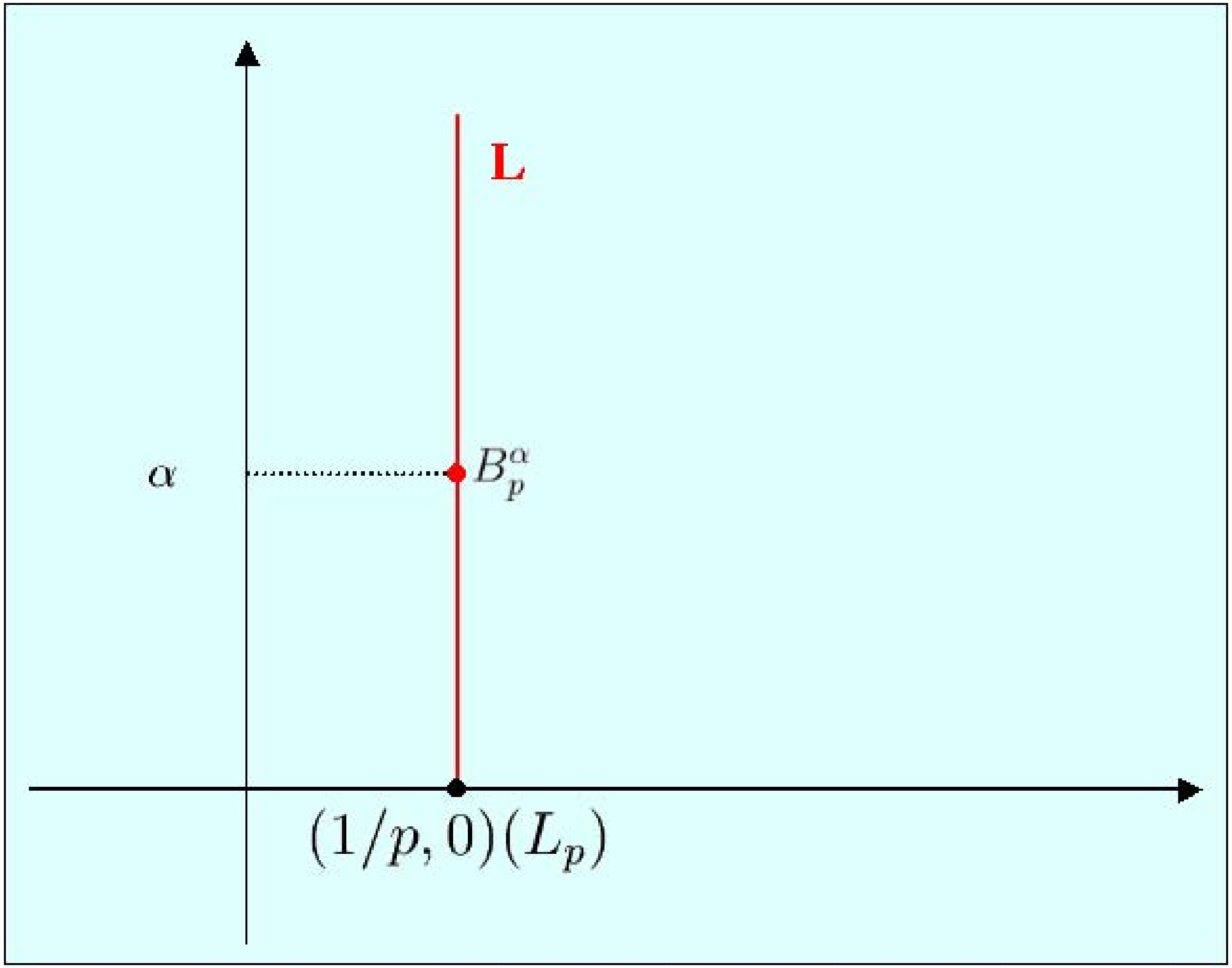
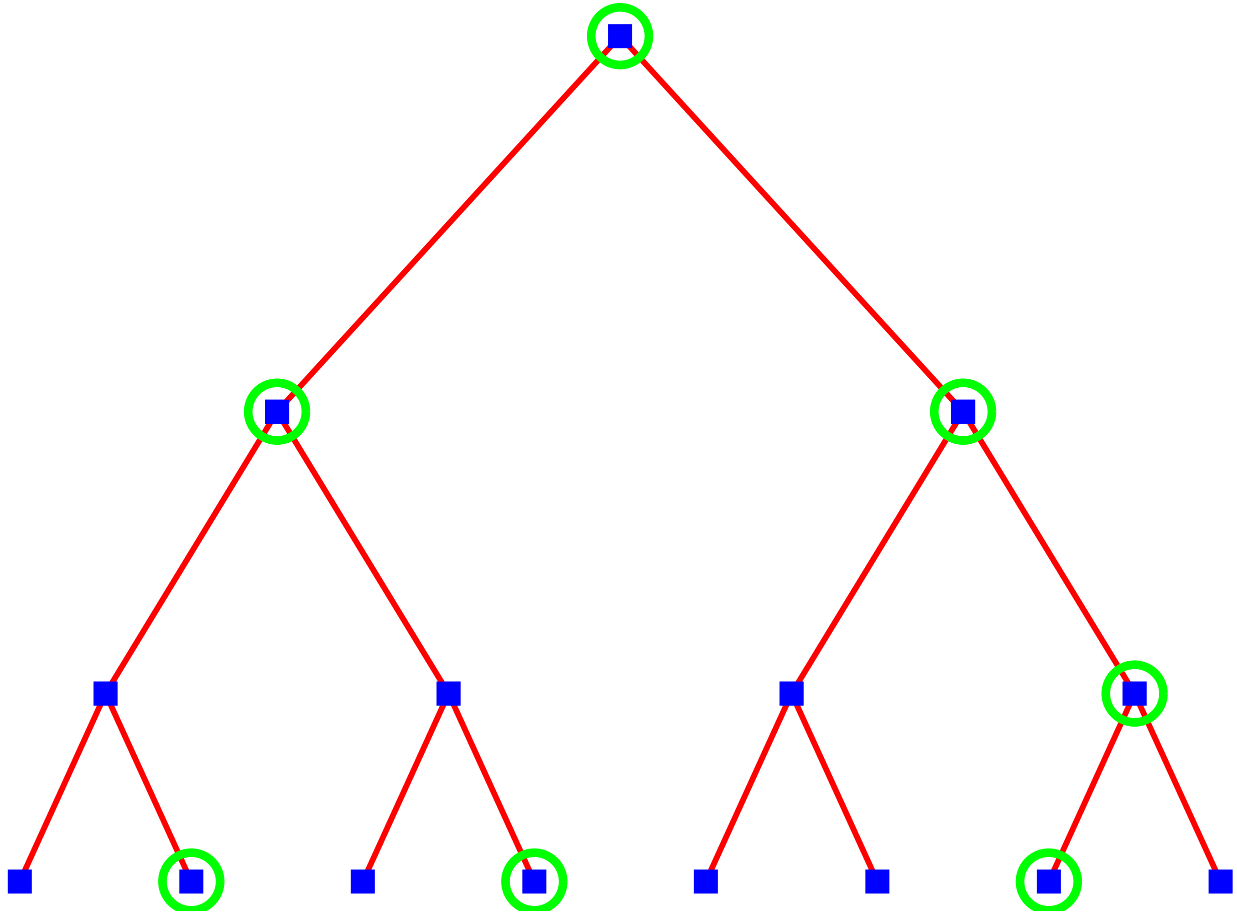
- The approximation classes for linear approximation with Haar wavelets are identical to those with piecewise constants.

# Linear Wavelet: $\mathcal{A}^s_\infty(L_p) = B^s_\infty(L_p)$



# $n$-term approximation



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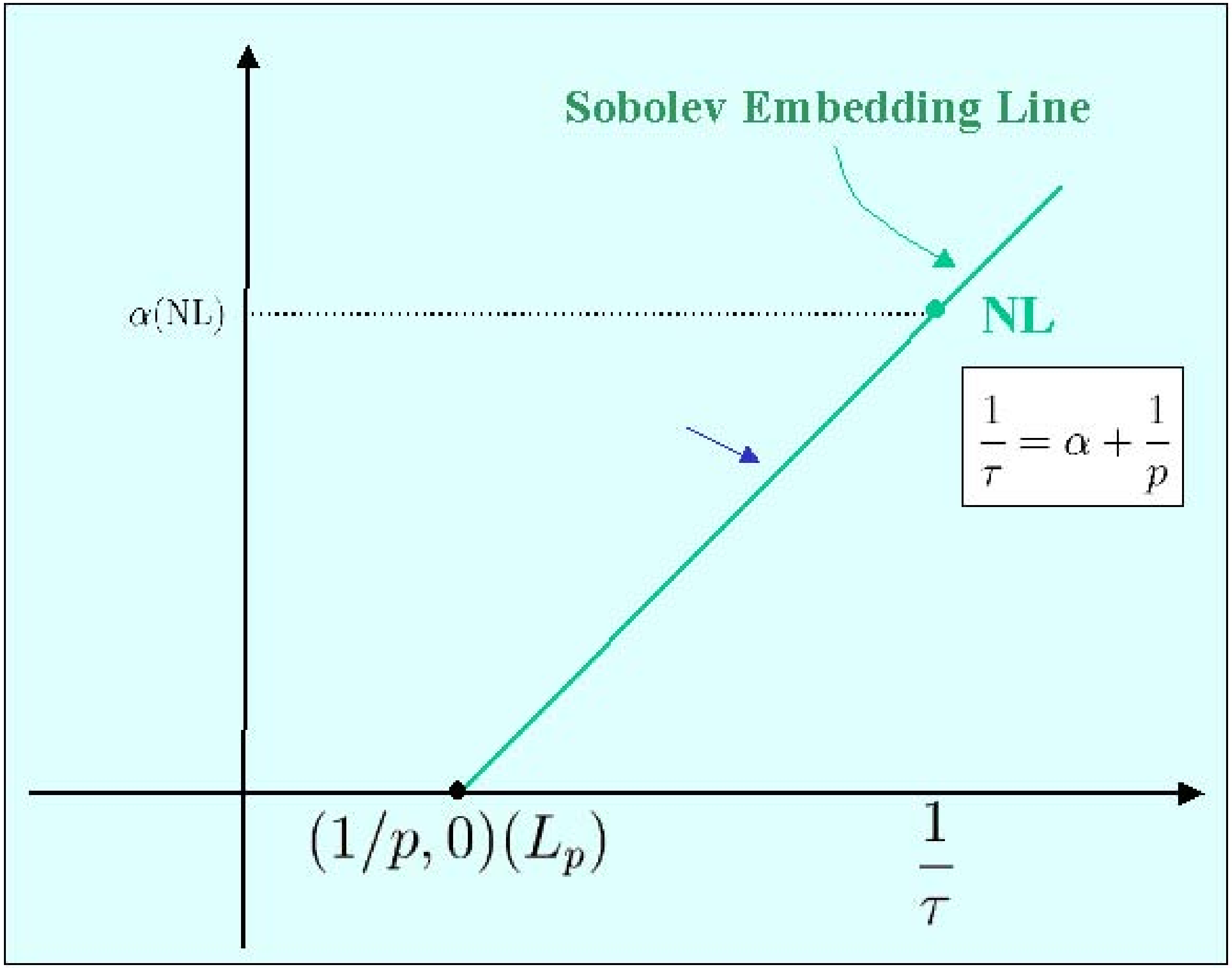

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- How can we find best $n$-term approximation??
- In $L_2$ take $n$ terms with largest coefficients
- DJP: Same strategy works in $L_p$, $1 < p < \infty$, and other spaces (Sobolev)

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- Greedy strategy is near optimal

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$$\frac{\|\sum_{I\in\Lambda}\psi_I\|_X}{\|\sum_{I\in\Lambda'}\psi_I\|_X} \leq C$$

whenever $\#(\Lambda) = \#(\Lambda')$

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- $\sum_{n=1}^\infty [n^\alpha \|f - G_n(f)\|_{L_p}]^\tau \frac{1}{n} \approx \|(c_j(f)\|_{\ell_\tau}^\tau$

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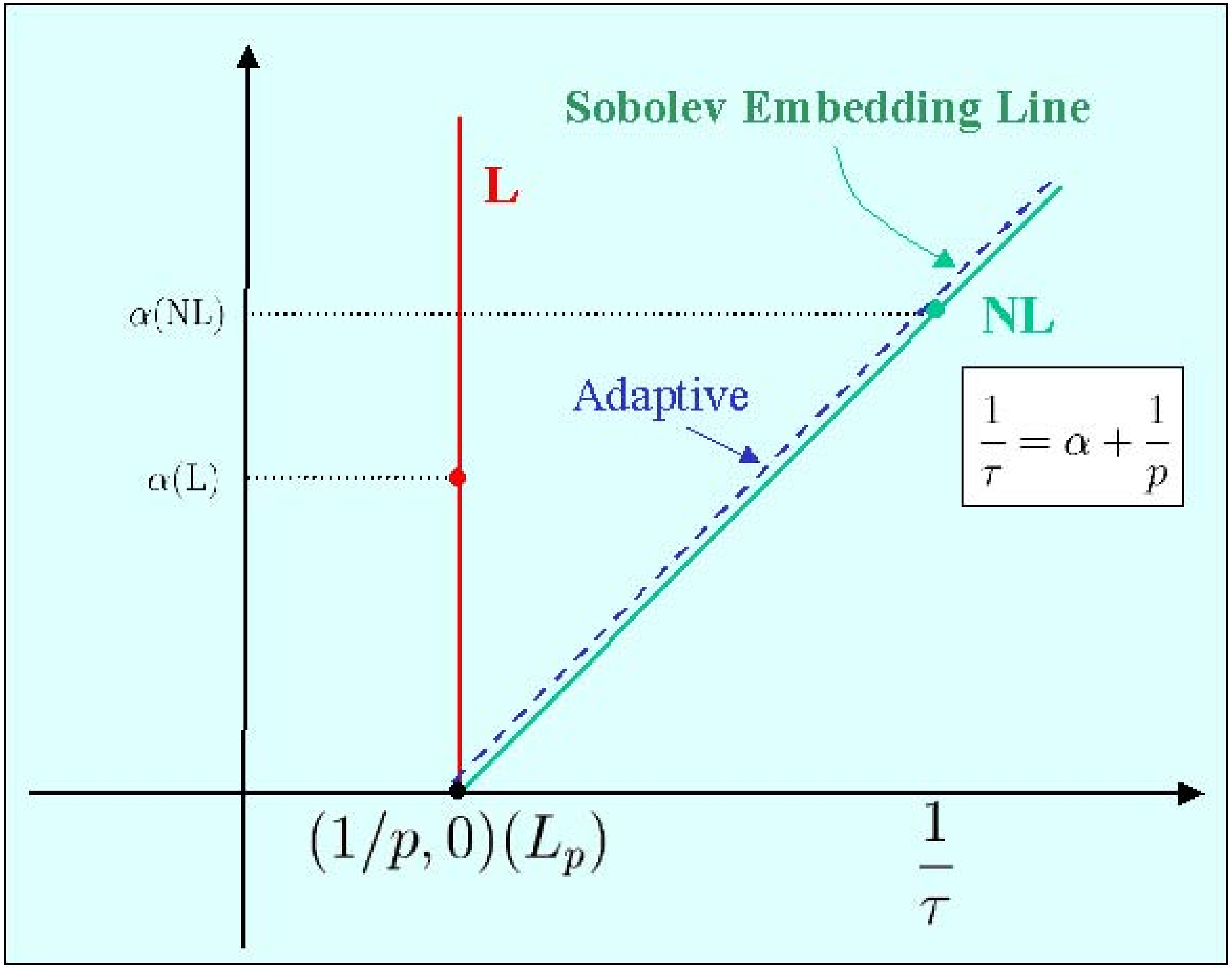
- Approximation properties analogous to adaptive approximation

# Tree approximation



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- $\mathcal{A}^{\alpha/d}_{\tau}(wavelets, L_p) = B^{\alpha}_{\tau}(L_p)$, $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$

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- Two essential ingredients

  a. metric $\rho$ to measure distortion

  b. Precise definition of classes $K_\alpha$ to be compressed

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- Game: Find encoder/decoder E/D: for all values of $n$ and all classes $K_\alpha$, encoder is near optimal

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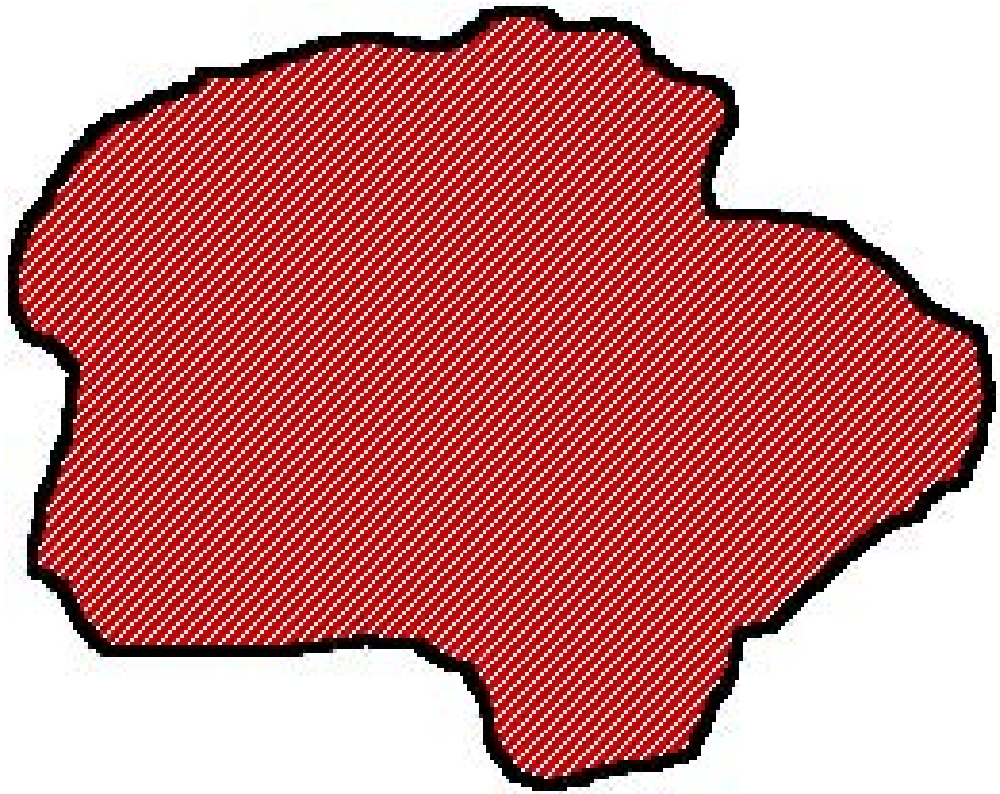

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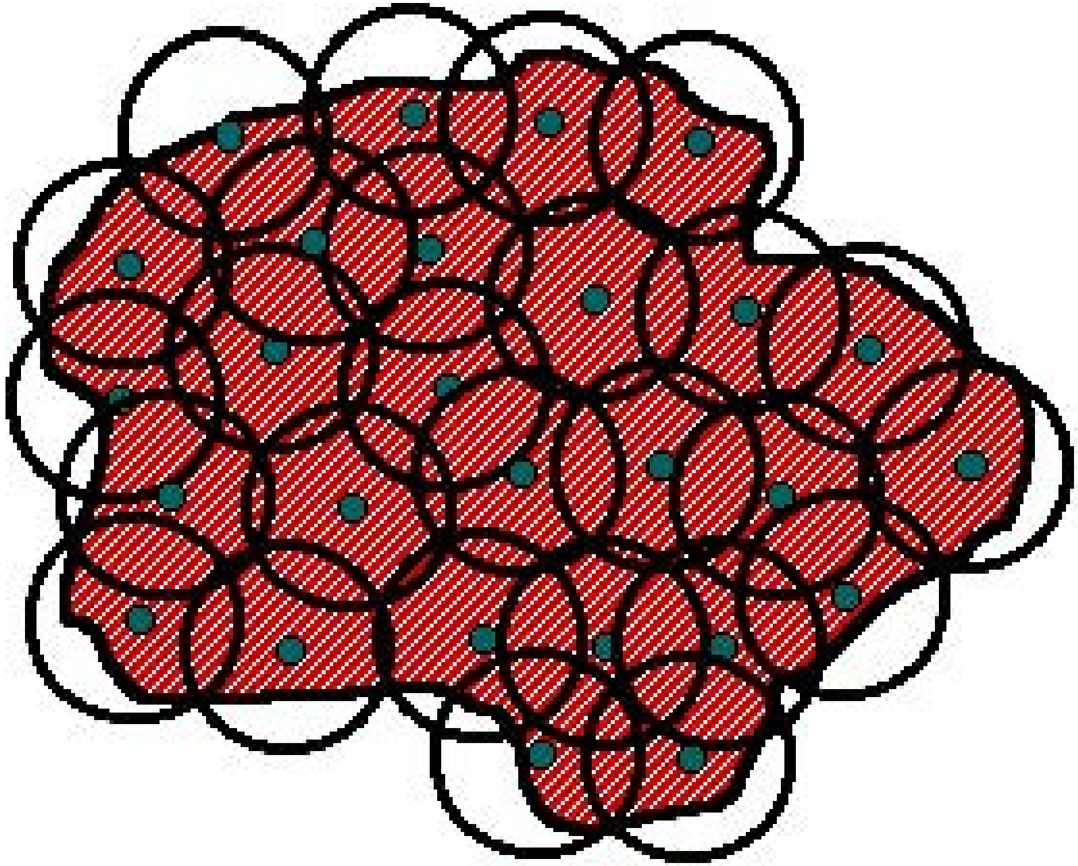

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- Minimal $\epsilon$ cover: $K \subset \cup_{i=1}^{N_\epsilon} \mathcal{B}(x_i, \epsilon)$
- $\delta_n(K) = \inf\{\epsilon : \ H_\epsilon(K) \leq n$
- Kolmogorov entropy of $K$ gives our benchmark
- Usually not practical encoder

# The Issues

1. The metric: least squares
2. The classes
3. Determine Entropy of Classes
4. Build near optimal Encoders/Decoders

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- EZW, Said-Pearlman,