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## COUPLED DETECTION RATES: AN INTRODUCTION

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# COUPLED DETECTION RATES: AN INTRODUCTION

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## Abstract

The case of two cooperative searchers is examined, and the effect of cueing on the probability of target detection is derived from first principles using a Markov chain analysis. There are two main results: first, that the effect of cueing can be quantified, and second, that there is an upper bound on the benefit of cueing. Both results are presented in closed form. The joint probability of detection for two independent searchers is derived from Koopman's formula for a single searcher, and is shown to be a special case of one of the results in this paper. Extensions of the model are discussed.

**Key Words:** Cooperative search, target detection, cueing, Markov chain

## 1. Introduction

In any system-of-systems analysis, consideration of dependencies between systems is imperative. In this paper, we consider a particular type of system interaction, called cueing. The interaction could be between similar systems, such as two or more wide area search munitions, or between dissimilar systems, such as a reconnaissance asset and a munition. In this introductory paper, we consider two identical search vehicles cooperatively interacting via cueing.

In Shakespeare's day, the word "cue" meant a signal (a word, phrase, or bit of stage business) to a performer to begin a specific speech or action [7]. The word is now used more generally for anything serving a comparable purpose. In this paper, we mean any information that provides focus to a search; e.g., information that limits the search area or provides a search heading.

Search theory is one of the oldest areas of operations research [10], with a solid foundation in mathematics, probability and experimental physics. Yet, search theory is clearly of more than academic interest. At times, a search can become an international priority, as in the 1966 search for the hydrogen bomb lost in the Mediterranean near Palomares, Spain.

That search was an immense operation involving 34 ships, 2,200 sailors, 130 frogmen and four mini-sub. The search took 75 days, but might have concluded much earlier if cueing had been utilized from the start. A Spanish fisherman had come forward quickly to say he'd seen something fall that looked like a bomb, but experts ignored him. Instead, they focused on four possible trajectories calculated by a computer, but for weeks found only airplane pieces. Finally, the fisherman, Francisco Simo, was

summoned back. He sent searchers in the right direction, and a two-man sub, the Alvin, located the 10-foot-long bomb under 2,162 feet of water [14].

Cueing is a current topic in vision research. For example, Arrington, et al. [2] study the role of objects in guiding spatial attention through a cluttered visual environment. Magnetic resonance imaging is used to measure brain activity during cued discrimination tasks requiring subjects to orient attention either to a region bounded by an object or to an unbounded region of space in anticipation of an upcoming target. Comparison between the two tasks revealed greater brain activity when an object cues the subject's attention.

Bernard Koopman pioneered the application of mathematical process to military search problems during World War II [10]. Koopman [4] discusses the case in which a searcher inadvertently provides information to the target, perhaps allowing the target to employ evasive action. The use of receivers on German U-boats to detect search radar signals in World War II is a classic example. Koopman referred to this type of cueing as "target alerting."

This paper uses a detection rate approach to examine the effect of cueing on probability of target detection. Koopman [5] used a similar approach in his discussion of target detection. In Koopman's terminology, a quantity  $\gamma$  was called the "instantaneous probability of detection." From this starting point, Koopman derived the probability of detection as a function of time. It is very clear that Koopman's instantaneous probability of detection is precisely the individual searcher detection rate used here. The main difference is that Koopman considered a single searcher, while we consider the case of two interdependent searchers.

Washburn [13] examines the case of a single searcher attempting to detect a randomly moving target at a discrete time. Given an effort distribution, bounded at each discrete time  $t$ , Washburn establishes an upper bound on the probability of target detection. It is noteworthy that Washburn mentions that the detection rate approach to computation of detection probabilities has proved to be more robust than approaches relying on geometric models.

In this paper, we use a Markov chain analysis to examine cueing as a coupling mechanism between two searchers. A Markov chain approach to target detection can be found in [10], which deals with the optimal allocation of effort to detect a target. A prior distribution of the target's location is assumed known to the searcher. Stone uses a Markov chain analysis to deal with the search for targets whose motion is Markovian. In Stone's formulation, the states correspond to cells that contain a target at a discrete time with a specified probability. In this research, the states correspond to detection states for individual search vehicles.

Alpern and Gal discuss the problem of searching for a submarine with a known initial location [1]. Thomas and Washburn [11] considered "dynamic search games" in which the hider starts moving at time zero from a location known to both a searcher and a hider, while the searcher starts with a time delay known to both players; for example, a helicopter attempts to detect a submarine that reveals its position by torpedoing a ship.

## 2. Problem Description

Consider two cooperative searchers, and assume that cueing increases an individual searcher's detection capability by a factor of  $k$ . That is, let the nominal detection rate for an individual searcher be given by  $\theta$  detections per unit of time, with the cued detection rate given by  $k\theta$  / time. We assume that once an individual searcher detects a target, it immediately cues the other searcher. This cue could take the form of a target coordinate, a search heading, or any other information that improves the second searcher's detection rate. We wish to examine the impact of cueing on the overall probability of target detection, denoted  $P_d$ .

## 3. Analysis

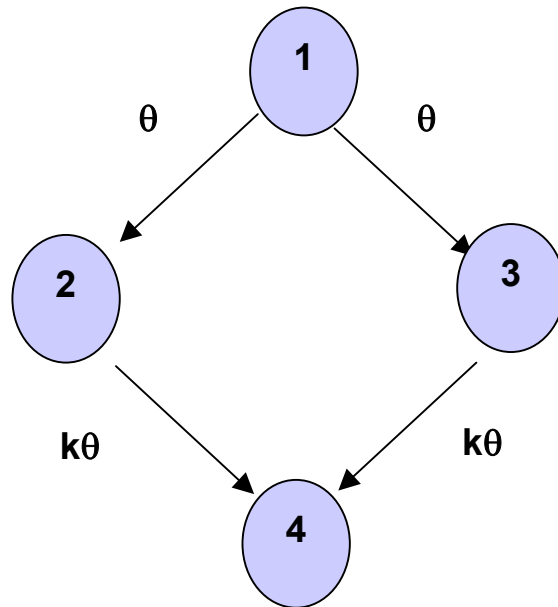
We first define four detection states for the two searchers, as shown in Table 1, in which "D" denotes detection, and "ND" denotes no detection.

*Table 1.* Detection States.

State	Searcher 1	Searcher 2
1	ND	ND
2	D	ND
3	ND	D
4	D	D

We will obtain the state probabilities using a Markov chain approach, and then derive the probability of target detection from the state probabilities. Professor Andrei A. Markov (1856 – 1922) is well known for his study of sequences of mutually dependent variables. Today, we use the term Markov process to denote a random process whose future state probabilities are determined only by its current state. A Markov process with a discrete state space is called a Markov chain [3]. In our analysis, we have a continuous time Markov chain, because transitions between the discrete states can occur at any time.

Figure 1 illustrates our four state Markov chain, with the transition rates between states. For example, the transition rate from state one (no detection by either searcher) to state two (detection by searcher 1 only) is given by  $\theta$ , the detection rate for searcher 1. Once searcher 1 detects a target, searcher 2 is immediately cued, so that the transition rate from state two to state four (detection by both searchers) is given by  $k\theta$ , the cued detection rate of searcher 2.



**Figure 1. Transition Rate Diagram**

We can use the transition rate diagram to write differential equations describing the change in states with respect to time. These equations are called Kolmogorov equations, after the Russian mathematician Andrei Kolmogorov (1903 – 1987), who was the first to derive these differential equations for continuous-time Markov chains.

$$\frac{d}{dt} P_1(t) = -2\theta \cdot P_1(t) \quad (1)$$

$$\frac{d}{dt} P_2(t) = +\theta \cdot P_1(t) - k\theta \cdot P_2(t) \quad (2)$$

$$\frac{d}{dt} P_3(t) = +\theta \cdot P_1(t) - k\theta \cdot P_3(t) \quad (3)$$

$$\frac{d}{dt} P_4(t) = +k\theta \cdot P_2(t) + k\theta \cdot P_3(t) \quad (4)$$

The initial conditions are defined by equations (5) through (8), based on the assumption that the process begins with no detections.

$$P_1(0) = 1 \quad (5)$$

$$P_2(0) = 0 \quad (6)$$

$$P_3(0) = 0 \quad (7)$$

$$P_4(0) = 0 \quad (8)$$

Given the four differential equations and the initial conditions defined by equations (1) through (8), we can find the state probability solutions using any technique familiar to the reader. To obtain the solutions below, we followed the approach of [6].

$$P_1(t) = e^{-2\theta t} \quad (9)$$

$$P_2(t) = [e^{-2\theta t} - e^{-k\theta t}] / (k - 2) \quad (10)$$

$$P_3(t) = [e^{-2\theta t} - e^{-k\theta t}] / (k - 2) \quad (11)$$

$$P_4(t) = [k - 2 - k e^{-2\theta t} + 2e^{-k\theta t}] / (k-2) \quad (12)$$

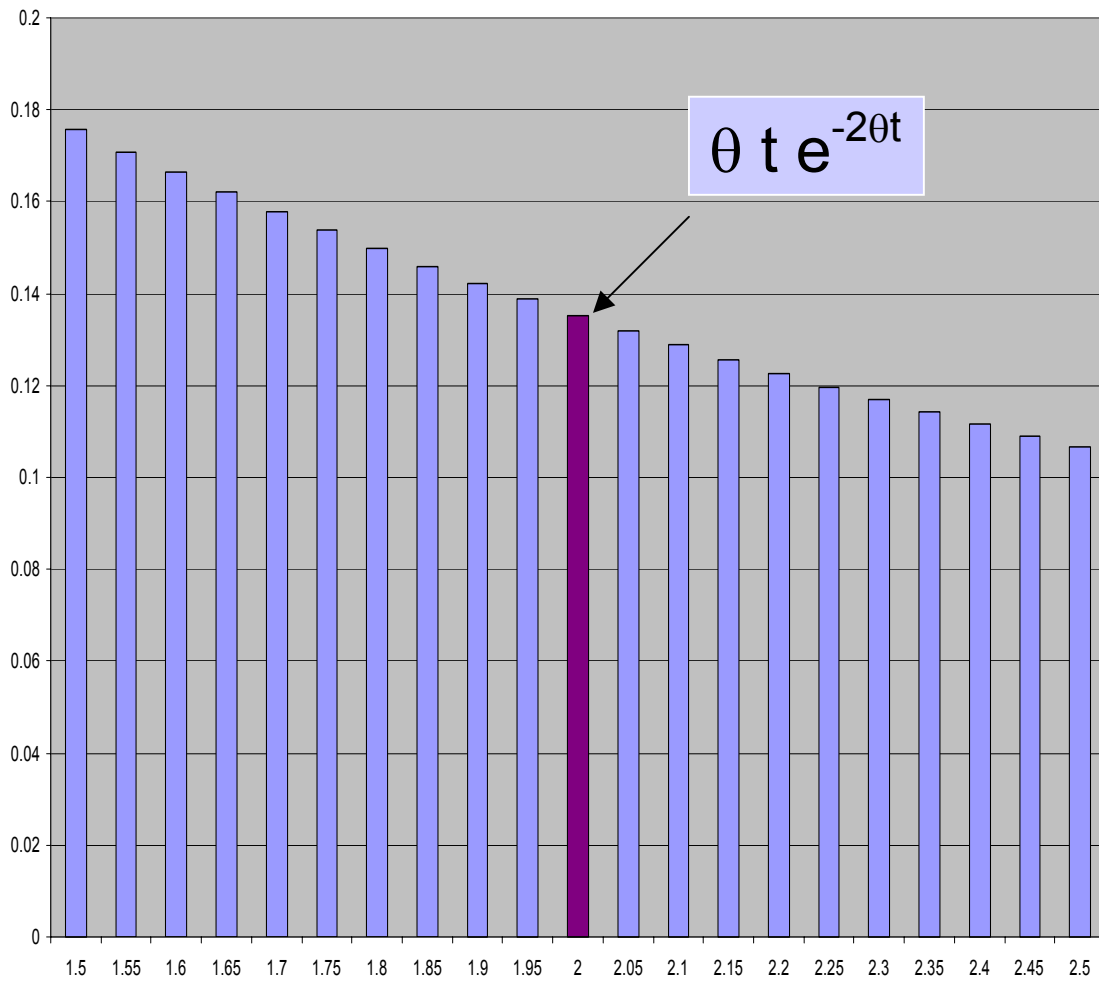
Note that all four functions are defined for any  $t \geq 0$ . Before moving to consideration of the probability of detection, we note with some concern that three of the state probabilities are not defined for  $k = 2$ . In particular,  $P_2(t)$ ,  $P_3(t)$ , and  $P_4(t)$  are indeterminate of the form  $0/0$  when  $k = 2$ . We can address this issue using L'Hopital's rule [9], shown in Eq. 13 for the particular case of  $k = 2$ .

$$\lim_{k \rightarrow 2} \frac{f(k)}{g(k)} = \lim_{k \rightarrow 2} \frac{f'(k)}{g'(k)} \quad (13)$$

Taking the appropriate derivatives and then evaluating the limit, we find that

$$P_2(t) = \theta t e^{-2\theta t} ; k = 2 \quad (14)$$

So,  $P_2(t)$  is defined for every nonnegative  $t$  and for every  $k \geq 1$ . Note that we have no interest here in values of  $k$  less than one, since that would imply a negative effect of cueing. As a quick check on Eq. (14), we can plot  $P_2$  as a function of  $k$ , using Eq. (10) for all values of  $k$  except  $k = 2$ . Figure 2 shows such a plot for  $\theta = 0.1$  and  $t = 10$ , with  $k$  ranging from 1.5 to 2.5. Although certainly not a proof, the plot gives us confidence that there is no problem at  $k = 2$ .



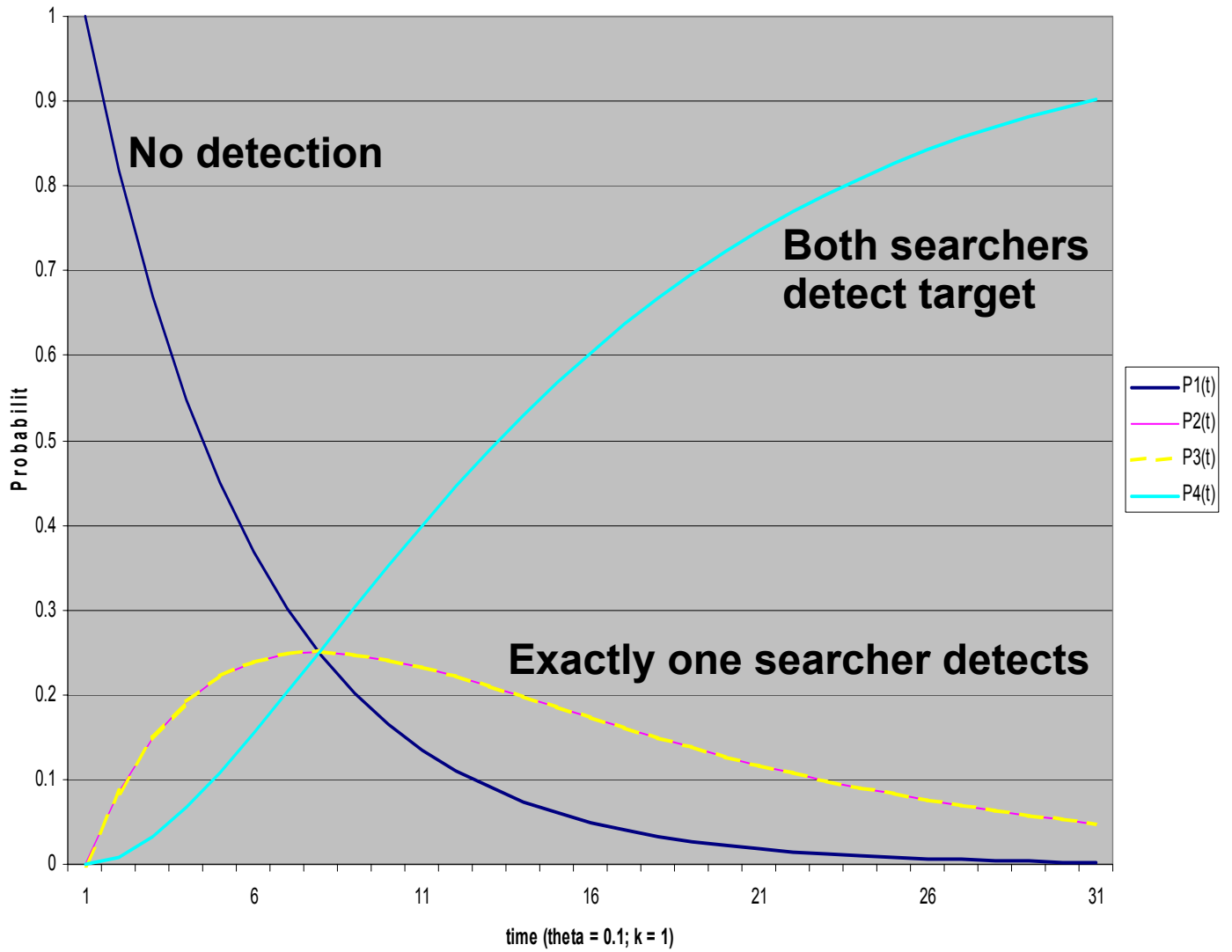
**Figure 2. P<sub>2</sub>(t,k) for θ = 0.1**

For the remainder of this paper, we will assume that P<sub>2</sub>(t) is continuous for all values of k ≥ 1. The situation is similar for P<sub>4</sub>(t). Again using equation (13), we find that

$$P_4(t) = 1 - e^{-2\theta t} [1 + 2\theta t]; \quad k = 2 \tag{14}$$

so that P<sub>4</sub>(t) is defined for every nonnegative t and every k ≥ 1.

Returning now to the state probabilities, Fig. 3 provides all four state probability plots for the particular case θ = 0.1 and k = 1; that is, with no cueing.



**Figure 3. State Probability Plots**

#### 4. The Probability of Detection

With the state probabilities in hand, we can turn to the probability of detection. For example, the probability of detection by at least one searcher is given by

$$P_d(t; \text{at least one searcher}) = P_2(t) + P_3(t) + P_4(t) \quad (15)$$

The probability of detection by both searchers is given by

$$P_d(t; \text{both searchers}) = P_4(t) \quad (16)$$

As an aside, we find for the case  $k=1$  that

$$P_4(t) = 1 + e^{-2\theta t} - 2e^{-\theta t} \quad (17)$$

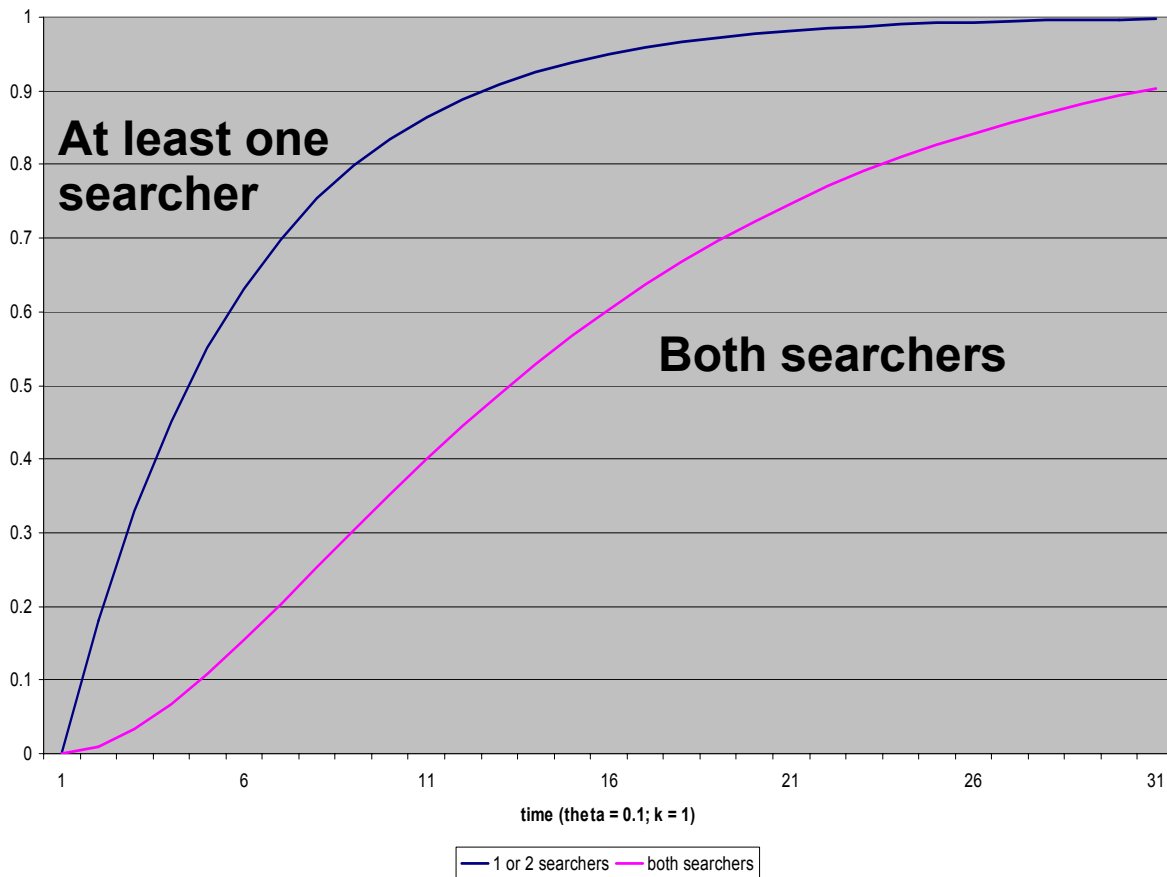
This is equivalent to the case of two independent searchers that derive no benefit from cueing. We can get the same result starting from Koopman's [5] single searcher formula for the probability of detection for a continuous search under unchanging conditions, where  $\gamma$  is the "instantaneous probability of detection."

$$p(t) = 1 - e^{-\gamma t} \quad (18)$$

The probability that two such searchers, working independently, would both find a target is given by  $[p(t)]^2$ , or

$$p(t) = 1 - 2e^{-\gamma t} + e^{-2\gamma t} \quad (19)$$

which is precisely equation (17) with the substitution  $\gamma = \theta$ . Fig. 4 shows two plots of probability of detection as a function of time, with  $\theta = 0.1$ , and  $k = 1$ ; i.e., no cueing.



**Figure 4. Probability of Detection**

## 5. The Effect of Cueing

Figure 5 shows the effect of cueing on the probability of detection for  $\theta = 1$ . In this case, the  $P_d$  represents the probability of detection by both searchers.

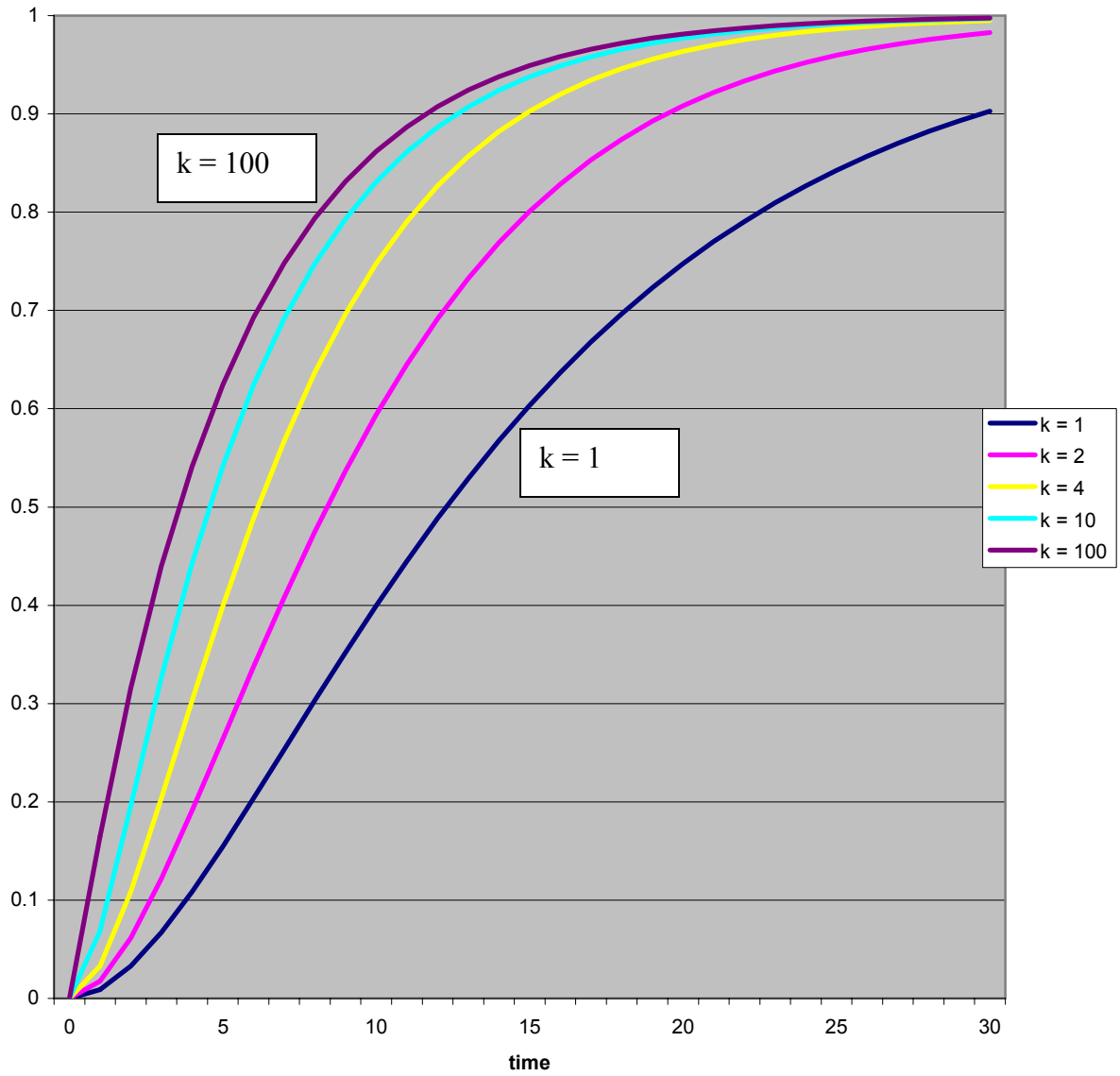


Figure 5. Effect of Cueing on  $P_d$

Note that cueing can dramatically increase the aggregate probability of detection for two searchers. For example, at  $t = 10$  and  $k = 4$ , we see that cueing essentially doubles the probability of detection (actual values are 0.4 for  $k = 1$  and 0.75 for  $k = 4$ ). Figure 5 also illustrates the diminishing return from cueing. The plots suggest that there is an upper bound to the benefit of cueing, at least for this problem. This can be verified by taking the limit of  $P_4(t)$  in Eq. (12) as  $k$  approaches infinity. Again using L'Hopital's rule, we obtain the result shown in Equation (13).

$$\lim_{k \rightarrow \infty} P_4(t) = 1 - e^{-2\theta t} \quad (13)$$

## 6. Extending the Model

Although an obvious next step is to examine larger problems, it is clear that the approach outlined here is limited by the difficulty of solving large sets of coupled differential equations. There are at least two approaches that may prove fruitful. One is to ignore the transient effects and to solve only for the steady-state probabilities. This can be done using a linear algebra approach. To illustrate the basic method, we construct the transition matrix  $Q$ , such that each off-diagonal element  $q_{ij}$ ,  $i \neq j$ , is the transition rate from state  $i$  to state  $j$ . The diagonal elements are defined to ensure that the elements in each row sum to zero. For our example problem,  $Q$  would be as shown in Figure 6.

$$Q = \begin{bmatrix} -2\theta & \theta & \theta & 0 \\ 0 & -k\theta & 0 & k\theta \\ 0 & 0 & -k\theta & k\theta \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Figure 6. The Transition Rate Matrix**

If we define  $P = [p_1, p_2, p_3, p_4]$  as the steady-state probability vector, then we can solve the set of linear equations in Equations (14) and (15) to find  $P$ .

$$PQ = 0 \quad (14)$$

$$\sum p_i = 1 \quad (15)$$

Solving these equations leads to the steady-state results  $p_1 = p_2 = p_3 = 0$ ; with  $p_4 = 1$ , as expected since  $p_4$  is clearly an absorbing state.

A second possible approach is matrix exponentiation, which has the potential to provide both transitional and steady state probabilities for large problems. Matrix exponentiation methods have been successfully applied to a broad class of problems in the theory of queues [8]. These methods exploit the structure of Markov chains to expedite numerical calculations.

## 7. Summary

We have shown that the effect of cueing on probability of detection can be quantified, and that cueing can dramatically affect the probability of detection over a fixed time interval. We have also shown that there is an upper bound on the steady-state benefit of cueing, at least for the problem defined. We have also introduced a line of inquiry into methods for addressing larger problems, which will be the subject of further research.

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