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Stein's Method and its Application in Radar Signal Processing

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ABSTRACT

In the analysis of radar signal processing, many approximations are made in order to simplify systems analysis. In the context of radar detection, environmental assumptions, such as in the modelling of clutter and noise, are often made. Similar assumptions are employed in the analysis of speckle in radar imaging systems. In addition to this, further statistical approximations of random processes of interest are made. In radar detection theory, an assumption on the distribution of a clutter statistic is often made. An important part of the model validation process is to assess the validity of employed assumptions. The validity of environmental assumptions is often assessed on the basis of field trials. In the case of approximations of random processes of interest, there are a number of approaches. This study introduces a general scheme, known as Stein's Method, for assesssing distributional approximations of random processes. It is described in general terms, and its application to the Poisson and Gaussian approximation of a given random variable is outlined. A new development of Stein's Method for Exponential approximation is included. The importance of the latter is that it can be used as an approximating distribution for a number of random variables of interest in radar.

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EXECUTIVE SUMMARY

An important component of the ongoing DSTO efforts in Radar Modelling and Analysis Group is model validation and verification. As a part of the efforts in a number of DSTO Tasks (AIR 01/217, AIR 02/232, NAV 01/299), radar clutter modelling and analysis work has been undertaken. Probability distributions, such as Rayleigh, Weibull and the K, have been found to be appropriate for the modelling of radar clutter, in various scenarios. Such modelling approximations for clutter can be validated via empirical studies through trials data. In the area of radar detection, further assumptions are made in the design of a radar detection scheme. In constant false alarm rate (CFAR) detectors, a function computing a measure of the clutter level is used. In order to assess the performance of a particular scheme, statistical assumptions are either imposed on this measure, or Monte Carlo estimation is necessary. There may in fact be sufficient grounds to assume that a clutter measure, in a given environment, has a particular statistical distribution. If such an approximation was valid, it could improve the estimation of radar performance measures such as the false alarm and detection probabilities.

Hence, it is desirable to be able to assess the validity of such approximations. The Stein-Chen Method, commonly referred to as Stein's Method, is a general scheme whereby such distributional approximations can be assesssed mathematically. The scheme allows a bound to be constructed, on the distributional approximation of one random variable by another. Stein's Method is illustrated in the case of the Poisson and Gaussian approximation of random variables. The latter distributions are frequently used in radar, and so these results may be applicable to radar performance analysis.

A new development of Stein's Method, for measuring the distributional approximation of a random variable by an Exponential distribution, is included. This work has direct application to a number of distributions of interest in radar. In particular, the time to detection in a radar detection scheme can be approximated by such a distribution. Additionally, some processes in radar image speckle modelling are asymptotically Exponential.

The purpose of this report is to introduce the mathematical framework for further work on the application of Stein's Method to radar systems analysis.

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Graham V. Weinberg is a specialist in mathematical analysis and applied probability, graduating in 2001 with a Doctor of Philosophy Degree from The University of Melbourne. His thesis examined random variable and stochastic process approximations, using a technique known as the Stein-Chen Method. The latter enables one to estimate the distributional approximation, under a probability metric, of a given random variable by a reference distribution.

Before joining DSTO in early 2002, he worked in a number of different industries, including finance and telecommunications. At DSTO he has been working in the area of radar signal processing, in the Microwave Radar Branch, predominantly in radar detection theory.

Contents

1 Introduction

A core responsibility of Electronic Warfare and Radar Division's Radar Modelling and Analysis Group is to provide technical advice to the Australian Defence Force on the performance of radar systems. This work has required the analysis and evaluation of radar systems in different environmental scenarios. As a result of this, many studies have been undertaken to classify the types of clutter where Australian assets are likely to be deployed. This has resulted in the studies of [Antipov 1998, 2001], which were undertaken as part of the Task AIR 98/084, which has been succeeded by AIR 01/217. This Task was designed to provide the RAAF with advice on the performance of the Israeli designed Elta EL/M-2022 maritime radar, used in the AP-3C Orion fleet. Also within the Microwave Radar Branch, [Dong 2003, 2004] has undertaken studies of clutter modelling, in support of the Task AIR 02/232. The latter is to provide technical support for the Airborne Early Warning and Control (AEW&C) acquisition under Project Wedgetail. A key component in all these studies of clutter is the search for a good mathematical model. In some cases, Rayleigh, Weibull and the K-distributions have been found to fit clutter well [Antipov 1998, 2001 and Dong 2004]. These clutter models are validated through trials. Models of the performance of radars are also validated by comparing a model's behaviour, in a prescribed clutter scenario, to the radar's performance in a similar real environment.

In the development of mathematical models of radar performance, many assumptions are made. As an example, in the case of radar detectors, a constant false alarm rate (CFAR) detector [Levanon 1988 and Minkler and Minkler 1990] estimates the level of clutter, in order to set an appropriate detection threshold for a desired probability of false alarm. This measure of clutter may be a simple arithmetic mean, or a complex function of other statistics of the clutter. A key issue with the design of an effective CFAR scheme is the best choice for the clutter measure. The latter will change depending on the type of clutter the radar is processing. For a given clutter scenario, the performance of a CFAR thus needs to be evaluated. Typical performance measures used are the probability of detection and false alarm. To evaluate these, it is often necessary to make statistical approximations. Monte Carlo Methods are often used, due to analytic intractibility of these probabilities. However, in some cases, it may be possible to apply a distributional approximation to the clutter measure before applying Monte Carlo techniques. The latter has the potential to speed up convergence of estimators. Thus it would be useful to have a framework that allows the errors in such approximations to be estimated.

The purpose of this report is to introduce a scheme that allows the mathematical analysis of such approximations to be assessed. This scheme, known as Stein's Method, will be discussed in considerable detail. Its application to a number of radar-related problems will also be outlined in general terms.

1.1 Stein's Method

A significant component of the theory of probability is concerned with the convergence of random variables and probability distributions [see Billingsley 1986, Durrett 1996 and

Ross 1996]. From a practical point of view, a modelling application of stochastic processes will often necessitate the estimation of probabilities of interest. An example of this is the analysis in [Weinberg 1999] on the modelling of stochastic queueing systems. Estimating probabilities is a part of the more general problem of *approximate computation of expectations*[1]. The work presented here is an examination of the methods introduced by Charles Stein, and his former student Louis Chen. Although the focus of the work presented here is the mathematics behind Stein's Method, its potential application to radar related problems will also be outlined.

In [Stein 1972] a general scheme was introduced, which provided a new proof of the classical Central Limit Theorem of statistics. The techniques provided Berry-Esseen type bounds [Esseen 1945, Feller 1968 and Chen and Shao 2003] on the Gaussian distributional approximation of a sum of random variables. In previous work, tools such as Fourier analysis were used to derive such estimates [Esseen 1945]. Stein's techniques differed because they used a characterisation of the Gaussian distribution to construct error rates.

Stein found a first order linear differential equation could be used to characterise the Gaussian distribution. This converted the problem of measuring distributional differences to one of bounding an expectation. This provided a new way of assessing distributional approximations. One of the interesting features of this method is that it is rather general. [Chen 1975] was the first to apply the technique to other distributions, and considered Poisson random variable approximations.

It was the discovery by [Barbour 1988] that explained the generality of Stein's Method. By a derivative transformation of the original Stein equation for Gaussian approximation, it was found that part of the Stein equation is the generator of an Ornstein-Uhlenbeck diffusion process [Aldous 1989]. This process has the Gaussian distribution as its stationary distribution. In the case of the Poisson Stein equation, the generator is that of an immigration-death process on the nonnegative integers [Ross 1983]. Again, this process has as its stationary distribution the Poisson distribution. The explanation for the presence of a generator in the Stein equation, and the reference distribution being the processes stationary distribution, can be found in [Ethier and Kurtz 1986]. Markov processes are the key to this generality. It can be shown that a Markov process is uniquely characterised by its generator, and that a distribution is the stationary distribution of a Markov process if and only if the expectation of its generator, under the stationary distribution, is identically zero. The implication of this is that if one is interested in developing a Stein equation for a reference distribution μ, then one needs to find a Markov process with μ as a stationary distribution. When this is done, the problem of measuring distributional differences translates to one of bounding an expectation of a generator of a Markov process. This generalisation of Stein's Method, and the manipulation of the underlying generator, is known as the *probabilistic interpretation or method.*

The probabilistic method does not always work well in practice. Compound Poisson approximation [Barbour and Chryssaphinouis 2001] is an example where the probabilistic approach is problematic. However, it can always provide a means of deriving an initial Stein equation for applications of interest.

[1] This is the title of Stein's Lecture Notes [Stein 1986].

Stein's Method has been applied extensively, in a wide range of contexts. In the Poisson setting, [Barbour, Holst and Janson 1992] is an encyclopaedia of applications, including random permutations, random graphs, occupancy and urn models, spacings and exceedances and extremes. Some other examples that can be found in the stochastic processes literature include the work of [Barbour 1988] on multivariate Poisson approximations, [Barbour 1990] on diffusion approximations, [Barbour and Brown 1992] on Poisson point process approximation, [Barbour and Brown 1996] on the application of Stein's Method to networks of queues, [Barbour, Chen and Loh 1992 and Barbour and Chryssaphinouis 2001] on compound Poisson approximations, [Phillips 1996] on negative binomial approximations, [Greig 1995] on Erlang fixed point approximations for telecommunications applications, [Loh 1992] on multinomial approximations, [Luk 1994] on gamma approximations and [Peköz 1996 and Phillips and Weinberg 2000] on geometric approximations.

The first application of Stein's Method to radar can be found in [Weinberg 2005]. In the latter, it is shown that the single pulse probability of detection of a target in Gaussian clutter and noise is equivalent to a comparison of two independent Poisson random variables. In order to undertake an asymptotic analysis of this detection probability, Stein's Method for Poisson approximation yielded a new expression for a comparison of Poisson variables. This enabled easy asymptotic analysis of the detection probability. However, the approach used in [Weinberg 2005] does not easily indicate how Stein's Method is used in general. Specifically, it does not indicate how a distributional approximation made for a clutter statistic can be evaluated.

This report will introduce the development of Stein's Method in a number of contexts. The case of the Poisson approximation of a sum of indicator random variables will be considered firstly. A bound on the Poisson approximation of a Binomial distribution will follow as a consequence. To illustrate the way Stein's Method works for nonatomic distributions, the Gaussian approximation of a sum of random variables will be considered, and a Berry-Essen type bound will be derived. A new development of Exponential approximation will be outlined, together with its usage and application to potential problems in radar.

A major objective of the report is to disseminate to a wide audience the potential usage and application of Stein's Method in defence scientific research applications. Although the applications pointed to here will be exclusively to radar, there is scope to apply the general methods to communications and weapon systems modelling.

2 Stein's Method

This section begins with the development of Stein's Method in general terms, using the probabilistic approach of [Barbour 1988]. It will be outlined how a Stein equation can be derived for a particular application. The specific cases of Poisson and Gaussian approximations will be used as examples of the usage of the scheme. The Gaussian distribution is of considerable importance in radar, and so these results may be of interest to radar analysts. Recent work on the single pulse probability of detection of a target in Gaussian clutter has shown the importance of the Poisson distribution in radar [Weinberg 2005]. Hence Poisson approximation is also of interest. Some of the mathematical notation and definitions can be found in Appendix A.

2.1 General Principles

In this Subsection, Stein's Method in outlined in general terms. In particular, the probabilistic method, which associates a distribution with a corresponding Markov process, and bounds the expectation of its generator, is introduced. This general development is important because it indicates how the scheme can be extended to practical applications of interest. However, the applications of Stein's Method to the Poisson, Gaussian and Exponential cases considered in this report, do not directly require this more abstract formulation. Hence, the less mathematically inclined reader can start with the specific examples in Subsections 2.2 and 2.3, without reference to this Subsection.

Suppose we have a probability distribution μ, associated with some random variable Z. Hence $\mathbb{P}(Z \in A) = \mu(A)$, for a measurable set A. We refer to μ as *the reference distribution.* Suppose we are interested in some other random variable X, and in particular, suppose we know empirically that the distribution of X can be approximated by μ. This means that

$$\mathbb{P}(X \in A) \approx \mathbb{P}(Z \in A) = \mu(A). \tag{1}$$

We refer to the distribution of X as *the target distribution.* From an analysis point of view, it is important to determine whether the distributional approximation implied by (1) is valid. Hence, it would be useful to be able to get an estimate of the difference of distributions

$$\left|\mathbb{P}(X \in A) - \mathbb{P}(Z \in A)\right|. \tag{2}$$

Stein's Method enables one to quantify the approximation in (1) by obtaining upper and lower bounds on (2).

The basic idea behind Stein's Method is to find an equation characterising a reference distribution, that enables one to obtain bounds on distributional differences, such as in (2). Stein's Method converts this problem to one of bounding an expectation of a generator of an associated Markov process.

The theory of Markov processes is a crucial component in the study of Stein's Method, and we begin by introducing some relevant facts. [Asmussen 1987, Dynkin 1965 and Ethier and Kurtz 1986] are useful references on the material to follow. All definitions used can be found in these references. Let $\{X_t, t \geq 0\}$ be a time homogeneous Markov process, on a state space $\mathcal{E}$, and suppose $\mathcal{A}$ is its generator. Define an operator on a suitable class of functions $h \in \mathcal{F}$ by $T_t h = \mathbb{E}[h(X_t)|X_0 = x)]$. Then the generator of the Markov process X_t is

$$\mathcal{A}h(x) = \lim_{t \to 0} \frac{[T_t h(x)] - h(x)}{t}.$$

A generator uniquely characterises a Markov process. The key to Stein's Method is the following principle. For a stationary distribution μ, $X \stackrel{d}{=} \mu$ if and only if $\mathbb{E}[\mathcal{A}h(X)] = 0$ for all h for which $\mathcal{A}h$ is defined [Ethier and Kurtz 1986, Lemma 4.9.1 and Proposition 4.9.2]. Thus, suppose we have a target random variable X, and we want to compare its distribution with μ, as in (1). Then the above implies that if $\mathbb{E}[\mathcal{A}h(X)] \approx 0$, then the distribution of X is approximately equal to μ, where $\mathcal{A}$ is the generator of a Markov process with stationary distribution μ.

In order to assess such distributional approximations, we attempt to find the solution $h(x)$ to the *Stein equation*

$$f(x) - \int f d\mu = \mathcal{A}h(x), \tag{3}$$

for some appropriate function f, and a reference distribution μ. As an example, we could choose $f(x) = \mathbb{I}_{[x \in A]}$, for some measurable set A, and where

$$\mathbb{I}_{[x \in A]} = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows, by applying expectations to both sides of (3), that if X is a random variable,

$$\mathbb{E}\left[\mathbb{I}_{[X \in A]} - \int \mathbb{I}_{[X \in A]} d\mu\right] = \mathbb{P}(X \in A) - \mu(A) = \mathbb{E}[\mathcal{A}h(X)]. \tag{4}$$

Hence, in order to assess the distributional approximation of X by μ, (4) implies we may attempt to estimate the generator component $\mathbb{E}[\mathcal{A}h(X)]$. Consequently, since this will depend on the function h, which is the solution to the Stein equation (3), we need to investigate whether we can in fact solve for h in (3).

It turns out that the equation (3) can be solved generally [Barbour, Holst and Janson 1992]. If X_t is an ergodic Markov process [Ethier and Kurtz 1986] with stationary distribution μ, then under some conditions on the functions f and h, the solution to (3) is

$$h(x) = -\int_0^\infty \mathbb{E}^x \left[f(X_t) - \int f d\mu\right] dt, \tag{5}$$

where $\mathbb{E}^x[f(X_t)] = \mathbb{E}[f(X_t)|X_0 = x]$. The required conditions on the functions f and h, so that (5) solves (3), can be established using the general theory of Feller processes [Ethier and Kurtz 1986]. One then attempts to write the generator component of the Stein equation in terms of h, and bounds on h are then used to obtain bounds on the generator. This is then applied to (4), providing upper bounds on the distributional approximation.

We provide some examples of Markov processes for some implementations of Stein's Method. When the reference distribution is Poisson with mean λ, the appropriate process is an immigration-death process, with immigration rate λ and unit per capita death rate [Ross 1983]. The generator is

$$\mathcal{A}h(x) = \lambda\Big(h(x+1) - h(x)\Big) - x\Big(h(x) - h(x-1)\Big). \tag{6}$$

In the case of Gaussian approximations, the process is the Ornstein-Uhlenbeck diffusion process [Aldous 1989], with generator

$$\mathcal{A}h(x) = h''(x) - xh'(x). \tag{7}$$

It can be shown in both cases that the reference distribution is the appropriate Markov process stationary distribution.

In many cases, the probabilistic method outlined above works very well, such as general random variable approximations [Weinberg 1999, 2000] and Poisson process approximations [Barbour and Brown 1992]. However, there are cases, such as compound Poisson approximation [Barbour, Chen and Loh 1992 and Barbour and Chryssaphinouis 2001], where it does not work. The probabilistic method is, however, not always essential. Stein's original work was not based on the generator approach outlined here, nor was the original work of Chen. It does, however, provide the framework for the construction of a Stein equation for a particular application.

In the following two subsections we introduce Stein's Method for Poisson and Gaussian approximations.

2.2 A Discrete Case: Poisson Approximation

A Poisson random variable is a model for counting rare events [Ross 1983]. It is as common a model for such phenomena as the Gaussian is for common events. As shown in [Weinberg 2005], it arises in detection probabilities for single pulses in Gaussian clutter. This subsection is concerned with the development of Stein's Method for the Poisson approximation of a sum of indicator random variables. This is done to illustrate the development of Stein's Method for discrete random variables. The general application of Stein's Method for arbitrary random variables can be found in [Barbour, Holst and Janson 1992 and Weinberg 1999, 2000].

The key to the method in this context is the following characterisation of a random variable W: W is Poisson with mean λ if and only if for all bounded functions g with $g(0) = 0$,

$$\mathbb{E}[\lambda g(W+1)] = \mathbb{E}[Wg(W)].$$

The proof of necessity is quite straightforward. To prove sufficiency, one attempts to find a unique solution to the equation

$$\lambda g(j+1) - jg(j) = \mathbb{I}_{[j\in A]} - \mathbf{Po}(\lambda)\{A\}, \tag{8}$$

for each $j \in \mathbb{N}$, where $A \subset \mathbb{N}$ and

$$\mathbb{I}_{[j\in A]} = \begin{cases} 1 & \text{if } j \in A; \\ 0 & \text{otherwise.} \end{cases}$$

Equation (8) is called the Stein equation for Poisson approximation. By introducing a functional transformation of $g(j) = h(j) - h(j-1)$, the left hand side of (8) transforms to the generator (6). If a unique solution to (8) can be found, then for a random variable W with mean λ, it follows that

$$\mathbb{E}[\lambda g(W+1) - Wg(W)] = \mathbb{P}(W \in A) - \mathbf{Po}(\lambda)\{A\}. \tag{9}$$

The idea is that the Poisson approximation of W can be determined by bounding the expectation in (9). As an example, suppose the random variable W is a sum of indicators, and so takes the form

$$W = \sum_{j=1}^{n} X_j, \tag{10}$$

where each X_j is independent and distributed according to

$$X_j = \begin{cases} 1 & \text{with probability } p_j; \\ 0 & \text{with probability } 1 - p_j. \end{cases}$$

Random variable W is a generalisation of a Binomial, and is called a Multinomial distribution. We are interested in the Poisson approximation of W. Let $\lambda = \mathbb{E}[W] = \sum_{j=1}^{n} p_j$, and define a random variable $W_j = W - X_j$. Then we have the following Theorem:

Theorem 2.1 *For the random variable W defined above, and for any set $A \subset \mathbb{N}$,*

$$|\mathbb{P}(W \in A) - \mathbf{Po}(\lambda)\{A\}| \le \frac{1 - e^{-\lambda}}{\lambda} \sum_{j=1}^{n} p_j^2.$$

Thus, the bound of Theorem 2.1 gives the rate of convergence of the distribution of W to a Poisson limit. We now outline the proof of Theorem 2.1. For technical reasons, we assume the solution to the Stein equation (8) satisfies $g(0) = 0$. It can be shown [Barbour, Holst and Janson 1992] that a unique solution exists to (8), and is given by

$$g(j+1) = \lambda^{-j-1} j! e^{\lambda} (\mathbf{Po}(\lambda)\{A \cap U_j\} - \mathbf{Po}(\lambda)\{A\}\mathbf{Po}(\lambda)\{U_j\}), \tag{11}$$

where $U_j := \{0, 1, \ldots, j\}$. Furthermore, the differences of g are uniformly bounded:

$$\sup_{j \in \mathbb{N}} |g(j+1) - g(j)| \le \frac{1 - e^{-\lambda}}{\lambda}. \tag{12}$$

A proof of (12) can also be found in [Barbour, Holst and Janson 1992], or as a consequence of the results in [Weinberg 2000]. We now manipulate the expectation of (9). Observe that

$$\mathbb{E}[\lambda g(W+1)] = \sum_{j=1}^{n} p_j \mathbb{E} g(W_j + X_j). \tag{13}$$

Also

$$\mathbb{E}[Wg(W)] = \sum_{j=1}^{n} \mathbb{E}[X_j g(W_j + X_j)] = \sum_{j=1}^{n} p_j \mathbb{E} g(W_j + 1). \tag{14}$$

Hence, an application of (13) and (14) to (9) yields

$$\begin{aligned}\mathbb{E}[\lambda g(W+1) - Wg(W)] &= \sum_{j=1}^{n} p_j \mathbb{E}[g(W+1) - g(W_j+1)] \\ &= \sum_{j=1}^{n} p_j \mathbb{E}[X_j[g(W_j+2) - g(W_j+1)]] \\ &= \sum_{j=1}^{n} p_j^2 \mathbb{E}[g(W_j+2) - g(W_j+1)]. \qquad (15)\end{aligned}$$

By applying the bound (12) to (15), we obtain

$$\begin{aligned}|\mathbb{P}(W \in A) - \mathbf{Po}(\lambda)\{A\}| &\leq \sup_{k\in\mathbb{N}} |g(k+1) - g(k)| \sum_{j=1}^{n} p_j^2 \\ &\leq \frac{1-e^{-\lambda}}{\lambda} \sum_{j=1}^{n} p_j^2, \qquad (16)\end{aligned}$$

which completes the proof of Theorem 2.1.

Note that by choosing $p_j = p$ for each j, and using the fact that $\lambda = np$,

$$|\mathbb{P}(W \in A) - \mathbf{Po}(\lambda)\{A\}| \leq p(1 - e^{-np}), \qquad (17)$$

implying the rate of convergence of a Binomial distribution with parameters n and p, to a Poisson distribution with mean np, is of order p.

In the following subsection we outline the way Stein's Method is used for continuous distributions.

2.3 A Continuous Case: Gaussian Approximation

The Gaussian distribution has been used widely by radar analysts as a model for clutter and noise [Levanon 1988]. Due to its importance, and as an indication of the way Stein's Method is used for continuous distributions, we consider the case of Gaussian approximation. The following development is taken from [Chen and Shao 2003 and Stein 1986]. In the Spirit of [Stein 1972, 1986], we derive a Berry-Esseen type bound [Esseen 1945, Feller 1968] on the Gaussian distributional approximation of a sum of independent random variables.

Throughout we will let $Z \stackrel{d}{=} \mathbf{Gau}(0,1)$ be a standard Gaussian/Normal random variable, with mean 0 and variance 1, with distribution function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}z^2} dz.$$

Following is the main result:

Theorem 2.2 *Let $\{X_j, j \in \mathbb{N}\}$ be a sequence of independent random variables with mean zero, and let $W = \sum_{j=1}^{n} X_j$. Suppose $\mathbb{E}|X_j|^3 < \infty$ and $\sum_{j=1}^{n} \mathbb{E}X_j^2 = 1$. Then for any absolutely continuous function h satisfying* $\sup_x |h'(x)| \leq c$,

$$|\mathbb{E}h(W) - \mathbb{E}h(Z)| \leq 3c \sum_{j=1}^{n} \mathbb{E}|X_j|^3.$$

Theorem 2.2 does not enable a central limit result to be obtained because it is a smoothness estimate for a class of absolutely continuous functions. Indicator functions are obviously not included in this class. However, it allows estimates to be made under metrics weaker than total variation based ones. A discussion of weaker metrics, together with their merits, can be found in [Weinberg 1999]. Considerably more analysis is required to extend the work presented here to derive total variation bounds. However, since the focus is on illustrating the application of Stein's Method, it is worth examining Theorem 2.2.

Based upon [Stein 1972], the Stein equation for Gaussian approximation is

$$g'(x) - xg(x) = h(x) - \Phi(z), \qquad (18)$$

where $\Phi(z)$ is the standard Gaussian cumulative distribution function, and $h(x)$ is a suitable test function. Notice by a transformation of $h'(x) = g(x)$, the left hand side of (18) becomes the generator (7). In the cases where we are interested in distributional approximations under a total variation norm, a suitable choice for h is $h(x) = \mathbb{I}_{[x \leq z]}$. If this is the case, then if W is a random variable, (18) implies

$$\mathbb{E}[g'(W) - Wg(W)] = \mathbb{P}(W \leq z) - \Phi(z). \qquad (19)$$

As in the Poisson case, we attempt to derive bounds on the expectation component of (19).

It can be shown that a unique solution exists to (18), and is given by

$$g(x) = e^{\frac{x^2}{2}} \int_{-\infty}^{x} [h(w) - \Phi(z)] e^{-\frac{w^2}{2}} dw. \qquad (20)$$

Let $\{X_j, j \in \mathbb{N}\}$ be a sequence of independent (continuous) random variables, with zero mean and also impose the condition that $\sum_{j=1}^{n} \mathbb{E}X_j^2 = 1$. As in the Poisson case, we consider $W = \sum_{j=1}^{n} X_j$, and define $W_j = W - X_j$. Also define $K_j(t) = \mathbb{E}[X_j[\mathbb{I}_{[0 \leq t \leq X_j]} - \mathbb{I}_{[X_j \leq t < 0]}]]$.

It can be shown that K_j satisfies the following three properties:

$$K_j(t) \text{ is nonnegative;}$$

$$\int_{-\infty}^{\infty} K_j(t) dt = \mathbb{E}X_j^2;$$

$$\int_{-\infty}^{\infty} |t| K_j(t) dt = \mathbb{E}|X_j^3|/2.$$

Let h be a measurable function that is absolutely integrable with respect to the standard Gaussian density, so that $\mathbb{E}|h(Z)| < \infty$. Since X_j and W_j are independent, and $\mathbb{E}X_j = 0$,

$$\begin{aligned} \mathbb{E}Wg(W) &= \sum_{j=1}^{n} \mathbb{E}X_j g(W) \\ &= \sum_{j=1}^{n} \mathbb{E}X_j[g(W) - g(W_j)] \\ &= \sum_{j=1}^{n} \mathbb{E}X_j \int_0^{X_j} g'(W_j + t)dt \\ &= \sum_{j=1}^{n} \mathbb{E}\int_{-\infty}^{\infty} g'(W_j + t)K_j(t)dt. \end{aligned} \tag{21}$$

Also, since

$$\sum_{j=1}^{n} \int_{-\infty}^{\infty} K_j(t)dt = \sum_{j=1}^{n} \mathbb{E}X_j^2 = 1,$$

it is clear that

$$\mathbb{E}g'(W) = \sum_{j=1}^{n} \mathbb{E}\int_{-\infty}^{\infty} g(W)K_j(t)dt. \tag{22}$$

Hence, by combining (21) and (22), it follows that

$$\mathbb{E}[g'(W) - Wg(W)] = \sum_{j=1}^{n} \mathbb{E}\int_{-\infty}^{\infty} [g'(W) - g'(W_j + t)]K_j(t)dt. \tag{23}$$

It is shown in [Chen and Shao 2003] that $|g''(x)| \leq 2c$. Hence, by an application of the Mean Value Theorem to the integrand of (23),

$$\begin{aligned} |\mathbb{E}[g'(W) - Wg(W)]| &\leq 2c\sum_{j=1}^{n} \mathbb{E}\int_{-\infty}^{\infty} |(|t| + |X_j|)K_j(t)dt \\ &= 2c\sum_{j=1}^{n} (\mathbb{E}|X_j^3|/2 + \mathbb{E}|X_j|\mathbb{E}X_j^2) \\ &\leq 3c\sum_{j=1}^{n} \mathbb{E}|X_j^3|. \end{aligned} \tag{24}$$

This completes the proof of Theorem 2.2.

Theorem 2.2 provides a Berry-Esseen type bound on the Gaussian approximation of a sum of random variables, but with a lot more effort, one can show the following using similar techniques:

Theorem 2.3 *Let $\{\xi_j, j \in \mathbb{N}\}$ be a sequence of independent random variables with mean zero, $\mathbb{E}|\xi_j|^3 < \infty$ and $\sum_{i=1}^{n} \mathbb{E}\xi_i^2 = 1$. Then*

$$\|\mathcal{L}(W) - \mathcal{L}(Z)\| = \sup_z |\mathbb{P}(W \leq z) - \Phi(z)| \leq 2.2\sqrt{4\beta_1 + 3\beta_2},$$

where $\mathcal{L}$ *denotes probability law or distribution,* $\|\cdot\|$ *is the total variation norm,* $\beta_1 = \sum_{j=1}^{n} \mathbb{E}[X_j^2 \mathbb{I}_{[|X_j|>1]}]$ *and* $\beta_2 = \sum_{j=1}^{n} \mathbb{E}[|X_j|^3 \mathbb{I}_{[|X_j|\leq 1]}]$.

Theorem 2.3 gives a bound on the rate of convergence of the distribution of a sum of random variables to a Gaussian limit, and so is a central limit theorem. Many other such bounds can be found in [Stein 1972, 1986].

3 Stein's Method for Exponential Approximation

The purpose of the following is to discuss the development of Stein's Method for distributional approximations of random variables by an Exponential distribution. There are many processes of interest that can be approximated by such a distribution. Examples include the time it takes for a stochastic process to enter a set of particular states [Aldous 1989], and the arrival time of an event in a stochastic process [Ross 1983]. In a radar context, there are a number of distributions of interest that are approximately Exponential. The time to a detection in a constant false alarm rate detection process is roughly Exponential, possibly dependent on the clutter assumptions. In radar imaging systems such as synthetic aperature radar, some speckle models are also approximately Exponential.

An Exponential random variable is a particular case of a Gamma, and Stein's Method has been developed for the latter in [Luk 1994]. The Stein equation for Gamma approximations can be found in [Reinert 2003], who considers the approximation by χ^2 distributions. The latter is also intimately related to the Gamma distribution. In the case of Exponential approximations, a much simpler form of the Stein equation is available. In the following we introduce this Stein equation, and show how it can be used in a simple context.

3.1 The Stein Equation

Recall that a continuous random variable W is said to have an Exponential distribution with positive parameter λ if its density is $f_W(x) = \lambda e^{-\lambda x}$, for $x \geq 0$. Its cumulative distribution function is $F_W(x) = 1 - e^{-\lambda x}$. It can be shown that the mean value of W is $\frac{1}{\lambda}$, while its variance is $\frac{1}{\lambda^2}$. We attempt to find an equation which characterises this distribution. In light of the form of the Stein equation for Gaussian approximations, we examine expressions for an expectation of derivatives of smooth functions in this variable. We also provide a more detailed discussion than the examples considered in the previous section. This is due to the fact that the work presented here is new, and still under development.

Suppose $\mathcal{C}$ is the class of all continuous and piecewise differentiable functions that map the real numbers to themselves. Since we will be looking at the solution of differential equations, we define a subset $\mathcal{C}^* \subset \mathcal{C}$, of all functions in $\mathcal{C}$ whose domain is the nonnegative real line, and that satisfy the two boundary conditions that $g(0) = 0$ and[2] $g(x) = o(e^{\lambda x})$ as $x \to \infty$. Then if W is an Exponentially distributed random variable, with parameter λ, and $g \in \mathcal{C}^*$,

$$\begin{aligned}\mathbb{E}g'(W) &= \int_0^\infty g'(x)\lambda e^{-\lambda x}dx \\ &= \Big[g(x)\lambda e^{-\lambda x}\Big]_0^\infty + \int_0^\infty g(x)\lambda^2 e^{-\lambda x}dx\end{aligned}$$

[2]The notation $f(x) = o(g(x))$ as $x \to \infty$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

$$= \lambda \int_0^\infty g(x) \lambda e^{-\lambda x} dx$$

$$= \lambda \mathbb{E} g(W). \qquad (25)$$

Hence, for any $g \in \mathcal{C}^*$, if $W \stackrel{d}{=} \mathbf{Exp}(\lambda)$, then $\mathbb{E}g'(W) = \lambda \mathbb{E} g(W)$. What is less apparent is that the converse of this statement is also true. That is, if (25) holds for all functions $g \in \mathcal{C}^*$, then this implies that W is Exponentially distributed.

To demonstrate this, consider the equation

$$g'(x) - \lambda g(x) = \mathbb{I}_{[x \le a]} - \mathbf{Exp}(\lambda)\{a\}, \qquad (26)$$

where the indicator function $\mathbb{I}_{[x \le a]}$ is defined as previously, and $\mathbf{Exp}(\lambda)\{a\}$ is the cumulative probability distribution of an Exponential random variable, with mean λ, on the interval $[0, a]$, for some $a \ge 0$. Hence $\mathbf{Exp}(\lambda)\{a\} = 1 - e^{-\lambda a}$. The differential equation (26) is the Stein equation for Exponential approximations. By a simple functional transformation, it can be shown to be equivalent to the corresponding Stein equation in [Reinert 2003]. We need to show there is a unique and well defined solution to (26). Then if W is some random variable with support the nonnegative real line,

$$\mathbb{E}[g'(W) - \lambda g(W)] = \mathbb{P}(W \le a) - \mathbf{Exp}(\lambda)\{a\}. \qquad (27)$$

As in the Poisson and Gaussian cases, bounding the expectation term in (27) will give bounds on the Exponential approximation of W. The next subsection is concerned with the existence and uniqueness of the solution to (26).

3.2 Solution to Stein Equation: Existence and Uniqueness

It is a well-known fact that a first order linear differential equation of the form $g'(x) - \lambda g(x) = k(x)$, with k a function, and boundary conditions $g(0) = \pi_1$ and $g'(0) = \pi_2$ has a unique solution [Kreyszig 1988, Theorem 1, Section 2.9]. To construct the solution to (26), multiply both sides of the Stein equation (26) by $e^{-\lambda x}$, and observe that

$$e^{-\lambda x} g'(x) - \lambda e^{-\lambda x} g(x) = \frac{d}{dx} e^{-\lambda x} g(x), \qquad (28)$$

from the product rule for differentiation. Hence, by integration and applying (28) to (26), we obtain

$$g(x) = e^{\lambda x} \int_0^x e^{-\lambda t} \left[\mathbb{I}_{[t \le a]} - \mathbf{Exp}(\lambda)\{a\} \right] dt. \qquad (29)$$

Note that (29) satisfies $g(0) = 0$ and $\lim_{x \to \infty} e^{-\lambda x} g(x) = 0$. In order to show that g defined by (29) is well defined, we show that it is bounded in the supremum norm on the nonnegative real numbers. It is useful to firstly simplify (29). It is not difficult to show that

$$g(x) = e^{\lambda x} \left[\int_0^{x \wedge a} e^{-\lambda t} dt - (1 - e^{-\lambda a}) \int_0^x e^{-\lambda t} dt \right]$$

$$= \frac{e^{\lambda x}}{\lambda} [1 - e^{-\lambda (x \wedge a)} - [1 - e^{-\lambda a}][1 - e^{-\lambda x}]], \qquad (30)$$

where $x \wedge a$ is the minimum of x and a. By considering the two cases where $x \wedge a = x$ and $x \wedge a = a$, it can be shown that (30) is equivalent to

$$g(x) = \frac{1}{\lambda}\left(e^{\left\{\lambda(x-a)\mathbb{I}_{[x \le a]}\right\}} - e^{-\lambda a}\right). \tag{31}$$

Consequently, it follows that

$$\|g\| = \sup_{x \ge 0} |g(x)| \le \frac{1 - e^{-\lambda a}}{\lambda}. \tag{32}$$

This shows that the solution (29) to the Stein equation (26) is well defined. Hence g in (31) is the uniquely well defined solution to the Stein equation (26).

The function g is continuous and piecewise differentiable, and so $g \in \mathcal{C}^*$, since it satisfies the appropriate boundary conditions. The only point on the nonnegative real line where it is not differentiable is at $x = a$. In fact, it is not difficult to show that

$$g'(x) = \begin{cases} e^{\lambda(x-a)} & \text{if } x < a; \\ 0 & \text{if } x > a; \\ \text{undefined} & \text{if } x = a. \end{cases} \tag{33}$$

Given a random variable W, in order to assess its closeness in distribution to an Exponential random variable, we can estimate the expectation $\mathbb{E}[g'(W) - \lambda g(W)]$ and see how close to zero it is. We illustrate this in the next subsection using a very simple application.

3.3 Rate of Convergence of a Truncated Exponential Distribution

Consider the truncated Exponential distribution, with density $f(x) = \frac{\lambda e^{-\lambda x}}{1-e^{-\lambda\theta}}$, for some $\theta > 0$, and for $x \in [0, \theta]$. Clearly, as $\theta \to \infty$, this density converges to that of an Exponential random variable. We use this example to illustrate the application of Stein's Method for Exponential approximations. We basically need to estimate the expectation $\mathbb{E}[g'(W) - \lambda g(W)]$, where W has the density f above.

Note that, by an application of integration by parts,

$$\begin{aligned} \mathbb{E}g'(W) &= \int_0^\infty g'(x)\frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda\theta}}dx \\ &= \frac{1}{1 - e^{-\lambda\theta}}\left[g(\theta)\lambda e^{-\lambda\theta} + \lambda(1 - e^{-\lambda\theta})\mathbb{E}g(W)\right]. \end{aligned} \tag{34}$$

Hence it follows that

$$\mathbb{E}[g'(W) - \lambda g(W)] = \frac{g(\theta)\lambda e^{-\lambda\theta}}{1 - e^{-\lambda\theta}}. \tag{35}$$

An application of the smoothness estimate (32) to (35) yields

$$\begin{aligned} |\mathbb{P}(W \le a) - \mathbf{Exp}(\lambda)\{a\}| &= |\mathbb{E}[g'(W) - \lambda g(W)]| \\ &\le \frac{\|g\|\lambda e^{-\lambda\theta}}{1-e^{-\lambda\theta}} \\ &\le \frac{e^{-\lambda\theta}}{1-e^{-\lambda\theta}}. \end{aligned} \tag{36}$$

The bound (36) gives the rate of convergence of the truncated Exponential distribution, to a standard Exponential distribution. As expected, this bound decreases to zero as θ increases without bound.

3.4 Towards a General Bound

This section outlines how the Exponential Stein equation (26) can be used to derive a general bound on the Exponential approximation of a continuous random variable W. At this stage, the work is in need of further development. However, some progress has been made toward establishing a useful result.

We are interested in the distributional approximation of W by an Exponential random variable with parameter λ. Hence, we assume $\mathbb{E}W = \frac{1}{\lambda}$. Firstly, as in the derivation of (21) for the Gaussian case, note that we can write

$$\begin{aligned} \mathbb{E}g(W) &= \mathbb{E}\int_0^W g'(t)dt \\ &= \mathbb{E}\int_0^\infty \mathbb{I}_{[W\ge t]}g'(t)dt \\ &= \int_0^\infty \mathbb{P}(W \ge t)g'(t)dt. \end{aligned} \tag{37}$$

It can be shown [Billingsley 1986] that for a continuous random variable W,

$$\mathbb{E}W = \int_0^\infty \mathbb{P}(W \ge t)dt.$$

Hence, an application of the above results in

$$\frac{1}{\lambda}g'(W) = [\mathbb{E}W]g'(W) = \int_0^\infty \mathbb{P}(W \ge t)g'(W)dt.$$

Taking expectations in the above yields

$$\mathbb{E}[\frac{1}{\lambda}g'(W)] = \int_0^\infty \mathbb{P}(W \ge t)\mathbb{E}[g'(W)]dt. \tag{38}$$

Consequently, by combining both (37) and (38), we obtain

$$\begin{aligned}\mathbb{E}[g'(W) - \lambda g(W)] &= \lambda \mathbb{E}[\frac{1}{\lambda} g'(W) - g(W)] \\ &= \lambda \int_0^\infty \mathbb{P}(W \geq t)\, [\mathbb{E}g'(W) - g'(t)]\, dt \\ &= \frac{\int_0^\infty \mathbb{P}(W \geq t)\, [\mathbb{E}g'(W) - g'(t)]\, dt}{\int_0^\infty \mathbb{P}(W \geq t) dt}. \qquad (39)\end{aligned}$$

What is now required is a bound on $\mathbb{E}g'(W) - g'(t)$, such that it does not depend on t. Suppose we have a bound of the form

$$|\mathbb{E}g'(W) - g'(t)| \leq \mathbb{E}\zeta(W, a), \qquad (40)$$

where ζ is some function, and $\mathbb{E}\zeta(W, a)$ is a bound on the moments of W. Then an application of (40) to (39) will yield

$$\mathbb{P}(W \leq a) - \mathbf{Exp}(\lambda)\{a\} = \mathbb{E}[g'(W) - \lambda g(W)] \leq \mathbb{E}\zeta(W, a). \qquad (41)$$

The function ζ may in fact be a uniform bound on the differences of the derivatives of the Stein equation (31). Specifically, based upon (33), it may be possible to establish a bound on $g'(x) - g'(y)$. As an example, it can be shown that

$$\mathbb{E}[g'(W) - g'(t)] \leq 1 \wedge (e^{-a}\mathbb{E}e^{\lambda W}) = 1 \wedge (e^{-a}\mathbb{M}_W(\lambda)),$$

where $\mathbb{M}_W(\lambda)$ is the moment generating function of W [Billingsley 1986]. It may then be possible to use properties of the moment generating function to produce a useful bound. It will also be necessary to extract lower bounds, of a similar form, in order to obtain bounds on a norm of $\mathbb{E}\zeta(W, a)$. These ideas will be pursued in future work.

4 Potential Radar Applications

The purpose of this final section is to identify a number of random variables and stochastic processes of interest, in a radar context, and to indicate how Stein's Method can be used to assess distributional approximations for these. It is hoped that these potential applications will generate some interesting radar related research. These applications fall under the two main categories of *radar detection* and *inverse synthetic aperture radar.* There are many more problems where Stein's Method could be employed, especially in the areas of clutter statistics and detection theory. As an aside, we point out that the ideas developed in the context of Exponential approximation could be applied to some applications in critical phenomena in physics. Specifically, there is scope to apply the development in Section 3 to [Abadi 2001].

4.1 Assessing Approximations of Clutter Statistics in Detection

Radar detection theory deals with the problems associated with deciding whether there is a target present in a noisy environment [Hippenstiel 2002]. An important type of detector is the constant false alarm rate (CFAR) detection scheme [Levanon 1988 and Minkler and Minkler 1990]. At each stage, a set of clutter statistics are extracted, from which an average measure of the clutter is obtained. This is then compared to a test cell observation. If this test cell observation exceeds a normalised clutter threshold, a target is declared present. Otherwise, there is no target in the cell under test. A performance measure of such a scheme is the probability of false alarm. For a particular CFAR scheme one would like to be able to estimate this probability. This is usually done by directly applying Monte Carlo techniques [see Weinberg 2004 and references contained therein]. In many cases Monte Carlo estimation is problematic, and so it may be useful to be able to approximate the clutter statistic firstly.

To illustrate this idea, we suppose we have m independent and identically distributed clutter statistics $X_1, X_2, \ldots, X_m$, and the cell under test statistic is X_0. If τ is the CFAR threshold parameter, then the probability of false alarm is

$$\mathbb{P}_{FA} = \mathbb{P}\left(X_0 > \frac{\tau}{m} f(X_1, X_2, \ldots, X_m)\right), \tag{42}$$

where f determines the level of the clutter, and X_0 has the same distribution as the clutter statistics [see Weinberg 2004]. There are many choices available for f. In the case of cell averaging CFAR, $f(X_1, X_2, \ldots, X_m) = X_1 + X_2 + \cdots + X_m$. Another common choice is the smallest-of CFAR, where $f(X_1, X_2, \ldots, X_m) = \min\{X_1 + \cdots + X_k, X_{k+1} + \cdots + X_m\}$, for some $1 \leq k < m$. In a Monte Carlo estimation of (42), one would find a suitable importance sampling estimator, since direct standard Monte Carlo estimation of a probability such as (42) requires an enormous number of simulations [Weinberg 2004]. In some cases, importance sampling can also be problematic. A possible remedy to this is to approximate the distribution of the clutter statistic $f(X_1, X_2, \ldots, X_m)$ and then use this together with

Monte Carlo estimation of (42). This idea is used in [Weinberg 2004] to a certain degree. The question of the validity of the distributional approximation of $f(X_1, X_2, \ldots, X_m)$ naturally arises. The clutter function f can be written, in many cases, as a sum of random variables. In the case of the cell averaging clutter measure, this is obvious. In the case of the smallest-of clutter statistic, we can write it as

$$\begin{aligned} f(X_1, X_2, \ldots, X_m) &= \sum_{j=1}^{k} X_j \mathbb{I}_{[X_1+\cdots+X_k<X_{k+1}+\cdots+X_m]} \\ &\quad + \sum_{j=k+1}^{m} X_j \mathbb{I}_{[X_1+\cdots+X_k\geq X_{k+1}+\cdots+X_m]}. \end{aligned} \tag{43}$$

The random variable sum in (43) is hence just a sum of dependent random variables. Hence, for a particular clutter scenario and specific clutter measure f, we are interested in the distributional approximation of a sum of random variables. As has been illustrated in this report, Stein's Method has been adapted to such problems, and there are many bounds on the accuracy on such distributional approximations. Thus Stein's Method could be used to assess the validity of approximations made to facilitate Monte Carlo estimation of radar performance measures.

4.2 Time to Detection in Constant False Alarm Rate Scheme

The time it takes for a detection in CFAR, or more specifically, the time between two detections, is of importance in radar detection because it is the correlation of such events that indicates the presence of a target. From a statistical distribution point of view, it would be a rather complex random variable, with a possibly long tailed distribution. Since detections, in a sense, are like rare events, the waiting time is roughly like the waiting time to a rare event. A Poisson process [Ross 1983] can be used as a model for counting the occurrences of rare events, and the waiting time in such a process has an Exponential distribution. Hence, as a first order approximation, an Exponential distribution could be used for the waiting time to a detection.

To formulate this idea mathematically, suppose we have a discrete time stochastic process $\{X_j, j \in \mathbb{N}\}$, where each X_j is an indicator random variable, determined at each CFAR detection decision, defined by

$$X_j = \begin{cases} 1 & \text{if } t_j < \tau_j; \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

For each j, τ_j is the CFAR threshold, and t_j is the cell under test statistic. Hence X_j is one if there has been no detection at the jth timepoint. Then we define a random variable W as

$$W = \inf_{j \in \mathbb{N}} \{X_j = 0\}. \tag{45}$$

Random variable W is the first time the stochastic process hits state 0, and so counts the discrete time it takes for a detection. Although W is a relatively simple random variable, it is distributionally complex. However, a number of approximations could be made. Note that W is the number of failures before the first success, in a series of dependent Bernoulli trials [Billingsley 1986], with a varying success probability. Hence, as a first approximation, it is roughly a Geometric random variable. Stein's Method could be used to assess this approximation, using the work of [Peköz 1996, Phillips 1996 and Phillips and Weinberg 2000].

Alternatively, since the random W can be viewed as the discretised version of the waiting time to a rare event, it could in turn, be approximated by an Exponential random variable. The work of Section 3, when developed further, could be used to assess this approximation.

4.3 Speckle Modelling in Inverse Synthetic Aperature Radar

Inverse Synthetic Aperature Radar (ISAR) is a useful technique for the acquisition of high resolution images of a target of interest [Curlander and McDonough 1991 and Oliver and Quegan 1998]. Speckle is a phenomenon found in coherent imaging systems, such as ISAR. In ISAR, both the amplitude and the phase of the backscattered radiation are recorded. Consequently, a grainy noise like characteristic is introduced. The latter is referred to as speckle. Each resolution cell of the system contains many scatterers. The phase of each returned signal from each scatterer is random, and the correlation between these cause speckle.

[Daba and Bell 1994] examine partially developed speckle, modelling the surface scattering statistics as a marked point process [Daley and Vere-Jones 1988]. Consequently, approximations are derived for the distributions of intensity of a single look and multi-look speckle, using orthonormal Laguerre polynomial representations.

In their model, it is assumed that a resolution cell contains a fixed number n of point scatterers, randomly distributed throughout the resolution cell, such that each scatterer's position is independent of that of other scatterers. Each backscatter electric field component has a constant amplitude A_j $(j \in \{1, 2, \ldots, n\})$ and a random phase Φ_j, where the latter is assumed to be uniformly distributed over the interval $[0, 2\pi)$. The overall single look speckle intensity measurement is

$$S_k = \left| \sum_{j=1}^{k} A_j e^{i\Phi_j} \right|^2. \tag{46}$$

Multi-look speckle is modelled as a noncoherent sum of L statistically independent single realisations of the single look speckle intensity:

$$T_{(n,L)} = \sum_{j=1}^{L} S_{nj}. \tag{47}$$

It is shown in [Daba and Bell 1994] that (46) has an asymptotic Exponential distribution, while (47) has an asymptotic Gamma distribution. An application of the work in Section 3, in conjunction with the development in [Luk 1994 and Reinert 2003], could yield conditions under which the limiting distributions exist, and the rate of convergence of (46) and (47) to their respective asymptotic distributions.

Point processes [Daley and Vere-Jones 1988] are a natural tool to use in the study of imaging problems, since they model the distribution of points within a region of space. Stein's Method has been developed extensively for Poisson point process approximation [Barbour and Brown 1992 and Barbour, Holst and Janson 1992], and since a Poisson point process is a model of completely random scattering of points, the Poisson approximation of processes such as (47) could also be investigated.

In addition, since (47) has a limiting Gamma distribution, the idea of Gamma point processes could be developed, together with Stein's Method for point process approximations in this setting. This would introduce a completely new process into the class of point processes, as well as a new development of Stein's Method.

5 Conclusions

This report introduced Stein's Method in general terms, and then developed its application to Poisson and Gaussian approximations. The ongoing development of Stein's Method for Exponential approximation was examined, and its application illustrated in a simple context. A significant amount of effort is still required to obtain useful general results.

The application of Stein's Method to two areas of radar research were outlined. Problems in radar detection and ISAR have been identified, where the techniques presented here could be applied. Stein's Method has the potential to provide some interesting results on the rates of convergence of radar related processes. It is hoped that this report instigates the development and application of Stein's Method to defence applications.

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References

1. Abadi, M. (2001), Exponential approximation for hitting times in mixing processes. Math. Phys. Elec. Journ., **7**, 1–19.

2. Aldous, D., (1989), *Probability Approximations via the Poisson Clumping Heuristic.* (Springer–Verlag, New York).

3. Antipov, I. (1998), Analysis of Sea Clutter Data. DSTO–TR–0647.

4. Antipov, I. (2001), Statistical Analysis of Northern Australian Coastline Sea Clutter Data. DSTO–TR–1236.

5. Asmussen, S., (1987), *Applied Probability and Queues.* (John Wiley and Sons).

6. Barbour, A. D., (1988), Stein's method and Poisson Process Convergence. J. Appl. Prob., **25** (A), 175–184.

7. Barbour, A. D., (1990), Stein's method for diffusion approximations. Prob. Theory Related Fields, **84**, 297–322.

8. Barbour, A. D. and Brown, T. C. (1992), Stein's method and point process approximation. Stoch. Proc. Appl., **43**, 9–31.

9. Barbour, A. D. and Brown, T. C. (1996), Approximate Versions of Melamed's Theorem. J. Appl. Prob., **33**, 472–489.

10. Barbour, A. D., Chen, L. H. Y. and Loh, W.–L. (1992), Compound Poisson approximation for nonnegative random variables via Stein's method, Ann. Prob., **20**, 1843–1866.

11. Barbour, A. D., and Chryssaphinou, O. (2001), Compound Poisson approximation: a user's guide. Ann. Appl. Prob., **11**, 964–1002.

12. Barbour, A. D., Holst, L. and Janson, S. (1992), *Poisson Approximation.* (Oxford University Press, Oxford).

13. Billingsley, P. (1986), *Probability and Measure.* (John Wiley and Sons).

14. Chen, L. H. Y., (1975), Poisson Approximation for Dependent Trials. Ann. Prob., **3**, 534–545.

15. Chen, L. H. Y. and Shao, Q. (2003), *Stein's Method and Normal approximation.* (Tutorial notes for the Workshop on Stein's Method and Applications).

16. Curlander, J. and McDonough, R. (1991), *Synthetic Aperature Radar: Systems and Signal Processing.*(Wiley, New York).

17. Daba, J. S. and Bell, M. R. (1994), Statistical Distributions of Partially Developed Speckle Based on a Small Number of Constant Scatterers with Random Phase. Geoscience and Remote Sensing Symposium, IGARSS '94: Surface and Atmospheric Remote Sensing: Technologies, Data Analysis and Interpretation, **4**, 2338 - 2341.

18. Daley, D. J. and Vere-Jones, D. (1988), *An Introduction to the Theory of Point Processes.* (Springer–Verlag).

19. Dong, Y. (2003), L-Band VV Clutter Analysis for Natural Land in Northern Territory Areas. DSTO–RR–0254.

20. Dong, Y. (2004), Clutter Spatial Distribution and New Approaches of Parameter Estimation for Weibull and K-Distributions. DSTO–RR–0274.

21. Durrett, R. (1996), *Probability: Theory and Examples.* (Wadsworth Publishing Company).

22. Dynkin, E. B. (1965), *Markov Processes I.* (Springer–Verlag.)

23. Esseen, C. G. (1945), Fourier analysis of distribution functions: a mathematical study of the Laplace-Gaussian Law. *Acta Math.* **77**, 1–125.

24. Ethier, S. and Kurtz, T., (1986), *Markov Processes: Characterization and Convergence.* (John Wiley and Sons).

25. Feller, W. (1968), On the Berry-Esseen Theorem. *Z. Wahrsch. Verw. Gebiete,* **10**, 261–268.

26. Greig, D. (1995), *Approximations in Telecommunication Networks.* (PhD. Thesis, University of Melbourne).

27. Hippenstiel, R. D. (2002), *Detection Theory: Applications and Digital Signal Processing.* (CRC Press, New York).

28. Kreyszig, E. (1988), Advanced Engineering Mathematics. (Wiley, New York).

29. Levanon, N. (1988), *Radar Principles.* (John Wiley & Sons, New York).

30. Lindvall, T. (1992), *Lectures on the Coupling Method.* (John Wiley and Sons).

31. Loh, W.–L. (1992), Stein's method and multinomial approximation, Ann. Appl. Prob., **2**, 536–554.

32. Luk, M. (1994), *Stein's method for the gamma distribution and related statistical applications.* (PhD. Thesis, University of Southern California).

33. Minkler, G. and Minkler, J. (1990), *CFAR: The Principles of Automatic Radar Detection in Clutter.* (Magellan Book Company, Baltimore, MD).

34. Oliver, C. and Quegan, S. (1998), *Understanding synthetic aperature radar images.* (Artech House, Boston).

35. Peköz, E. A. (1996), Stein's method for geometric approximation, J. Appl. Prob., **33**, 707–713.

36. Phillips, M. J. (1996), *Stochastic process approximation and network applications.* (PhD. Thesis, University of Melbourne).

37. Phillips, M. J. and Weinberg, G. V., (2000), Non–Uniform Bounds for Geometric Approximation, Stat. Prob. Lett., **49**, 305–311

38. Reinert, G. (2003), *Stein's method for* χ^2 *Approximations, Weak Law of Large Numbers, and Discrete Distributions from a Gibbs View Point.* (Tutorial notes for the Workshop on Stein's Method and Applications).

39. Ross, Sheldon (1983), *Stochastic Processes.* (Wiley, New York).

40. Stein, C. (1972), A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. Proc. Sixth Berkeley Symp. Math. Stat. Prob., **2**, 583–602.

41. Stein, C. (1986), *Approximate Computation of Expectations.* (Institute of Mathematical Statistics Lecture Notes, Vol. 7).

42. Weinberg, G. V. (1999), *On the role of non-uniform smoothness parameters, and the probabilistic method, in applications of the Stein-Chen Method.* (PhD Thesis, University of Melbourne).

43. Weinberg, G. V. (2000), Stein factor bounds for random variables, J. Appl. Prob., **37**, 1181–1187.

44. Weinberg, G. V. (2004), Estimation of False Alarm Probabilities in Cell Averaging Constant Rate False Alarm Detectors via Monte Carlo Methods. DSTO-TR-1624.

45. Weinberg, G. V. (2005), Single pulse probability of detection in Gaussian clutter: A Poisson equivalence and asymptotic analysis via Stein's Method. DSTO Research Report (in vetting).

Appendix A: Mathematical Notation

$\mathbb{N}$	Natural numbers $\{0, 1, 2, \ldots\}$.
$\mathbb{R}$	Real numbers.
$\mathbb{R}^+$	Positive real numbers.
$\mathbb{P}$	Probability.
$\mathbb{E}$	Expectation with respect to $\mathbb{P}$.
$\mathcal{C}$	Class of all real valued continuous and piecewise differentiable functions.
$\mathcal{C}^*$	Subclass of $\mathcal{C}$, defined on page 12.
$\mathbb{I}$	Indicator function: $\mathbb{I}_{[x \in A]} = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$
$\stackrel{d}{=}$	Equality in distribution: $X \stackrel{d}{=} Y$ is equivalent to $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all sets A.
$\mathbb{M}$	Moment generating function / Laplace transform, defined on page 16.
$\mathbf{Po}(\lambda)$	Poisson Distribution with mean λ: if $X \stackrel{d}{=} \mathbf{Po}(\lambda)$, then $\mathbb{P}(X = j) = \frac{e^{-\lambda}\lambda^j}{j!}$, for all $j \in \mathbb{N}$.
$\mathbf{Po}(\lambda)\{A\}$	Cumulative Poisson probability on set $A \subset \mathbb{N}$.
$\mathbf{Exp}(\lambda)$	Exponential Distribution with mean $\frac{1}{\lambda}$: if $X \stackrel{d}{=} \mathbf{Exp}(\lambda)$, then $\mathbb{P}(X \in A) = \int_A \lambda e^{-\lambda x} dx$, for $A \subset \mathbb{R}^+ \cup \{0\}$.
$\mathbf{Exp}(\lambda)\{a\}$	Cumulative Exponential probability on interval $[0, a]$.
$\mathbf{Gau}(0, 1)$	Standard Gaussian (or Normal) distribution, with cumulative distribution function defined on page 8.

$\mathcal{A}$	Generator of a Markov Process, defined on page 5.
$\sup_A$	Supremum on set A.
$\inf_A$	Infimum on set A.
$\Vert \cdot \Vert$	Total variation norm: $\Vert g \Vert = \sup_{x \in \mathbb{R}} \vert g(x) \vert$.
$a \vee b$	Maximum of a and b.
$a \wedge b$	Minimum of a and b.

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Graham V. Weinberg

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19. ABSTRACT

In the analysis of radar signal processing, many approximations are made in order to simplify systems analysis. In the context of radar detection, environmental assumptions, such as in the modelling of clutter and noise, are often made. Similar assumptions are employed in the analysis of speckle in radar imaging systems. In addition to this, further statistical approximations of random processes of interest are made. In radar detection theory, an assumption on the distribution of a clutter statistic is often made. An important part of the model validation process is to assess the validity of employed assumptions. The validity of environmental assumptions is often assessed on the basis of field trials. In the case of approximations of random processes of interest, there are a number of approaches. This study introduces a general scheme, known as Stein's Method, for assesssing distributional approximations of random processes. It is described in general terms, and its application to the Poisson and Gaussian approximation of a given random variable is outlined. A new development of Stein's Method for Exponential approximation is included. The importance of the latter is that it can be used as an approximating distribution for a number of random variables of interest in radar.