

A wavelet-based generalization of the multifractal formalism from scalar to vector valued d-dimensional random fields : from theoretical concepts to experimental applications

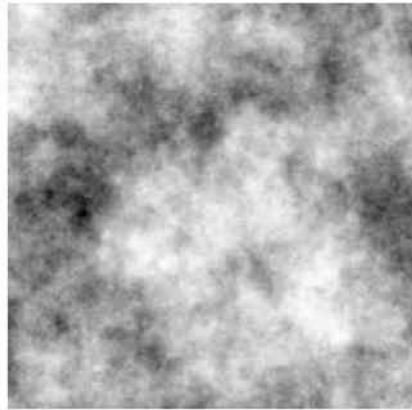
P. Kestener and A. Arneodo

Laboratoire de Physique
Ecole Normale Supérieure de Lyon
46, allée d'Italie
69364 Lyon cedex 07, FRANCE

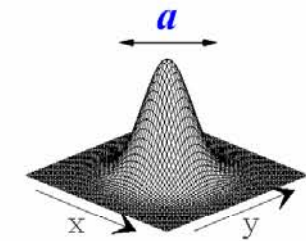
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2D WTMM Methodology : PhD work of N. Decoster

2D data : I



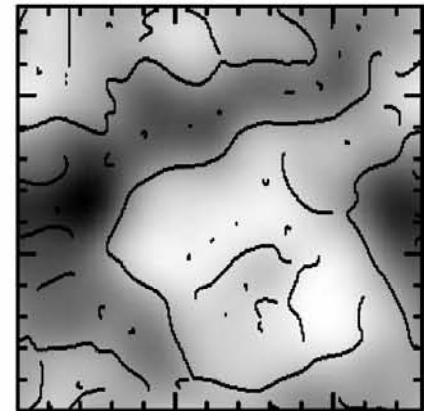
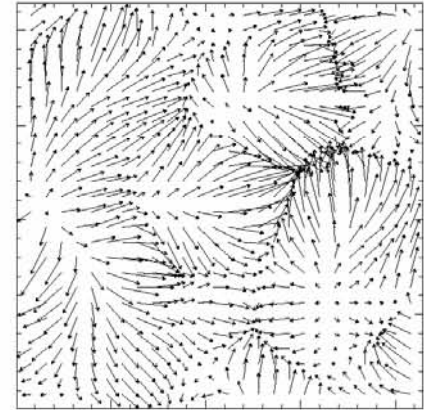
Smoothing
scale a



Gradient ∇

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right]$$

Wavelet Transform

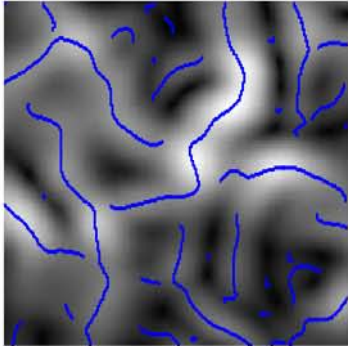


$$T_{\psi}(r, a) = \begin{pmatrix} I * \frac{\partial \phi_a}{\partial x}(r) \\ I * \frac{\partial \phi_a}{\partial y}(r) \end{pmatrix}$$

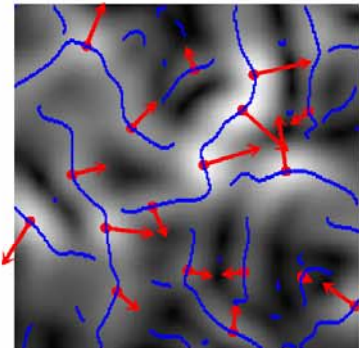
$$T_{\psi}(r, a) = \nabla (I * \phi_a)(r) = (\mathcal{M}_{\psi}(r, a), \mathcal{A}_{\psi}(r, a))$$

WTMM Methodology : Skeleton

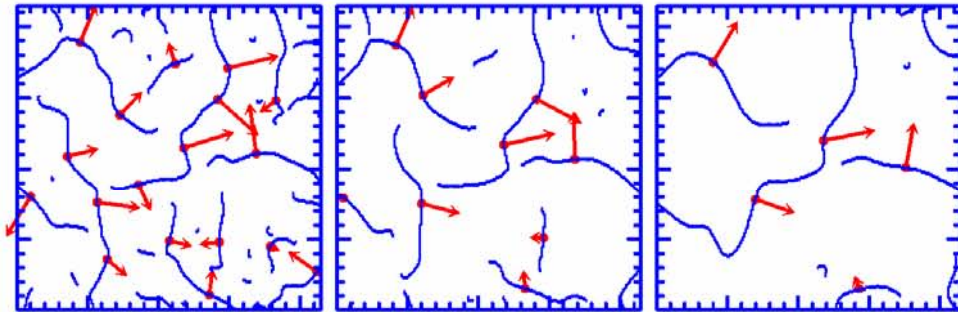
WTMM Chains



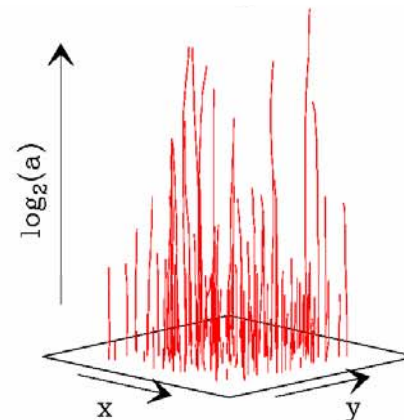
WTMMM



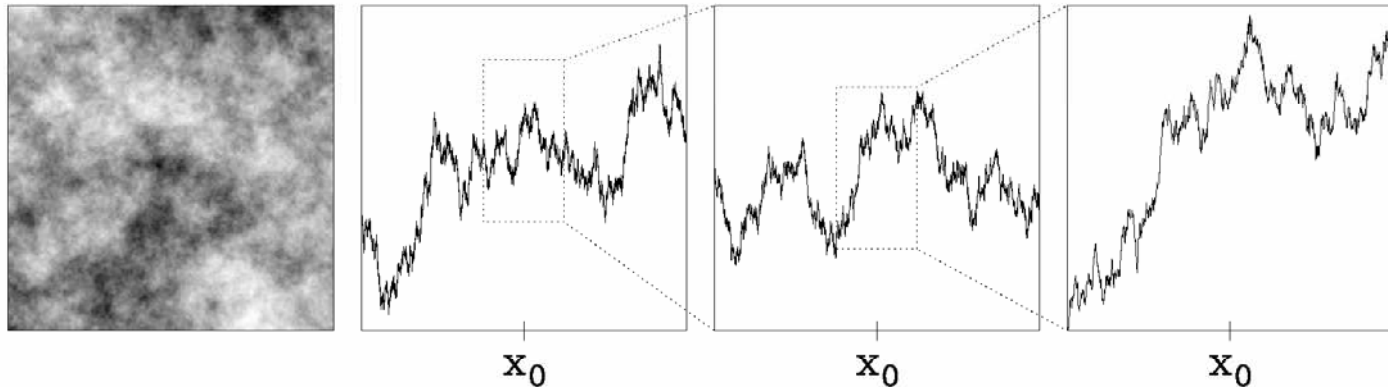
WTMM Chains at 3 different scales



Maxima lines : WT skeleton

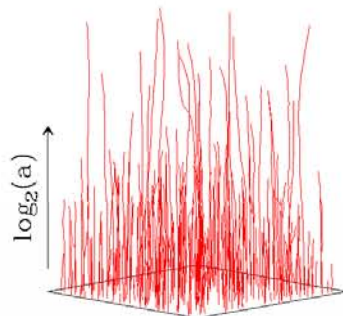


Local roughness characterization : Hölder exponent



$$f(x_0 + \lambda u) - f(x_0) \sim \lambda^{h(x_0)} (f(x_0 + u) - f(x_0))$$

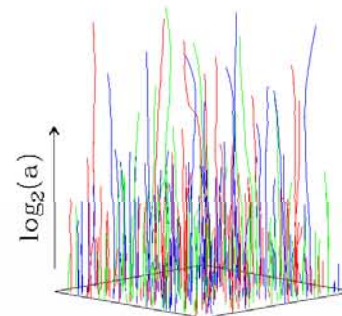
● Monofractal Image



$$\mathcal{M} \sim a^h,$$

single h

● Multifractal Image



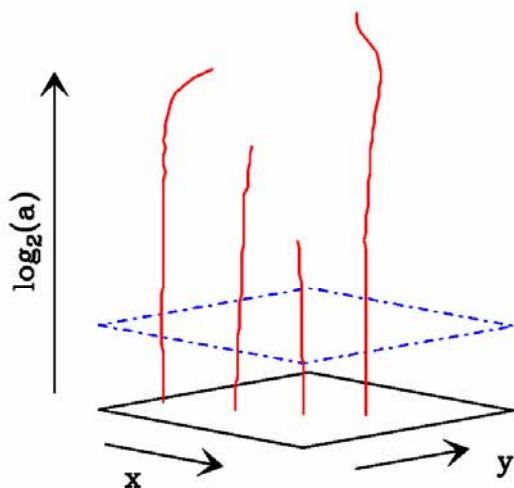
$$\mathcal{M} \sim a^h, a^h, a^h,$$

$$h \in [h_{\min}, h_{\max}]$$

WTMM Method : multifractal formalism

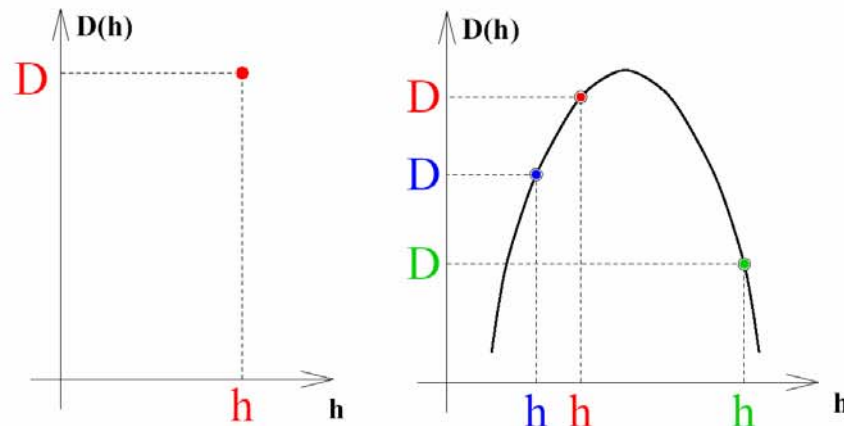
Singularity spectrum :

$$D(h) = d_H\{r \in R^d, h(r) = h\}$$



Legendre transform :

$$D(h) = \min_q (qh - \tau(q))$$



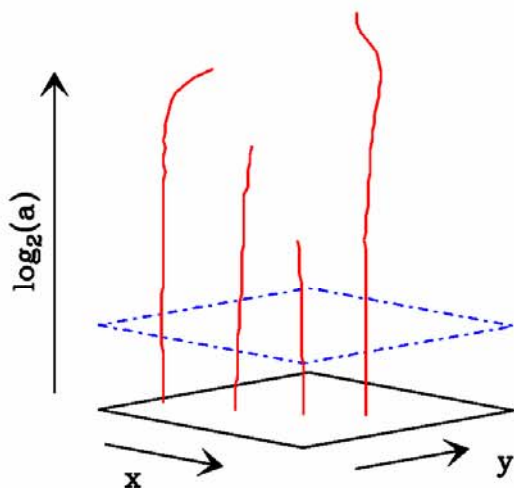
Analogy with statistical physics : compute
partition functions

$$\mathcal{Z}(q, a) = \sum_{\mathcal{L}(a)} (\mathcal{M}_\psi(r, a))^q \sim a^{\tau(q)}$$

WTMM Method : multifractal formalism

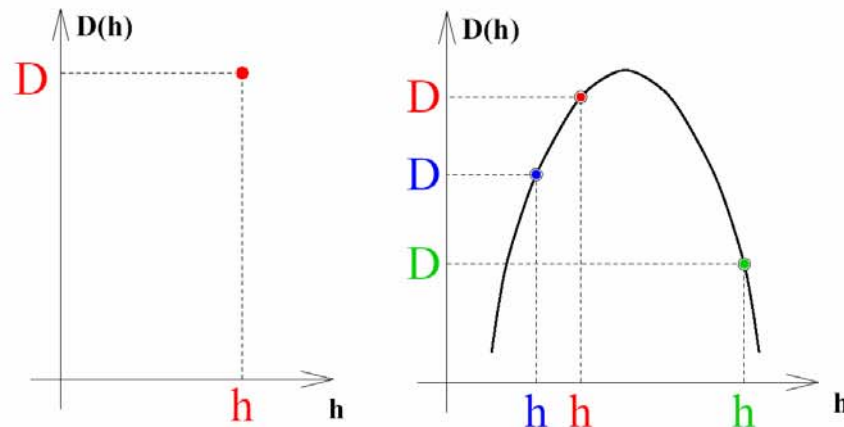
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$$\mathcal{Z}(q, a) = \sum_{\mathcal{L}(a)} (\mathcal{M}_\psi(r, a))^q \sim a^{\tau(q)}$$

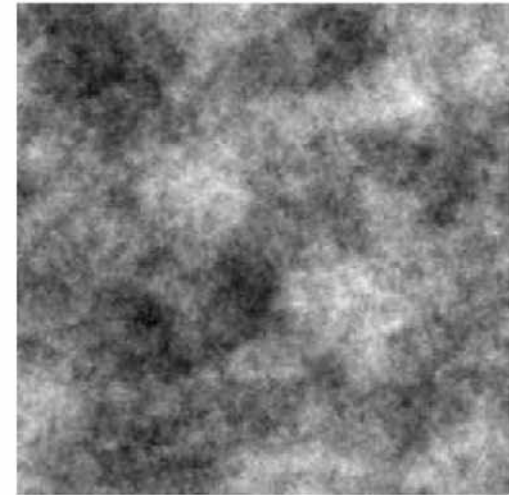
$$\mathcal{H}(q, a) = \sum_{\mathcal{L}(a)} \ln |\mathcal{M}_\psi(r, a)| \mathcal{W}_\psi(r, a) \sim a^{h(q)}$$

$$\mathcal{D}(q, a) = \sum_{\mathcal{L}(a)} \ln |\mathcal{W}_\psi(r, a)| \mathcal{W}_\psi(r, a) \sim a^{D(q)}$$

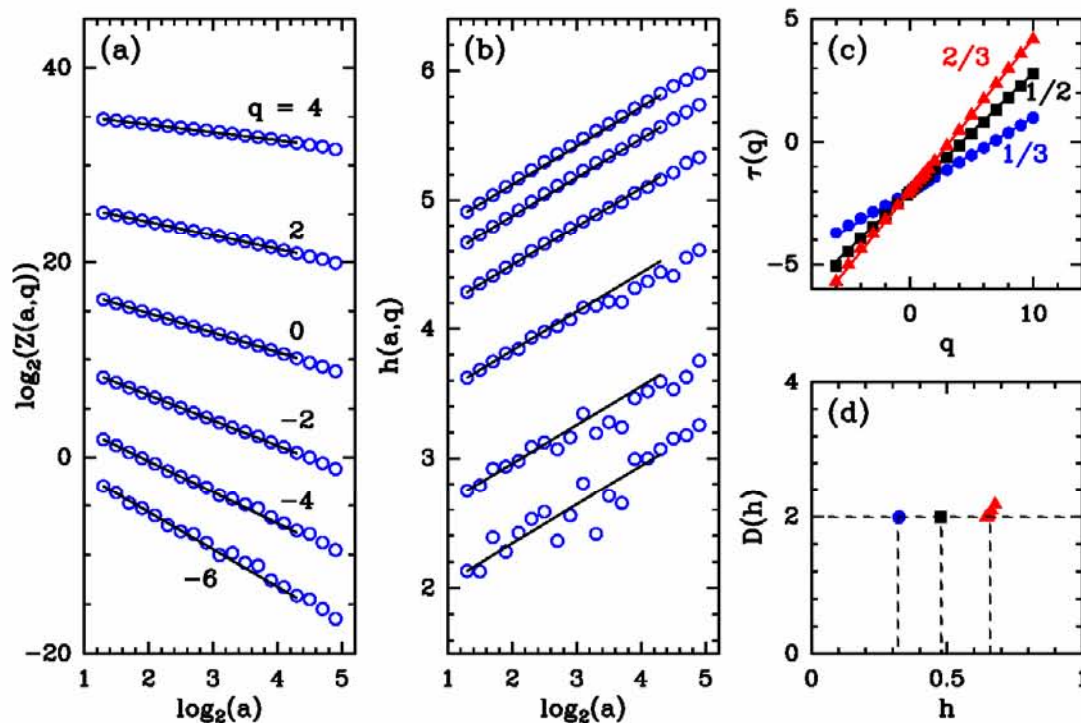
Application to synthetic **monofractal** surfaces

fractional Brownian surfaces : $B_H(r)$

$$H = 1/3$$



- $H < 0.5$: anti-correlated increments
- $H = 0.5$: uncorrelated increments
- $H > 0.5$: correlated increments



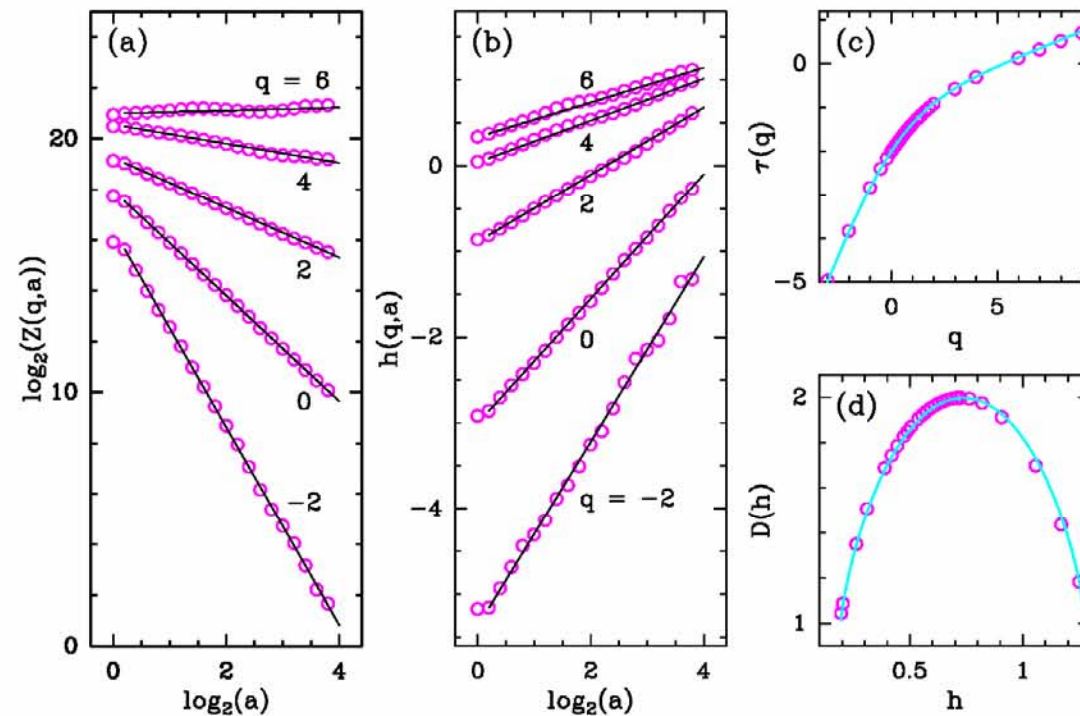
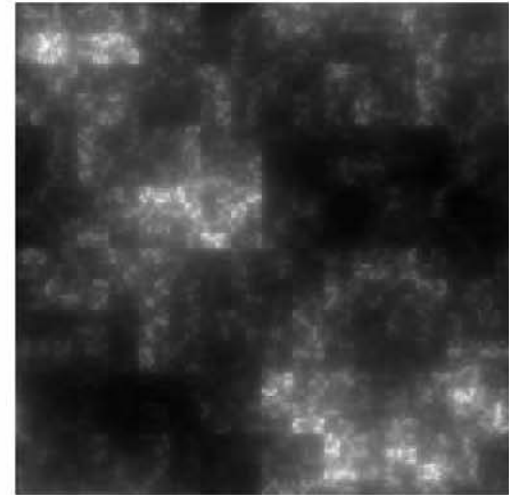
Theoretical Predictions :

- $\tau(q)$ is linear :
 $\tau(q) = qH - 2$
- multifractal spectrum is degenerated :
 $D(h = H) = 2$

Application to synthetic **multifractal** surfaces

Multifractal (Fractionally Integrated
Singular Cascades) surfaces

FISC



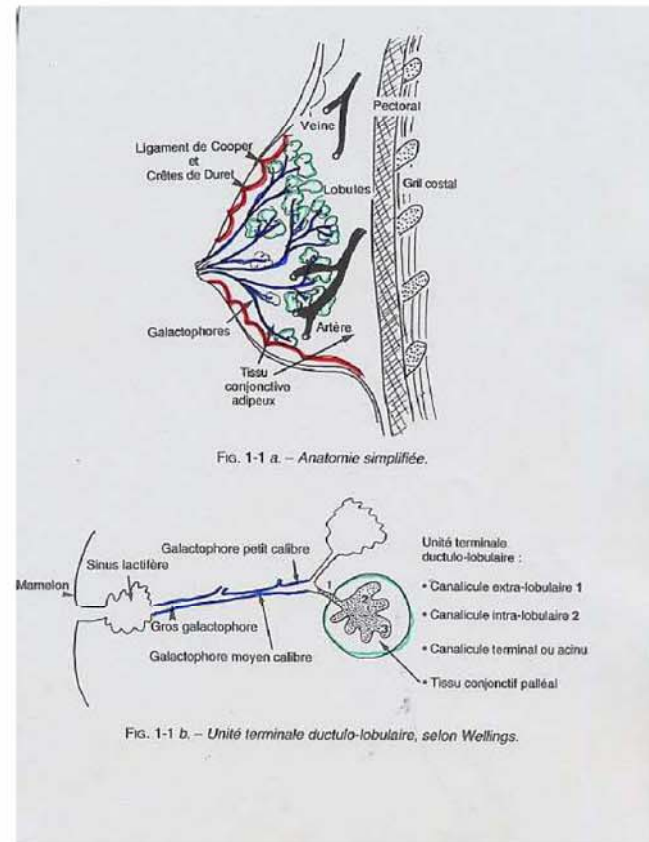
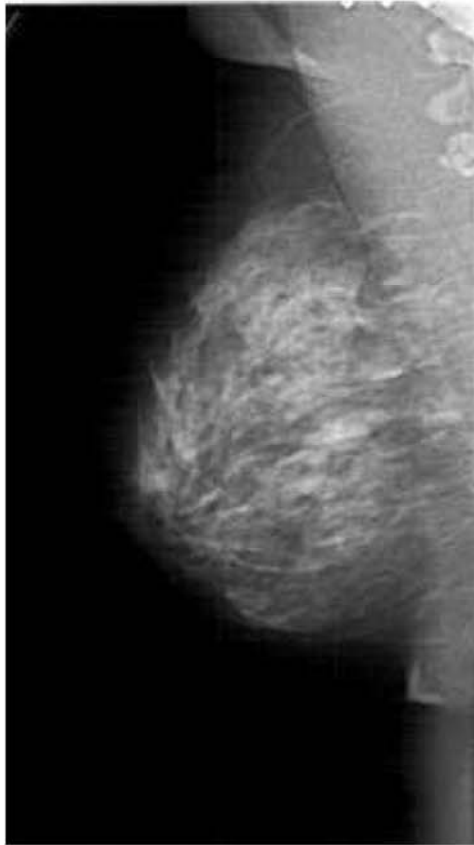
Theoretical predictions :

- $\tau(q)$ is non-linear

$$\tau(q) = -2 - q(1 - H^*) - \log_2(p_1^q + p_2^q)$$
- singularity spectrum is a non-degenerated convex curve

Mammography and breast anatomy

Goals : using WTMM method to help in diagnosis of breast cancer



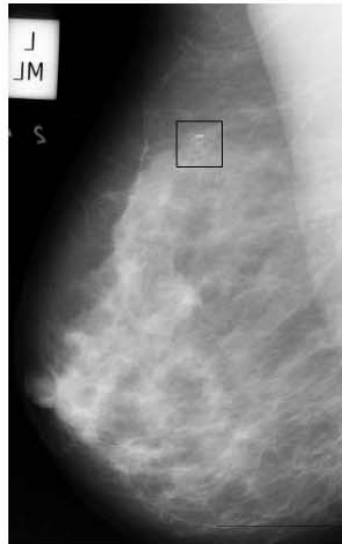
What is breast cancer ?

- malignant tumor of mammal gland
 - incidence : 30000 new case each year in France
 - prevention is very difficult (as opposed to lung cancer)
 - hereditary : 5 to 10 % only (BRCA1/2 genes)
 - forecast depends on the tumoral volume at diagnosis
- ⇒ **SCREENING** using mammography

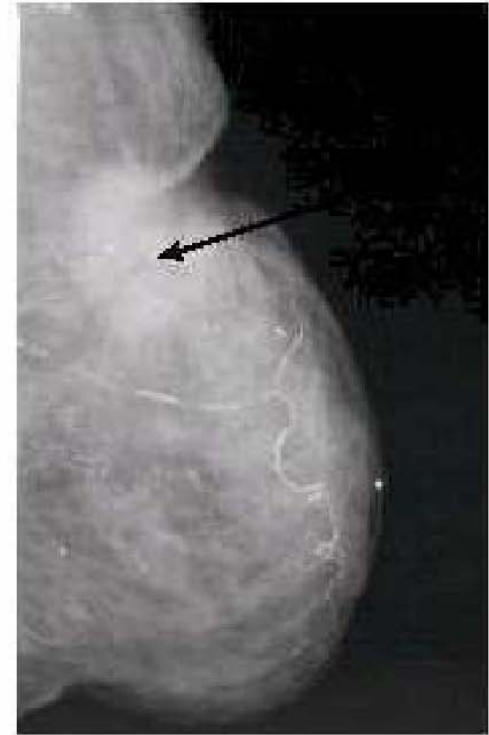
Radiological anomalies



● Opacities



● Calcifications



● Architectural distortions

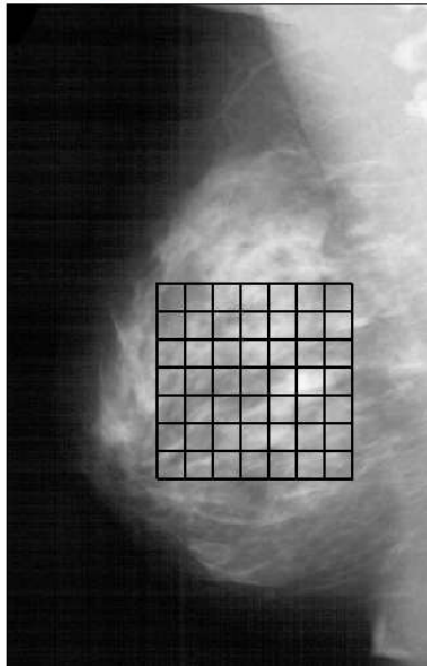
Digitalized mammographies : texture analysis

- 📍 dense breasts : more difficult to diagnose
- 📍 only 2 classes of monofractal properties

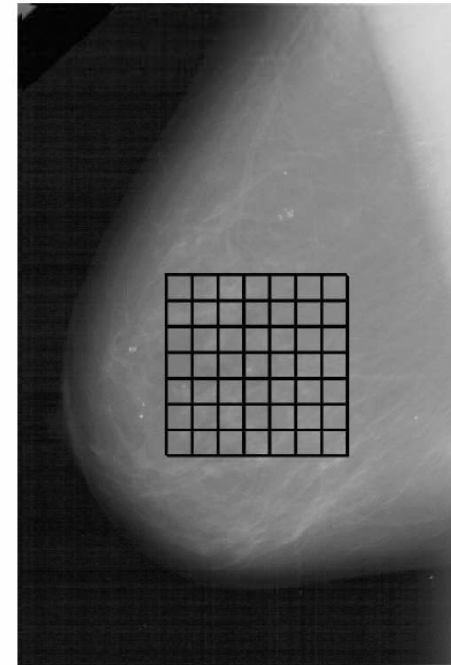
Digital Database for Screening Mammography:

<http://marathon.csee.usf.edu/Mammography/Database.html>

Dense breast

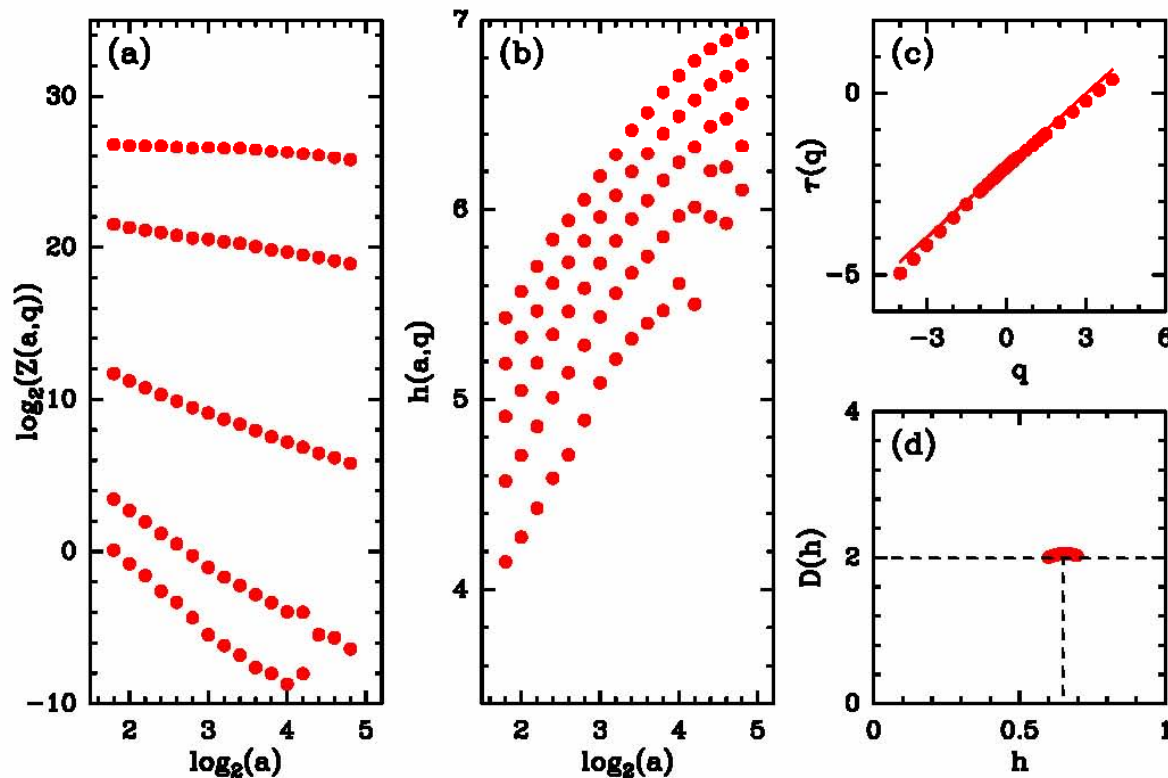


Fatty breast



Application of 2D WTMM methodology in mammography

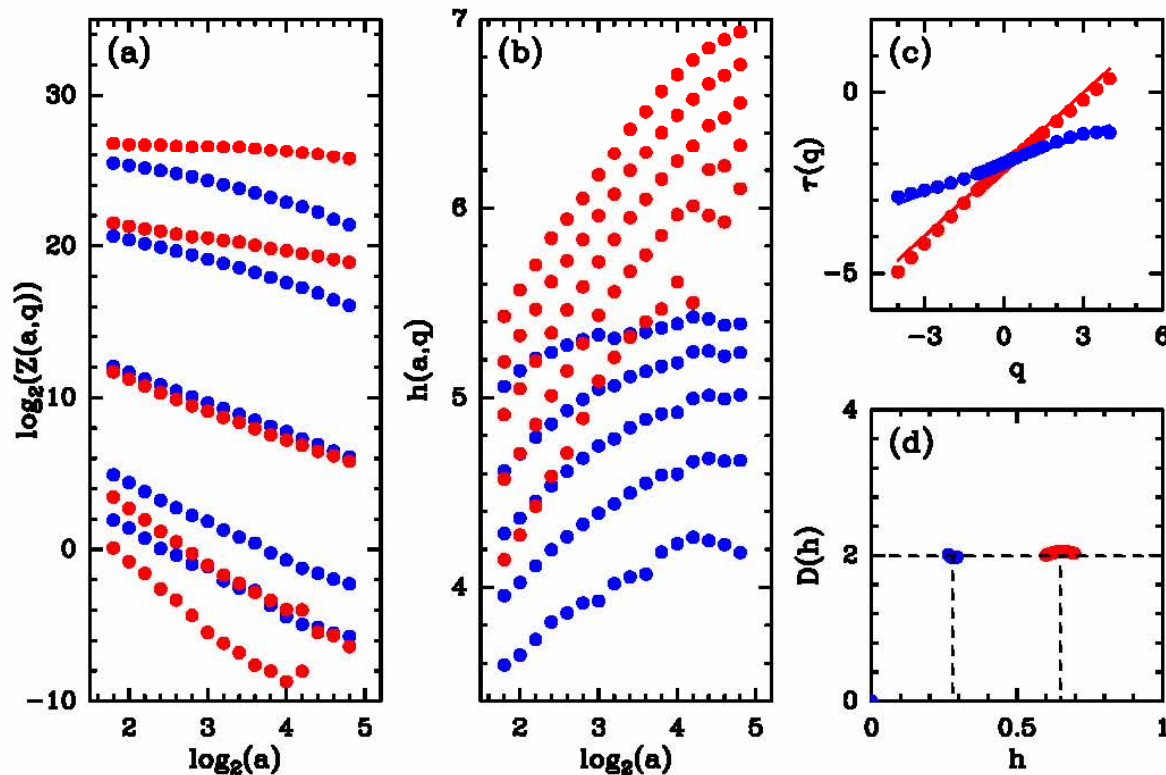
Tissue classification : **dense**



 **Dense breast :**
monofractal, $H = 0.65$
persitent correlations

Application of 2D WTMM methodology in mammography

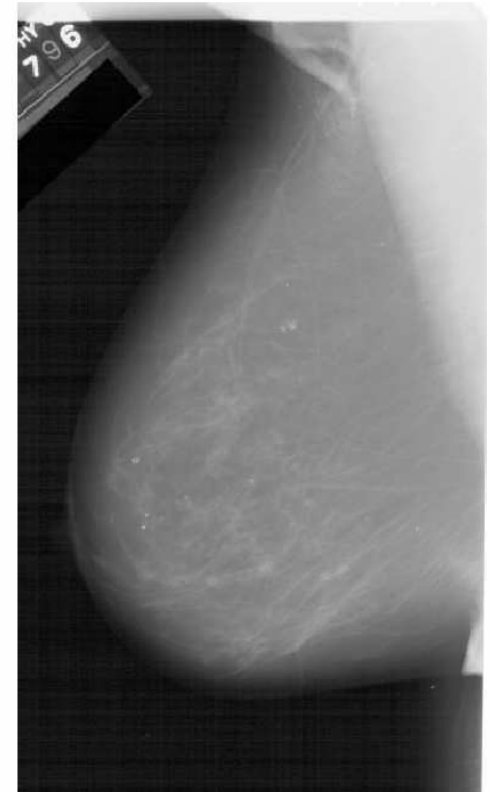
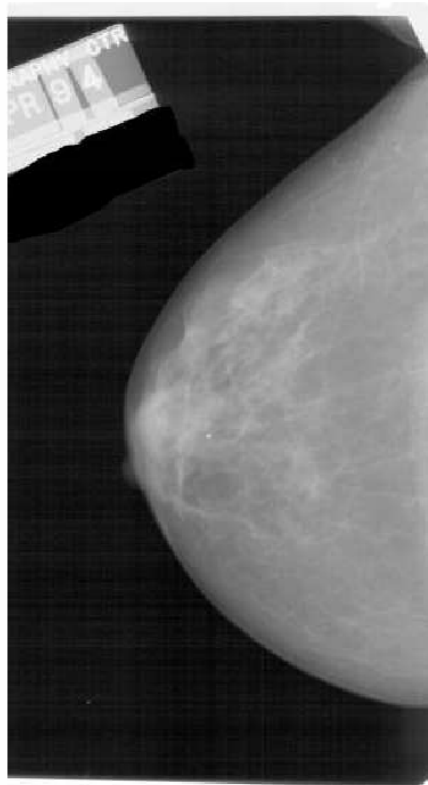
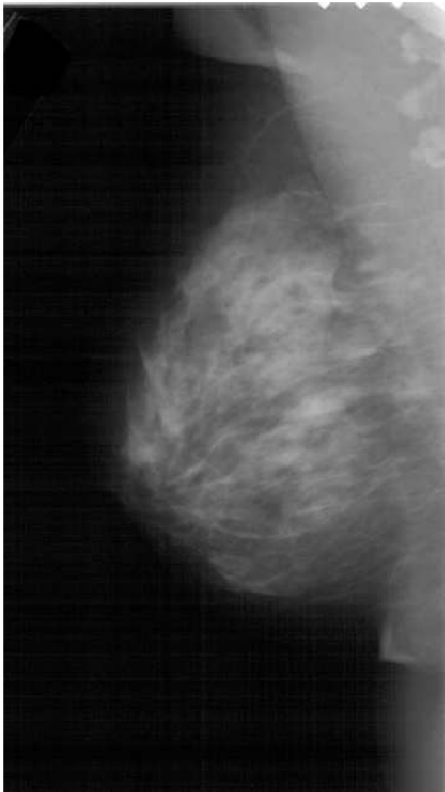
Tissue classification : **dense** vs **fatty**



- Dense breast :**
monofractal, $H = 0.65$
persitent correlations
- Fatty breast :**
monofractal, $H = 0.30$
anti-persitent correlations

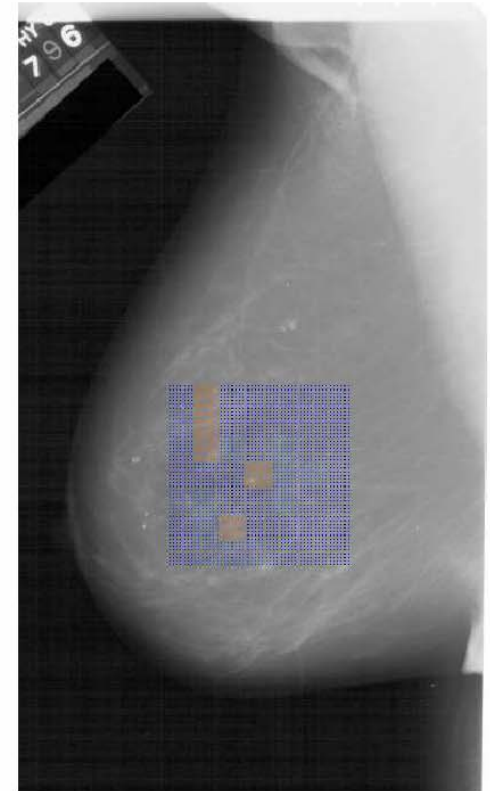
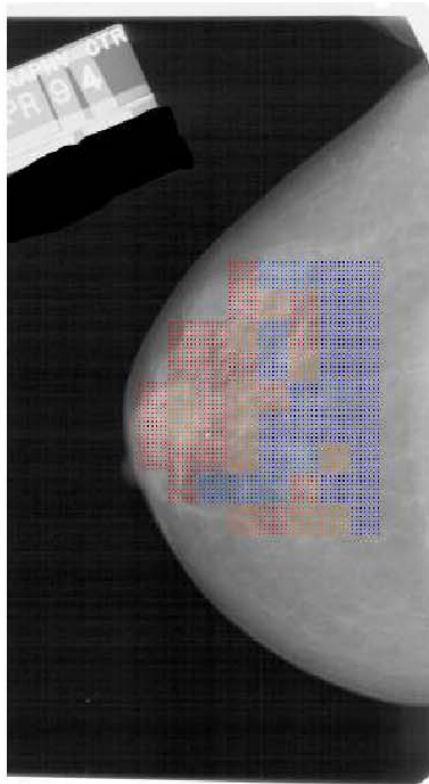
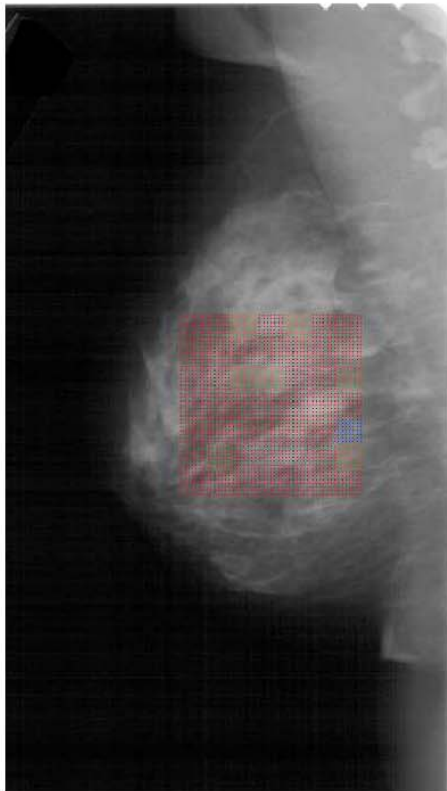
Application to digitalized mammographies

Colored Maps :
segmentation of **dense** $h > 0.52$ areas and **fatty** $h < 0.38$ areas



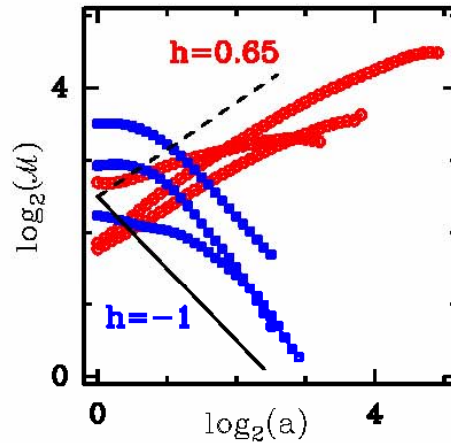
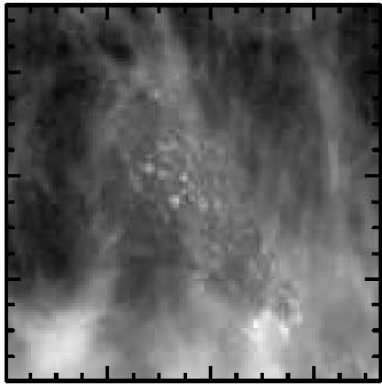
Application to digitalized mammographies

Colored Maps :
segmentation of **dense** $h > 0.52$ areas and **fatty** $h < 0.38$ areas



Microcalcifications detection

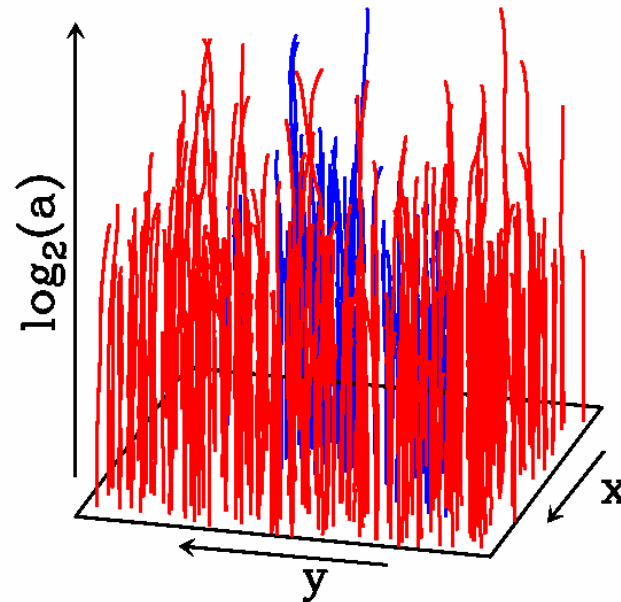
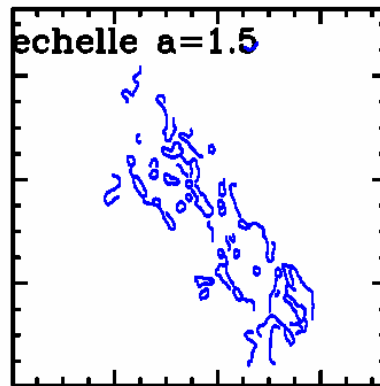
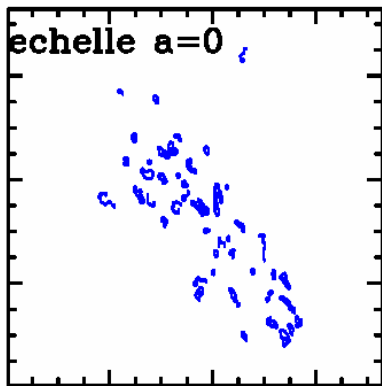
Segmentation of WT skeleton lines :
microcalcifications vs background texture



● Background lines

● Microcalcifications

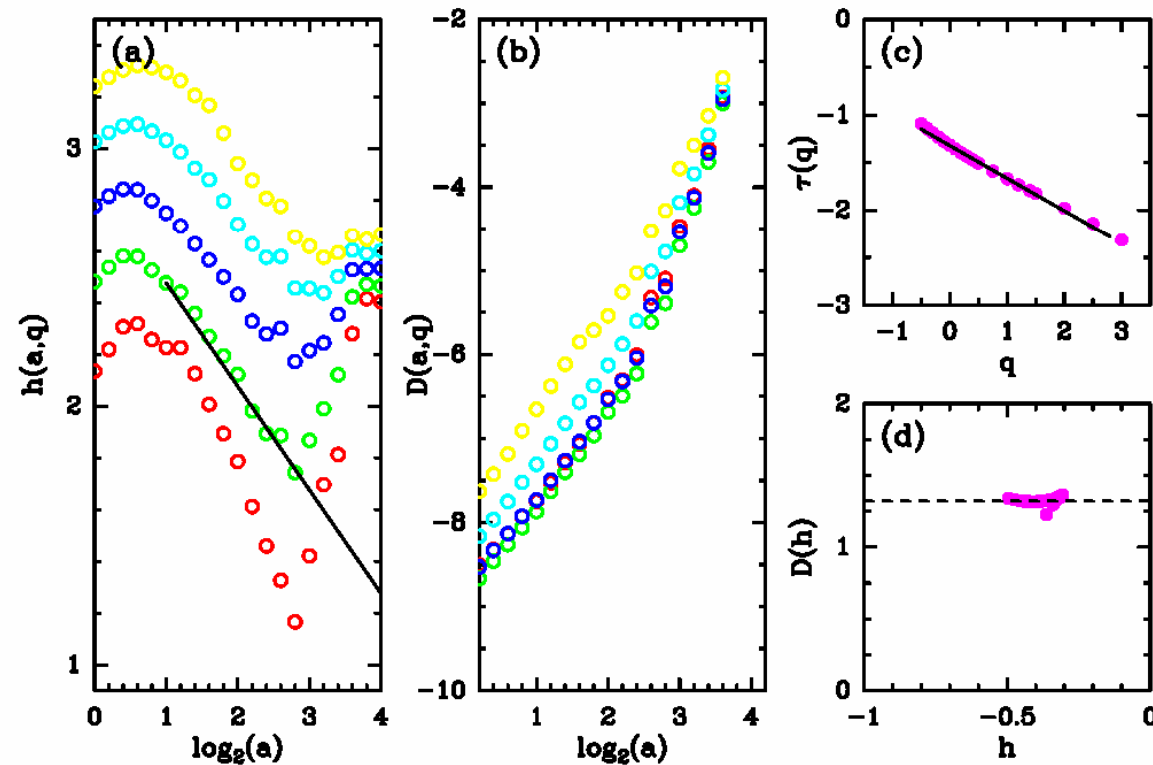
almost-punctual objects behave like
'Dirac' shapes ($h = -1$)



Cluster of microcalcifications

Study of microcalcification spatial distribution

Partition functions :



$$D_F = 1.3$$

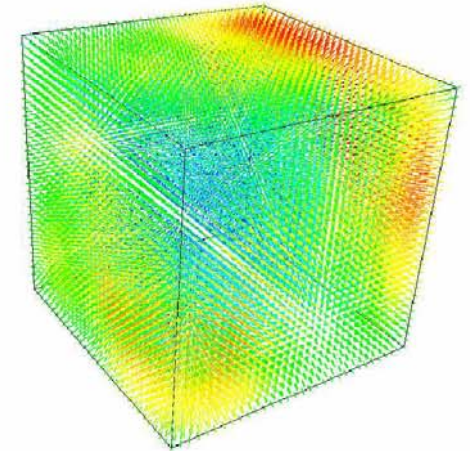
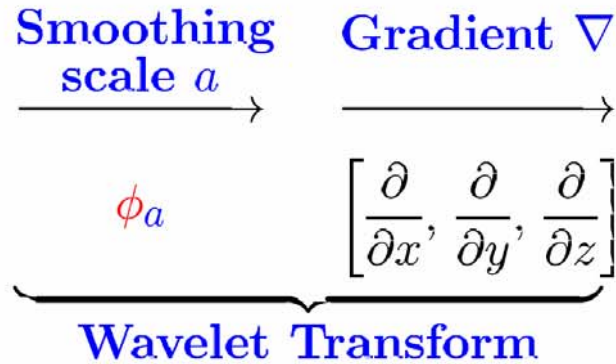
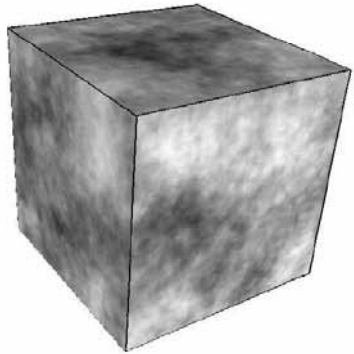
observation :
fractal ramification of
cluster of
microcalcifications
($1 < D_F < 2$) seems to be
correlated to the
pathology's malignancy

Conclusions and prospects (1)

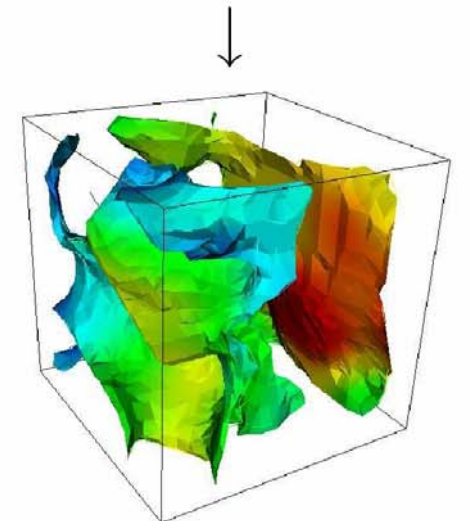
- the **2D WTMM method** provides a framework for an automated measure of the breast **radio-density** and for studying the **fractal geometry of clusters of microcalcifications**.
- further study is necessary to validate quantitatively how far measuring the fractal dimension D_F could improve computer-aided diagnosis systems **benign/malignant**

3D scalar WTMM method

3D data

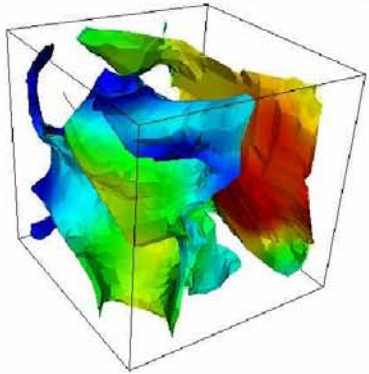


$$\mathbf{T}_\psi(\mathbf{r}, a) = \begin{pmatrix} \mathbf{I} * \frac{\partial \phi_a}{\partial x}(\mathbf{r}) \\ \mathbf{I} * \frac{\partial \phi_a}{\partial y}(\mathbf{r}) \\ \mathbf{I} * \frac{\partial \phi_a}{\partial z}(\mathbf{r}) \end{pmatrix} = \nabla (\mathbf{I} * \phi_a)(\mathbf{r})$$

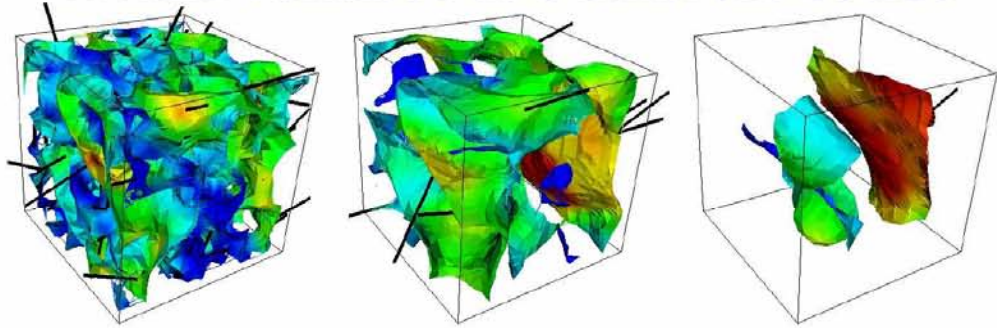


3D scalar WTMM method : skeleton

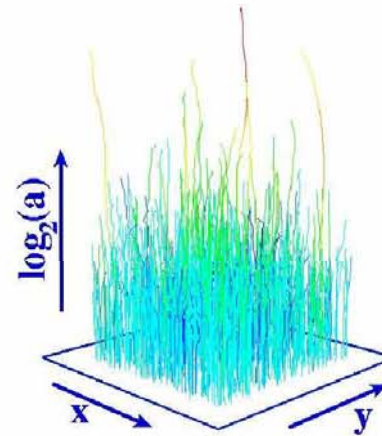
WTMM surfaces



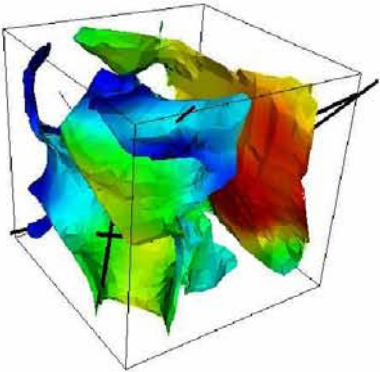
WTMM surfaces at 3 different scales



Maxima lines : WT Skeleton (projection along z)



WTMMM points



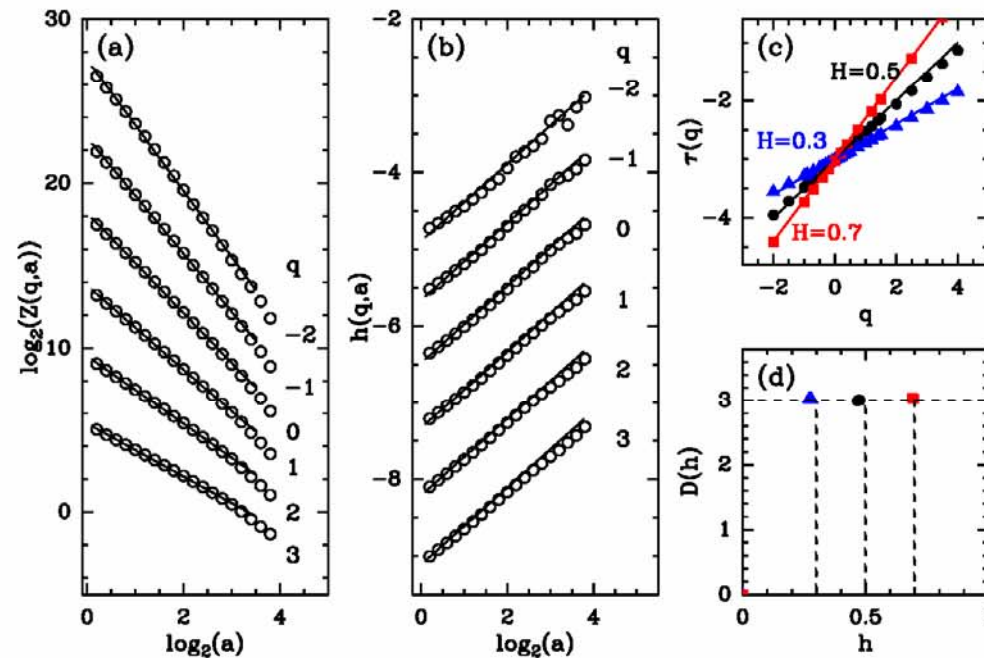
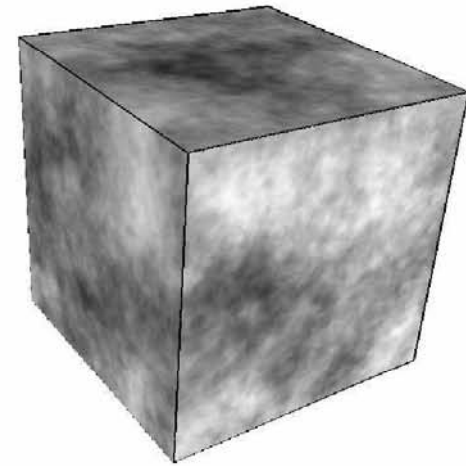
Test-application to synthetic 3D monofractal fields

fractional Brownian fields : $B_H(r)$

⇒ $H < 0.5$: anti-correlated increments

⇒ $H = 0.5$: uncorrelated increments

⇒ $H > 0.5$: correlated increments



Theoretical predictions :

⇒ $\tau(q)$ is linear:

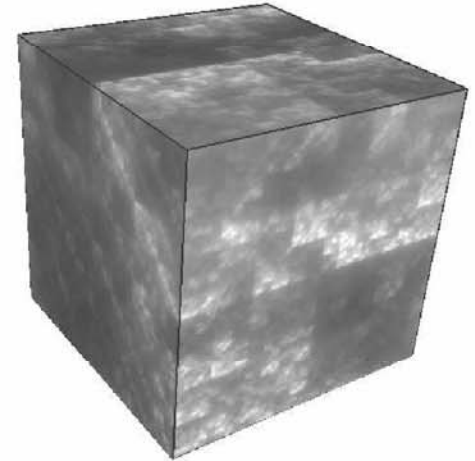
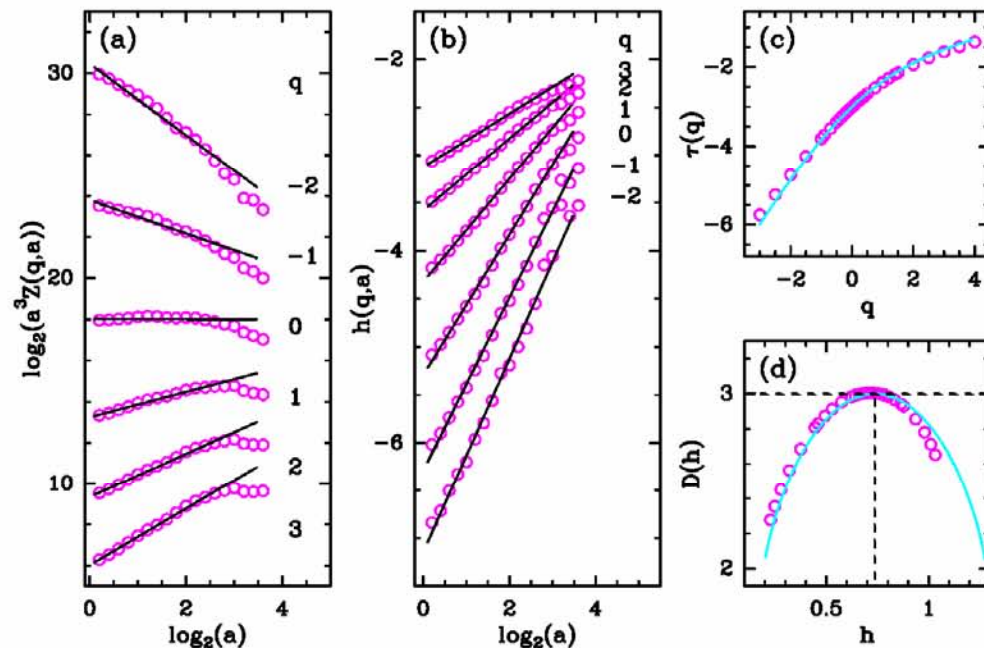
$$\tau(q) = qH - 3$$

⇒ multifractal spectrum is degenerated:

$$D(h = H) = 3$$

Test-application to synthetic 3D multifractal fields

3D multifractal fields (Fractionally Integrated Singular Cascades)



Theoretical predictions :

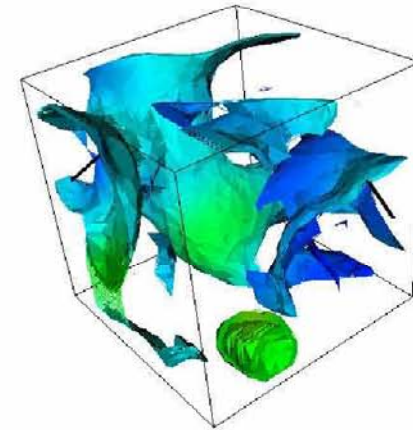
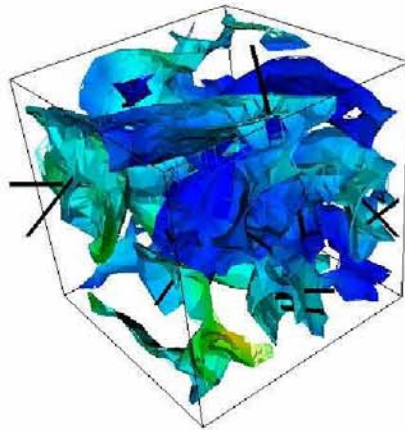
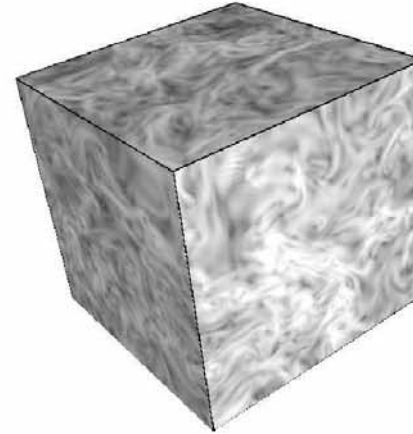
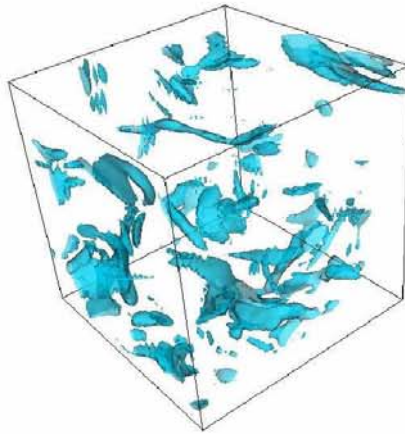
$$\tau(q) = -2 - q(1 - H^*) - \log_2(p_1^q + p_2^q).$$

with $p_1 + p_2 = 1$

singularity spectrum is a non-degenerated convex curve

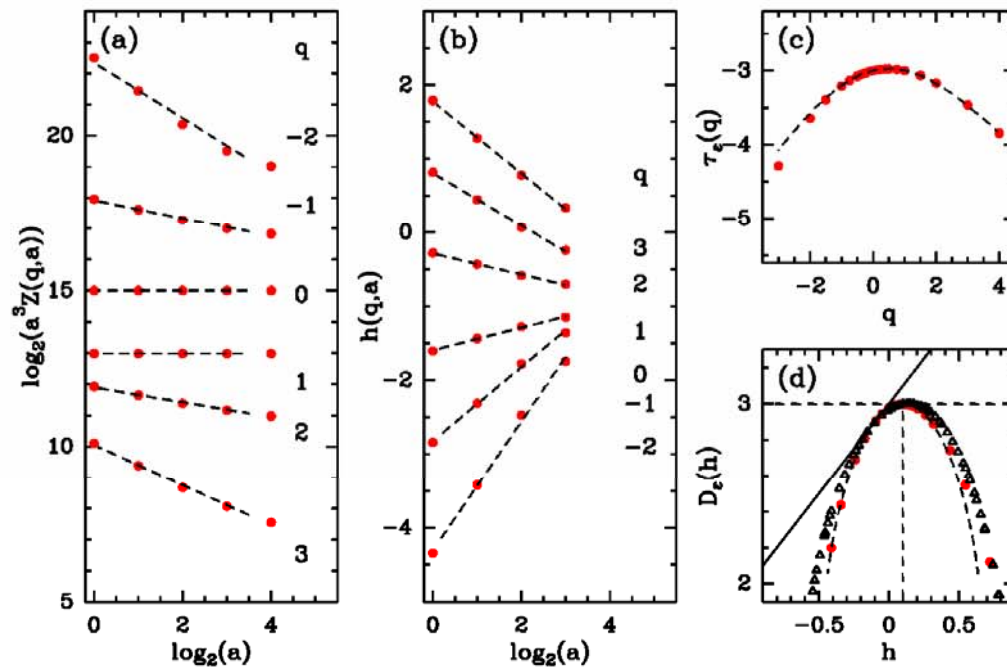
3D dissipation field : isotropic turbulence DNS

pseudo-spectral code, $(512)^3$ grid, $R_\lambda = 216$ (M. Meneguzzi)



3D WTMM methodology vs Box-Counting algorithms

☞ “**Box-Counting**” algorithm, binomial fit with $p_1 = 0.3$ and $p_2 = 0.7 \rightarrow$
 $p_1 + p_2 = 1$: diagnoses a **conservative** multiplicative structure



dissipation

binomial model :

$$\tau(q) = -2 - q - \log_2(p_1^q + p_2^q)$$

3D WTMM methodology vs Box-Counting algorithms

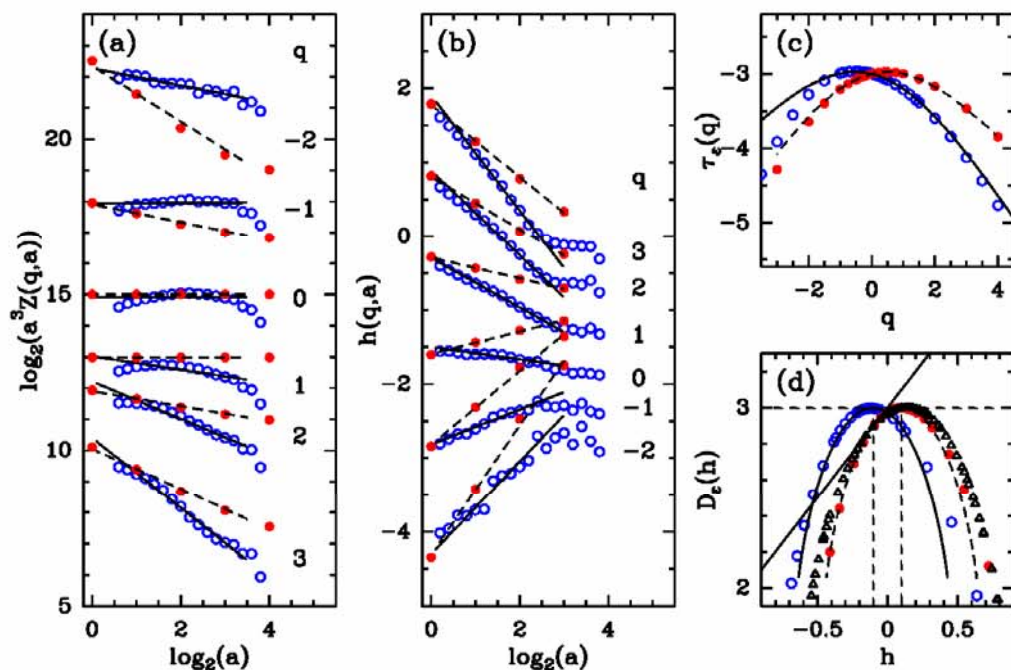
☞ “**Box-Counting**” algorithm, binomial fit with $p_1 = 0.3$ and $p_2 = 0.7 \rightarrow$

$p_1 + p_2 = 1$: diagnoses a **conservative** multiplicative structure

☞ “**3D WTMM**” method reveals a **non-conservative** multiplicative structure :

binomial fit with $p_1 = 0.36$ and $p_2 = 0.80 \Rightarrow p_1 + p_2 \neq 1$

dissipation



binomial model :
 $\tau(q) = -2 - q - \log_2(p_1^q + p_2^q)$

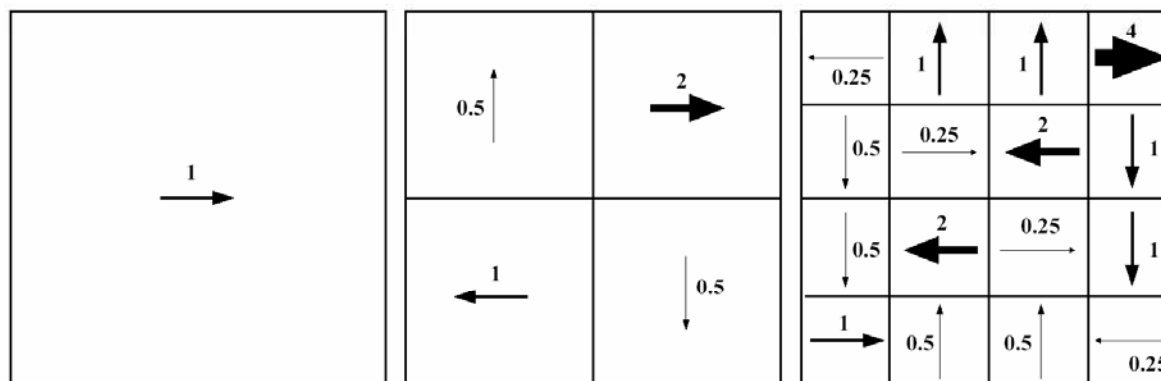
remark:

$$p_1 = \frac{p_1}{p_1 + p_2}$$

$\Rightarrow$ inconsistent

box-counting !

Self-similar multifractal vector-valued measure (2D case)



👉 Falconer et O'Neil (1995)

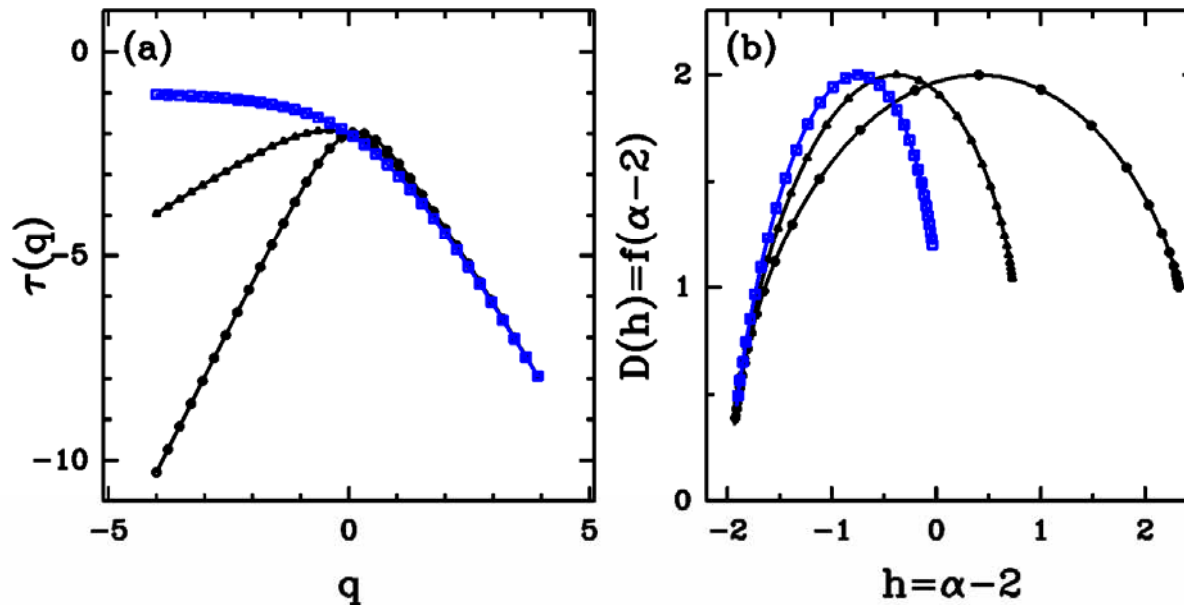
👉 scalar measure $\{r : \lim_{l \rightarrow 0} \frac{\log \mu(B(r, l))}{\log l} = \alpha\}, \quad \alpha = h + 2$

$$\mathcal{Z}(\mathbf{q}, l) = \sum_i \mu_i^{\mathbf{q}}(l) \sim l^{\tau_{\mu}(\mathbf{q})}$$

👉 vector-valued measure $\left\{ r : \lim_{l \rightarrow 0} \frac{\log \int_{B(r, l)} \|\Phi_l \mu(s)\| d\mathcal{L}_d(s)}{\log l} = \alpha \right\}, \quad \alpha = h + 2$

$$\mathcal{Z}(\mathbf{q}, l) = \sum_i \|\Phi_l \mu\|_i^{\mathbf{q}} \sim l^{\tau_{\mu}(\mathbf{q})}$$

Self-similar multifractal vector-valued measure (2D case)



$$\tau_{\mu}(\mathbf{q}) = -\frac{\log(p_1^{\mathbf{q}} + p_2^{\mathbf{q}} + p_3^{\mathbf{q}} + p_4^{\mathbf{q}})}{\log 2}$$

$$D_{\mu}(h) = f_{\mu}(\alpha - 2) = \inf_{\mathbf{q}}(\mathbf{q}h - \tau_{\mu}(\mathbf{q})).$$

Tensorial wavelet transform (2D case)

1. Tensorial wavelet transform of field $V = (V_1, V_2)$:

$$\mathbb{T}_{\psi}[V](b, a) = (T_{\psi_i}[V_j](b, a)) = \begin{pmatrix} T_{\psi_1}[V_1] & T_{\psi_1}[V_2] \\ T_{\psi_2}[V_1] & T_{\psi_2}[V_2] \end{pmatrix}$$

$$T_{\psi_i}[V_j](b, a) = a^{-3} \int d^3r \psi_i(a^{-1}(r - b)) V_j(r), j = 1, 2$$

2. Direction of greatest variation of vector field :

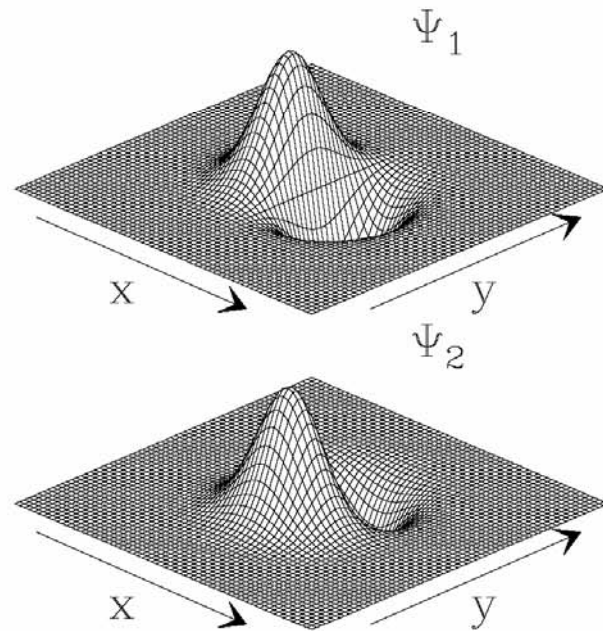
$$|\mathbb{T}_{\psi}[V]| = \sup_{C \neq 0} \frac{||\mathbb{T}_{\psi}[V] \cdot C||}{||C||}$$

3. Singular value decomposition of WT tensor:

$$\mathbb{T}_{\psi}[V] = (G) \cdot \begin{pmatrix} \sigma_{\max} & 0 \\ 0 & \sigma_{\min} \end{pmatrix} \cdot (D)^T$$

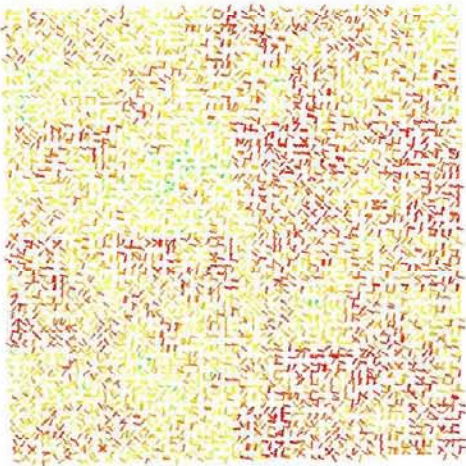
4. Tensorial wavelet transform :

$$T_{\psi, \max}[V](b, a) = \sigma_{\max} G_{\sigma_{\max}}$$



Tensorial 2D WTMM methodology

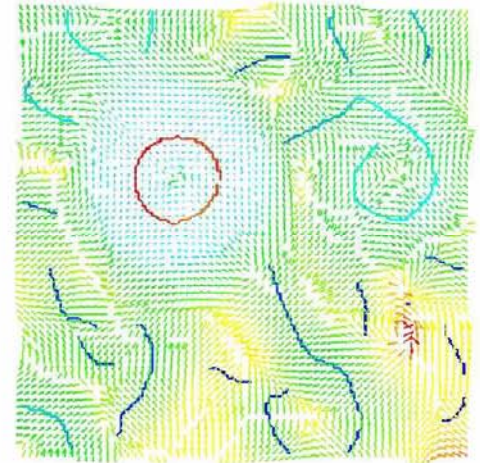
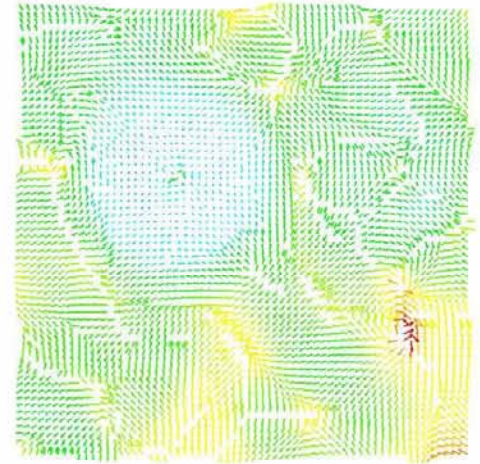
Data



Tensorial wavelet transform



$$\mathbf{T}_{\psi, \max}[\mathbf{V}](\mathbf{b}, a) = \sigma_{\max} \mathbf{G}_{\sigma_{\max}}$$

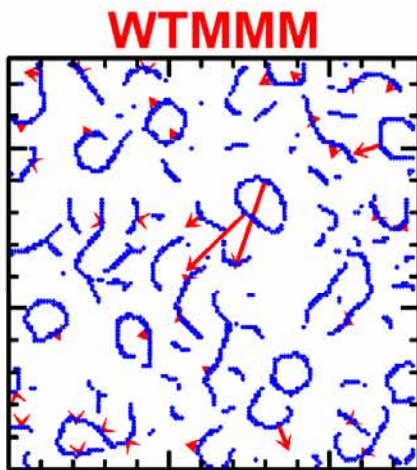
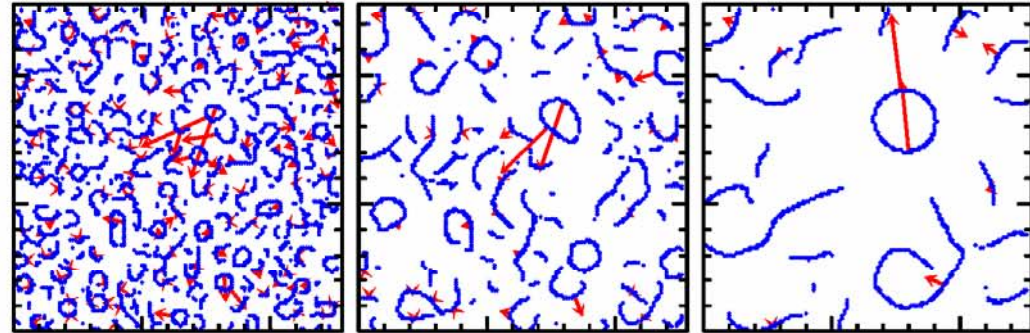
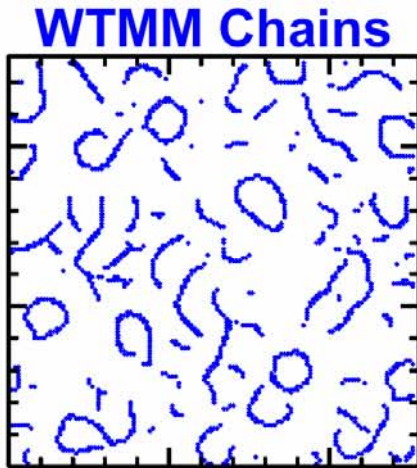


Modulus Maxima $\sigma_{\max}$ chains of tensorial wavelet transform at scale a :

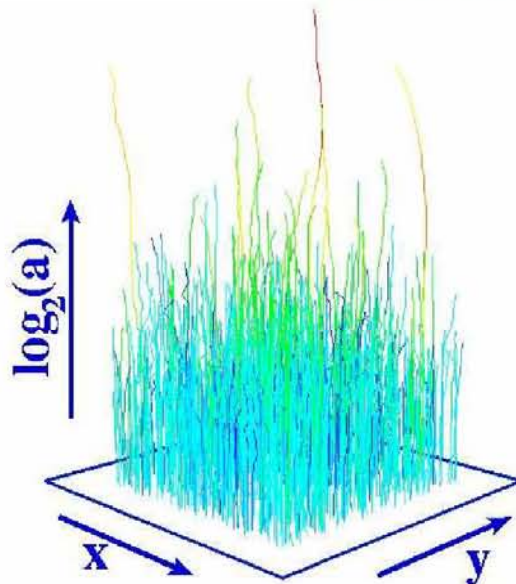
$$\left\{ (b, a) / \frac{\partial \sigma_{\max}}{\partial G_{\max}} = 0 \quad \text{et} \quad \frac{\partial^2 \sigma_{\max}}{\partial G_{\max}^2} < 0 \right\}$$

Tensorial 2D WTMM methodology: Skeleton

WTMM Chains at 3 different scales



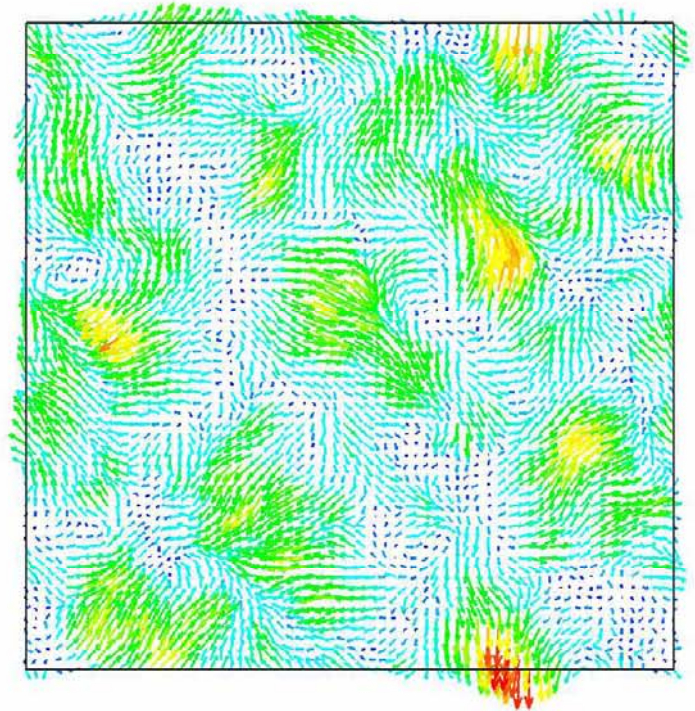
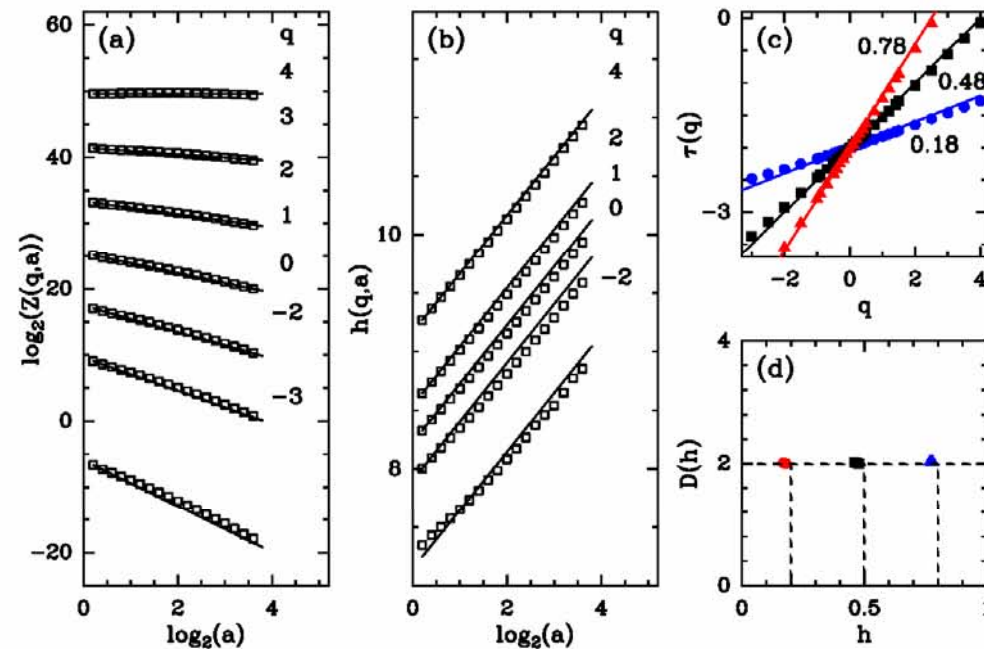
Maxima lines : **WT Skeleton**



Monofractal 2D vector fields

fractional Brownian fields : $B_H(r)$

👉 Spectral method simulation



Theoretical predictions :

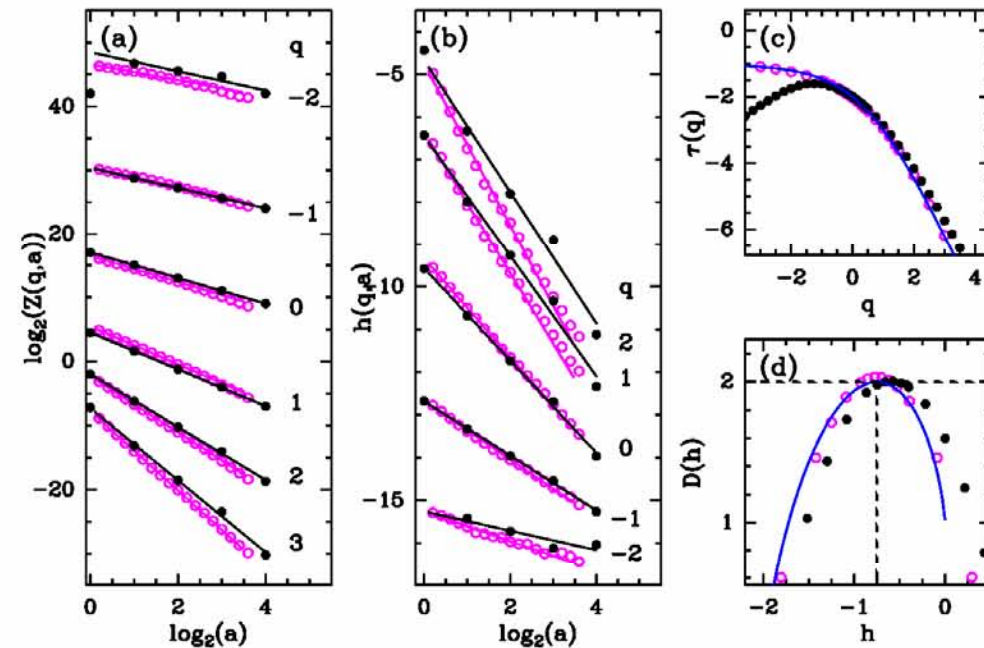
● linear $\tau(q)$: $\tau(q) = qH - 2$

● degenerated singularity spectrum:

$$D(h = H) = 2$$

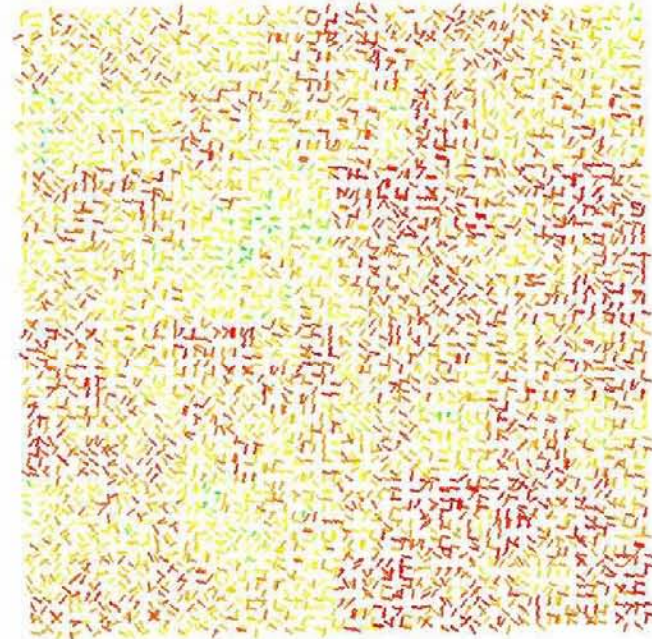
2D self-similar multifractal vector-valued measures

Self-similar **multifractal** vector-valued measures
(Falconer and O'Neil's **model**)



● vectorial box-counting

○ vectorial 2D WTMM method



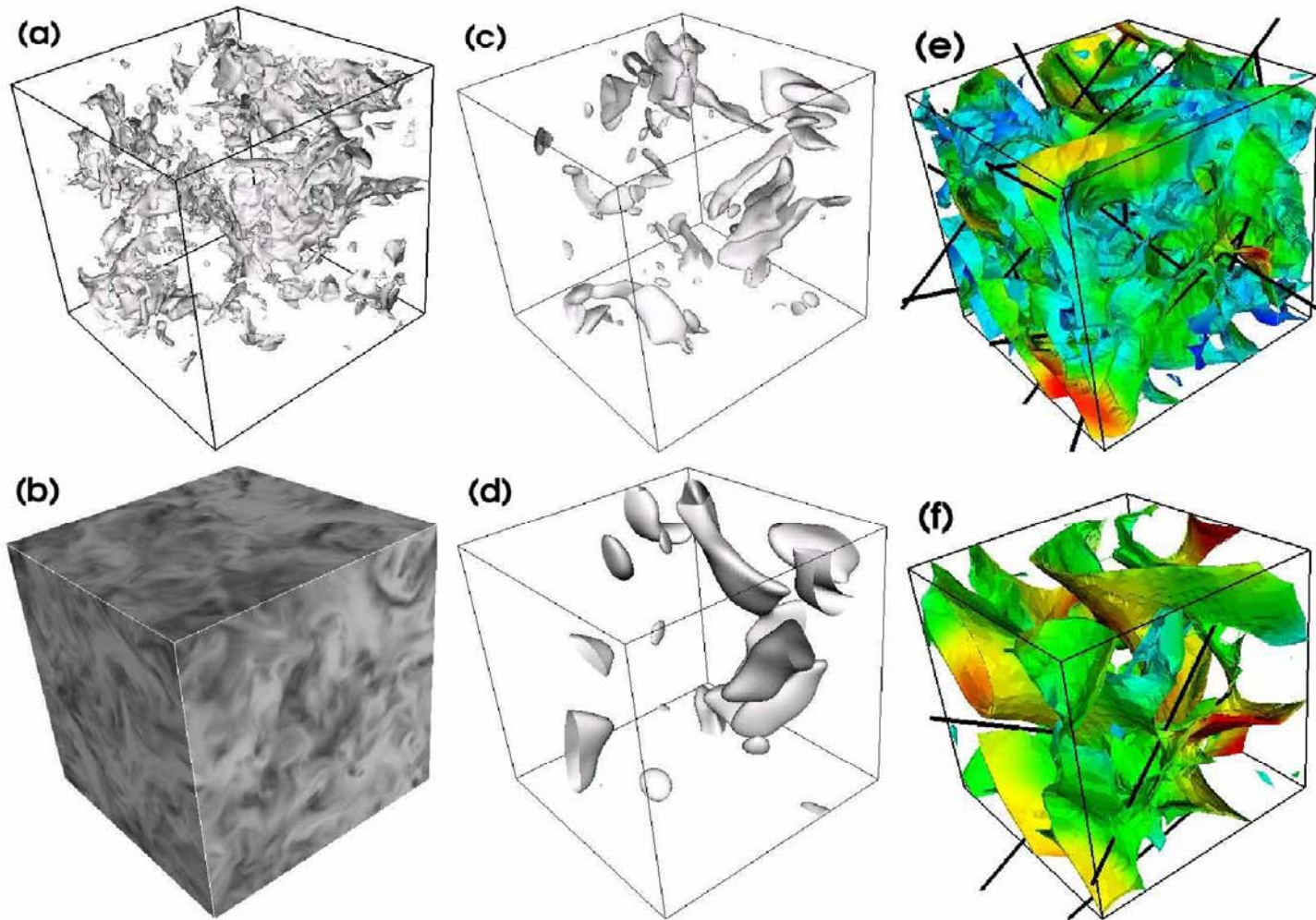
Theoretical predictions:

● $\tau(q) = -\log_2(p_1^q + p_2^q + p_3^q + p_4^q)$

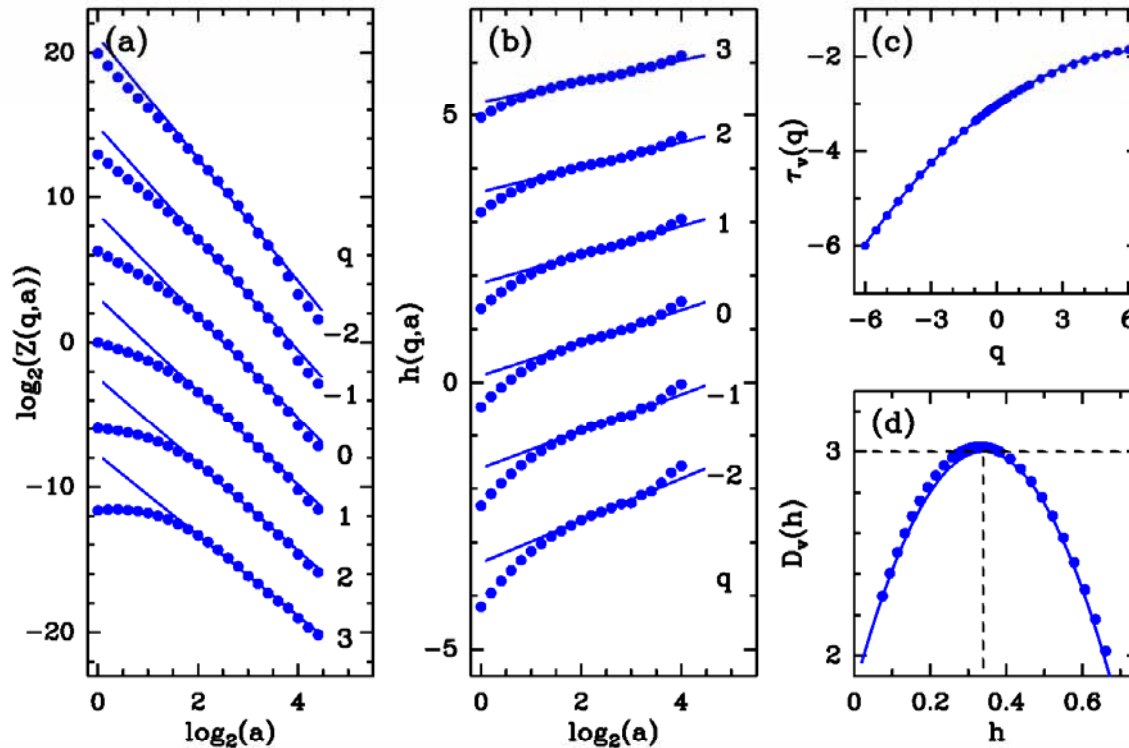
$p_1 = p_4 = 0.5, p_2 = 2$ and $p_3 = 1$

● vectorial box-counting is less accurate

Tensorial 3D WTMM method: turbulent velocity field ($R_\lambda = 140$)



Tensorial 3D WTMM method: singularity spectrum of velocity

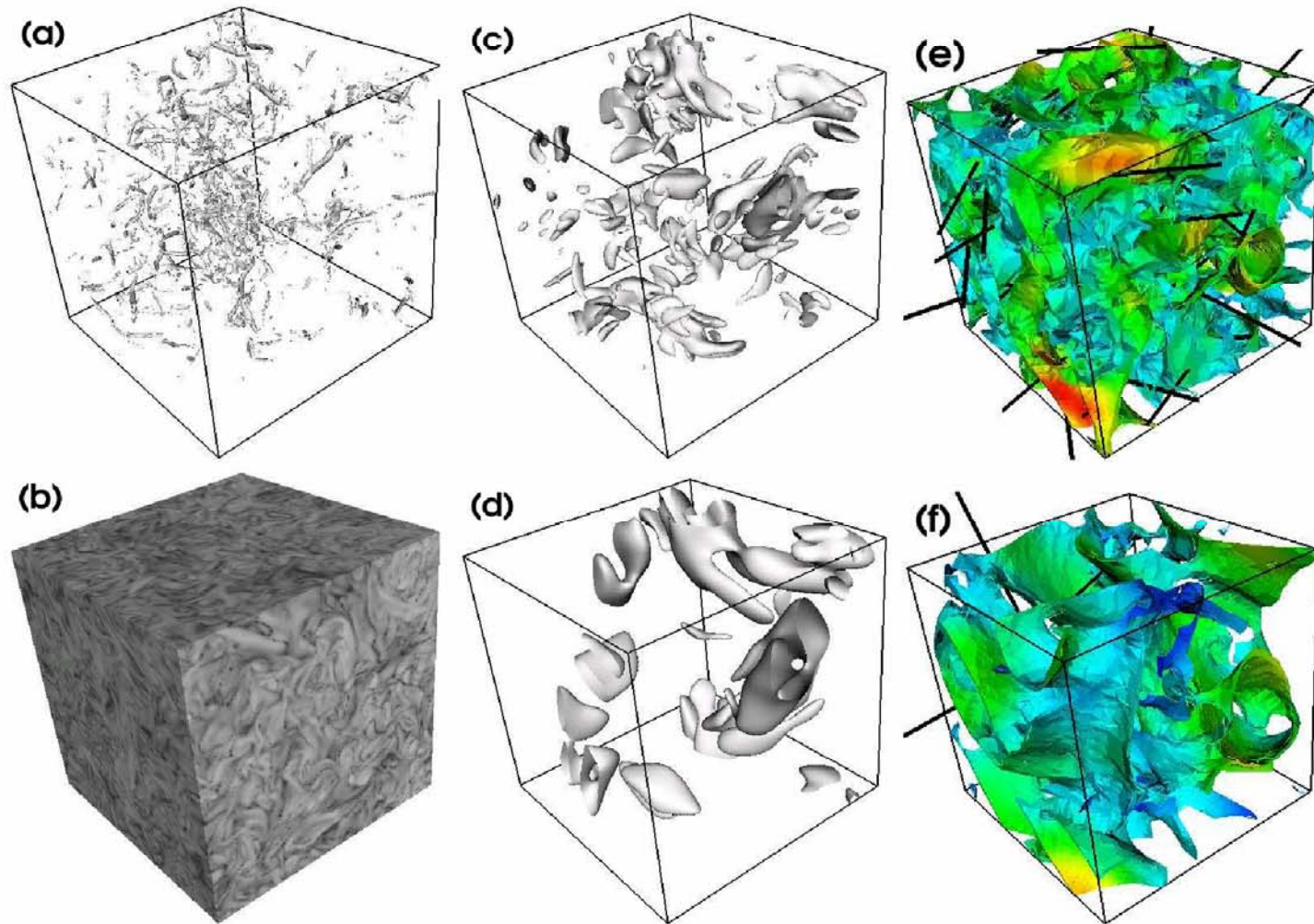


- parabolic fit: $\tau(q) = -C_0 - C_1 q - C_2 \frac{q^2}{2}$
- intermittency coefficient $C_2 = 0.049 \pm 0.004$

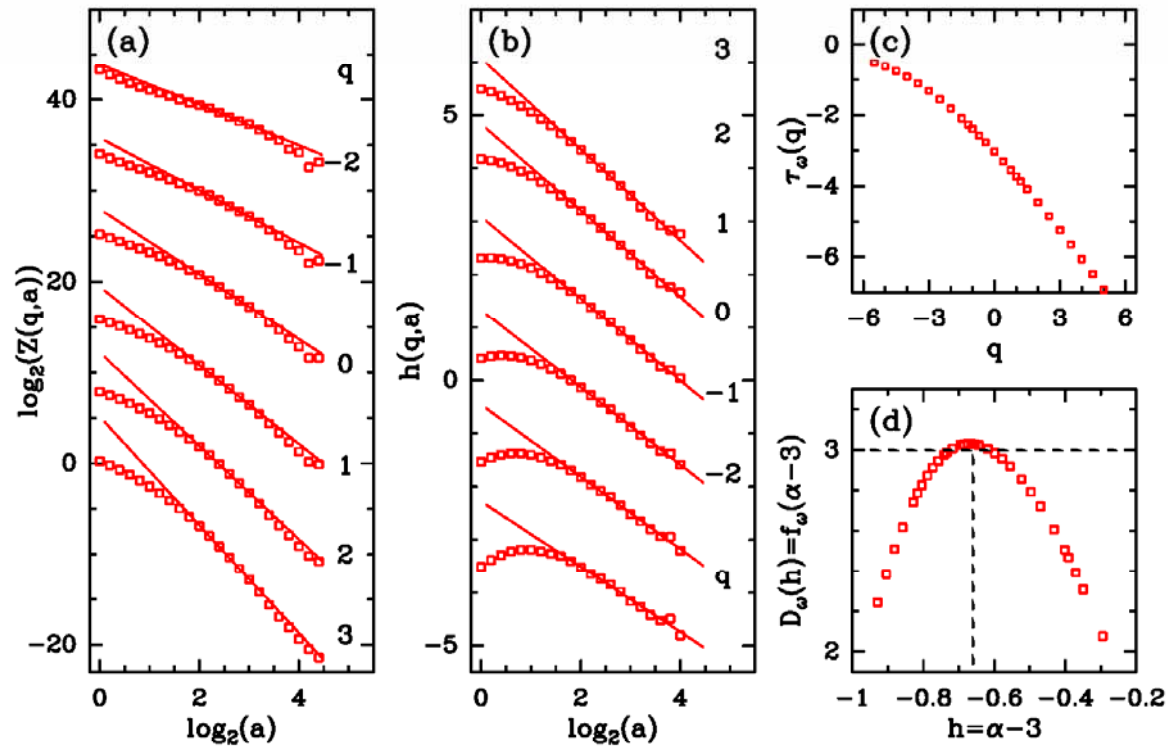
1D increments method:

- longitudinal : $C_2(\delta v_L) \sim 0.025$
- transverse : $C_2(\delta v_T) \sim 0.040$

Tensorial 3D WTMM method: turbulent vorticity field ($R_\lambda = 140$)

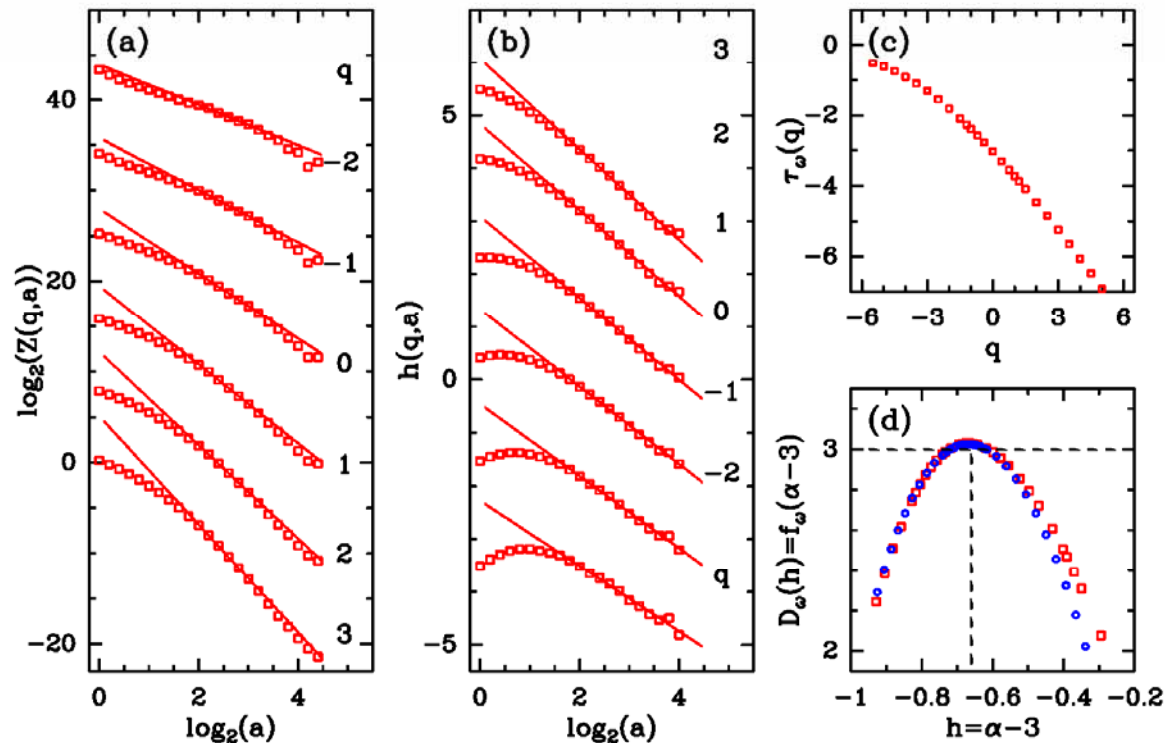


Tensorial 3D WTMM method: singularity spectrum of vorticity



□ vorticity

Tensorial 3D WTMM method: singularity spectrum of vorticity



□ vorticity

○ $D_v(h+1)$ spectrum translated velocity

⇒ same 3D intermittency coefficient !

assessment:

- 👉 WTMM multifractal analysis: moving towards **vector fields**

outlooks :

- 👉 better understanding of the information embedded in the **WT tensor**.
- 👉 **identification of coherent structures** in turbulence using **WT tensor**'s smallest singular value: vorticity filaments or sheets.
- 👉 others applications : astrophysics (interstellar medium, interstellar turbulence), MHD, geophysics, ...

Thanks :

- 👉 E. Lévêque, Laboratoire de Physique, ENS Lyon (Turbulent flows DNS).

PHYSICS TODAY

JANUARY 1990



Generalizing the wavelet-based multifractal formalism to vector-valued random fields : application to turbulent velocity and vorticity 3D numerical data.

P. KESTENER & A. ARNEODO, Phys. Rev. Lett. (2004), to appear

A wavelet-based method for multifractal analysis of 3D random fields : application to turbulence simulation data.

P. KESTENER & A. ARNEODO, in Proc. 4th Pacific Symposium on Flow visualization and Image Processing, edited by J. Prenel and Y. Bailly (2004), to appear

Three-dimensional wavelet-based multifractal method : the need for revisiting the multifractal description of turbulence dissipation data.

P. KESTENER & A. ARNEODO, Phys. Rev. Lett. **91**, 194501 (2003)

A wavelet-based method for multifractal image analysis : from theoretical concepts to experimental applications.

A. ARNEODO, N. DECOSTER, P. KESTENER & S.G. ROUX, Advances in Imaging and Electron Physics **126**, 1 (2003)

Wavelet-based multifractal formalism to assist in diagnosis in digitized mamograms.

P. KESTENER, J.M. LINA, P. SAINT-JEAN & A. ARNEODO, Image Anal. Stereol. **20**, 169 (2001)

A wavelet-based method for multifractal image analysis. I. Methodology and test applications on isotropic rough surfaces.

A. ARNEODO, N. DECOSTER & S.G. ROUX, Eur. Phys. J. B **15**, 567 (2000)

A wavelet-based method for multifractal image analysis. II. Application to synthetic multifractal rough surfaces.

N. DECOSTER, S.G. ROUX & A. ARNEODO, Eur. Phys. J. B **15**, 739 (2000)

A wavelet-based method for multifractal image analysis. III. Application to high resolution satellite images of cloud structure.

A. ARNEODO, S.G. ROUX & N. DECOSTER, Eur. Phys. J. B **15**, 765 (2000)

Intermittency, log-normal statistics and multifractal cascade process in high-resolution satellite images of cloud structure.

A. ARNEODO, N. DECOSTER & S.G. ROUX, Phys. Rev. Lett. **83**, 1255 (1999)