

A Posteriori Error Estimation for a New Stabilized Discontinuous Galerkin Method

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1 Introduction

In recent years, several variations of the Discontinuous Galerkin Methods (DGM), for second order elliptic boundary value problems have been proposed which exhibit special convergence, conservation and local approximation properties attractive for parallel adaptive hp -approximations. An account of several types of DGM's can be found in the book edited by Cockburn, Karniadakis and Shu [4].

In previous work [9], we introduced a new stabilized DGM formulation. Existence and uniqueness of stable solutions were established and *a priori* convergence estimates on the approximation error were derived for the case of a reaction-diffusion type model problem.

We continue our error analysis of this stabilized DGM here by deriving global implicit *a posteriori* estimates of the approximation error. Results of our previous analysis enable us to prove equivalence between the error norm and the norm of a residual functional that characterizes the accuracy of the approximate solution. Hence, upon estimating the norm of this residual, we indirectly obtain an estimate of the error. To verify the efficiency of our estimates, we present 1D and 2D numerical experiments on a reaction-diffusion type model problem.

In Chapter 2 we introduce the notations, the model problem, and the stabilized DGM formulation. Error estimators are derived in Chapter 3 and numerical verifications are presented in Chapter 4. Finally, concluding remarks are summarized in Chapter 5.

2 Model Problem and Notations

2.1 The Reaction-Diffusion Equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary $\partial\Omega$ and let $\{\mathcal{P}_h\}$ be a family of regular partitions of Ω into open elements K , with diameters h_K , such that (see Fig. 2.1):

$$\Omega = \text{int} \left(\bigcup_{K \in \mathcal{P}_h} \overline{K} \right).$$

The maximum diameter in the partition is denoted h . The set of all edges of the partition $\mathcal{P}_h$ is given by $\mathcal{E}_h = \{\gamma_k\}, k = 1, \dots, N_{\text{edge}}$, where N_{edge} represents the number of edges in the partition $\mathcal{P}_h$. The interior interface Γ_{int} is then defined as the union of all common edges shared by elements of the partition $\mathcal{P}_h$:

$$\Gamma_{\text{int}} = \bigcup_{k=1}^{N_{\text{edge}}} \gamma_k \setminus \partial\Omega.$$

The definition of the unit normal vector $\mathbf{n}$ on each γ_k is related to the numbering of the elements in the partition, such that $\mathbf{n}$ is defined outward with respect to the element with the highest index number (see Fig. 2.1). The normal vector μ is defined outward to each element individually. Within this setting, the following reaction-diffusion problem is considered:

$$\begin{aligned} -\Delta u + u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where f is a real-valued function in $L^2(\Omega)$. For the sake of clarity in the notation, the jump and average operators on each $\gamma_k \in \Gamma_{\text{int}}$ are respectively defined as (see Fig 2.1):

$$\left. \begin{aligned} [v] &= v|_{\gamma_k \subset \partial K_i} - v|_{\gamma_k \subset \partial K_j} \\ \langle v \rangle &= \frac{1}{2}(v|_{\gamma_k \subset \partial K_i} + v|_{\gamma_k \subset \partial K_j}) \end{aligned} \right\} \gamma_k = \text{int}(\partial K_i \cap \partial K_j), \quad i > j. \tag{2.2}$$

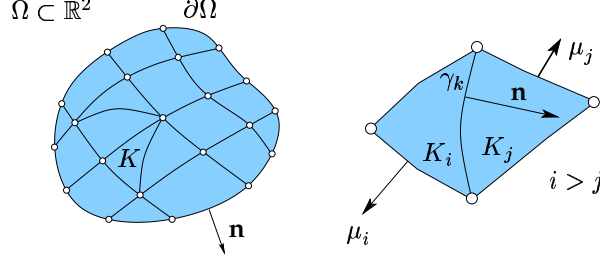


Figure 2.1: Geometrical definitions.

2.2 The Variational Formulation

In this section, we present our weak discontinuous formulation of the model problem (2.1). First, we introduce the following *broken space*:

$$\mathcal{M}(\mathcal{P}_h) = \left\{ v \in L^2(\Omega) : v|_K \in H(\Delta, K), \forall K \in \mathcal{P}_h, [\nabla v \cdot \mathbf{n}] \in L^2(\Gamma_{\text{int}}) \right\},$$

where

$$H(\Delta, K) = \left\{ v \in L^2(K) : \nabla \cdot \nabla v \in L^2(K) \right\} \subset H^1(K).$$

Notice that $v \in H(\Delta, K)$ implies $\nabla v \cdot \mu \in H^{-1/2}(\partial K)$ (see [2, 7]). The norm $\|\cdot\|$ on $\mathcal{M}(\mathcal{P}_h)$, is defined as:

$$\begin{aligned} \|v\|^2 = \sum_{K \in \mathcal{P}_h} & \left\{ \|v\|_{H^1(K)}^2 + \frac{h^\nu}{p^\theta} \|\nabla v \cdot \mu\|_{H^{-1/2}(\partial K)}^2 \right\} \\ & + \sigma \frac{h^\lambda}{p^\zeta} \|[\nabla v \cdot \mathbf{n}]\|_{L^2(\Gamma_{\text{int}})}^2. \end{aligned} \quad (2.3)$$

where the parameters ν , λ , θ , and ζ are all positive numbers. At this stage of our development, p is merely a non-zero positive integer arbitrarily assigned to each element. Later, this parameter will be taken as the polynomial degree p_K of the approximations on each element K , or globally, as the minimum value of all p_K 's in the partition $\mathcal{P}_h$ (see Section 2.3). Finally the norms in (2.3) are defined as:

$$\begin{aligned} \|g\|_{H^{-1/2}(\partial K)} &= \sup_{\varphi \in H^{1/2}(\partial K)} \frac{|\langle g, \varphi \rangle_{-1/2 \times 1/2, \partial K}|}{\|\varphi\|_{H^{1/2}(\partial K)}}, \\ \|\varphi\|_{H^{1/2}(\partial K)} &= \inf_{\substack{w \in H^1(K) \\ \gamma_0 w = \varphi}} \|w\|_{H^1(K)}, \end{aligned} \quad (2.4)$$

where $\langle \cdot, \cdot \rangle_{-1/2 \times 1/2, \partial K}$ is the duality pairing in $H^{-1/2}(\partial K) \times H^{1/2}(\partial K)$ and where γ_0 denotes the trace operator:

$$\gamma_0 : H^1(K) \longrightarrow H^{1/2}(\partial K).$$

The choice for the space of test functions V for the weak formulation is the completion of $\mathcal{M}(\mathcal{P}_h)$ with respect to the norm $\|\cdot\|$. The discontinuous variational formulation can then be stated as follows:

Find $w \in V$:

$$\mathcal{B}(w, v) = \mathcal{L}(v), \quad \forall v \in V,$$

(2.5)

where the bilinear form $\mathcal{B} : V \times V \longrightarrow \mathbb{R}$ and linear form $\mathcal{L} : V \longrightarrow \mathbb{R}$ are defined as follows:

$$\begin{aligned} \mathcal{B}(w, v) = & \sum_{K \in \mathcal{P}_h} \left\{ \int_K \{ \nabla w \cdot \nabla v + wv \} d\mathbf{x} \right\} \\ & - \sum_{K \in \mathcal{P}_h} \left\{ \int_{\partial K} \{ v(\nabla w \cdot \boldsymbol{\mu}) - (\nabla v \cdot \boldsymbol{\mu})w \} ds \right\} \\ & + \int_{\Gamma_{\text{int}}} \{ \langle v \rangle [\nabla w \cdot \mathbf{n}] - \langle w \rangle [\nabla v \cdot \mathbf{n}] \} ds \\ & + \int_{\Gamma_{\text{int}}} \sigma \frac{h^\lambda}{p^\zeta} [\nabla w \cdot \mathbf{n}] [\nabla v \cdot \mathbf{n}] ds, \\ \mathcal{L}(v) = & \int_{\Omega} f v d\mathbf{x}. \end{aligned} \tag{2.6}$$

The following notation is used to denote the duality pairing in $H^{-1/2}(\partial K) \times H^{1/2}(\partial K)$:

$$\int_{\partial K} (\nabla w \cdot \boldsymbol{\mu}) v ds = \langle \nabla w \cdot \boldsymbol{\mu}, v \rangle_{H^{-1/2}(\partial K) \times H^{1/2}(\partial K)}.$$

As noted in [9], this weak formulation is closely related to the DGM formulation by Oden, Babuška and Baumann [6]. A distinctive difference is the addition of the stabilization term on the jumps of the normal fluxes across the element interfaces. This stabilization allows us to prove the well-posedness of the weak problem while having the advantage of not polluting the *local conservation property*, which is one of the appealing properties of DGM's. Indeed, by taking $v = 1$ in (2.6) on an element $K \in \mathcal{P}_h$, we obtain:

$$\int_K w d\mathbf{x} - \int_{\partial K} \langle \nabla w \cdot \boldsymbol{\mu} \rangle ds = \int_K f d\mathbf{x},$$

and local conservation is satisfied in terms of the average fluxes on the element edges. The well-posedness of the problem (2.5) is established in [9].

2.3 The Discrete Problem

Let $\{\mathbf{F}_K\}$ be a family of invertible maps defined for a regular partition $\mathcal{P}_h$ such that every element $K \in \mathcal{P}_h$ is the image of $\mathbf{F}_K$ acting on a master element $\hat{K}$, *i.e.*

$$\mathbf{F}_K : \hat{K} \longrightarrow K, \quad \mathbf{x} = \mathbf{F}_K(\hat{\mathbf{x}}). \quad (2.7)$$

In the computational model, a finite dimensional space of real-valued piecewise polynomial functions of degree less than p_K is introduced as:

$$V^{hp} = \left\{ v \in L^2(\Omega) : v|_K = \hat{v} \circ \mathbf{F}_K^{-1}, \hat{v} \in P^{p_K}(\hat{K}), \forall K \in \mathcal{P}_h \right\}. \quad (2.8)$$

We note that V^{hp} is a subspace of V . Using the Galerkin method, an approximation u_h of u is sought as the solution of the following discrete problem:

Find $u_h \in V^{hp}$:

$$\mathcal{B}(u_h, v_h) = \mathcal{L}(v_h), \quad \forall v_h \in V^{hp}.$$

(2.9)

3 A Posteriori Error Estimation

In this section, we derive implicit *a posteriori* estimators of the approximation error. These are *global* in the sense that they estimate the error in a global norm defined on the whole domain Ω .

We present two types of estimators. Section 3.3 introduces the estimator χ_h which is the solution of a global problem. This type of estimator will be referred to as a *coupled* estimator, since it requires solving a global system of linear equations. On the other hand, the estimator ψ_h proposed in Section 3.4 is obtained by solving a set of local problems and is referred to as a *decoupled*-type estimator.

3.1 The Error Problem

Let $u \in V$ and $u_h \in V^{hp}$ be the exact and approximate solutions to (2.5) and (2.9), respectively. Then by using the linearity of the bilinear form $\mathcal{B}(\cdot, \cdot)$, it follows easily that the approximation error $e = u - u_h$ is governed by:

$$\boxed{\mathcal{B}(e, v) = \underbrace{\mathcal{L}(v) - \mathcal{B}(u_h, v)}_{\mathcal{R}^h(v)}, \quad \forall v \in V} \quad (3.1)$$

where $\mathcal{R}^h : V \rightarrow \mathbb{R}$ is the *residual functional*. Note that due to (2.9) the residual satisfies the following orthogonality property on V^{hp} :

$$\mathcal{R}^h(v) = \mathcal{B}(e, v_h) = 0, \quad \forall v_h \in V^{hp}. \quad (3.2)$$

Lemma 3.1.1 *The residual functional is continuous on V , i.e.:*

$$\exists C > 0 : \quad |\mathcal{R}^h(v)| \leq C \|v\|, \quad \forall v \in V.$$

Proof: This lemma follows trivially from the continuity properties of the variational forms $\mathcal{L}(\cdot)$ and $\mathcal{B}(\cdot, \cdot)$ on V (see [9]). ■

3.2 Equivalence between the Residual and the Error

Since $\mathcal{R}^h$ is continuous on V with respect to the norm $\|\cdot\|$, we define the norm of the residual (dual norm) as:

$$\|\mathcal{R}^h\|_* = \sup_{v \in V \setminus \{0\}} \frac{|\mathcal{R}^h(v)|}{\|v\|}$$

Lemma 3.2.1 *The norm of the error and the norm of the residual functional are equivalent, i.e. there exist two positive constants C_1 and C_2 which depend on the mesh parameters h and p such that:*

$$C_1 \|e\| \leq \|\mathcal{R}^h\|_* \leq C_2 \|e\|.$$

Proof: Firstly, by recalling the Inf-Sup property of $\mathcal{B}(\cdot, \cdot)$ in Theorem 3.3.2 in [9], we know that there exists $\gamma(h, p) > 0$ such that:

$$\gamma(h, p) \|e\| \leq \sup_{v \in V \setminus \{0\}} \frac{|\mathcal{B}(e, v)|}{\|v\|}.$$

Using (3.1) and the definition of the norm $\|\mathcal{R}^h\|_*$, we obtain:

$$\gamma(h, p) \|e\| \leq \sup_{v \in V \setminus \{0\}} \frac{|\mathcal{R}^h(v)|}{\|v\|} = \|\mathcal{R}^h\|_*.$$

Conversely, by recalling the continuity property of the bilinear form, as stated in Theorem D.1, we get:

$$\|\mathcal{R}^h\|_* = \sup_{v \in V \setminus \{0\}} \frac{|\mathcal{B}(e, v)|}{\|v\|} \leq M(h, p) \|e\|,$$

$$\text{where } M(h, p) = \max \left\{ 3, \sqrt{\frac{p^\theta}{h^\nu}}, \sqrt{\frac{p^\zeta}{2\sigma h^\lambda}} \right\}. \quad \blacksquare$$

In principle, this lemma justifies estimation of the error norm indirectly by estimating $\|\mathcal{R}^h\|_*$. However, it is essential that the constants $C_1(h, p)$ and $C_2(h, p)$ do not degenerate or diverge as $h \rightarrow 0$ and $p \rightarrow \infty$. This implies that both the Inf-Sup and continuity constants of $\mathcal{B}(\cdot, \cdot)$ have to be independent of h and p . According to Corollary 3.1 in [9] and Theorem D.1, both coefficients are constant if $\lambda = \nu = \zeta = \theta = 0$. In the following, these parameters are set equal to zero so that the norm $\|\cdot\|$ is now defined as:

$$\|v\|^2 = \sum_{K \in \mathcal{P}_h} \left\{ \|v\|_{H^1(K)}^2 + \|\nabla v \cdot \mu\|_{H^{-1/2}(\partial K)}^2 \right\} + \sigma \|\nabla v \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}^2. \quad (3.3)$$

3.3 A Coupled Error Estimator

A natural approach for error estimation could be to approximate (3.1) using a space W^{hp} such that $V^{hp} \subset W^{hp} \subset V$. A possible candidate for W^{hp} is the finite element space of functions W^{hp} of piecewise continuous polynomials that are of degree one higher than the functions in V^{hp} :

$$W^{hp} = \left\{ v \in L^2(\Omega) : v|_K = \hat{v} \circ \mathbf{F}_K^{-1}, \hat{v} \in P^{p_K+1}(\hat{K}) \quad \forall K \in \mathcal{P}_h \right\}.$$

In this case, an approximation $\phi_h \in W^{hp}$ of the error is obtained as the solution to the error problem (3.1) in the space W^{hp} , i.e.:

$$\mathcal{B}(\phi_h, v_h) = \mathcal{R}^h(v_h), \quad \forall v_h \in W^{hp} \quad (3.4)$$

and a global estimate of the error is given by the norm of ϕ_h , i.e. $\|\phi_h\| \approx \|e\|$, as indicated in the following lemma:

Lemma 3.3.1 *Let $\phi_h \in W^{hp}$ be the unique solution to (3.4), $u_h \in V^{hp}$ the solution to (2.9), $u \in V$ the solution to (2.5), and let $u|_K \in H^2(K)$ for every $K \in \mathcal{P}_h$. Then if the family of mappings (2.7) is affine and invertible, there exists a constant $C(\sigma)$ independent of h and p , such that:*

$$\|e - \phi_h\| \leq C(\sigma) \frac{h^{\mu-3/2}}{p^{r-7/2}} \sqrt{\sum_{K \in \mathcal{P}_h} \|u\|_{H^r(K)}^2}, \quad p \geq 1, r_K \geq 2,$$

where $\mu = \min(p+1, r)$ and $r = \min_{K \in \mathcal{P}_h} (r_K)$.

Proof: See Appendix A. ■

For $p \geq 1$, $\|\phi_h\|$ provides an *asymptotic* estimate of the error norm. The predicted convergence rates are suboptimal, but convergence to the error norm is guaranteed even for $p = 1$. Note that in this case the error itself does not necessarily converge as h tends to zero (see Figure 5.1a in [9]).

However, although this approach generally provides for accurate error estimates, solving the resulting system of equations is far more expensive than solving the original problem for u_h . Moreover, for linear equations, this approach is equivalent to computing a new finite element solution $\tilde{u}_h \in W^{hp}$ of (2.5) and taking the difference $\tilde{u}_h - u_h$. In order to reduce

the computational cost, it is desirable in *a posteriori* error estimation to set up local problems defined on each element or patch of elements. The DG formulation seems well suited to do so. Indeed, let the test function v be zero everywhere except in element K . The equation of problem (3.1) thus reduces to

$$\begin{aligned} \int_K \{ \nabla e \cdot \nabla v + ev \} d\mathbf{x} &- \int_{\partial K} \{ v(\nabla e \cdot \boldsymbol{\mu}) - (\nabla v \cdot \boldsymbol{\mu})e \} ds \\ &+ \int_{\partial K \setminus \partial\Omega} \left\{ \frac{v}{2} [\nabla e \cdot \mathbf{n}] - \langle e \rangle (\nabla v \cdot \mathbf{n}) \right\} ds \\ &+ \int_{\partial K \setminus \partial\Omega} \sigma [\nabla e \cdot \mathbf{n}] (\nabla v \cdot \mathbf{n}) ds = \mathcal{R}_K^h(v_h) \end{aligned} \quad (3.5)$$

and is thus defined locally except for the coupling terms $[\nabla e \cdot \mathbf{n}]$ and $\langle e \rangle$.

As an intermediate step toward full decoupling of the system, we introduce the function $\chi \in V$ that is governed by:

$$\boxed{\mathcal{A}(\chi, v) = \mathcal{R}^h(v), \quad \forall v \in V} \quad (3.6)$$

where the bilinear form $\mathcal{A} : V \times V \rightarrow \mathbb{R}$ is now defined as:

$$\begin{aligned} \mathcal{A}(w, v) &= \sum_{K \in \mathcal{P}_h} \left\{ \int_K \{ \nabla w \cdot \nabla v + wv \} d\mathbf{x} \right\} \\ &- \sum_{K \in \mathcal{P}_h} \left\{ \int_{\partial K} \{ v(\nabla w \cdot \boldsymbol{\mu}) - (\nabla v \cdot \boldsymbol{\mu})w \} ds \right\} \\ &+ \int_{\Gamma_{\text{int}}} \sigma [\nabla w \cdot \mathbf{n}] [\nabla v \cdot \mathbf{n}] ds. \end{aligned}$$

Comparison of $\mathcal{A}(\cdot, \cdot)$ with the bilinear form $\mathcal{B}(\cdot, \cdot)$ reveals that the terms involving $\langle v \rangle [\nabla w \cdot \mathbf{n}]$ and $\langle w \rangle [\nabla v \cdot \mathbf{n}]$ have been eliminated from the formulation. We show in the following that $\mathcal{A}(\cdot, \cdot)$ satisfies the Inf-Sup condition and is continuous.

Lemma 3.3.2 (*Inf-Sup condition*) *There exists $\gamma_{\mathcal{A}} > 0$, independent of h and p such that:*

$$\sup_{v \in V \setminus \{0\}} \frac{|\mathcal{A}(u, v)|}{\|v\|} \geq \gamma_{\mathcal{A}} \|u\|, \quad \forall u \in V.$$

Proof: See Appendix B. ■

Lemma 3.3.3 *The bilinear form $\mathcal{A}(\cdot, \cdot)$ is continuous on $V \times V$, i.e.:*

$$\exists M_{\mathcal{A}} > 0 : \quad |\mathcal{A}(u, v)| \leq M_{\mathcal{A}} \|u\| \|v\|, \quad \forall u, v \in V,$$

where $M_{\mathcal{A}} = 2$.

Proof: By recalling the definition of $\mathcal{A}(\cdot, \cdot)$ and applying the Schwarz Inequality, we get:

$$\begin{aligned} \mathcal{A}(u, v) \leq & \sum_{K \in \mathcal{P}_h} \left\{ \|u\|_{H^1(K)} \|v\|_{H^1(K)} + \|\nabla u \cdot \mu\|_{H^{-1/2}(\partial K)} \|v\|_{H^{1/2}(\partial K)} \right. \\ & \left. + \|\nabla v \cdot \mu\|_{H^{-1/2}(\partial K)} \|u\|_{H^{1/2}(\partial K)} \right\} \\ & + \sigma \|\nabla u \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})} \|\nabla v \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}. \end{aligned} \quad (3.7)$$

By definition of the $H^{1/2}(\partial K)$ -norm, we know that:

$$\|v\|_{H^{1/2}(\partial K)} \leq \|v\|_{H^1(K)}, \quad v \in H^1(K).$$

Substitution of this inequality into (3.7) finishes the proof. $\blacksquare$

Theorem 3.3.1 *Let $\chi \in V$ be the unique solution to (3.6). Then the norm $\|\chi\|$ and the norm of the residual functional $\|\mathcal{R}^h\|_*$ are equivalent:*

$$\gamma_{\mathcal{A}} \|\chi\| \leq \|\mathcal{R}^h\|_* \leq M_{\mathcal{A}} \|\chi\|.$$

Proof: By substituting (3.6) into the definition of $\|\mathcal{R}^h\|_*$, one obtains:

$$\|\mathcal{R}^h\|_* = \sup_{v \in V/\{0\}} \frac{|\mathcal{A}(\chi, v)|}{\|v\|}. \quad (3.8)$$

The assertion is then established by applying Lemmas 3.3.2 and 3.3.3. $\blacksquare$

Hence, by estimating $\|\chi\|$ we indirectly estimate $\|\mathcal{R}^h\|_*$ and, therefore, the error norm itself. The coupled error estimator η_c is then defined by the approximate solution $\chi_h \in W^{hp}$ of χ such that:

$$\eta_c = \|\chi_h\| \quad (3.9)$$

and

$$\mathcal{A}(\chi_h, v_h) = \mathcal{R}^h(v_h), \quad \forall v_h \in W^{hp} \quad (3.10)$$

We prove in the following that η_c provides for an acceptable error estimate of $\|\chi\|$ and, *a fortiori*, of the error norm itself:

Lemma 3.3.4 *Let $\chi \in V$ be the solution to (3.6), $\chi|_K$ be in $H^2(K)$ for all $K \in \mathcal{P}_h$, and $\chi_h \in W^{hp}$ be the unique solution to (3.10). Then if the family of mappings (2.7) is affine and invertible, there exists a constant $C(\sigma) > 0$ independent of h and p , such that:*

$$\|\chi - \chi_h\| \leq C(\sigma) \frac{h^{\mu-3/2}}{p^{r-7/2}} \sqrt{\sum_{K \in \mathcal{P}_h} \|\chi\|_{H^r(K)}^2}, \quad p \geq 1, \quad r_K \geq 2,$$

where $\mu = \min(p+1, r)$ and $r = \min_{K \in \mathcal{P}_h} (r_K)$.

Proof: See Appendix C. ■

Although the estimator η_c is global, it can still be decomposed into a sum of element-wise contributions $\eta_{c,K}$ such that:

$$\eta_{c,K}^2 = \|\chi_h\|_{H^1(K)}^2 + \|\nabla \chi_h \cdot \mu\|_{H^{-1/2}(\partial K)}^2 + \frac{1}{2} \sigma \|\nabla \chi_h \cdot \mathbf{n}\|_{L^2(\partial K \cap \Gamma_{\text{int}})}^2$$

so that

$$\eta_c^2 = \sum_{K \in \mathcal{P}_h} \eta_{c,K}^2.$$

The contributions $\eta_{c,K}$ can serve as refinement indicators for each element in an mesh adaptation strategy.

3.4 A Decoupled Error Estimator

In this section we seek to derive an estimator which involves solving local problems only on each element of the mesh in order to reduce the computational cost. Decoupling the global equation supposes to eliminate the interactions between neighboring elements through the element boundaries. In doing so, a loss of accuracy with respect to the estimator η_c is therefore expected.

To establish the decoupling, we decompose the finite dimensional space W^{hp} into the family of local spaces $\{W_K^{hp}\}$:

$$W_K^{hp} = \left\{ v = \hat{v} \circ F_K^{-1}, \quad \hat{v} \in P^{p_K+1}(\hat{K}) \right\}.$$

with associated norms:

$$\|v\|_{W_K}^2 = \|v\|_{H^1(K)}^2 + \|\nabla v \cdot \mu\|_{L^2(\partial K)}^2 + 2\sigma \|\nabla v \cdot \mu\|_{L^2(\partial K \cap \Gamma_{\text{int}})}^2. \quad (3.11)$$

Using this function space setting, we then consider the following local problem for every $K \in \mathcal{P}_h$:

$$\boxed{\begin{aligned} \text{Find } \psi_K^h \in W_K^{hp} : \\ \mathcal{D}_K(\psi_K^h, v_h) = \mathcal{R}_K^h(v_h), \quad \forall v_h \in W_K^{hp} \end{aligned}} \quad (3.12)$$

where $\mathcal{R}_K^h(\cdot)$ denotes the restriction of $\mathcal{R}^h(\cdot)$ to K and $\mathcal{D}_K(\cdot, \cdot)$ is the bilinear form defined on $W_K^{hp} \times W_K^{hp}$ such that:

$$\begin{aligned} \mathcal{D}_K(\psi_K^h, v_h) = \int_K \left\{ \nabla \psi_K^h \nabla v_h + \psi_K^h v_h \right\} d\mathbf{x} + \int_{\partial K} (\nabla \psi_K^h \cdot \mu)(\nabla v_h \cdot \mu) ds \\ + 2\sigma \int_{\partial K \cap \Gamma_{\text{int}}} (\nabla \psi_K^h \cdot \mu)(\nabla v_h \cdot \mu) ds. \end{aligned}$$

Remark 3.4.1 *There exists a unique solution $\psi_K^h \in W_K^{hp}$ to (3.12). Indeed, since W_K^{hp} is finite dimensional, existence and uniqueness of ψ_K^h follows by application of the Fredholm alternative and by noting that $\mathcal{D}(\cdot, \cdot)$ is positive definite on $W_K^{hp} \times W_K^{hp}$.*

Once again we can consider the global error estimator η_d in terms of local contributions $\eta_{d,K}$:

$$\eta_{d,K} = \|\psi_K^h\|_{W_K} = \sqrt{\mathcal{D}_K(\psi_K^h, \psi_K^h)} \quad (3.13)$$

so that:

$$\eta_d^2 = \sum_{K \in \mathcal{P}_h} \eta_{d,K}^2. \quad (3.14)$$

Theorem 3.4.1 *Let $\{\psi_K\}$ be the set of unique solutions to (3.12) and let η_d be the error estimator as defined in (3.14). Then the following inequality holds:*

$$\boxed{\eta_d \leq \|\mathcal{R}^h\|_*}$$

Proof: Starting from the definition of η_d and the formulation of the local problems (3.12), we get

$$\eta_d^2 = \sum_{K \in \mathcal{P}_h} \mathcal{D}_K(\psi_K^h, \psi_K^h) = \sum_{K \in \mathcal{P}_h} \mathcal{R}_K^h(\psi_K^h) = \mathcal{R}^h\left(\sum_{K \in \mathcal{P}_h} \psi_K^h\right) \quad (3.15)$$

where we assume that each local function ψ_K^h is extended to the whole domain Ω such that $\psi_K^h = 0$ outside of K . For simplicity we introduce the global function $\psi^h \in W^{hp}$ such that $\psi^h = \sum_{K \in \mathcal{P}_h} \psi_K^h$. It follows that:

$$\eta_d^2 = \mathcal{R}^h(\psi^h) \leq \|\mathcal{R}^h\|_* \|\psi^h\|.$$

Let v_h be an arbitrary function of W^{hp} . Now, applying the Cauchy-Schwarz inequality, we observe that:

$$\begin{aligned} \|v_h\|^2 &= \sum_{K \in \mathcal{P}_h} \left\{ \|v_h\|_{H^1(K)}^2 + \|\nabla v_h \cdot \mu\|_{H^{-1/2}(\partial K)}^2 \right\} + \sigma \|\nabla v_h \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}^2 \\ &\leq \sum_{K \in \mathcal{P}_h} \left\{ \|v_h\|_{H^1(K)}^2 + \|\nabla v_h \cdot \mu\|_{H^{-1/2}(\partial K)}^2 \right\} \\ &\quad + 2\sigma \sum_{K \in \mathcal{P}_h} \left\{ \|\nabla v_h \cdot \mu^+\|_{L^2(\partial K \cap \Gamma_{\text{int}})}^2 + \|\nabla v_h \cdot \mu^-\|_{L^2(\partial K \cap \Gamma_{\text{int}})}^2 \right\} \\ &\leq \sum_{K \in \mathcal{P}_h} \left\{ \|v_h\|_{H^1(K)}^2 + \|\nabla v_h \cdot \mu\|_{L^2(\partial K)}^2 + 2\sigma \|\nabla v_h \cdot \mu\|_{L^2(\partial K \cap \Gamma_{\text{int}})}^2 \right\} \\ &= \sum_{K \in \mathcal{P}_h} \|v_h\|_{W_K}^2. \end{aligned}$$

Replacing v_h by ψ^h in the above inequality, we get:

$$\eta_d^2 \leq \|\mathcal{R}^h\|_* \|\psi^h\| \leq \|\mathcal{R}^h\|_* \sqrt{\sum_{K \in \mathcal{P}_h} \|\psi_K^h\|_{W_K}^2} \leq \eta_d \|\mathcal{R}^h\|_*$$

which allows to conclude the proof. ■

4 Numerical Results

4.1 One-Dimensional Experiments

We consider the one-dimensional version of (2.1):

$$\begin{aligned} -\frac{d^2u}{dx^2} + u &= 1, \quad \text{for } 0 < x < 1, \\ u(0) &= u(1) = 0. \end{aligned} \tag{4.1}$$

The exact solution of this problem is given by:

$$u(x) = 1 - \frac{e^x + e^{1-x}}{1 + e}. \tag{4.2}$$

In the following experiments, σ is set equal to one and the quality of a given error estimator η is measured in terms of the effectivity index, namely the ratio of η and $\|e\|$. We show in Fig. 4.1 the effectivity indices for η_c and η_d in the case of uniform approximation degrees $p = 1, 2, 3$ and 4, and for uniform mesh refinement. For comparison, we also show the effectivity index for $\|\phi_h\|$, that is the ratio $\|\phi_h\|/\|e\|$. We observe that the estimate η_c is very accurate for $p \geq 2$, with effectivity indices ranging between 89 and 100 percent. For $p = 1$ however, the estimator is less accurate as $\|e\|$ does not necessarily converge in this case. Nevertheless, the effectivity index remains bounded, as expected since $\|e\|$ and $\|\chi\|$ are equivalent and the fact that χ_h converges to χ .

The results for the decoupled estimator η_d show that the effectivity indices are still accurate despite the decoupling. They are within 1 or 2 percent range of the results for the coupled estimator η_c when $p \geq 2$. However, for $p = 1$, the decoupled estimator is not as accurate, certainly because once again the norm of the error does not converge to zero as h is decreased.

4.2 Two-Dimensional Experiments

For our 2D example problem we consider the following BVP, given on the unit square $\Omega = (0, 1) \times (0, 1)$ with prescribed Dirichlet boundary condi-

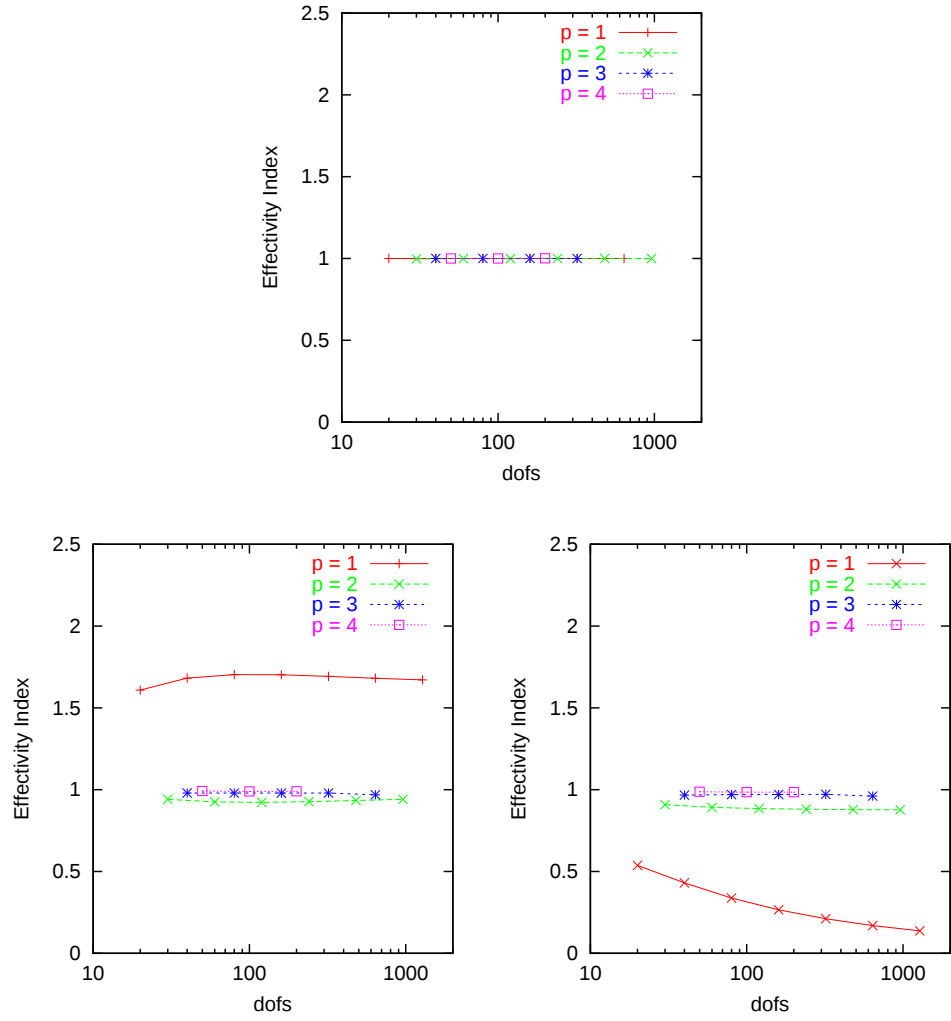


Figure 4.1: Effectivity indices versus the number of degrees of freedom for $\|\phi_h\|$ (top), for the coupled estimator η_c (bottom left) and the decoupled estimator η_d (bottom right) - 1D results.

tions on $\partial\Omega$:

$$\begin{aligned}
-\Delta u + u &= 0, & \text{in } \Omega, \\
u(0, y) &= 0, \quad u(1, y) = 0, & y \in [0, 1], \\
u(x, 0) &= 0, \quad u(x, 1) = \frac{1}{2} \sin(\pi x) \sinh \sqrt{1 + \pi^2}, & x \in [0, 1].
\end{aligned} \tag{4.3}$$

The exact solution to this problem is given by:

$$u(x, y) = \frac{1}{2} \sin(\pi x) \sinh(\sqrt{1 + \pi^2} y). \tag{4.4}$$

As before the parameters λ , ν , ζ and θ were set equal to zero, σ was set equal to one, and computations for uniform orders of approximation $p = 1, 2, 3$ and 4 were performed while uniformly refining the mesh. However, the calculation of the $H^{-1/2}(\partial K)$ -norm in two dimensions is a rather complicated task. Therefore, we choose here to approximate the norm $\|\cdot\|$ as follows:

$$\|v\|^2 \approx \sum_{K \in \mathcal{P}_h} \left\{ \|v\|_{H^1(K)}^2 + \|\nabla v \cdot \mu\|_{L^2(\partial K)}^2 \right\} + \sigma \|\nabla v \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}^2.$$

Figure 4.2 shows that the coupled estimator η_c is again very accurate in two-dimensional computations. Its effectivity index ranges between 85 and 110 percent for all orders of approximation. The results are actually very similar to the results obtained for $\|\phi_h\|$. On the other hand, for $p = 1$, we again observe an overshoot in the effectivity index, but the overshoot remains bounded and is far less important than is the case for the one-dimensional results. Finally, Figure 4.2 shows a dramatic loss in accuracy for the uncoupled estimator η_d in 2D. The effectivity indices of η_d ranges between 15 and 20 percent for $p = 1, 2$ and 4 and are slightly better for $p = 3$ with values converging around 40 percent.

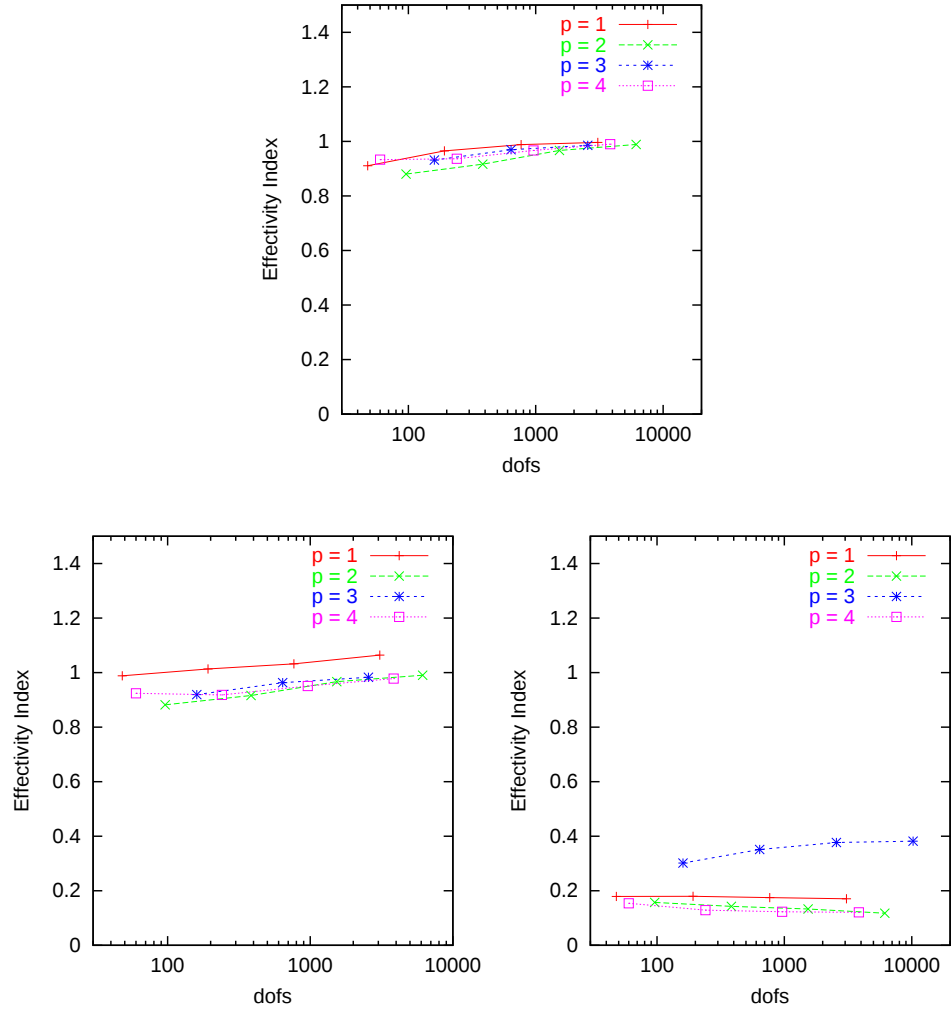


Figure 4.2: Effectivity indices versus the number of degrees of freedom for $\|\phi_h\|$ (top), for the coupled estimator η_c (bottom left) and the decoupled estimator η_d (bottom right) - 2D results.

5 Concluding Remarks

A posteriori global error estimates are derived for a Stabilized Discontinuous Galerkin Method (SDGM). By first proving an equivalence relation between the norm of the residual and the norm of the error, we compute the error estimates by estimating the norm of the residual functional. Two types of estimators are proposed: 1) *Coupled* estimator. This estimator involves the solution of a global problem on the entire domain due to the presence of coupling terms in the formulation. It is obviously expensive to compute; however it was investigated as an intermediate step between the approximation of the error problem and a fully decoupled error estimator. One- and two-dimensional numerical experiments show that this coupled estimator is still accurate (although certain terms have been eliminated from the error equation) and generally yields effectivity indices ranging between 85 and 110 percent. 2) *Decoupled* estimator. This estimator involves the solution of a set of local problems. One-dimensional results show good accuracy, with values of the effectivity index varying between 80 and 100 percent. However, in two dimensions the decoupling leads to a dramatic loss in accuracy with effectivity indices ranging between 15 and 40 percent.

More numerical experiments would be necessary to understand what is happening during the decoupling procedure in two dimensions. However, we believe that an accurate decoupled error estimator can be derived for the present version of the Discontinuous Galerkin method. An alternative approach would be to set up the local problems on patches of elements, in which case partial exchange of information would be allowed, following a subdomain-type residual method proposed for continuous finite element approximations (see *e.g.* [3, 5, 8]).

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Appendix A

Proof of Convergence Theorem for $\|\phi_h\|$

The existence and uniqueness of $\phi_h \in W^{hp}$ to (3.4) follows by application of the Fredholm alternative and remarking that the bilinear form $\mathcal{B}(\cdot, \cdot)$ is positive definite on $W^{hp} \times W^{hp}$. Now, if we define the approximation of u in the augmented finite dimensional space W^{hp} as u_h^* , i.e.:

$$\mathcal{B}(u_h^*, v_h) = \mathcal{L}(v_h), \quad \forall v_h \in W^{hp},$$

and substitute this equation into (3.4), then we observe by the linearity of the bilinear form that $\phi_h = u_h^* - u_h$. Therefore,

$$\|e - \phi_h\| = \|u - u_h^*\|.$$

We now recall the global interpolation operator $\Pi_{hp}(\cdot)$ that we introduced in our previous work [9]:

$$\Pi_{hp} : V \longrightarrow W^{hp}, \quad \Pi_{hp}(v) = \sum_{K \in \mathcal{P}_h} \pi_{hp}^K(u|_K), \quad (\text{A.1})$$

where the local interpolants $\pi_{hp}^K(\cdot)$ are defined as follows [1]:

$$\begin{aligned} \pi_{hp}^K : H^{r_K}(K) &\longrightarrow P^{p_K+1}(K), \\ \pi_{hp}^K(v_h) &= v_h, \quad \forall v_h \in P^{p_K+1}(K). \end{aligned}$$

Given the interpolant $\Pi_{hp}(u)$, we split $u - u_h^*$ such that $u - u_h^* = \eta - \xi$, where $\eta = u - \Pi_{hp}u$ and $\xi = u_h^* - \Pi_{hp}u$. Notice that $\xi \in W^{hp}$ and that the interpolation error $\eta \in H^2(\mathcal{P}_h)$. Thus, by using the triangle inequality, one obtains:

$$\|e - \phi_h\| \leq \|\eta\| + \|\xi\|. \quad (\text{A.2})$$

By applying the discrete Inf-Sup property of the bilinear form $\mathcal{B}(\cdot, \cdot)$ in Theorem 3.3.3 in [9], and by recalling that $\nu = \theta = 0$, we know there exists $C > 0$ such that:

$$\|\xi\| \leq C \frac{p^2}{h} \sup_{v_h \in W^{hp}/\{0\}} \frac{|\mathcal{B}(\xi, v_h)|}{\|v_h\|}. \quad (\text{A.3})$$

From (3.1) and (3.4), one easily observes that $e - \phi_h$ satisfies an orthogonality property on W^{hp} :

$$\mathcal{B}(e - \phi_h, v_h) = 0, \quad \forall v_h \in W^{hp}.$$

By employing this property, we can rewrite (A.3) as:

$$\|\xi\| \leq C \frac{p^2}{h} \sup_{v_h \in W^{hp}/\{0\}} \frac{|B(\eta, v_h)|}{\|v_h\|}.$$

If we apply the continuity property of $\mathcal{B}(\cdot, \cdot)$ and note that the continuity coefficient M is a constant due to our choice of values for ν and θ (see Section 3.2), then we can bound $\|\xi\|$ by the interpolation error as follows:

$$\|\xi\| \leq C(\sigma) \frac{p^2}{h} \|\eta\|.$$

Thus, returning to (A.2) we conclude that:

$$\|e - \phi_h\| \leq C(\sigma) \frac{p^2}{h} \|\eta\|.$$

By recalling the interpolation Theorem 4.2.2 in [9] and remarking that $\nu = \lambda = \theta = \zeta = 0$ and that the order of approximation in W^{hp} is $p + 1$, we get:

$$\|\eta\| \leq C(\sigma) \frac{h^{\mu-1/2}}{p^{r-3/2}} \sqrt{\sum_{K \in \mathcal{P}_h} \|u\|_{H^{r_K}(K)}^2}, \quad r_K \geq 2.$$

By combining the last two results we establish the assertion in Lemma 3.3.1.

Appendix B

Inf-Sup Property of Bilinear Form $\mathcal{A}(\cdot, \cdot)$

The proof of Lemma 3.3.2 is analogous to the proof of Theorem 3.3.2 in [9]. Given a $u \in V$, then we can construct a $\hat{u} \in$, such that:

$$\hat{u} = u + \beta \sum_{K \in \mathcal{P}_h} z_K, \quad (\text{B.1})$$

where $\beta \in \mathbb{R}$ and the local functions $z_K \in H^1(K) \cap V$ have the following properties:

$$\begin{aligned} \nabla z_K \cdot \mu &= \nabla u \cdot \mu, \quad \text{on } \partial K, \\ (z, v)_{1,K} &= \int_{\partial K} (\nabla u \cdot \mu) v \, ds, \\ \|z_K\|_{H^1(K)} &= \|\nabla u \cdot \mu\|_{H^{-1/2}(\partial K)}, \end{aligned} \quad (\text{B.2})$$

where $(\cdot, \cdot)_{1,K}$ denotes the $H^1(K)$ inner product. For more details on the proofs of these properties and the construction of $\hat{u}$, the reader is referred to [2, 7, 9]. Now, by the definition of the supremum we get:

$$\sup_{v \in V/\{0\}} \frac{|\mathcal{A}(u, v)|}{\|v\|} \geq \frac{|\mathcal{A}(u, \hat{u})|}{\|\hat{u}\|}. \quad (\text{B.3})$$

If we choose $\beta = -1/4$, for instance, then by simply applying the definition of $\mathcal{A}(\cdot, \cdot)$, we can expand the term in the nominator as follows:

$$\begin{aligned} \mathcal{A}(u, \hat{u}) &= \sum_{K \in \mathcal{P}_h} \left\{ \|u\|_{H^1(K)}^2 - \frac{1}{4} (u, z_K)_{1,K} \right. \\ &\quad \left. + \frac{1}{4} \int_{\partial K} \{z_K (\nabla u \cdot \mu) - (\nabla z_K \cdot \mu) u\} \, ds \right\} \\ &\quad + \|\nabla u \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}^2 - \frac{1}{4} \sigma \int_{\Gamma_{\text{int}}} [\nabla u \cdot \mathbf{n}] [\nabla z_K \cdot \mathbf{n}] \, ds. \end{aligned}$$

Substitution of the first two properties in (B.2), gives:

$$\begin{aligned} \mathcal{A}(u, \hat{u}) &= \sum_{K \in \mathcal{P}_h} \left\{ \|u\|_{H^1(K)}^2 - \frac{1}{2}(u, z_K)_{1,K} + \frac{1}{4}\|z_K\|_{H^1(K)}^2 \right\} \\ &\quad + \frac{3}{4} \|\nabla u \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}^2 \end{aligned} \quad (\text{B.4})$$

By recalling Young's Inequality:

$$(u, z_K)_{1,K} \leq \frac{\varepsilon}{2} \|u\|_{H^1(K)}^2 + \frac{1}{2\varepsilon} \|z_K\|_{H^1(K)}^2, \quad \varepsilon > 0,$$

and choosing $\varepsilon = 2$, we can rewrite (B.4) as:

$$\mathcal{A}(u, \hat{u}) \geq \sum_{K \in \mathcal{P}_h} \left\{ \frac{1}{2} \|u\|_{H^1(K)}^2 + \frac{1}{8} \|z_K\|_{H^1(K)}^2 \right\} + \frac{3}{4} \|\nabla u \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}^2$$

By substituting the third property in (B.2) into this expression, one obtains:

$$\mathcal{A}(u, \hat{u}) \geq \frac{1}{8} \|u\|^2. \quad (\text{B.5})$$

Conversely, Lemma 3.3.1 in [9], yields the following inequality for the denominator of (B.3):

$$\|\hat{u}\| \leq \sqrt{2} \|u\|. \quad (\text{B.6})$$

Finally, substitution of (B.5) and (B.6) into (B.3) yields:

$$\sup_{v \in V/\{0\}} \frac{|\mathcal{A}(u, v)|}{\|v\|} \geq \frac{1}{8\sqrt{2}} \|u\|.$$

■

Appendix C

Proof of Convergence Theorem for $\|\chi_h\|$

The existence and uniqueness of $\chi_h \in W^{hp}$ to (3.6) follows by application of the Fredholm alternative and remarking that the bilinear form $\mathcal{A}(\cdot, \cdot)$ is positive definite on $W^{hp} \times W^{hp}$. Now, to prove the lemma, we will need an discrete Inf-Sup condition of the bilinear form $\mathcal{A}(\cdot, \cdot)$ on $W^{hp} \times W^{hp}$.

Lemma C.1 *Let $\{\mathbf{F}_K\}$ define a family of affine invertible mappings. Then there exists a $\gamma_h = \gamma_h(h, p) > 0$, such that:*

$$\sup_{v_h \in V^{hp}/\{0\}} \frac{|\mathcal{A}(u_h, v_h)|}{\|v_h\|} \geq \gamma_h \|u_h\|, \quad \forall u_h \in V^{hp}/\{0\},$$

and $\gamma_h = C \frac{h}{p^2} > 0$.

Proof: By definition of the supremum, we get:

$$\sup_{v_h \in V^{hp}/\{0\}} \frac{|\mathcal{A}(u_h, v_h)|}{\|v_h\|} \geq \frac{\mathcal{A}(u_h, u_h)}{\|u_h\|}.$$

By applying Corollary 3.3.3 in [9], it is clear that there exists $C > 0$, such that

$$\begin{aligned} \mathcal{A}(u_h, u_h) \geq C \Bigg\{ \sum_{K \in \mathcal{P}_h} \left(\|u_h\|_{H^1(K)}^2 + \frac{h}{p^2} \|\nabla u_h \cdot \mathbf{n}\|_{H^{-1/2}(\partial K)}^2 \right) \\ + \sigma \|\nabla u_h \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}^2 \Bigg\}, \end{aligned}$$

which concludes the proof. ■

Given the interpolant $\Pi_{hp}(\cdot)$ in (A.1), we split $\chi - \chi_h$ such that $\chi - \chi_h = \eta - \xi$, where $\eta = \chi - \Pi_{hp}\chi$ and $\xi = \chi_h - \Pi_{hp}(\chi)$. Notice that $\xi \in W^{hp}$ and that

the interpolation error $\eta \in H^2(\mathcal{P}_h)$. Thus, by using the triangle inequality, one obtains:

$$\|\chi - \chi_h\| \leq \|\eta\| + \|\xi\|. \quad (\text{C.1})$$

By applying the discrete Inf-Sup property in Lemma C.1, we know there exists $C > 0$ such that:

$$\|\xi\| \leq C \frac{p^2}{h} \sup_{v_h \in W^{hp}/\{0\}} \frac{|\mathcal{A}(\xi, v_h)|}{\|v_h\|}. \quad (\text{C.2})$$

From (3.6) and (3.10), one observes that $\chi - \chi_h$ satisfies an orthogonality property on W^{hp} :

$$\mathcal{A}(\chi - \chi_h, v_h) = 0, \quad \forall v_h \in W^{hp}.$$

By employing this property, we can rewrite (C.2) as:

$$\|\xi\| \leq C \frac{p^2}{h} \sup_{v_h \in W^{hp}/\{0\}} \frac{|\mathcal{A}(\eta, v_h)|}{\|v_h\|}.$$

If we apply Lemma 3.3.3, then we can bound $\|\xi\|$ by the interpolation error as follows:

$$\|\xi\| \leq C(\sigma) \frac{p^2}{h} \|\eta\|.$$

Thus, returning to (C.1) we conclude that:

$$\|\chi - \chi_h\| \leq C(\sigma) \frac{p^2}{h} \|\eta\|.$$

By recalling the interpolation Theorem 4.2.2 in [9] and remarking that $\nu = \lambda = \theta = \zeta = 0$ and that the order of approximation in W^{hp} is $p + 1$, we get:

$$\|\eta\| \leq C(\sigma) \frac{h^{\mu-1/2}}{p^{r-3/2}} \sqrt{\sum_{K \in \mathcal{P}_h} \|\chi\|_{H^{r_K}(K)}^2}, \quad r_K \geq 2.$$

By combining the last two results we establish the assertion of Lemma 3.3.4.

Appendix D

Continuity Property of Bilinear Form $\mathcal{B}(\cdot, \cdot)$

Theorem D.1 *Let $\mathcal{B}(\cdot, \cdot)$ be the bilinear form as defined in (2.6). If $\sigma > 0$, then:*

$$\boxed{\exists M > 0 : \quad |\mathcal{B}(u, v)| \leq M \|u\| \|v\| \quad \forall u, v \in V}$$

where $M = \max \left\{ 3, \sqrt{\frac{p^\theta}{h^\nu}}, \sqrt{\frac{p^\zeta}{2\sigma h^\lambda}} \right\}.$

Proof: From the definition of $\mathcal{B}(\cdot, \cdot)$ and observing that for $u, v \in V$ we can write:

$$\int_{\Gamma_{\text{int}}} \langle v \rangle [\nabla u \cdot \mathbf{n}] ds = \frac{1}{2} \sum_{K \in \mathcal{P}_h} \int_{\partial K \cap \Gamma_{\text{int}}} v [\nabla u \cdot \mathbf{n}] ds, \quad (\text{D.1})$$

which yields:

$$\begin{aligned} \mathcal{B}(u, v) = & \sum_{K \in \mathcal{P}_h} \left\{ \int_K \{ \nabla u \cdot \nabla v + uv \} d\mathbf{x} \right. \\ & - \int_{\partial K} v (\nabla u \cdot \mu) ds + \int_{\partial K} u (\nabla v \cdot \mu) ds \\ & + \frac{1}{2} \int_{\partial K \cap \Gamma_{\text{int}}} v [\nabla u \cdot \mathbf{n}] ds - \frac{1}{2} \int_{\partial K \cap \Gamma_{\text{int}}} u [\nabla v \cdot \mathbf{n}] ds \left. \right\} \\ & + \sigma \frac{h^\lambda}{p^\zeta} \int_{\Gamma_{\text{int}}} [\nabla u \cdot \mathbf{n}] [\nabla v \cdot \mathbf{n}] ds. \end{aligned}$$

By applying the Schwarz Inequality and by using the definition of the $H^{1/2}(\partial K)$ norm (2.4), we can bound the above as follows:

$$\begin{aligned}
\mathcal{B}(u, v) &\leq \max \left\{ 1, \sqrt{\frac{p^\theta}{h^\nu}}, \sqrt{\frac{p^\zeta}{2\sigma h^\lambda}} \right\} \\
&\times \left\{ \sum_{K \in \mathcal{P}_h} \left(3\|u\|_{H^1(K)}^2 + \frac{h^\nu}{p^\theta} \|\nabla u \cdot \mu\|_{H^{-1/2}(\partial K)}^2 \right) \right. \\
&\quad \left. + 2\sigma \frac{h^\lambda}{p^\zeta} \|\nabla u \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}^2 \right\}^{1/2} \\
&\times \left\{ \sum_{K \in \mathcal{P}_h} \left(3\|v\|_{H^1(K)}^2 + \frac{h^\nu}{p^\theta} \|\nabla v \cdot \mu\|_{H^{-1/2}(\partial K)}^2 \right) \right. \\
&\quad \left. + 2\sigma \frac{h^\lambda}{p^\zeta} \|\nabla v \cdot \mathbf{n}\|_{L^2(\Gamma_{\text{int}})}^2 \right\}^{1/2},
\end{aligned}$$

which establishes the assertion. ■