

INTERIOR NUMERICAL APPROXIMATION OF BOUNDARY VALUE PROBLEMS WITH A DISTRIBUTIONAL DATA

IVO BABUŠKA AND VICTOR NISTOR

ABSTRACT. We study the approximation properties of a harmonic function $u \in H^{1-k}(\Omega)$, $k > 0$, on a relatively compact subset A of Ω , using the Generalized Finite Element Method (GFEM). If $\Omega = \mathcal{O}$, for a smooth, bounded domain $\mathcal{O}$, we obtain that the GFEM-approximation $u_S \in S$ of u satisfies $\|u - u_S\|_{H^1(A)} \leq Ch^\gamma \|u\|_{H^{1-k}(\mathcal{O})}$, where h is the typical size of the “elements” defining the GFEM-space S and $\gamma \geq 0$ is such that the local approximation spaces contain all polynomials of degree $k + \gamma$. The main technical ingredient is an extension of the classical super-approximation results of Nitsche and Schatz [20, 21]. In addition to the usual “energy” Sobolev spaces $H^1(\mathcal{O})$, we need also the duals of the Sobolev spaces $H^m(\mathcal{O})$, $m \in \mathbb{Z}_+$.

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INTRODUCTION

Let us consider the Neumann problem

$$(1) \quad \begin{cases} \Delta u = 0 & \text{on } \mathcal{O}, \\ \partial_\nu u = g \in H^{r-3/2}(\partial\mathcal{O}) & \text{on } \partial\mathcal{O}, \end{cases}$$

where $\mathcal{O}$ is a *smooth, bounded* open subset of $\mathbb{R}^n$, $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$, and ∂_ν is the directional derivative in the direction of the outer unit normal ν to $\partial\mathcal{O}$. In this paper, we are interested mainly in the case $r \leq 1$, $r \in \mathbb{Z}$, and we are looking for the approximation properties of the solution $u \in H^r(\mathcal{O})$ on suitable subsets of $\mathcal{O}$.

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For $v \in H^r(\mathcal{O})$, $r \in \mathbb{R}$, $r > 3/2$, the boundary values (or *traces*) $v|_{\partial\mathcal{O}}$ and $\partial_\nu v|_{\partial\mathcal{O}}$ are defined classically, because the restriction to the boundary extends by continuity to maps $H^r(\mathcal{O}) \ni v \rightarrow v|_{\partial\mathcal{O}} \in H^{r-1/2}(\partial\mathcal{O})$ and $H^r(\mathcal{O}) \ni v \rightarrow \partial_\nu v|_{\partial\mathcal{O}} \in H^{r-3/2}(\partial\mathcal{O})$, see [11, 31] for example. For $r \leq 3/2$, this is no longer true in general, but for $v = u$, we can take advantage of the fact that u satisfies an elliptic equation, so it is still possible to define $\partial_\nu u \in H^{r-3/2}(\partial\mathcal{O})$ [15] (see also [18, 28]). We can assume, without loss of generality, that $\mathcal{O}$ is connected, that is, that $\mathcal{O}$ is a *domain*.

It is then known [15, 26, 28] that a solution u of Equation (1) exists for any g such that $\langle g, 1 \rangle_{\partial\mathcal{O}} = 0$ and that this solution satisfies

$$(2) \quad \|u\|_{H^r(\mathcal{O})} \leq C \|g\|_{H^{r-3/2}(\partial\mathcal{O})},$$

with a constant C that may depend on r , but is independent of g . (Here $\langle g, 1 \rangle_{\partial\mathcal{O}}$ is the value of the distribution g on the function constant equal to 1, thus $\langle g, 1 \rangle_{\partial\mathcal{O}} = \int_{\partial\mathcal{O}} g(x) dS(x)$ if g is a function.) In particular, to solve the boundary value problem (1), it is enough to do that for $g_n \in C^\infty(\partial\mathcal{O})$, $g_n \rightarrow g \in H^{r-3/2}(\partial\mathcal{O})$. The estimate of Equation (2) will be discussed in detail in [6], where more references will be given. In that paper, the case $\Delta u \neq 0$ will also be considered. In this paper, however, we avoid all together the issue of defining the traces $u|_{\partial\mathcal{O}}$ by introducing in Section 4 a variational formulation of the boundary value problem (1). This also leads to a quick derivation of Equation (2). See also [18].

The consideration of the case when $g \in H^{r-3/2}(\mathcal{O})$, $r \leq 3/2$, when g is a distribution rather than a function, is important in order to be able to handle the case of “concentrated moments” and “concentrated loads,” see for example [27]. The concentrated loads and moments are distributions concentrated at one point, so they are obtained by taking derivatives of δ -distributions. See Example 1.4 and below for the definition of the delta distributions and of their derivatives. Most of our results work without any reference to boundary conditions. However, the definition of the numerical (or discrete solution) is more difficult for the Dirichlet problem, so we do not address this problem explicitly in this paper.

Let $\Omega \subset \mathbb{R}^n$ denote a bounded, connected, open subset of $\mathbb{R}^n$ (*i.e.*, Ω will be a bounded domain). We do not assume that Ω is smooth, unless explicitly mentioned (if Ω is assumed to be smooth, then we shall use the notation $\mathcal{O}$ instead of Ω). Recall that $A \Subset B$ means that A is bounded, is contained in the interior of B and ∂A and ∂B are disjoint (*i.e.*, A is a *relatively compact* subset of B). If $A \Subset \Omega$ is an open subset, then the solution u of Equation (1) will be smooth on A for any r and

$$(3) \quad \|u\|_{H^m(A)} \leq C \|u\|_{H^r(\Omega)},$$

with a constant C that depends on A , Ω , r , and m , but is independent of u satisfying $\Delta u = 0$. An important problem, with potential practical applications, is to approximate on A the solution u of Equation (1).

In this paper, we prove several results on the approximation of the solution u on subsets $A \Subset \mathcal{O}$ using the Generalized Finite Element Method. Let $S_\nu \subset H^m(\mathcal{O})$, $m \geq k+1$, be a sequence of Generalized Finite Element Spaces associated to a sequence $\Sigma_\nu = \{\omega_j^\nu, \phi_j^\nu, \Psi_j^\nu, \omega_j^{*\nu}\}_{j=1}^{N_\nu}$, $\nu \in \mathbb{Z}_+$, of GFEM-data with typical size h_ν satisfying the assumptions of Subsection 2.3 (so, in particular, $h_\nu \rightarrow 0$). Then Theorem 5.8 gives that the sequence of GFEM-approximations $u_\nu := u_{S_\nu} \in S_\nu$ of the solution u of the boundary value problem (1) satisfies

$$(4) \quad \|u - u_\nu\|_{H^1(A)} \leq Ch_\nu^\gamma \|u\|_{H^{1-k}(\mathcal{O})},$$

provided that our local approximation spaces contain all polynomials of degree $k + \gamma$, $\gamma \geq 0$.

The idea of considering partitions of unity and of the Generalized Finite Element Method was introduced by Babuška, Caloz, and Osborn [3]. It was further developed in [2, 5, 16], and [4]. The Generalized Finite Element Method is used today in Engineering under various names, such as: the method of “clouds,” the method of “finite spheres,” the “X-finite element method,” and others. The Generalized Finite Element Method is a generalization of the mesh free (meshless) methods which use as a paradigm the idea of partition of unity introduced in [3, 5, 14] and [16]. See [2] and [14] for further references.

We stress that our results require not just the energy Sobolev space H^1 , but also negative order Sobolev spaces H^{-l} , defined in this paper as the duals of H^l , $l \in \mathbb{Z}_+$. One of the main reasons for the need to consider the negative order Sobolev spaces is that the solution u is in $H^{1-k}(\Omega)$, and not in $H^{1-k}(\Omega)$, in general. Moreover, even if we approximate the boundary data g and the solution u with functions in H^1 , then it will still be important to use the norm on a negative order Sobolev space in the estimate of the error.

Here is now a brief description of the contents of the paper. *We continue to assume that Ω is a bounded domain, but we do not assume that Ω is smooth, except when explicitly mentioned. A domain that is assumed to be smooth will be usually denoted by $\mathcal{O}$.* In Section 1, we set up the notation and we establish our conventions on Sobolev spaces. Section 2 contains a quick review of the necessary definitions involving the Generalized Finite Element Method (GFEM). Our main results are statements on a sequence S_ν of GFEM-spaces satisfying the assumptions of Subsection 2.3. Our assumptions are formulated in terms of four general conditions (Conditions A(h), B, C, and D), formulated in Subsection 2.2. The spaces S_ν will contain the sequence $u_\nu \in S_\nu$ of approximations of the solution u to our boundary value problem (Equation (1)). The following section, Section 3, contains the calculations necessary to establish our interior estimates for the sequence $u_\nu \in S_\nu$, $\nu \in \mathbb{Z}_+$. Our approach follows, to a certain extend, that in the article of Nitsche and Schatz [21], relying also on Wahlbin’s survey article [33]. The main differences between the approach in Section 3 of our paper and the approach in [21, 33] to interior estimates are due mostly to the fact that several assumptions from those papers are not fully satisfied in our approach. As in those articles, the main step is a superapproximation property, Proposition 3.7. The proof in [21, 33] cannot be used to obtain Proposition 3.7 because the property “ $\partial^\alpha w = 0$ if $|\alpha|$ is large,” is not satisfied in general for $w \in S_\nu$. For the results of Sections 4 and 5, we assume that $\Omega = \mathcal{O}$ is smooth. In Section 4, we introduce a weak formulation of the Neumann problem (1) and the Galerkin approximation (or GFEM-approximation) $u_S \in S$ of the solution u . This is based on an extension $\tilde{B}$ of the form $B(w, v) := \int_{\mathcal{O}} \nabla u(x) \cdot \nabla v(x) dx$ to the case when $v \in H^{k-1}(\mathcal{O})$ is arbitrary and $u \in H^{1-k}(\mathcal{O})$ can be written as $u = u_1 + u_2$, where $\Delta u_1 = 0$ in distribution sense and $u_2 \in H^1(\mathcal{O})$. We also establish a well posedness result for (1) by establishing that the extension $\tilde{B}$ satisfies the Babuška–Brezzi condition. In Section 5 we exploit the definition and properties of the Galerkin approximations $u_\nu := u_{S_\nu} \in S_\nu$ and of the form $\tilde{B}$. Several estimates for $u \in H^{1-k}(\mathcal{O})$ and its approximations u_ν are established in this section, including the main theorem, Theorem 5.8 (whose main conclusion was summarized in Equation (4) above). The last section, Section 6 contains, in particular, a proof

that, for a domain Ω with piecewise C^1 -boundary, we can construct a family of partitions of unity S_ν with typical size of supports $h_\nu \rightarrow 0$ that satisfies the assumptions of our main results (*i.e.*, the assumptions of Subsection 2.3) for a fixed choice of the structural constants (*i.e.*, of A , C_j , σ , κ , λ , and m). For this construction, we assume that the local approximation spaces are $\Psi_j = \mathcal{Q}_\lambda$, the space of polynomials of degree at most λ . By contrast, it is not possible to find a family of partitions of unity as above for domains with cusps, see Remark 6.7. For suitable $g \in H^{-1/2-k}(\partial\mathcal{O})$, we plan to perform some concrete numerical simulations in a future paper [7], where we shall also give more examples of domains and partitions of unity satisfying the Assumption (iv) of Subsection 2.3.

We shall write $x := y$ if x is *defined* by y . By $\hat{C}$ we shall denote a constant that may depend only on the dimension n ($\mathcal{O} \subset \mathbb{R}^n$). By A , C_j , σ , κ , λ , and m , we shall denote the so called “structural constants” introduced in Conditions A–D (Subsection 2.2). The structural constants will remain fixed throughout our discussion. By contrast, C will denote a generic constant that may depend only on the structural constants (and, occasionally, on subsets $A, A', \dots \subset \Omega$, when explicitly mentioned).

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1. PRELIMINARIES

We begin by fixing the notation and terminology. We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{C} := \{a + bi, a, b \in \mathbb{R}\}$ the set of complex numbers. Also, $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$. By $L^2(\Omega)$ we shall denote the space of bounded, measurable functions $f : \Omega \rightarrow \mathbb{C}$ with norm $\|f\|_{L^2(\Omega)}^2 := \int_\Omega |f(x)|^2 dx$. (As usual, we identify two functions that coincide outside a set of Lebesgue measure zero.)

We shall use the multi-index notation for partial derivatives. Namely, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ will denote a multi-index and $\partial^\alpha u := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} u$, as usual. Also, $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n$, is the order of the partial derivative ∂^α .

From now on we shall assume that Ω is a domain, that is, an open, connected subset of $\mathbb{R}^n$.

1.1. Sobolev spaces. We now introduce the *integral Sobolev spaces* on Ω . This is enough for our purposes. Let $s \in \mathbb{Z}_+$. Then $H^s(\Omega)$ is the space of functions $f \in L^2(\Omega)$ such that

$$(5) \quad \|f\|_s^2 = \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(\Omega)}^2 < \infty.$$

The space $H_0^s(\Omega)$ is defined as the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H^s(\Omega)$.

We define the negative order Sobolev spaces by duality. Namely, $H^{-s}(\Omega) := H^s(\Omega)^*$, the dual of $H^s(\Omega)$, $s \in \mathbb{N}$. We shall denote by $\langle w, \phi \rangle = w(\phi)$ the value of the linear functional $w \in H^{-s}(\Omega)$ on $\phi \in H^s(\Omega)$. Then the norm on $H^{-s}(\Omega)$ is given by

$$\|w\|_{H^{-s}(\Omega)} = \sup_{\phi} \frac{|\langle w, \phi \rangle|}{\|\phi\|_{H^s(\Omega)}}, \quad 0 \neq \phi \in H^s(\Omega).$$

Our definition of negative order Sobolev spaces by duality follows [10, 22, 25], for example. Note, however, that the negative order Sobolev spaces are often

also defined by restriction from $\mathbb{R}^n$, as in [11, 15, 31], for example. The space of restrictions to Ω of distributions in $H^{-s}(\mathbb{R}^n)$ is the dual of $H_0^s(\Omega)$, and will be denoted $H_0^{-s}(\Omega)$. The spaces $H_0^{-s}(\Omega) = H_0^s(\Omega)^*$, $s \geq 0$, will also be used below. When $\Omega = \mathbb{R}^n$, these two approaches yield the same spaces, but for general Ω they may lead to different “negative order” Sobolev spaces. The spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, can also be defined using the Fourier transform.

1.2. Distributions. Since $\mathcal{C}_c^\infty(\Omega) \subset H^s(\Omega)$ for any $s \in \mathbb{Z}_+$, we obtain that every $w \in H^{-s}(\Omega)$, $s \in \mathbb{Z}_+$, defines a distribution on Ω . For $s < 0$, the spaces $H^s(\Omega)$ consist, in general, of distributions and not of functions. We shall not use any nontrivial results on distributions, but we shall use the terminology related to distributions, so we now review a few needed definitions. Let $B_R(0)$ denote the open ball of radius R centered at the origin. Also, let $\mathcal{C}_c^\infty(\mathbb{R}^n)$ be the set of infinitely differentiable, complex valued functions that vanish outside a ball $B_R(0)$, for some large $R > 0$. The elements of this space are sometimes called *test functions*. A linear map $u : \mathcal{C}_c^\infty(\mathbb{R}^n) \rightarrow \mathbb{C}$ is called a *distribution on $\mathbb{R}^n$* [12, 13, 31] if, for any $R > 0$, there exists $m \in \mathbb{Z}_+$ and $C > 0$ such that

$$(6) \quad |u(\phi)| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \phi\|_{L^2}, \quad \text{if } \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ and } \phi = 0 \text{ outside } B_R(0).$$

This definition does not exclude the case when larger and larger values of m and C have to be chosen as $R \rightarrow \infty$, and in fact this situation actually occurs in specific examples. The set of distributions on $\mathbb{R}^n$ will be denoted $\mathcal{D}'(\mathbb{R}^n)$.

We now fix more notation and terminology. If f is a function, then the closure of the set $\{f \neq 0\}$ is called the *support of f* and will be denoted $\text{supp}(f)$. Therefore, any $\phi \in \mathcal{C}_c^\infty(\Omega)$ has compact support. We shall also write $\langle u, \phi \rangle := u(\phi)$ for the value of the distribution u on the function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. The *support of a distribution u* is the smallest closed set F such that $\langle u, \phi \rangle = 0$ for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus F)$.

Here are some examples of distributions and constructions based on distributions that are useful below.

Example 1.1. If f is a measurable function on $\mathbb{R}^n$ that is integrable on any closed ball in $\mathbb{R}^n$ (i.e., it is *locally integrable*, or $f \in L_{\text{loc}}^1(\mathbb{R}^n)$), then we can define

$$(7) \quad \langle f, \phi \rangle := \int_{\mathbb{R}^n} f(x)\phi(x)dx,$$

for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Thus any $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ defines a distribution on $\mathbb{R}^n$, that is, $L_{\text{loc}}^1(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.

The derivatives of distributions are defined by duality.

Example 1.2. The derivatives $\partial^\alpha u$ of a distribution u are defined by

$$\langle \partial^\alpha u, \phi \rangle := (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle.$$

1.3. Sobolev spaces on the boundary. We shall also need the definition of the spaces $H^{m+1/2}(\partial\mathcal{O})$ for $\mathcal{O}$ a smooth, bounded domain and $m \in \mathbb{Z}_+$. Then $H^{m+1/2}(\partial\mathcal{O})$ consists of the restrictions to $\partial\mathcal{O}$ of the functions in $H^{m+1}(\mathcal{O})$ with norm

$$(8) \quad \|v\|_{H^{m+1/2}(\partial\mathcal{O})} = \|w\|_{H^{m+1}(\mathcal{O})},$$

where $w \in H^{m+1}(\mathcal{O})$ is the unique solution of $\Delta w = 0$ and $w|_{\partial\mathcal{O}} = v$ ($m \in \mathbb{Z}_+$). An equivalent norm is given by $\inf \|u\|_{H^{m+1}(\mathcal{O})}$, where $u \in H^{m+1}(\mathcal{O})$ satisfies $u = v$ on $\partial\mathcal{O}$. If $r = m + 1/2$, $m \in \mathbb{Z}_+$, the space $H^{-r}(\partial\mathcal{O})$ is defined as the dual of $H^r(\partial\mathcal{O})$. We shall denote by $\langle v, \phi \rangle_{\partial\mathcal{O}} = v(\phi)$, the value of $v \in H^{-r}(\partial\mathcal{O})$ on $\phi \in H^r(\partial\mathcal{O})$.

Example 1.3. If $n = 2$, then each connected component of $\partial\mathcal{O}$ is diffeomorphic to S^1 , the unit circle. It is therefore enough to define $H^r(S^1)$ in this case, which has a more concrete description. Let $f \in \mathcal{C}^\infty(S^1)$. Then, up to a multiplicative constant

$$(9) \quad \|f\|_{H^r(S^1)}^2 = \pi \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 (1 + |n|)^{2r},$$

where $2\pi \hat{f}(n) = \int_0^1 f(e^{2\pi i \theta}) e^{-2\pi i n \theta} d\theta$, $n \in \mathbb{Z}$, are the Fourier coefficients of f . Then $H^r(S^1)$ is the closure of $\mathcal{C}^\infty(S^1)$ in the norm $\|f\|_{H^r(S^1)}$.

The following example is relevant for the discussion of concentrated loads and moments.

Example 1.4. The *Dirac measure* (or distribution) concentrated at $a \in \partial\mathcal{O}$ is the distribution δ_a defined by

$$\langle \delta_a, \phi \rangle_{\partial\mathcal{O}} := \phi(a), \quad \phi \in \mathcal{C}^\infty(\partial\mathcal{O}).$$

An explicit calculation shows that $\delta_a \in H^{-(n-1)/2-\epsilon}(\partial\mathcal{O})$ and the resulting norms behave as $\|\delta_a\|_{H^{-(n-1)/2-\epsilon}} \approx \epsilon^{-1/2} \rightarrow \infty$ as $\epsilon \rightarrow 0$.

In Section 4, it will be convenient to use another definition of fractional Sobolev spaces on $\partial\mathcal{O}$, which yields a more suitable form of the inner product. Namely, let Δ be the Laplace operator on $\partial\mathcal{O}$. Recall that Δ is defined by

$$-(\Delta v, v) = \int_{\partial\mathcal{O}} |\nabla v|^2 dS(x).$$

Then let $u_j \in L^2(\partial\mathcal{O})$, $j \in \mathbb{Z}_+$, be an orthonormal basis of $L^2(\partial\mathcal{O})$ consisting of eigenfunctions of $-\Delta$, that is, $-\Delta u_j = \lambda_j u_j$. We can assume that $\lambda_j \leq \lambda_{j+1}$ for all $j \in \mathbb{Z}_+$. In particular, it follows that $\lambda_0 = 0$ and that u_0 is constant on each connected component of $\partial\mathcal{O}$. We then define $H^{s/2}(\partial\mathcal{O})$ to be the set of those functions $u = \sum_j a_j u_j \in L^2(\partial\mathcal{O})$ for which

$$\|u\|_{H^{s/2}(\partial\mathcal{O})}^2 := \sum_j (1 + \lambda_j)^s |a_j|^2 < \infty.$$

If also $v = \sum_j b_j u_j \in H^{s/2}(\partial\mathcal{O})$, then the inner product in $H^{s/2}(\partial\mathcal{O})$ is given by

$$(u, v)_{H^{s/2}(\partial\mathcal{O})} = \sum_j (1 + \lambda_j)^s a_j \bar{b}_j.$$

The advantage of this formula is that it is immediately seen that

$$(10) \quad (u, 1)_{H^{s/2}(\partial\mathcal{O})} = \int_{\partial\mathcal{O}} u(x) dS(x),$$

for all $s \geq 0$. For $s < 0$, the same conclusion follows by duality.

2. THE GENERALIZED FINITE ELEMENT METHOD

We now recall a few basic facts about the Generalized Finite Element Method [2, 5, 16]. This method is quite convenient when one needs finite element spaces with high regularity. Most of the results of this section work for a general bounded open set Ω , except the application to the boundary value problems, Section 4, in which case we shall need to assume that Ω has a smooth boundary.

2.1. Basic facts. Let $k \in \mathbb{Z}_+$. We shall denote as usual

$$|u|_{W^{k,\infty}(\Omega)} := \max_{|\alpha|=k} \|\partial^\alpha u\|_{L^\infty(\Omega)}, \quad \|u\|_{W^{k,\infty}(\Omega)} := \max_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)},$$

$W^{k,\infty}(\Omega) := \{u, \|u\|_{W^{k,\infty}(\Omega)} < \infty\}$, and $\|\nabla \omega\|_{W^{k,\infty}(\Omega)} := \sum_j \|\partial_j \omega\|_{W^{k,\infty}(\Omega)}$. In particular, $|u|_{W^{0,\infty}(\Omega)} = \|u\|_{W^{0,\infty}(\Omega)} = \|u\|_{L^\infty(\Omega)}$.

We shall need the following slight generalization of a definition from [5, 16]:

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and $\{\omega_j\}_{j=1}^N$ be an open cover of Ω such that any $x \in \Omega$ belongs to at most κ of the sets ω_j . Also, let $\{\phi_j\}$ be a partition of unity consisting of $W^{m,\infty}(\Omega)$ functions and subordinated to the covering $\{\omega_j\}$ (i.e., $\text{supp } \phi_j \subset \overline{\omega_j}$). If

$$(11) \quad \|\partial^\alpha \phi_j\|_{L^\infty(\Omega)} \leq C_k / (\text{diam } \omega_j)^k, \quad k = |\alpha| \leq m,$$

for any $j = 1, \dots, N$, then $\{\phi_j\}$ is called a $(\kappa, C_0, C_1, \dots, C_m)$ *partition of unity*.

Assume also that we are given linear subspaces $\Psi_j \in H^m(\omega_j)$, $j = 1, 2, \dots, N$. The spaces Ψ_j will be called *local approximation spaces* and will be used to define the space

$$(12) \quad S = S_{GFEM} := \left\{ \sum_{j=1}^N \phi_j v_j, v_j \in \Psi_j \right\} \subset H^m(\cup \omega_j),$$

which will be called the *GFEM-space*. The set $\{\omega_j, \phi_j, \Psi_j\}$ will be called the *set of data defining the GFEM-space* S .

A basic approximation property of the GFEM-spaces is the following Theorem from [5].

Theorem 2.2 (Babuška-Melenk). *We shall use the notations and definitions of Definition 2.1 and after. Let $\{\phi_j\}$ be a (κ, C_0, C_1) partition of unity. Also, let $v_j \in \Psi_j \subset H^1(\omega_j)$, $u_{ap} := \sum_j \phi_j v_j \in S$, and $d_j = \text{diam } \omega_j$, the diameter of ω_j . Then*

$$(13) \quad \begin{aligned} \|u - u_{ap}\|_{L^2(\Omega)}^2 &\leq \kappa C_0^2 \sum_j \|u - v_j\|_{L^2(\omega_j)}^2 \quad \text{and} \\ \|\nabla(u - u_{ap})\|_{L^2(\Omega)}^2 &\leq 2\kappa \sum_j \left(\frac{C_1^2 \|u - v_j\|_{L^2(\omega_j)}^2}{(d_j)^2} + C_0^2 \|\nabla(u - v_j)\|_{L^2(\omega_j)}^2 \right). \end{aligned}$$

2.2. Conditions on GFEM data. Our main results will involve a sequence of GFEM-spaces S_ν . Let $\{\omega_j, \phi_j, \Psi_j\}_{j=1}^N$ be a *single*, fixed data defining a GFEM-space S , as in the previous subsection, and let $\Sigma := \{\omega_j, \phi_j, \Psi_j, \omega_j^*\}$, where $\omega_j^* \subset \omega_j$. We now introduce some conditions on Σ that will be used in the next subsection to formulate our assumptions on the *sequence* $\Sigma_\nu = \{\omega_j^\nu, \phi_j^\nu, \Psi_j^\nu, \omega_j^{*\nu}\}_{j=1}^{N_\nu}$ defining GFEM-spaces S_ν , $\nu \in \mathbb{Z}_+$, studied in this paper.

Recall that ω is *star-shaped* with respect to $\omega^* \subset \omega$ if for every $x \in \omega$ and every $y \in \omega^*$, the segment with end points x and y is completely contained in ω .

Condition A(h). We have that $\Omega = \cup_{j=1}^N \omega_j$ and for each $j = 1, 2, \dots, N$, the set ω_j is open of diameter $d_j \leq h \leq 1$ and $\omega_j^* \subset \omega_j$ is an open ball of diameter $\geq \sigma h$ such that ω_j is star-shaped with respect to ω_j^* .

Condition B. The family $\{\phi_j\}_{j=1}^N$ is a $(\kappa, C_0, C_1, \dots, C_m)$ partition of unity.

Condition C. For each $j = 1, 2, \dots, N$, the space Ψ_j contains all polynomials of degree λ and

$$(14) \quad \|w\|_{H^l(\omega_j)} \leq A \|w\|_{H^l(\omega^*)}$$

for any $w \in \Psi_j$, any $0 \leq l \leq m$, and any ball $\omega^* \subset \omega_j$ of diameter $\geq \sigma h$.

We shall need to use “admissible” subsets of Ω , a class of subsets that we now define.

Definition 2.3. Let $\Sigma = \{\omega_j, \phi_j, \Psi_j, \omega_j^*\}_{j=1}^N$ be as above and $U \subset \Omega$ be an open subset. Denote by $J(U)$ the set of those indices j such that $\omega_j^* \subset U$. We shall say that U is *admissible* for Σ if, for all $j = 1, \dots, N$,

- (i) $\phi_j = 1$ on ω_j^* and
- (ii) $\sum_{j \in J(U)} \phi_j = 1$ on U .

We can now formulate our last condition.

Condition D. The domain Ω is admissible for the set $\Sigma = \{\omega_j, \phi_j, \Psi_j, \omega_j^*\}_{j=1}^N$ defining the GFEM-space S .

Let us notice that, in view of our previous conditions, Condition D amounts to the fact that $\phi_j = 1$ on ω_j^* for all $j = 1, \dots, N$.

2.3. Assumptions. We are now ready to formulate our assumptions.

- (i) We assume that we are given a sequence $\Sigma_\nu = \{\omega_j^\nu, \phi_j^\nu, \Psi_j^\nu, \omega_j^{*\nu}\}_{j=1}^{N_\nu}$, $\nu \in \mathbb{Z}_+$, defining GFEM-spaces S_ν .
- (ii) There exists constants A , C_j , σ , κ , λ , and m and a sequence $h_\nu \rightarrow 0$, as $\nu \rightarrow \infty$, such that Σ_ν satisfies Conditions A(h_ν), B, C, and D for each $\nu \in \mathbb{Z}_+$.

The constants A , C_j , σ , κ , λ , and m will be called *structural constants*. Note that we must have $N_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$. For simplicity, we shall say below that the sequence Σ_ν satisfies the Conditions A–D instead of saying that it satisfies the conditions A(h_ν), B, C, and D.

Let us recall now the following standard lemma.

Lemma 2.4. Let ψ_j be measurable functions defined on an open set W . Assume that there exists an integer κ such that a point $x \in W$ can belong to no more than κ of the sets $\text{supp}(\psi_j)$. Let $f = \sum_j \psi_j$. Then there exists a constant $C > 0$, depending only on κ , such that $\|f\|_{H^1(W)}^2 \leq C \sum_j \|\psi_j\|_{H^1(W)}^2$.

Proof. The inequality

$$(15) \quad |a_1 + a_2 + \dots + a_M|^2 \leq M(|a_1|^2 + |a_2|^2 + \dots + |a_M|^2),$$

with $\kappa \leq M$, gives the desired result. $\square$

We can now establish the following proposition.

Proposition 2.5. *Under the assumptions of this subsection, we have that there exists a constant $B > 0$ such that*

$$(16) \quad |w|_{H^t(U)} \leq B h_\nu^{-t} \|w\|_{L^2(U)},$$

for all $0 \leq t \leq m$, $\nu \in \mathbb{Z}_+$, $w \in S_\nu$, and $U \subset \Omega$ admissible for Σ_ν . The constant B may depend only on the structural constants A , C_j , κ , λ , σ , and m (in particular, it is independent of ν , u , and the admissible set U).

Proof. Let us denote $h = h_\nu$, for simplicity. Let $w \in S$. Since U is admissible, we have that $w = \sum \phi_j w_j$ on U , with $w_j \in \Psi_j$, the sum being taken over all j such that $\omega_j^* \subset U$. Then Lemma 2.4 and Assumptions A–D give

$$\begin{aligned} |w|_{H^t(U)}^2 &\leq C \sum_j \sum_{l=0}^t |\phi_j|_{W^{l,\infty}(\omega_j)}^2 |w_j|_{H^{t-l}(\omega_j)}^2 \\ &\leq C \sum_j \sum_{l=0}^t C_l^2 h^{-2l} B^2 h^{2l-2t} |w_j|_{L^2(\omega_j)}^2 \leq C A^2 h^{-2t} \sum_j |w_j|_{L^2(\omega_j^*)}^2 \\ &\leq C h^{-2t} \|w\|_{L^2(U)}^2, \end{aligned}$$

where all the structural constants C above depend only on the structural constants A , C_j , σ , κ , λ , and m . This proves the result. $\square$

We obtained right away that, under the same assumptions as those in the above proposition, that

$$\|w\|_{H^t(U)} \leq B h_\nu^{-t} \|w\|_{L^2(U)}, \quad w \in S.$$

In fact, this equation is equivalent to the proposition. Recall that the constant m is the fixed integer appearing in Assumptions A–D.

2.4. Remarks. We now include a few simple remarks that will help clarify the above conditions and assumptions.

Remark 2.6. Condition A(h) implies that the diameters of ω_j are comparable with h . Indeed, $d_j := \text{diam}(\omega_j) \geq \sigma h$.

Remark 2.7. The explicit inequalities implied by Conditions A and B are

$$(17) \quad \|\phi_j\|_{W^{l,\infty}(\Omega)} \leq C_l h^{-l}, \quad l = 0, 1, \dots, m,$$

by Definition 2.1.

Remark 2.8. Condition C is satisfied, for example, if $\Psi_j = \mathcal{Q}_\lambda$, the set of polynomials of degree at most $\lambda \geq 1$, see Section 6.

Remark 2.9. A typical example of an admissible set is obtained as follows. Fix a subset J of indices j and let G be the set of points where $\sum_{j \in J} \phi_j = 1$. Then the interior of G is an admissible open subset of Ω .

Remark 2.10. If $U \subset \Omega$ is an admissible open set, then the sets $\omega_j^* \subset U$, must be disjoint. Moreover, the boundary of U cannot be arbitrary, see Remark 6.7.

Remark 2.11. Let, for each fixed j , $\{w_{ji}\}$ be a basis of Ψ_j . Assume that Ω is admissible. Then $\{\phi_j w_{ji}\}$ is a basis of the GFEM-space S . This is an important, non-trivial consequence that is not always satisfied, see for example [29, 30].

Finally, let $\mathcal{F}$ be a family of open subsets of Ω . We shall say that $\mathcal{F}$ satisfies the ν -chain condition if for any $\emptyset \neq A_0 \in \mathcal{F} \subset \Omega$, there exist $A_0 \in B_1 \in B_2 \in \dots \in B_{\nu-1} \in \mathcal{F}$, with all $B_j \in \mathcal{F}$. The family of subsets of Ω admissible for Σ_ν satisfies the ν -chain condition for h_ν small, see Lemma 6.5 (the proof of this fact requires that Ω be admissible for all Σ_ν , namely Condition D). It is likely that all our approximation results remain true if one replaces Condition D with the conditions that the family of subsets of Ω admissible for Σ_ν satisfies the ν -chain condition for h_ν small enough.

3. INTERIOR ESTIMATES FOR THE GFEM

From now on, we shall assume that $\Sigma_\nu = \{\omega_j^\nu, \phi_j^\nu, \Psi_j^\nu, \omega_j^{*\nu}\}$, $\nu \in \mathbb{N}$, is a sequence satisfying all assumptions formulated in Subsection 2.3. Also, S_ν will be the resulting Generalized Finite Element Space of Equation (12) associated to the data and $h_\nu \rightarrow 0$ will be the corresponding “small parameters.” In this section, the set Ω will be a bounded, connected open subset of $\mathbb{R}^n$ (i.e., Ω will be a bounded domain). We shall not require in this section that Ω have a smooth boundary.

In the following, we shall occasionally drop the index ν , in order not to overburden the notation. For instance, we shall denote $S = S_\nu$, $h = h_\nu$, $\omega_j = \omega_j^\nu$, and so on, when the index ν is understood. Also, recall that the structural constants A , C_j , σ , κ , λ , and m appearing in Conditions A–D, except $h = h_\nu$, will be fixed in what follows.

3.1. H^s -approximation. We shall need a basic result on the approximation of functions in $H^s(\Omega)$ with elements in the GFEM-space S , extending Theorem 2.2. Only the case $s = 1$ will be needed in this section, but later on we shall also need the general case.

For the following result, we shall need the well known Bramble–Hilbert lemma in the form given in [9]. Let us recall this basic result in the form that we need below, as well as the relevant definitions. Let $\omega \subset \mathbb{R}^n$ be a bounded, open set and let

$$\rho_{\max}(\omega) = \sup\{\rho, \Omega \text{ is star-shaped with respect to a ball } B \subset \omega \text{ with radius } \rho\}.$$

The *chunkiness parameter* $\gamma(\omega)$ of ω is then $\gamma(\omega) := \text{diam}(\omega)/\rho_{\max}(\omega)$.

If f is a smooth function, let

$$Q_{y,f,n}(x) = f(y) + \sum_{j=1}^n \partial_j f(y)(x_j - y_j) + \dots + \sum_{|\alpha|=t} \frac{f^{(\alpha)}(y)}{\alpha!} (x - y)^\alpha, \quad \alpha! = \alpha_1! \dots \alpha_n!,$$

be the Taylor polynomial of f at y of degree t . If $B \subset \omega$ is an open ball, then $Q^t f$, the *Taylor polynomial of degree $t \in \mathbb{Z}_+$ of f averaged over B* is given by

$$(18) \quad Q^t f(x) = \int_B Q_{y,f,n}(x) \phi_B(y) dy,$$

with $\phi_B \in C_c^\infty(B)$ a function with integral 1. (In [9], this polynomial is called the Taylor polynomial of order $t + 1$ of f averaged over B .) Note that, by integration

by parts, we can extend the definition of $Q^t f$ to $f \in L^2(\omega)$. We assume that all the functions ϕ_B are affine equivalent to a fixed given function.

We shall need the following lemma (Lemma 4.3.8 from [9]).

Theorem 3.1 (Bramble–Hilbert). *Let ω be an open set with chunkiness parameter $\gamma(\omega) \leq \gamma$. Also, let $B \subset \omega$ be an open ball with radius $\geq \rho_{\max}(\omega)/2$ such that ω is star-shaped with respect to B . Let $Q^{t-1}f$ be the Taylor polynomial of degree $t-1$ of f averaged over B . Then*

$$|f - Q^{t-1}f|_{H^s(\omega)} \leq C_{t,n,\gamma} \text{diam}(\omega)^{t-s} |f|_{H^t(\omega)}, \quad 0 \leq s \leq t.$$

Here $C_{t,n,\gamma} > 0$ is a constant depending only on t , n , and γ .

Recall that the local approximation spaces Ψ_j contain all polynomials of degree $\leq \lambda$. We are ready now to prove the following theorem.

Theorem 3.2. *Let $U \subset \Omega$ be an admissible subset and $0 \leq s \leq t \leq \lambda + 1$, $s \leq m$. Let $U' = \cup_j \omega_j$, where j satisfies $\omega_j^* \subset U$. Then, for any $v \in H^t(U')$, there exists $w \in S_\nu$ such that*

$$\|v - w\|_{H^s(U)} \leq C h_\nu^{t-s} \|v\|_{H^t(U')}$$

for a constant C that depends only on the structural constants, and is, in particular, independent of ν , v , and U .

Let us notice that, by taking $s = t$ in the above theorem, we immediately obtain that, using the same notation, that

$$(19) \quad \|w\|_{H^t(U)} \leq C \|v\|_{H^t(U')}.$$

Also, observe that $\Omega = \cup_{j=1}^N \omega_j$ implies that $U = U'$ if $U = \Omega$.

Proof. Let $h = h_\nu$, $\omega_j = \omega_j^\nu$, and so on. We shall use the notation and the results from [9][Chapter 4] introduced before Theorem 3.1. Let $w_j(f) = Q^{t-1}f$ be the Taylor polynomial of degree $t-1$ of f averaged over ω_j^* , for $\omega_j^* \subset U$. We set $w_j(f) = 0$ if $t = 0$ or if $\omega_j^* \not\subset U$. Then, by the Bramble–Hilbert Lemma recalled above (Theorem 3.1), we have

$$(20) \quad |f - w_j(f)|_{H^s(\omega_j)} \leq C h^{t-s} |f|_{H^t(\omega_j)},$$

with a constant C depending only on s , t , and σ (this is due to the fact that ω_j has diameter $\leq h_\nu$ and is star-shaped with respect to the ball ω_j^* of diameter $\geq \sigma h$, so the chunkiness parameter of ω_j satisfies $\gamma(\omega_j) \leq \sigma^{-1}$).

Fix now $v \in H^t(U')$ arbitrary. Also, let $w_j = w_j(v) \in \Psi_j$ and $w = \sum_j \phi_j w_j \in S$, the sum being taken over all indices such that $\omega_j^* \subset U$. Then, using also Condition B, Lemma 2.4, and $\sum_j \phi_j = 1$ on U , we obtain

$$\begin{aligned} (21) \quad |v - w|_{H^r(U)}^2 &\leq C \sum_j |\phi_j(v - w_j)|_{H^r(\omega_j)}^2 \\ &\leq C \sum_j \sum_{i=1}^r |\phi_j|_{W^{i,\infty}(\omega_j)}^2 |v - w_j|_{H^{r-i}(\omega_j)}^2 \leq C \sum_j \sum_{i=1}^r C_i^2 h^{-2i} h^{2t-2r+2i} |v|_{H^t(\omega_j)}^2 \\ &\leq C \kappa h^{2t-2r} |v|_{H^t(U')}^2 \end{aligned}$$

Summing over $0 \leq r \leq s$ and using $0 < h \leq 1$ gives the desired result. $\square$

Let us also record, for further use, the following well known Poincaré–Friedrichs inequality [9], Lemma (4.3.8). (See also [9], Lemma (4.3.14), and [10], Equation (2.2), Theorem 14.1, and Theorem 15.3., or [11, 31].) The precise statement that we need is the following.

Theorem 3.3. *Assume ω is an open set of diameter $\text{diam}(\omega)$ star-shaped with respect to a ball $\omega^* \subset \omega$ of diameter $\geq \sigma \text{diam}(\omega)$. Let $0 \leq \psi \leq 1$ be a measurable function on ω , $\psi = 1$ on ω^* , and $\bar{v} = \int_{\omega} v(x)\psi(x)dx / \int_{\omega} \psi(x)dx$ be the weighted average of v over ω . Then there exists a constant C_P that depends only on σ and the dimension n , but not on ψ or ω , such that*

$$(22) \quad \|v - \bar{v}\|_{L^2(\omega)} \leq C_P \text{diam}(\omega) |v|_{H^1(\omega)},$$

for all $v \in H^1(\omega)$.

When $\psi = \phi_B$ from Equation (18), the result above reduces to the Bramble–Hilbert lemma. Except for the fact that C_P depends only on n and h , this result is well known when $\bar{v}$ is the average of v over ω . The more general form of the above theorem may be useful when checking that the assumptions of Subsection 2.3 are satisfied for a suitable sequence S_ν of GFEM–spaces.

Throughout this paper, $\hat{C}$ will denote a generic constant that depends only on the dimension n .

Proof. Using a dilation and the homogeneity properties of the norms in the statement, we see that we can reduce to the case $h = \sigma^{-1}$ and $\omega^* = B_1$, the ball of radius 1 centered at the origin. Then $\omega \subset B_{2h} = B_{2\sigma^{-1}}$. Let ϕ be the function appearing in the definition of the averaged Taylor polynomial $Q^k f$, Equation (18). Thus ϕ has support in the unit ball $\omega^* = B_1$.

Recall that $\hat{C}$ denotes a generic constant that depends only on the dimension n . Let $v \in H^1(\omega)$ be arbitrary. We subtract from v various constants to obtain, using also the Bramble–Hilbert lemma, Theorem 3.1, for $l = 1$ and $k = 1$,

$$\begin{aligned} \|v - \bar{v}\|_{L^2(\omega)} &\leq \|v - \int_{\omega} \phi(x)v(x)dx\|_{L^2(\omega)} + \left\| \int_{\omega} \phi(x)v(x)dx - \bar{v} \right\|_{L^2(\omega)} \\ &\leq \hat{C}\|v\|_{H^1(\omega)} + \left\| \int_{\omega} \phi(x)v(x)dx - \bar{v} \right\|_{L^2(\omega)} \leq \hat{C}(\|v\|_{H^1(\omega_j)} + \left| \int_{\omega} \phi(x)v(x)dx - \bar{v} \right|), \end{aligned}$$

where by $\int_{\omega} \phi(x)v(x)dx - \bar{v}$ we mean the constant function on ω with this value (so its L^2 norm is a multiple of this constant, and in our case this multiple can be bounded by a constant depending only on the dimension n).

Let $a := \int_{\omega} \psi(x)dx$ and $g = \phi - a^{-1}\psi$, so that, in particular, $\int_{\omega} g(x)dx = 0$. Since $\bar{v} = a^{-1} \int_{\omega} v(x)\psi(x)dx$, it is enough to show that

$$(23) \quad \left| \int_{\omega} g(x)v(x)dx \right| \leq \hat{C}|v|_{H^1(\omega)}.$$

Let $S^{n-1} = \partial B_1$ denote the unit sphere of radius 1 in $\mathbb{R}^n$ (the boundary of the unit ball B_1). Let ϕ_1 have support in the closure of $B_1 = \omega^*$, be constant on the ray $\mathbb{R}x' \cap B_1$, and satisfy $\int_0^\infty (\phi_1(rx') - a^{-1}\psi(rx'))r^{n-1}dr = 0$, almost everywhere in $x' \in S^{n-1}$. (Note that the function to be integrated vanishes if $r \geq 2\sigma^{-1}$.) Then ϕ_1 is bounded by a constant that depends only on the dimension. Let $g_1 = \phi_1 - a^{-1}\psi$

and let $G(tx') = \int_0^t g_1(rx')r^{n-1}dr$, so that $|G(tx')| \leq \hat{C}t^n$. An integration by parts then shows that

$$\int_{\omega} g_1(x)v(x)dx = \int_{\omega} G(x)\partial_r v(x)|x|^{1-n}dx,$$

where ∂_r is the derivative in the radial direction. Since $G(x)|x|^{1-n}$ is bounded by a constant that depends only on the dimension n , we obtain that

$$\left| \int_{\omega} g_1(x)v(x)dx \right| \leq \hat{C}|v|_{H^1(\omega)},$$

which is the desired Equation (23), but with g replaced by g_1 . It is therefore enough to prove

$$\left| \int_{\omega} [g_1(x) - g(x)]v(x)dx \right| = \left| \int_{B_1} [g_1(x) - g(x)]v(x)dx \right| \leq \hat{C}|v|_{H^1(\omega)},$$

where the first equality is due to the fact that $g_1 - g = \phi_1 - \phi$ has support in B_1 .

Let ∂_{ν} denote the derivative in the direction of the outer normal to $\partial\mathcal{O}$. Since $\int_{B_1} [g_1(x) - g(x)]dx = 0$, we can find V be such that $\Delta V = g_1 - g$ and $\partial_{\nu}V = 0$ on the boundary of B_1 . Moreover, $\|\nabla V\|_{L^2} \leq \hat{C}\|g_1 - g\|_{L^2} \leq \hat{C}$, where, we recall, $\hat{C}$ is a generic constant that may depend only on the dimension n . Then

$$\left| \int_{\omega} [g_1(x) - g(x)]v(x)dx \right| = \left| \int_{B_1} \nabla V \cdot \nabla v(x)dx \right| \leq \hat{C}|v|_{H^1(\omega)}.$$

The proof is now complete. $\square$

For $k = 1$, we shall need the following consequence of Theorem 3.2, which replaces Assumption 9.5 of [33] and does not require the open sets involved, except Ω , to be admissible. Define

$$(24) \quad S_{\nu}^{<}(\Omega) := S_{\nu} \cap \mathcal{C}_c(\Omega).$$

That is, $S_{\nu}^{<}(\Omega)$ consists of the elements of the GFEM-space S_{ν} with compact support inside Ω .

Recall that $A \Subset B$ means that the closure of A is a compact set contained in the interior of B (i.e., A is a relatively compact subset of B). Also, recall that the family $\mathcal{F}$ of subsets of Ω admissible for Σ_{ν} satisfies the ν -chain condition for h_{ν} small enough, see Subsection 2.4 and Lemma 6.5.

Our main goal in this section is to prove Theorem 3.12.

Proposition 3.4. *Let $U \Subset \Omega_1 \subset \Omega$ subsets of Ω and θ be the distance from ∂U to $\partial\Omega_1$. Assume that U is admissible. Then there exists $C > 0$, independent of θ , ν , and U , with the following property. For any $u \in H^2(\Omega)$ with support in U , there exists $w \in S_{\nu}^{<}(\Omega_1)$ such that*

$$\|u - w\|_{H^1(\Omega_1)} \leq Ch_{\nu}\|u\|_{H^2(\Omega_1)},$$

if $h_{\nu} < \theta$.

Proof. Choose w_j and w as in the proof of Theorem 3.2. We shall continue to use the notation of that Theorem. In particular, $h = h_{\nu}$. If $h < \theta$, then $w_j = 0$ unless ω_j intersects U , which gives that the closure of ω_j is completely contained in Ω_1 . In particular, $U' \subset \Omega_1$ and the support of w constructed above is compact

and contained in Ω_1 . Then we can replace U with U' in Equation (21) and we thus obtain

$$\|u - w\|_{H^1(\Omega_1)} = \|u - w\|_{H^1(U')} \leq Ch\|u\|_{H^2(U')} = Ch\|u\|_{H^2(\Omega_1)}.$$

This completes the proof. $\square$

Remark 3.5. By taking $C = 1$ and $w = 0$ for $h \geq \theta$, we obtain

$$\|u - w\|_{H^1(\Omega_1)} \leq C\theta^{-1}h_\nu\|u\|_{H^2(\Omega_1)},$$

for all ν (not just for $h_\nu < \theta$).

3.2. The super-approximation property. The assumptions of Subsection 2.3 continue to remain valid, in particular, all the structural constants below will be independent on ν . Also, recall that we have considered in the Introduction the bilinear form

$$(25) \quad B(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in H^1(\Omega).$$

Our approach follows the approach from [21], as presented in [33][Section 9]. See [8, 20, 23, 24] for related results on approximation in the “sup”-norm.

Lemma 3.6. *Let ρ be a smooth function on ω_j and $w \in \Psi_j$. Then there exists $\tilde{w} \in \Psi_j$ such that*

$$(26) \quad \|\rho w - \tilde{w}\|_{H^1(\omega_j)} \leq \hat{C}h\|\rho\|_{W^{2,\infty}(\omega_j)}\|w\|_{H^1(\omega_j)},$$

where, we recall, $\hat{C} > 0$ may depend only on the dimension n (in particular, it is independent of w , ρ , and j).

Proof. We shall use the inner product induced from $H^1(\omega_j)$. Let $\rho \in W^{2,\infty}(\omega_j)$ be given.

To prove the lemma, we shall assume first that $w \in \Psi_j$ is a constant. Let L be the degree one Taylor polynomial approximation of ρ at the center of the ball ω_j^* . Then $L \in \Psi_j$, because first order polynomials are in Ψ_j (Condition C). Since $h \leq 1$, we obtain

$$\|\rho - L\|_{W^{1,\infty}(\omega_j)} \leq \hat{C}h\|\rho\|_{W^{2,\infty}(\omega_j)}.$$

Choose $\tilde{w} = Lw$. Then

$$\begin{aligned} \|\rho w - \tilde{w}\|_{H^1(\omega_j)} &= \|\rho w - Lw\|_{H^1(\omega_j)} \leq \|\rho - L\|_{W^{1,\infty}(\omega_j)}\|w\|_{H^1(\omega_j)} \\ &\leq \hat{C}h\|\rho\|_{W^{2,\infty}(\omega_j)}\|w\|_{H^1(\omega_j)}. \end{aligned}$$

Assume now that $w \in \Psi_j$ is such that $(w, 1) = 0$, that is, w is orthogonal in $H^1(\omega_j)$ to the subspace generated by constants. We then write

$$\rho = \tilde{\rho} + \rho^*,$$

where $\tilde{\rho}$ is a constant function (say the value of ρ at the center of ω^*) and

$$(27) \quad \|\rho^*\|_{L^\infty(\omega_j)} \leq \hat{C}h\|\nabla \rho\|_{L^\infty(\omega_j)}.$$

We shall choose then $\tilde{w} = \tilde{\rho}w \in \Psi_j$, which makes sense since Ψ_j is a vector space. Then

$$\begin{aligned} \|\rho w - \tilde{w}\|_{H^1(\omega_j)} &= \|\rho^*w\|_{H^1(\omega_j)} \leq C\|\nabla \rho^*\|_{L^\infty(\omega_j)}\|w\|_{L^2(\omega_j)} \\ &\quad + C\|\rho^*\|_{L^\infty(\omega_j)}\|w\|_{H^1(\omega_j)} \leq \hat{C}h\|\nabla \rho\|_{L^\infty(\omega_j)}\|w\|_{H^1(\omega_j)}, \end{aligned}$$

where in the last step we have used the Poincaré–Friedrichs inequality for ω_j (Theorem 3.3) to estimate $\|w\|_{L^2(\omega_j)}$ and Equation (27) above to estimate the second term.

For a general $w \in \Psi_j$, we decompose $w = w_1 + w_2$ with w_1 a constant and w_2 orthogonal to the space of constants and choose $\tilde{w}_1$ and $\tilde{w}_2$ as above. Then

$$\begin{aligned} \|\rho w - \tilde{w}_1 - \tilde{w}_2\|_{H^1(\omega_j)} &\leq \|\rho w_1 - \tilde{w}_1\|_{H^1(\omega_j)} + \|\rho w_2 - \tilde{w}_2\|_{H^1(\omega_j)} \\ &\leq \hat{C}h \|\rho\|_{W^{2,\infty}(\omega_j)} \|w_1\|_{H^1(\omega_j)} + \hat{C}h \|\rho\|_{W^{2,\infty}(\omega_j)} \|w_2\|_{H^1(\omega_j)} \\ &\leq \hat{C}h \|\rho\|_{W^{2,\infty}(\omega_j)} \|w\|_{H^1(\omega_j)}. \end{aligned}$$

The lemma is now proved. $\square$

Recall that we denote $A \Subset B$ if $\bar{A}$, the closure of A in $\mathbb{R}^2$, is a compact subset of the interior of B . Also, recall from Equation (24) that $S_\nu^<(A)$ denotes the set of elements in S_ν with compact support contained in A , for any open subset $A \subset \Omega$.

An important technical step in our proof of the Theorem 3.12 is the following “super-approximation” result.

Proposition 3.7. *Let $\rho \in W^{2,\infty}(\Omega)$ and $w \in S_\nu$. Then there exists $\tilde{w} \in S_\nu$ such that*

$$(28) \quad \|\rho w - \tilde{w}\|_{H^1(\Omega)} \leq Ch_\nu \|\rho\|_{W^{2,\infty}(\Omega)} \|w\|_{H^1(\Omega)},$$

where C is independent of ν . If w has support in $\Omega_1 \Subset \Omega_2$ and θ is the distance from $\partial\Omega_1$ to $\partial\Omega_2$, then $w \in S_\nu^<(\Omega_2)$ for $h_\nu < \theta$.

As explained above, the constant C may depend on the structural constants A , C_j , κ , m , σ , and λ , but is independent of $h = h_\nu$ and of the number N_ν of sets $\{\omega_j\}$. In particular, it is independent of the GFEM-space S_ν used.

Proof. Let $h = h_\nu$ and

$$(29) \quad w = \sum_{j=1}^N \phi_j w_j \in S_\nu, \quad w_j \in \Psi_j.$$

Let $\tilde{w}_j$ be the orthogonal projection of ρw_j onto Ψ_j in the inner product of $H^1(\omega_j)$. Lemma 3.6 then shows that

$$(30) \quad \|\rho w_j - \tilde{w}_j\|_{H^1(\omega_j)} \leq \hat{C}h \|\rho\|_{W^{2,\infty}(\Omega)} \|w_j\|_{H^1(\omega_j)}.$$

Moreover, we have that $\int_{\omega_j} (\rho w_j - \tilde{w}_j) dx = 0$ because the constant functions are in Ψ_j and $\rho w_j - \tilde{w}_j$ is orthogonal to Ψ_j .

Let $\tilde{w} := \sum_{j=1}^N \phi_j \tilde{w}_j$. Then $\|\nabla \phi_j\|_{L^\infty(\omega_j)} \leq C_1/h$ by Equation (11) and

$$\|\rho w_j - \tilde{w}_j\|_{L^2(\omega_j)} \leq C_P h \|\rho w_j - \tilde{w}_j\|_{H^1(\omega_j)} \leq C_P \hat{C} h^2 \|\rho\|_{W^{2,\infty}(\Omega)} \|w_j\|_{H^1(\omega_j)},$$

by the Poincaré–Friedrichs inequality (Theorem 3.3), and hence

$$\begin{aligned} \|\rho w - \tilde{w}\|_{H^1(\Omega)}^2 &= \left\| \sum_{j=1}^N \phi_j (\rho w_j - \tilde{w}_j) \right\|_{H^1(\Omega)}^2 \leq C \sum_{j=1}^N \left(\|\phi_j\|_{L^\infty(\omega_j)}^2 \|\rho w_j - \tilde{w}_j\|_{H^1(\omega_j)}^2 \right. \\ &\quad \left. + \|\nabla \phi_j\|_{L^\infty(\omega_j)}^2 \|\rho w_j - \tilde{w}_j\|_{L^2(\omega_j)}^2 \right) \leq Ch^2 \|\rho\|_{W^{2,\infty}(\Omega)}^2 \sum_{j=1}^N \|w_j\|_{H^1(\omega_j)}^2, \end{aligned}$$

where for the first inequality we have used also Lemma 2.4. The result will follow now if we can prove that $\sum_{j=1}^N \|w_j\|_{H^1(\omega_j)}^2 \leq C \|w\|_{H^1(\Omega)}^2$, for any $w = \sum_{j=1}^N \phi_j w_j$, as above and C a constant independent of $S = S_\nu$. Indeed, since $\phi_j = 1$ on ω_j^* , by Condition D, we have $w = w_j$ on ω_j^* , and hence

$$\|w\|_{H^1(\Omega)}^2 \geq \sum_{j=1}^N \|w_j\|_{H^1(\omega_j^*)}^2 \geq A^{-2} \sum_{j=1}^N \|w_j\|_{H^1(\omega_j)}^2,$$

by Condition C (A is the constant appearing in that assumption).

The proof of the last part is completed as in Proposition 3.4. $\square$

3.3. Estimates on “discrete-harmonic” functions. We shall also need the following “inverse property,” which is somewhat similar to Assumption A.3. in [21] or Assumption 9.2 in [33].

The rest of this section follows closely the approach in the paper of Nitsche and Schatz [21], relying also from the survey paper [33] (which in turn is based on the paper by Nitsche and Schatz). There are, however, some differences in the assumptions that we are using, so we include complete proofs for the convenience of the reader. For instance, the following lemma, Lemma 3.8, plays the role of Assumption A.3. in the Nitsche–Schatz article [21], respectively, of the Assumption 9.2 (Inverse assumption) in Wahlbin’s article. Also, the following lemma is an analog of Lemma 5.2 of [21], respectively, of Lemma 9.1 of [33].

Lemma 3.8. *There exists a constant $C > 0$, depending only on the structural constants, such that*

$$\|w\|_{L^2(U)} \leq Ch_\nu^{-j} \|w\|_{H^{-j}(U)},$$

for all $0 \leq j \leq m$, $\nu \in \mathbb{Z}_+$, $w \in S_\nu$, and $U \subset \Omega$ admissible for Σ_ν .

Proof. Let $h = h_\nu$. We have

$$\|w\|_{H^{-j}(\Omega)} = \sup_{\phi} \frac{|\langle w, \phi \rangle|}{\|\phi\|_{H^j(\Omega)}} \geq \frac{(w, w)}{\|w\|_{H^j(\Omega)}} = \frac{\|w\|_{L^2(\Omega)}^2}{\|w\|_{H^j(\Omega)}} \geq C^{-1} h^j \|w\|_{L^2(\Omega)},$$

by Proposition 2.5. $\square$

We now prove the following crucial lemma.

Lemma 3.9. *Let $A \Subset A' \Subset \Omega$ be open subsets. Then there exists $C > 0$ with the following property. If $w \in S_\nu$ and*

$$(31) \quad B(w, \chi) = 0, \quad \text{for all } \chi \in S_\nu^<(A'),$$

then, for h_ν small enough, $\|w\|_{H^1(A)} \leq C \|w\|_{L^2(A')}$, with C depending on the distance θ from ∂A and $\partial A'$, but not on ν or w .

Proof. The proof is very similar to the one in [21, 33], using Proposition 2.5 in place of Condition A.3, respectively Assumption 9.2 (“Inverse assumption”), and Proposition 3.7 in place of Assumption A.2, respectively Assumption 9.1 (“Super-approximation”), of [21], respectively [33].

Let $h = h_\nu$ be small enough. Let us chose open sets $A \Subset A_0 \Subset A_1 \Subset A'$ with A_1 admissible for Σ_ν and with the distances between the boundaries comparable with θ . Also, let $\omega \in C_c^\infty(A_0)$, with $\omega = 1$ on A and $\omega \geq 0$ on A_0 such that all the norms

$\|\omega\|_{W^{k,\infty}}$ are bounded by a constant depending only on θ . Then, by Equation (31), we obtain

$$\begin{aligned}\|\nabla w\|_{L^2(A)}^2 &\leq (\nabla w, \omega \nabla w) = (\nabla w, \nabla(\omega w)) - (\nabla w, \nabla(\omega)w) \\ &= (\nabla w, \nabla(\omega w - \psi)) + \frac{1}{2}(w, (\Delta \omega)w),\end{aligned}$$

where the inner products are in $L^2(A_0)$ and $\psi \in S_\nu^<(A_0)$. Proposition 3.7 then gives

$$\|\nabla w\|_{L^2(A)}^2 \leq Ch\|\omega\|_{W^{2,\infty}(\Omega)}\|w\|_{H^1(A_0)}^2 + C\|w\|_{L^2(A_0)}^2,$$

which, in turn, implies

$$(32) \quad \|w\|_{H^1(A)} \leq C(h^{1/2}\|w\|_{H^1(A_0)} + \|w\|_{L^2(A_0)}),$$

where all the constants depend only on θ . We now repeat the argument for $A_0 \Subset A_1 \Subset \Omega$ (and A replaced by A_0 and A_0 replaced by A_1), which gives

$$(33) \quad \|w\|_{H^1(A_0)} \leq C(h^{1/2}\|w\|_{H^1(A_1)} + \|w\|_{L^2(A_1)}).$$

Combining Equations (32) and (33) and using also $h \leq 1$, we obtain

$$(34) \quad \|w\|_{H^1(A)} \leq C(h\|w\|_{H^1(A_1)} + C\|w\|_{L^2(A_1)}).$$

Then we use Proposition 2.5 to obtain that $h\|w\|_{H^1(A_1)} \leq C\|w\|_{L^2(A_1)}$, with C independent of w and $\nu \in \mathbb{Z}_+$. This finally gives

$$(35) \quad \|w\|_{H^1(A)} \leq C\|w\|_{L^2(A_1)} \leq C\|w\|_{L^2(A')},$$

with a constant depending only on θ . The proof is now complete. $\square$

We shall need the following simple estimate.

Lemma 3.10. *Let $\Phi(x) = \log|x|$ if $n = 2$, $\Phi(x) = |x|^{2-n}$, if $n \neq 2$. Let $U \subset \Omega$ be an open subset. Then there exists $C > 0$, independent of U , such that*

$$\Phi * u(x) := \int_U \Phi(x-y)u(y)dy$$

satisfies

$$\|\Phi * u\|_{H^{l+2}(U)} \leq C\|u\|_{H^l(U)},$$

for any $l \in \mathbb{Z}$ and any $u \in C_c^\infty(U)$. The constant C will depend on Ω , however.

Proof. Fix $U \subset \Omega$ and let $u \in C_c^\infty(U) \subset C_c^\infty(\Omega)$ and $v = \Phi * u$. We have $\Delta v = c_n u$ for some constant c_n depending only on the dimension n (this well known fact is proved, for example, in [11]). The generic constants below, denoted C , are allowed to depend only on R and the dimension n .

Let $\mathcal{O} = B_R(0)$ be the ball of radius $R > 3 \text{diam}(\Omega)$ centered at the origin, where $\text{diam}(\Omega)$ is the diameter of Ω . We shall assume that R is very large. In particular, we shall assume that $\Omega \subset B_{R/3}(0)$. Let $\eta : [0, \infty) \rightarrow \mathbb{R}$ be a smooth function such that $\eta(t) = 1$ if $t \leq \text{diam}(\Omega)$ and $\eta(t) = 0$ if $t \geq 2 \text{diam}(\Omega)$. Let $\Phi_1(x) = \eta(|x|)\Phi(x)$ and define $v = \Phi_1 * u$.

Then $v(x)$ vanishes if the distance from x to Ω is greater than $2 \text{diam}(\Omega) < 2R$, so $v = 0$ on $\partial\mathcal{O} = \partial B_R(0)$. Also, $\Phi(x) = \Phi_1(x)$ for $|x| \leq \text{diam}(\Omega)$, and hence $v(x) = \Phi * u(x)$ for $x \in \Omega$. Moreover, $\Delta v = (\Delta \Phi_1) * u = c_n u + \phi * u$, where $\phi(x) = \Delta \Phi_1(x)$ for $x \neq 0$ and $\phi(x) = 0$ if $|x| \leq \text{diam}(\Omega)$. Since $\phi \in L^1(\mathbb{R}^n)$, we have that $\|\phi * u\|_{H^l(\mathbb{R}^n)} \leq C\|u\|_{H^l(\mathbb{R}^n)}$, and hence $\|\Delta v\|_{H^l(\mathbb{R}^n)} \leq C\|u\|_{H^l(\mathbb{R}^n)}$. Finally, since $v = 0$ on $\partial\mathcal{O}$, we obtain that $\|v\|_{H^{l+2}(\Omega)} \leq C\|u\|_{H^l(\mathbb{R}^n)}$, by standard

estimates on elliptic boundary value problems (see [11] or [31], for instance). This proves that

$$\|\Phi * u|_U\|_{H^{l+2}(U)} = \|v|_U\|_{H^{l+2}(U)} \leq C\|u\|_{H^l(\mathbb{R}^n)} = C\|u\|_{H^l(U)}.$$

The proof is complete. $\square$

An alternative, shorter proof of this lemma can be obtained using the fact that, for any smooth, compactly supported function, $u \rightarrow \omega(\Phi * u)$ is a pseudodifferential operator of order -2 with compact distribution kernel [32].

We define

$$(36) \quad \|u\|_{H_0^{-s}(U)} = \sup \frac{|(u, v)|}{\|v\|_{H^s(U)}} \leq \|u\|_{H^{-s}(U)}, \quad 0 \neq v \in \mathcal{C}_c^\infty(U)$$

for any open set U , any $u \in L^2(U)$, and any $s \in \mathbb{N}$. We define $H_0^{-s}(U)$ to be the completion of $L^2(U)$ in the norm $\|u\|_{H_0^{-s}(U)}$. Then $H_0^{-s}(U)$, $s \in \mathbb{N}$, identifies with the dual of $H_0^s(U)$.

Let θ be the distance from ∂A and $\partial A'$, as before. We now prove the following lemma.

Lemma 3.11. *We keep the notation and assumption of Lemma 3.9. In particular, we assume that $w \in S_\nu$ satisfies Equation (31). Then, for h_ν small enough,*

$$(37) \quad \|w\|_{L^2(A)} \leq C\|w\|_{H^{-m}(A')},$$

where C is a constant depending θ and the structural constants, but not on ν .

Combining Lemmas 3.9 and 3.11, we obtain

$$(38) \quad \|w\|_{H^1(A)} \leq C\|w\|_{H^{-m}(A')},$$

for h_ν small enough and any $w \in S_\nu$ satisfying the assumptions of Lemma 3.9.

Proof. Let $h = h_\nu$ and $A \Subset B_0 \Subset B_1 \Subset A'$ be open sets with B_0 admissible. Note that Lemma 3.9 gives

$$(39) \quad \|w\|_{H^1(B_1)} \leq C\|w\|_{L^2(A')}.$$

For any $v \in \mathcal{C}_c^\infty(A)$, let $V := c_n \Phi * v \in H^{l+2}(B_1)$, where c_n is chosen such that $\Delta V = v$ (see [11]). Lemma 3.10 then gives

$$(40) \quad \|V\|_{H^{l+2}(B_1)} \leq C\|v\|_{H^l(B_1)} = C\|v\|_{H^l(A)}, \quad l \in \mathbb{Z}_+,$$

for some constant C that depends only on Ω .

Let $\omega \in \mathcal{C}_c^\infty(B_0)$ with $\omega = 1$ on A . Since $\omega V \in \mathcal{C}_c^\infty(B_0)$ and B_0 is admissible, we know from Proposition 3.4 that, for h small enough, there exists $\chi \in S_\nu^<(B_1)$ such that

$$(41) \quad \|\omega V - \chi\|_{H^1(B_1)} \leq Ch\|\omega V\|_{H^2(B_1)} \leq Ch\|V\|_{H^2(B_1)} \leq Ch\|v\|_{L^2(A)}.$$

Then, still assuming that $v \in \mathcal{C}_c^\infty(A)$ is arbitrary and taking into account also Equation (31), we obtain

$$\begin{aligned} (w, v)_A &= (\omega w, v)_A = \int_A \omega w \Delta V dx = \int_{B_0} \omega w \Delta V dx = - \int_{B_0} \nabla(\omega w) \cdot \nabla V dx \\ &= - \int_{B_0} w(2\nabla \omega \cdot \nabla V + V \Delta \omega) dx - \int_{B_1} \nabla w \cdot \nabla(\omega V - \chi) dx, \end{aligned}$$

for any $\chi \in S_\nu^<(B_1) \subset S_\nu^<(A')$, where the inner products are calculated on the indicated sets. Then, by combining Equations (39), (40), and (41), as well as Lemma 3.9, we obtain, for all $l \geq 0$,

$$\begin{aligned} |(w, v)_A| &\leq C \|w\|_{H_0^{-l-1}(B_0)} \|V\|_{H^{l+2}(B_1)} + Ch \|w\|_{H^1(B_1)} \|V\|_{H^2(B_1)} \\ &\leq C (\|w\|_{H_0^{-l-1}(A')} + h \|w\|_{L^2(A')}) \|v\|_{H^l(A)}, \end{aligned}$$

and hence

$$(42) \quad \|w\|_{H_0^{-l}(A)} \leq C (\|w\|_{H_0^{-l-1}(A')} + h \|w\|_{L^2(A')}).$$

Next, let us choose a sequence of open sets $A \Subset B_1 \Subset B_2 \Subset \dots \Subset B_m \Subset A'$ with the distances between the boundaries comparable with θ . Changing notation and iterating Equation (42), using also $h \leq 1$, we obtain

$$\begin{aligned} (43) \quad \|w\|_{L^2(A)} &\leq C (\|w\|_{H_0^{-1}(B_1)} + h \|w\|_{L^2(B_1)}) \\ &\leq C (\|w\|_{H_0^{-2}(B_2)} + h \|w\|_{L^2(B_2)}) \leq \dots \\ &\leq C (\|w\|_{H_0^{-m}(B_m)} + h \|w\|_{L^2(B_m)}). \end{aligned}$$

We now repeat the above reasoning. We change notation again, so that, this time, B_m becomes B_1 , then we chose as before a sequence of open sets

$$(44) \quad A \Subset B_1 \Subset B_2 \Subset \dots \Subset B_m \Subset A'.$$

We can assume that the distances between the boundaries are comparable with θ . Also, we can assume that B_m is admissible, since the family of admissible sets satisfies the ν -chain condition for h small enough, see Lemma 6.5. Then we iterate Equation (43) using again $h \leq 1$, and obtain,

$$\begin{aligned} \|w\|_{L^2(A)} &\leq C (\|w\|_{H_0^{-m}(B_1)} + h \|w\|_{L^2(B_1)}) \\ &\leq C (\|w\|_{H_0^{-m}(B_2)} + h^2 \|w\|_{L^2(B_2)}) \leq \dots \leq C (\|w\|_{H_0^{-m}(B_m)} + h^m \|w\|_{L^2(B_m)}) \\ &\leq C (\|w\|_{H_0^{-m}(B_m)} + \|w\|_{H^{-m}(B_m)}) \leq C \|w\|_{H^{-m}(B_m)}, \end{aligned}$$

where at the end we have used the inverse property $\|w\|_{L^2(U)} \leq Ch^{-m} \|w\|_{H^{-m}(U)}$ for any admissible open subset $U \subset \Omega$, (see Lemma 3.8). The proof is complete. $\square$

3.4. The interior error estimate. The following result, the main result of this section, is an analog of [21][Theorem 5.1] and of [33][Theorem 9.2].

Let S_ν be the GFEM-spaces associated to the data $\Sigma_\nu = \{\phi_j^\nu, \omega_j^\nu, \Psi_j^\nu, \omega_j^{*\nu}\}$ satisfying the Assumptions of Subsectio 2.3.

Theorem 3.12. *Let $A \Subset B \subset \Omega$ be open subsets. Then there exists $C > 0$ with the following property. If $u \in H^1(\Omega)$ and $u_\nu \in S_\nu$, $\nu \in \mathbb{Z}_+$, are such that $B(u - u_\nu, \chi) = 0$ for all $\chi \in S_\nu^<(\Omega)$, then for h_ν small enough,*

$$\|u - u_\nu\|_{H^1(A)} \leq C \left(\inf_{\chi \in S_\nu} \|u - \chi\|_{H^1(B)} + \|u - u_\nu\|_{H^{-m}(B)} \right).$$

The constant C depends only on the distance from ∂A to ∂B and the structural constants, but not on $\nu \in \mathbb{Z}_+$.

For example, u_ν in the above theorem could be the GFEM-approximation of u (see Equation (54)), but our assumptions on u_ν are in fact somewhat weaker.

Proof. Let $A \Subset A_1 \Subset A_2 \Subset B \subset \Omega$. Choose $\omega \in \mathcal{C}_c^\infty(A_2)$ such that $\omega = 1$ on A_1 . Let P_1 be the $H^1(\Omega)$ orthogonal projection onto $S_\nu^<(A_1) \subset S \subset H^1(\Omega)$. Then on A_1

$$(45) \quad u - u_\nu = (\omega u - P_1(\omega u)) + (P_1(\omega u) - u_\nu).$$

Then, by the general properties of orthogonal projections, we have

$$(46) \quad \|\omega u - P_1(\omega u)\|_{H^1(\Omega)} \leq \|\omega u\|_{H^1(\Omega)} \leq C\|u\|_{H^1(B)}.$$

Hence

$$(47) \quad \|\omega u - P_1(\omega u)\|_{H^{-m}(A_1)} \leq \|\omega u - P_1(\omega u)\|_{H^1(\Omega)} \leq C\|u\|_{H^1(B)}.$$

Let $w = P_1(\omega u) - u_\nu$. Then $B(w, \chi) = B(\omega u - u_\nu, \chi) = B(u - u_\nu, \chi) = 0$, for all $\chi \in S_\nu^<(A_1)$, and hence w satisfies the assumptions of Lemmas 3.9 and 3.11. From this, using also Equations (38), (45), and (47), we obtain

$$(48) \quad \begin{aligned} \|w\|_{H^1(A)} &\leq C\|w\|_{H^{-m}(A_1)} \leq \|\omega u - P_1(\omega u)\|_{H^{-m}(A_1)} + \|u - u_\nu\|_{H^{-m}(A_1)} \\ &\leq \|u\|_{H^1(B)} + \|u - u_\nu\|_{H^{-m}(A_1)}. \end{aligned}$$

Equations (45–48) then give

$$\begin{aligned} \|u - u_\nu\|_{H^1(A)} &\leq \|\omega u - P_1(\omega u)\|_{H^1(A)} + \|w\|_{H^1(A)} \quad \text{by (45)} \\ &\leq C\|u\|_{H^1(B)} + \|u - u_\nu\|_{H^{-m}(A_1)} \quad \text{by (46)–(48)}. \end{aligned}$$

The desired result follows by replacing u and u_ν with $u - \chi$ and, respectively, $u_\nu - \chi$, with χ in $S = S_\nu$. $\square$

4. DISCRETE SOLUTIONS

We assume in this and next section that $\Omega = \mathcal{O}$, a smooth, bounded domain. We discuss in this section a variational formulation of the boundary value problem (1). We also prove the existence, uniqueness of the solution of this problem as well as the estimate (2). (So we establish the well posedness of this boundary value problem.) The discrete solutions, or GFEM–approximations to u are also defined using the weak formulation of (1). We begin with the definition of the discrete solutions. The results extend to the case of g with low regularity the usual results and definitions of the weak solution and its discretization, thus, the reader willing to accept that the theory is essentially the same as in the case $g \in L^2(\partial\mathcal{O})$, can skip this section at a first reading.

Let us fix, in this section, a finite dimensional subspace $S \subset H^m(\mathcal{O})$ containing all constant functions. In applications, $S = S_\nu$, for some $\nu \in \mathbb{Z}_+$, but that is not required in this section.

4.1. Discrete solutions. Consider the usual bilinear form

$$(49) \quad B(\phi, \psi) := \int_{\mathcal{O}} \nabla \phi \cdot \nabla \psi \, dx,$$

where $\phi, \psi \in H^1(\mathcal{O})$. Assume, for the purpose of this discussion, that $\Omega = \mathcal{O}$ is smooth, so that the boundary value problem (1) is defined.

Recall that we write $\langle u, v \rangle_{\partial\mathcal{O}} := u(v) \in \mathbb{C}$ for any $u \in H^{1-k}(\mathcal{O})$ and $v \in H^{k-1}(\mathcal{O})$ for the “value of u evaluated at v .”

Lemma 4.1. Assume $\langle g, 1 \rangle_{\partial\mathcal{O}} = 0$ and $g \in H^{-1/2-k}(\partial\mathcal{O})$, $k \leq m-1$. Then there exists a unique $u_S \in S$ such that $\langle u_S, 1 \rangle = 0$ and

$$(50) \quad B(u_S, v_S) = \langle g, v_S|_{\partial\mathcal{O}} \rangle_{\partial\mathcal{O}}, \quad \text{for all } v_S \in S,$$

for all $v_S \in S$. (Recall that $S \subset H^m(\mathcal{O})$.)

Note that $S \subset H^m(\mathcal{O})$ and $g \in H^{-1/2-k}(\partial\mathcal{O})$, $k \leq m-1$, guarantees that $\langle g, v_S|_{\partial\mathcal{O}} \rangle_{\partial\mathcal{O}}$ is defined.

Proof. Let S_0 be the subspace of the GFEM-space S consisting of functions $\chi_0 \in S$ with $\langle \chi_0, 1 \rangle = \int_{\mathcal{O}} \chi_0(x) dx = 0$. The bilinear form B is non-degenerate on S_0 (that is, if $\phi \in S_0$ is such that $B(\phi, \psi) = 0$ for all $\psi \in S_0$, then $\phi = 0$). This gives by standard linear algebra the existence of a unique $u_S \in S_0$ such that Equation (50) is satisfied for all $v_S \in S_0$. $\square$

Definition 4.2. If $u_S \in S$ is as in the above Lemma, then we shall say that u_S is the GFEM-approximation of the solution of Equation (1).

Similarly, if $g \in H^{1/2}(\mathcal{O})$ and $\langle g, 1 \rangle_{\partial\mathcal{O}} = \int_{\partial\mathcal{O}} g(x) dS(x) = 0$, then the solution u of Equation (1) is in $H^2(\mathcal{O})$, $\langle u, 1 \rangle = 0$ and,

$$(51) \quad B(u, v) = \langle g, v|_{\partial\mathcal{O}} \rangle_{\partial\mathcal{O}},$$

for all v smooth enough. Moreover, u is uniquely determined by these conditions. This is of course, nothing but the weak formulation of the boundary value problem of Equation (1). We now extend the formulation to the case when the data g is a distribution.

Lemma 4.1 states only the existence and uniqueness of the discrete solution u_S of Equation (1). It claims nothing about the relation between u_S and the exact solution u of Equation (1). This will be discussed in the remaining of this section and in the following section.

4.2. The weak solution. Let us begin with a remark that will justify the following constructions.

Remark 4.3. When trying to extend the weak formulation of Equation (1) to g with lower regularity (i.e., g a distribution), we face the following difficulty. Let $v \in H^{1+k}(\mathcal{O})$ and $w \in H^{1-k}(\mathcal{O}) = H^{k-1}(\mathcal{O})^*$. We can choose $v_n, w_n \in H^1(\mathcal{O})$, $v_n \rightarrow v$ in the topology of $H^{1+k}(\mathcal{O})$ and $w_n \rightarrow w$ in the topology of $H^{1-k}(\mathcal{O})$. Then

$$B(w_n, v_n) = - \int_{\mathcal{O}} w_n \Delta v_n dx + \int_{\partial\mathcal{O}} (w_n|_{\partial\mathcal{O}}) \partial_\nu v_n dS(x).$$

Next, we notice that $\Delta v_n \rightarrow \Delta v$ in $H^{k-1}(\mathcal{O})$, and hence $\int_{\mathcal{O}} w_n \Delta v_n dx \rightarrow \langle w, \Delta v \rangle$. Similarly, $\partial_\nu v_n \rightarrow \partial_\nu v$ in $H^{k-1/2}(\partial\mathcal{O})$. Now, if the sequence $B(w_n, v_n)$ had a limit that depended only on w and v , then it would follow that the sequence of traces $w_n|_{\partial\mathcal{O}}$ would have a limit depending only on w . In turn, this would provide by a continuous trace map $H^{1-k}(\mathcal{O}) \rightarrow H^{1/2-k}(\partial\mathcal{O})$, $k \geq 1$, which is known not to be possible.

For the reasons just explained, we introduce, following an idea from [22], the space $\tilde{H}^{-s}(\mathcal{O}) := H^{-s}(\mathcal{O}) \oplus H^{-s-1/2}(\partial\mathcal{O})$, where $s \in \mathbb{Z}$, $s \geq 2$. Intuitively, the

second component ζ of an element $\tilde{u} = (u, \zeta)$ in $\tilde{H}^{-s}(\mathcal{O})$ should be thought of as some sort of trace at the boundary of u . We then define

$$(52) \quad \begin{aligned} &\tilde{B} : \tilde{H}^{1-k}(\mathcal{O}) \times H^{1+k}(\mathcal{O}) \rightarrow \mathbb{C} \quad \text{by} \\ &\tilde{B}(\tilde{u}, V) = -\langle u, \Delta V \rangle + \langle \zeta, \partial_\nu V|_{\partial\mathcal{O}} \rangle_{\partial\mathcal{O}}, \quad \text{where} \\ &\tilde{u} = (u, \zeta) \in H^{1-k}(\mathcal{O}) \oplus H^{1/2-k}(\partial\mathcal{O}) = \tilde{H}^{1-k}(\mathcal{O}), \quad k \in \mathbb{N}. \end{aligned}$$

With this definition, we can now introduce weak solutions of Equation (1) for $r = 1 - k$, $k \in \mathbb{Z}_+$.

Definition 4.4. Let $g \in H^{-1/2-k}(\partial\Omega)$, $k \in \mathbb{Z}_+$. We say that $u \in \tilde{H}^{1-k}(\Omega)$ satisfies the Equation (1) (i.e., $-\Delta u = 0$, $\partial_\nu u = g \in H^{-1/2-k}(\partial\Omega)$) in *weak sense* (or that u is a *weak solution* of the Equation (1)) if, and only if, there exists $\zeta \in H^{1/2-k}(\partial\Omega)$ such that

$$\tilde{B}(\tilde{u}, V) = \langle g, V \rangle_{\partial\mathcal{O}}, \quad \text{where } \tilde{u} = (u, \zeta)$$

for all $V \in H^{1+k}(\Omega)$.

The pair $\tilde{u} = (u, \zeta)$ above will also be called a weak solution of Equation (1).

Remark 4.5. The above definition of weak solutions of the Neumann problem generalizes the classical definition. Indeed, if g is regular enough so that $u \in H^2(\mathcal{O})$ is a *classical* solution of the Neumann problem with data g (i.e., of Equation (1)) then

$$B(u, V) = \tilde{B}(\tilde{u}, V) = \langle g, V \rangle_{\partial\mathcal{O}}$$

for all $V \in H^{1+k}(\Omega)$, where $\tilde{u} := (u, u|_{\partial\Omega})$. Hence u is a weak solution of Equation (1) also in the sense of Definition 4.4.

We shall need the following simple observation.

Lemma 4.6. Let $\tilde{u}, \tilde{u}_1 \in \tilde{H}^{1-k}(\mathcal{O}) = H^{1-k}(\mathcal{O}) \oplus H^{1/2-k}(\partial\mathcal{O})$, $\tilde{u} = (u, \zeta)$ and $\tilde{u}_1 = (u_1, \zeta_1)$. Then $\tilde{B}(\tilde{u}, V) = \tilde{B}(\tilde{u}_1, V)$ for all $V \in H^{1+k}(\Omega)$ if, and only if, $u - u_1 = c$ and $\zeta - \zeta_1 = c$, where c is a constant.

Proof. The subspace $\mathcal{V} := \{(-\Delta V, \partial_\nu V)\} \subset H^{k-1}(\mathcal{O}) \oplus H^{k-1/2}(\partial\Omega)$ has codimension one, by the solvability conditions of the Neumann problem. Moreover, the solvability conditions for the Neumann problem (recalled in the proof of Proposition 4.9) show that the annihilator of $\mathcal{V}$ is the vector $(1, 1)$. Our assumption is equivalent to the fact that $w := \tilde{u}_1 - \tilde{u}$ satisfies $\langle w, w_1 \rangle$ for any $w_1 \in \mathcal{V}$. Since $\tilde{H}^{1-k}(\mathcal{O})$ is the dual of $H^{k-1}(\mathcal{O}) \oplus H^{k-1/2}(\partial\Omega)$, it follows that $\tilde{u}_1 - \tilde{u}$ must be a multiple of the non-zero vector $(1, 1)$ spanning the annihilator of $\mathcal{V}$. $\square$

From the above lemma, Lemma 4.6, we obtain the following corollary.

Corollary 4.7. If $\tilde{u} = (u, \zeta)$ is a solution of Equation (1) in weak sense (Definition 4.4) then ζ is uniquely determined by u . Moreover, if $\tilde{u}_1 = (u_1, \zeta_1)$ is another solution of Equation (1), then $u_1 - u$ is a constant.

The usual properties of the solutions of the Neumann problem are therefore satisfied. The above corollary also allows us to define $u|_{\partial\mathcal{O}} := \zeta$ if (u, ζ) is a weak solution of Equation (1) for some $g \in H^{-1/2-k}(\partial\mathcal{O})$. By abuse of terminology, we shall say that $u \in H^{1-k}(\mathcal{O})$ is a *weak solution* of Equation (1) if there exists $\zeta \in$

$H^{-1/2-k}(\partial\mathcal{O})$ (uniquely determined by the above Corollary) such that $\tilde{u} = (u, \zeta)$ is a weak solution of that equation. In this case, we shall also write

$$(53) \quad B(u + u_1, V) := \tilde{B}(\tilde{u}, V) + B(u_1, V)$$

for any $u_1 \in H^1(\mathcal{O})$. This is needed in Equation (54) below. See also (63).

Let $u_S \in S$ be the GFEM-approximation of the solution of Equation (1) (as defined by Lemma 4.1). Also, let $\tilde{u} = (u, \zeta)$ be a weak solution of Equation (1). The extension (53) of the definition of the bilinear form B is useful because, for example, it allows us write that

$$(54) \quad B(u - u_S, v_S) = 0, \quad \text{for all } v_S \in S.$$

The relation (54) is then equivalent to the Definition 4.2.

Let X and Y be normed spaces with norms $\|x\|_X$ and $\|y\|_Y$ and let $B_1 : X \times Y \rightarrow \mathbb{C}$ be a bilinear form. Recall that B_1 is said to be *continuous* if, and only if, there exists $C < \infty$ such that $|B_1(x, y)| \leq C\|x\|_X\|y\|_Y$ for all $x \in X$ and $y \in Y$. We shall need the following result.

Theorem 4.8. *Let X and Y be reflexive Banach spaces with norms $\|x\|_X$ and $\|y\|_Y$. Also, let $B_1 : X \times Y \rightarrow \mathbb{C}$ be a bilinear form. Assume that*

- (i) B_1 is continuous;
- (ii) There exists $\gamma > 0$ such that

$$\inf_{\|x\|_X=1} \sup_{\|y\|_Y \leq 1} |B_1(x, y)| \geq \gamma;$$

- (iii) $\sup_{\|x\|_X \leq 1} |B_1(x, y)| > 0$ whenever $y \neq 0$.

Then for any continuous functional $F : Y \rightarrow \mathbb{C}$ there exists a unique $x \in X$ such that $F(y) = B_1(x, y)$, for all $y \in Y$. Moreover, we have $\|x\| \leq \|F\|/\gamma$.

This theorem is a generalization of the well known Lax–Milgram Lemma. For a proof, see Theorem 5.2.1, page 112, of [1]. The proof in that book is an adaptation of the proof in [17] and [19]. See also [18], page 294.

We now check that our form $\tilde{B}$ satisfies the conditions of Theorem 4.8.

Proposition 4.9. *Let $X \subset \tilde{H}^{1-k}(\mathcal{O}) := H^{1-k}(\mathcal{O}) \oplus H^{1/2-k}(\partial\mathcal{O})$ consist of the pairs $\tilde{u} = (u, \zeta)$ satisfying $\langle u, 1 \rangle + \langle \zeta, 1 \rangle_{\partial\mathcal{O}} = 0$. Also, let $Y \subset H^{1+k}(\mathcal{O})$ consist of the functions V such that $\langle V, 1 \rangle := \int_{\mathcal{O}} V(x) dx = 0$. Then the restriction of the form $\tilde{B}$ of Equation (52) to $X \times Y$ satisfies the conditions of Theorem 4.8 for $k \geq 1$.*

Proof. The bilinear form $\tilde{B}$ is immediately seen to be continuous by definition of $\tilde{B}$ and its definition (Equation (52)) and by the definition of our negative order Sobolev spaces. Therefore Condition (i) in Theorem 4.8 is satisfied.

To check that Condition (ii) in Theorem 4.8 is satisfied, we shall use the well posedness of the Neumann problem on $\mathcal{O}$. Namely, we shall use the fact that, for any $v \in H^{k-1}(\mathcal{O})$ and any $g \in H^{k-1/2}(\partial\mathcal{O})$ satisfying

$$\langle 1, v \rangle + \langle 1, g \rangle_{\partial\mathcal{O}} := \int_{\Omega} v(x) dx + \int_{\partial\Omega} g(x) dS(x) = 0,$$

there exists a unique $V \in H^{k+1}(\mathcal{O})$ such that

$$(55) \quad -\Delta V = v, \quad \partial_\nu V = g, \quad \text{and} \quad \langle V, 1 \rangle = 0.$$

Moreover, there exists $C_\Omega > 0$ such that

$$(56) \quad \|V\|_{H^{k+1}(\mathcal{O})} \leq C_\Omega (\|v\|_{H^{k-1}(\mathcal{O})} + \|g\|_{H^{k-1/2}(\partial\mathcal{O})}).$$

Let now $\tilde{u} = (u, \zeta) \in X \subset H^{1-k}(\mathcal{O}) \oplus H^{1/2-k}(\partial\mathcal{O}) = (H^{k-1}(\mathcal{O}) \oplus H^{k-1/2}(\partial\mathcal{O}))^*$. By definition, there exist $v \in H^{k-1}(\mathcal{O})$ and $g \in H^{k-1/2}(\partial\mathcal{O})$, not both zero, such that

$$(57) \quad \langle \tilde{u}, (v, g) \rangle := \langle u, v \rangle + \langle \zeta, g \rangle_{\partial\mathcal{O}} \geq \frac{1}{2} \|\tilde{u}\| \left(\|v\|_{H^{k-1}(\mathcal{O})}^2 + \|g\|_{H^{k-1/2}(\partial\mathcal{O})}^2 \right)^{1/2},$$

where $\|\tilde{u}\|^2 := \|u\|_{H^{1-k}(\mathcal{O})}^2 + \|\zeta\|_{H^{1/2-k}(\partial\mathcal{O})}^2$, and the last paranthesis stands for the norm of the pair (v, g) in the Hilbert space $H^{k-1}(\mathcal{O}) \oplus H^{k-1/2}(\partial\mathcal{O})$.

If we replace v with $v + \lambda$ and g with $g + \lambda$, where λ denotes the constant function equal to $\lambda \in \mathbb{C}$, then the pairing $\langle u, v \rangle + \langle \zeta, g \rangle_{\partial\mathcal{O}}$ does not change, since $\tilde{u} \in X$. Moreover, choosing λ such that $\|v + \lambda\|_{H^{k-1}(\mathcal{O})}^2 + \|g + \lambda\|_{H^{k-1/2}(\partial\mathcal{O})}^2$ is minimal means replacing the pair (v, g) with its projection onto the orthogonal complement of $(1, 1)$ (we have used here Equation (10)). This choice will not affect Equation (57). We can thus assume that $(v, g) \perp (1, 1)$. Therefore $\langle 1, v \rangle + \langle 1, g \rangle_{\partial\mathcal{O}} = 0$, and hence the Neumann problem with data (v, g) is solvable. Let us chose then V as in Equation (55).

With $\tilde{u} = (u, \zeta) \in X \subset H^{1-k}(\mathcal{O}) \oplus H^{1/2-k}(\partial\mathcal{O})$, as above, we obtain

$$\begin{aligned} \tilde{B}(\tilde{u}, V) &= -\langle u, \Delta V \rangle + \langle \zeta, \partial_\nu V|_{\partial\mathcal{O}} \rangle_{\partial\mathcal{O}} = \langle u, v \rangle + \langle \zeta, g \rangle_{\partial\mathcal{O}} \\ &\geq \frac{1}{2} \|\tilde{u}\| (\|v\|_{H^{k-1}(\mathcal{O})}^2 + \|g\|_{H^{k-1/2}(\partial\mathcal{O})}^2)^{1/2} \geq C \|V\|_{H^{k+1}(\mathcal{O})}^2. \end{aligned}$$

This verifies Condition (ii) of Theorem 4.8.

Finally, to check Condition (iii) of Theorem 4.8, let $V \in Y \subset H^{1+k}(\mathcal{O})$ be such that $\tilde{B}(\tilde{u}, V) = 0$ for all $\tilde{u} \in X \subset \tilde{H}^{1-k}(\mathcal{O})$. Then the definition of the space X as the orthogonal of $(1, 1)$ shows that $-\Delta V = c$ and $\partial_\nu V = c$, for some $c \in \mathbb{C}$. Green's formula gives

$$0 = -\langle 1, \Delta V \rangle + \langle 1, \partial_\nu V|_{\partial\mathcal{O}} \rangle_{\partial\mathcal{O}} = c(\text{vol}(\Omega) + \text{vol}(\partial\Omega)),$$

and hence $c = 0$. From this we next obtain that V is a constant. Since $V \in Y$, this constant must also be zero. $\square$

We therefore obtain from Theorem 4.8 that the Neumann problem, Equation (1) has a weak solution for any $g \in H^{-1/2-k}(\partial\mathcal{O})$.

Theorem 4.10. *Let $g \in H^{-1/2-k}(\partial\mathcal{O})$ satisfy $\langle g, 1 \rangle_{\partial\mathcal{O}} = 0$. Then there exists $\tilde{u} = (u, \zeta) \in \tilde{H}^{1-k}(\mathcal{O})$ satisfying the Equation (1) in weak sense. This solution is uniquely determined if $\langle u, 1 \rangle = 0$ and then it satisfies*

$$\|u\|_{H^{1-k}(\mathcal{O})} + \|\zeta\|_{H^{1/2-k}(\partial\mathcal{O})} \leq C_\mathcal{O} \|g\|_{H^{-1/2-k}(\partial\mathcal{O})},$$

for a constant that depends only on $\mathcal{O}$.

Conversely, let $\tilde{u} = (u, \zeta)$ be a weak solution of Equation (1). By taking $\Delta v = 0$ and $v|_{\partial\mathcal{O}}$ arbitrary, it follows from the definition of the weak solutions, that $\|g\|_{H^{-1/2-k}(\partial\mathcal{O})} \leq C \|\zeta\|_{H^{1/2-k}(\partial\mathcal{O})}$. Then, by taking $v = 0$ on $\partial\mathcal{O}$ but Δv arbitrary, we again obtain from the definition that

$$(58) \quad \|\zeta\|_{H^{1/2-k}(\partial\mathcal{O})} \leq C \|u\|_{H^{1-k}(\mathcal{O})}.$$

Finally, this gives that

$$(59) \quad \|g\|_{H^{-1/2-k}(\partial\mathcal{O})} \leq C\|u\|_{H^{1-k}(\mathcal{O})},$$

as in the case $k < 0$ (note however that in the case $k \geq 0$, we also have $\Delta u = 0$ in distribution sense).

5. APPROXIMATE SOLUTION OF THE LAPLACE EQUATION WITH DISTRIBUTION BOUNDARY CONDITIONS USING THE GFEM

We shall consider the same setting as in the previous sections. For instance, $S = S_\nu$, $\nu \in \mathbb{Z}_+$, will be the GFEM-space associated to any of the data Σ_ν satisfying the assumptions of Subsection 2.3. Also, $\Omega = \mathcal{O}$, a smooth, bounded domain, as in the previous section.

From now on, we shall fix the weak solution $u \in H^{1-k}(\mathcal{O})$, $k \in \mathbb{N}$, $k \leq m-1$, of the Equation (1) satisfying $\langle u, 1 \rangle = 0$. Recall from Definition 4.4 that this means that there exists $\zeta \in H^{1/2-k}(\partial\mathcal{O})$ such that $\tilde{u} = (u, \zeta)$ is a weak solution of the Equation (1) in the sense that

$$(60) \quad \tilde{B}(\tilde{u}, V) = \langle g, V \rangle_{\partial\mathcal{O}}, \quad \text{for all } V \in H^{1+k}(\mathcal{O}),$$

where $\tilde{B}(\tilde{u}, V) = -\langle u, \Delta V \rangle + \langle \zeta, \partial_\nu V|_{\partial\mathcal{O}} \rangle_{\partial\mathcal{O}}$, see Equation (52). Here the data $g \in H^{-1/2-k}(\mathcal{O})$ is also fixed.

With u, ζ , and g as in the paragraph above, we shall define

$$(61) \quad u|_{\partial\Omega} := \zeta \quad \text{and} \quad \partial_\nu u|_{\partial\Omega} := g.$$

We shall think of $\partial_\nu u$ as the normal derivative of u in the direction of the outer normal at the boundary. Recall from Equations (58) and (59), that ζ and g above depend continuously on u . In the forthcoming paper [6], we shall compare our definition of a solution of Equation (1) with the definitions in [15, 26] or [28].)

We shall also assume that

$$(62) \quad \langle u, 1 \rangle = \langle g, 1 \rangle_{\partial\mathcal{O}} = 0,$$

which guarantees the existence of u and that

$$\|u\|_{H^{1-k}(\mathcal{O})} + \|\zeta\|_{H^{1/2-k}(\partial\mathcal{O})} \leq C_{\mathcal{O}}\|g\|_{H^{-1/2-k}(\partial\mathcal{O})},$$

by Theorem 4.10.

These condition of Equation (60) imply that $\langle u, \Delta\phi \rangle = 0$ for all $\phi \in \mathcal{C}_c^\infty(\mathcal{O})$, that is, that $\Delta u = 0$ in the sense of distributions on $\mathcal{O}$. Since ζ is determined by u , we can write $B(u, v) := \tilde{B}(\tilde{u}, v)$. More generally, we shall write

$$(63) \quad B(u + u_1, v) = \tilde{B}(\tilde{u}, v) + B(u_1, v),$$

whenever $u_1 \in H^1(\mathcal{O})$. See also Equation (53). In particular,

$$B(u - u_\nu, v) = 0,$$

where $u_\nu = u_{S_\nu} \in S_\nu$ is the GFEM-approximation of u , see Definition 4.2 and Equation 54. This is in agreement with the results in [15] (especially Theorem 6.5) on traces of functions w such that Δw is regular enough.

We have the following estimate.

Lemma 5.1. *With u as in Equation (60) above, we have*

$$|B(u, v)| := |\tilde{B}(\tilde{u}, v)| \leq C\|u\|_{H^{1-k}(\mathcal{O})}\|v\|_{H^{1+k}(\mathcal{O})},$$

for any $v \in H^{1+k}(\mathcal{O})$ and a constant C depending only on $\mathcal{O}$.

Proof. By definition, using also Equation (58), we have

$$\begin{aligned}
 (64) \quad |B(u, v)| &= |-\langle u, \Delta v \rangle + \langle u|_{\partial\mathcal{O}}, \partial_\nu v \rangle_{\partial\mathcal{O}}| \leq |\langle u, \Delta v \rangle| + |\langle u|_{\partial\mathcal{O}}, \partial_\nu v \rangle_{\partial\mathcal{O}}| \\
 &\leq \|u\|_{H^{1-k}(\mathcal{O})} \|\Delta v\|_{H^{-1+k}(\mathcal{O})} + \|u\|_{H^{1/2-k}(\partial\mathcal{O})} \|\partial_\nu v\|_{H^{-1/2+k}(\partial\mathcal{O})} \\
 &\leq C \|u\|_{H^{1-k}(\mathcal{O})} \|v\|_{H^{1+k}(\mathcal{O})}.
 \end{aligned}$$

This completes the proof. $\square$

We continue with more lemmas. We have the following “inverse property.”

Lemma 5.2. *We have that*

$$(65) \quad \|w\|_{H^t(\mathcal{O})} \leq B h_\nu^{s-t} \|w\|_{H^s(\mathcal{O})},$$

for $0 \leq s \leq t \leq m$ and $w \in S_\nu$, for a constant B independent of ν .

Proof. For $s = t$, the result is tautologically true with $B = 1$. For $s = 0$, the result is given by Proposition 2.5. Since $\mathcal{O}$ is smooth, the general case by interpolation using the results of [15] on the interpolation properties of Sobolev spaces on smooth, bounded domains. $\square$

In the proofs below, we shall occasionally denote $h = h_\nu$.

Lemma 5.3. *The GFEM-approximations u_ν satisfy*

$$\|u_\nu\|_{H^1(\mathcal{O})} \leq C h_\nu^{-k} \|u\|_{H^{1-k}(\mathcal{O})}$$

for a constant C depending only on $\mathcal{O}$ (so C is independent of ν).

Proof. The Poincaré-Friedrichs inequality, Lemma 5.1, and Lemma 5.2 give

$$\begin{aligned}
 \|u_\nu\|_{H^1(\mathcal{O})}^2 &\leq CB(u_\nu, u_\nu) = CB(u, u_\nu) \leq C \|u\|_{H^{1-k}(\mathcal{O})} \|u_\nu\|_{H^{1+k}(\mathcal{O})} \\
 &\leq C h^{-k} \|u\|_{H^{1-k}(\mathcal{O})} \|u_\nu\|_{H^1(\mathcal{O})},
 \end{aligned}$$

where in the last inequality we have used Lemma 5.2 for $s = 1$ and $h = h_\nu$. $\square$

This gives the following corollaries.

Corollary 5.4. *Let $k \leq \lambda$. Then the GFEM-approximations u_ν satisfy*

$$\|u_\nu\|_{H^{1-k}(\mathcal{O})} \leq C \|u\|_{H^{1-k}(\mathcal{O})}$$

for a constant C depending only on $\mathcal{O}$. In particular, u_ν depends continuously on u and $\|u - u_\nu\|_{H^{1-k}(\mathcal{O})} \leq C \|u\|_{H^{1-k}(\mathcal{O})}$.

Proof. The result is well known for $k = 0$ since u_ν is the B -orthogonal projection of u onto S (this is C  a’s Lemma, see [9, 10]). We shall therefore assume that $k \geq 1$. Let $h = h_\nu$.

Let $v \in H^{k-1}(\mathcal{O})$ be arbitrary. Let $c \in \mathbb{C}$ be such that $\int_{\mathcal{O}} (v - c) dx = 0$. Then we can find $V \in H^{k+1}(\mathcal{O})$ such that

$$-\Delta V = v - c, \quad \int_{\mathcal{O}} V dx = 0, \quad \partial_\nu V = 0, \quad \text{and} \quad \|V\|_{H^{k+1}(\mathcal{O})} \leq C \|v\|_{H^{k-1}(\mathcal{O})},$$

where C is a constant depending only on $\mathcal{O}$. Also, chose $w \in S$ such that

$$\|w\|_{H^{k+1}(\mathcal{O})} \leq C \|V\|_{H^{k+1}(\mathcal{O})} \quad \text{and} \quad \|V - w\|_{H^1(\mathcal{O})} \leq C h^k \|V\|_{H^{k+1}(\mathcal{O})}.$$

This is possible by Theorem 3.2. Then

$$\begin{aligned}\langle u_\nu, v \rangle &= \langle u_\nu, v - c \rangle = -\langle u_\nu, \Delta V \rangle = -\langle u_\nu, \Delta V \rangle + \langle u_\nu|_{\partial\mathcal{O}}, \partial_\nu V \rangle_{\partial\mathcal{O}} = B(u_\nu, V) \\ &= B(u_\nu, w) + B(u_\nu, V - w) = B(u, w) + B(u_\nu, V - w).\end{aligned}$$

Using also Lemmas 5.1 and 5.3, this gives

$$\begin{aligned}|\langle u_\nu, v \rangle| &\leq C\|u\|_{H^{1-k}(\Omega)}\|w\|_{H^{1+k}(\Omega)} + \|u_\nu\|_{H^1(\Omega)}\|V - w\|_{H^1(\Omega)} \\ &\leq C\|u\|_{H^{1-k}(\Omega)}\|V\|_{H^{k+1}(\Omega)} + Ch^{-k}\|u\|_{H^{1-k}(\Omega)}h^k\|V\|_{H^{k+1}(\Omega)} \\ &\leq C\|u\|_{H^{1-k}(\Omega)}\|V\|_{H^{k+1}(\Omega)} \leq C\|u\|_{H^{1-k}(\Omega)}\|v\|_{H^{k-1}(\Omega)}.\end{aligned}$$

This gives the result since $\|u_\nu\|_{H^{1-k}(\Omega)} := \sup |\langle u_\nu, v \rangle|/\|v\|_{H^{k-1}(\Omega)}$, $v \neq 0$. $\square$

Similarly,

Corollary 5.5. *We have $\|u_\nu|_{\partial\mathcal{O}}\|_{H^{1/2-k}(\partial\mathcal{O})} \leq C\|u\|_{H^{1-k}(\mathcal{O})}$ for a constant C depending only on $\mathcal{O}$. In particular, $\|(u - u_\nu)|_{\partial\mathcal{O}}\|_{H^{1/2-k}(\partial\mathcal{O})} \leq C\|u\|_{H^{1-k}(\mathcal{O})}$.*

Proof. The proof is similar to that of the previous corollary. Let $v \in H^{-1/2+k}(\partial\mathcal{O})$ be arbitrary. Let $c \in \mathbb{C}$ be a constant such that $\int_{\partial\mathcal{O}} v dS = \int_{\mathcal{O}} c dx$. Then we can find a unique $W \in H^{1+k}(\mathcal{O})$ satisfying

$$\Delta W = c, \quad \int_{\mathcal{O}} W dx = 0, \quad \partial_\nu W = v, \quad \text{and} \quad \|W\|_{H^{k+1}(\Omega)} \leq C\|v\|_{H^{-1/2+k}(\partial\mathcal{O})},$$

for a constant $C > 0$ depending only on $\mathcal{O}$. Using also Theorem 3.2, we choose $w \in S$ such that $\|w\|_{H^{k+1}(\Omega)} \leq C\|W\|_{H^{k+1}(\Omega)}$ and $\|W - w\|_{H^1(\Omega)} \leq Ch^k\|W\|_{H^{k+1}(\Omega)}$.

Then, using also $\langle u_\nu, 1 \rangle = 0$, we obtain

$$\begin{aligned}\langle u_\nu|_{\partial\mathcal{O}}, v \rangle_{\partial\mathcal{O}} &= \langle u_\nu|_{\partial\mathcal{O}}, \partial_\nu W \rangle_{\partial\mathcal{O}} = \langle u_\nu, \Delta W \rangle + B(u_\nu, W) = B(u_\nu, W) \\ &= B(u_\nu, w) + B(u_\nu, W - w) = B(u, w) + B(u_\nu, W - w).\end{aligned}$$

Using Lemmas 5.1 and 5.3, we then obtain

$$\begin{aligned}|\langle u_\nu|_{\partial\mathcal{O}}, v \rangle_{\partial\mathcal{O}}| &\leq C\|u\|_{H^{1-k}(\Omega)}\|w\|_{H^{1+k}(\Omega)} + \|u_\nu\|_{H^1(\Omega)}\|W - w\|_{H^1(\Omega)} \\ &\leq C\|u\|_{H^{1-k}(\Omega)}\|W\|_{H^{1+k}(\Omega)} + Ch^{-k}\|u\|_{H^{1-k}(\Omega)}h^k\|W\|_{H^{1+k}(\mathcal{O})} \\ &\leq C\|u\|_{H^{1-k}(\Omega)}\|W\|_{H^{1+k}(\mathcal{O})} \leq C\|u\|_{H^{1-k}(\Omega)}\|v\|_{H^{-1/2+k}(\partial\mathcal{O})},\end{aligned}$$

which completes the proof in view of the definition of $\|u_\nu\|_{H^{1/2-k}(\Omega)}$. $\square$

We now establish our main approximation results. Recall that $u \in H^{1-k}(\mathcal{O})$ is the weak solution of $\Delta u = 0$, $\partial_\nu u = g \in H^{-1/2-k}(\partial\mathcal{O})$ (Equation (60)) satisfying $\langle u, 1 \rangle = 0$ and that u_ν , $\nu \in \mathbb{Z}_+$, are the GFEM-approximations of u , where $S = S_\nu$ are the GFEM-finite element spaces associated to a sequence of data $\Sigma_\nu = \{\omega_j^\nu, \phi_j^\nu, \Psi_j^\nu, \omega_j^{*\nu}\}$ satisfying the assumptions of Subsection 2.3. The constants $C > 0$ below are assumed to be independent of S_ν .

Proposition 5.6. *Assume that $k + \gamma \leq \lambda$ and $k \leq m - 1$, $\gamma \in \mathbb{Z}_+$. Then the error $u - u_\nu \in H^{1-k}(\mathcal{O})$ satisfies*

$$\|u - u_\nu\|_{H^{1-k-\gamma}(\Omega)} \leq Ch_\nu^\gamma \|u\|_{H^{1-k}(\Omega)},$$

with a constant C independent of ν .

Proof. Let $v \in H^{-1+k+\gamma}(\mathcal{O})$ be arbitrary. Let c be a constant such that $\langle v - c, 1 \rangle = 0$. Then there exists a unique $V \in H^{1+k+\gamma}(\mathcal{O})$ such that

$$-\Delta V = v - c, \quad \int_{\mathcal{O}} V dx = 0, \quad \partial_{\nu} V = 0, \quad \text{and} \quad \|V\|_{H^{1+k+\gamma}(\mathcal{O})} \leq C \|v\|_{H^{-1+k+\gamma}(\mathcal{O})},$$

for a constant $C > 0$ depending only on $\mathcal{O}$.

Then, for any $w \in S$,

$$\begin{aligned} (66) \quad \langle u - u_{\nu}, v \rangle &= \langle u - u_{\nu}, v - c \rangle = -\langle u - u_{\nu}, \Delta V \rangle \\ &= -\langle u - u_{\nu}, \Delta V \rangle + \langle (u - u_{\nu})|_{\partial\mathcal{O}}, \partial_{\nu} V \rangle_{\partial\mathcal{O}} = B(u - u_{\nu}, V) = B(u - u_{\nu}, V - w) \\ &= -\langle u - u_{\nu}, \Delta(V - w) \rangle + \langle (u - u_{\nu})|_{\partial\mathcal{O}}, \partial_{\nu} w \rangle_{\partial\mathcal{O}}. \end{aligned}$$

Since $k + \gamma \leq \lambda$ and $k \leq m - 1$, the assumptions of Theorem 3.2 are satisfied, so we can choose $w \in S$ such that $\|w\|_{H^{1+k+\gamma}(\Omega)} \leq C \|V\|_{H^{1+k+\gamma}(\Omega)}$ and $\|V - w\|_{H^{1+k}(\Omega)} \leq Ch^{\gamma} \|V\|_{H^{1+k+\gamma}(\Omega)}$. In particular,

$$\|\partial_{\nu} w\|_{H^{-1/2+k}(\partial\mathcal{O})} = \|\partial_{\nu}(V - w)\|_{H^{-1/2+k}(\partial\mathcal{O})} \leq h^{\gamma} \|V\|_{H^{1+k+\gamma}(\Omega)}.$$

From $\|V\|_{H^{1+k+\gamma}(\Omega)} \leq C \|v\|_{H^{-1+k+\gamma}(\mathcal{O})}$, Corollaries 5.4 and 5.5, and Equation (66), we then obtain,

$$\begin{aligned} |\langle u - u_{\nu}, v \rangle| &\leq \|u - u_{\nu}\|_{H^{1-k}(\Omega)} \|V - w\|_{H^{1+k}(\Omega)} + \|u - u_{\nu}\|_{H^{1-k}(\Omega)} \|\partial_{\nu} w\|_{H^{-1/2+k}(\partial\mathcal{O})} \\ &\leq Ch^{\gamma} \|u\|_{H^{1-k}(\Omega)} \|v\|_{H^{-1+k+\gamma}(\mathcal{O})}. \end{aligned}$$

The proof is complete. $\square$

We then obtain.

Proposition 5.7. *Assume that the local approximation spaces Ψ_j contain the polynomials of degree $\lambda \geq k + \gamma$, $\gamma \in \mathbb{Z}_+$, and let $A \Subset B \Subset \mathcal{O}$ be open subsets. Then for h_{ν} small enough and $k + \gamma \leq m + 1$, $k \leq m - 1$, and $\lambda \geq l \geq 1$, we have*

$$\|u - u_{\nu}\|_{H^1(A)} \leq Ch_{\nu}^l \|u\|_{H^{l+1}(B)} + Ch_{\nu}^{\gamma} \|u - u_{\nu}\|_{H^{1-k}(B)},$$

where the constant C depends on the structural constants and the distance from ∂A to ∂B , but not on ν .

Proof. Assume first that $u \in H^1(\mathcal{O})$ and let us replace m with $k + \gamma - 1$ in Theorem 3.12, which is possible since $k + \gamma \leq m + 1$ and we can always decrease m in Theorem 3.12. The use Theorem 3.12 for $A \Subset B_1 \subset B$, for some admissible open subset B_1 such that $B'_1 \subset B$. The result then follows from Proposition 5.6 and from

$$\inf_{\chi \in S} \|u - \chi\|_{H^1(B_1)} \leq Ch^l \|u\|_{H^{l+1}(B'_1)} \leq Ch^l \|u\|_{H^{l+1}(B)}.$$

In general, for $u \in H^{1-k}(\mathcal{O})$, the result follows by continuity, using Corollary 5.4 and Equation (3). $\square$

We keep all our previous assumptions, the most important of which were recalled before Proposition 5.6. In particular, the assumptions of Subsection 2.3 remain valid. Also, recall that the local approximation spaces Ψ_j are assumed to contain the polynomials of degree λ . By taking $l = \gamma$ and using again Equation (3), we obtain the following error estimate for the GFEM-approximation u_{ν} .

Theorem 5.8. *Let $u \in H^{1-k}(\mathcal{O})$ be a weak solution of the Neumann problem $\Delta u = 0$, $\partial_\nu u = g \in H^{-1/2-k}(\partial\mathcal{O})$ and $u_\nu \in S_\nu$ be its GFEM-approximation, where the sequence S_ν satisfies the assumptions of Subsection 2.3 with $k \leq \min\{\lambda, m-1\}$. Also, let $A_0 \Subset \mathcal{O}$ be an open subset of the smooth, bounded domain $\mathcal{O}$. Then for h_ν small enough.*

$$\|u - u_\nu\|_{H^1(A_0)} \leq Ch_\nu^\gamma \|u\|_{H^{1-k}(\mathcal{O})}, \quad \gamma = \min\{\lambda, m+1\} - k.$$

The constant C above depends only the structural constants and the distance from ∂A_0 to $\partial\mathcal{O}$, but not on ν .

Remark 5.9. The constant $C > 0$ above may depend on γ and on the structural constants A , C_j , κ , σ , λ , and m , as well as on A_0 and $\mathcal{O}$.

In particular, since $\|u\|_{H^{1-k}(\mathcal{O})} \leq C\|g\|_{H^{-1/2-k}(\partial\mathcal{O})}$, by Theorem 4.10, we obtain the following corollary.

Corollary 5.10. *Under the assumptions of Theorem 5.8, we have*

$$\|u - u_\nu\|_{H^1(A_0)} \leq Ch_\nu^\gamma \|g\|_{H^{-1/2-k}(\partial\mathcal{O})},$$

with $C > 0$ independent of ν and h_ν small enough.

Remark 5.11. The condition that h_ν be small enough is not an essential restriction. Indeed, by increasing the constants C in the above results, we can drop the requirement that h_ν be small enough.

6. POLYNOMIAL LOCAL APPROXIMATION SPACES

In this section we shall verify that the Conditions A–D are verified if we choose $\Psi_j = \mathcal{Q}_m$, to be the space of polynomials of degree $\leq m$, $1 \leq m$, and the boundary of Ω is piecewise smooth, for a suitable covering and subordinated partition of unity. Some results in this section are either elementary or well known. We include them nevertheless for the benefit of the reader and for completeness.

For any ball B of radius r , we shall denote by tB the ball with the same center as B and radius tr .

Lemma 6.1. *There exists a constant $C > 0$, depending only on n , m , and M , such that for any ball $B \subset \mathbb{R}^n$, for any $Q \in \mathcal{Q}_m$, and for any $t \in (0, M]$, we have $\|Q\|_{L^2(tB)} \leq C\|Q\|_{L^2(B)}$.*

Proof. For any fixed B , $Q \mapsto \|Q\|_{L^2(tB)}$ and $Q \mapsto \|Q\|_{L^2(B)}$ are two norms on the finite dimensional space $\mathcal{Q}_m$ of polynomials of degree $\leq m$, and hence they are equivalent. This gives the result, except the independence of C on B and t . But all balls are affine equivalent and the L^2 -norm is scaled by the (square root of the) determinant of the matrix of the affine transformation. Thus the constant C can be chosen to be the same for all balls B . $\square$

This gives immediately the following corollary.

Corollary 6.2. *There exists a constant $C > 0$, depending only on n , m , and M such that for any ball $B \subset \mathbb{R}^n$, for any polynomial $Q \in \mathcal{Q}_m$, and for any $t \in (0, M]$, we have $|Q|_{H^l(tB)} \leq C|Q|_{H^l(B)}$ and $\|Q\|_{H^l(tB)} \leq C\|Q\|_{H^l(B)}$, $0 \leq l \leq m$.*

Proof. Use Lemma 6.1 for all derivatives $Q^{(\alpha)}$, where $|\alpha| \leq l$. $\square$

We now establish to the following “inverse property.”

Lemma 6.3. *There exists a constant $C > 0$, depending only on n , m , and α , such that $\|Q^{(\alpha)}\|_{L^2(B)} \leq Cr^{l-|\alpha|}\|Q\|_{H^1(B)}$ for any $l \leq |\alpha| \leq m$, any $Q \in \mathcal{Q}_m$, and any ball B of radius r .*

Proof. Let us prove first the result for $l = 0$. That is, we need to prove that $\|Q^{(\alpha)}\|_{L^2(B)} \leq Cr^{-|\alpha|}\|Q\|_{L^2(B)}$.

Let $B_1 = B_1(0)$ be the unit ball centered at 0. Then $Q \mapsto \|Q^{(\alpha)}\|_{L^2(B_1)}$ is a semi-norm on $\mathcal{Q}_m$, the space of polynomials of degree at most m , and hence it is bounded by the norm $Q \mapsto \|Q\|_{L^2(B_1)}$. Thus $\|Q^{(\alpha)}\|_{L^2(B_1)} \leq C_1\|Q\|_{L^2(B_1)}$. Let L be an affine transformation mapping B_1 onto the ball B of radius r consisting of the composition of a translation and a dilation of ratio r (so $\det(L) = r^n$). Then

$$\begin{aligned} \|Q^{(\alpha)}\|_{L^2(B)} &= \det(L)^{1/2} \|Q^{(\alpha)} \circ L\|_{L^2(B_1)} = \det(L)^{1/2} r^{-|\alpha|} \|(Q \circ L)^{(\alpha)}\|_{L^2(B_1)} \\ &\leq C_1 \det(L)^{1/2} r^{-|\alpha|} \|Q \circ L\|_{L^2(B_1)} = Cr^{-|\alpha|} \|Q\|_{L^2(B)}, \end{aligned}$$

for any $Q \in \mathcal{Q}_m$.

Assume now that $|\alpha| \geq l > 0$. Choose $\beta \leq \alpha$, $|\beta| = l$. Then

$$\|D^{\alpha-\beta} D^\beta Q\|_{L^2(B)} \leq Cr^{l-|\alpha-\beta|} \|D^\beta Q\|_{L^2(B)} \leq Cr^{l-|\alpha|} \|Q\|_{H^1(B)}.$$

This completes the proof. $\square$

The relevant “inverse property,” implying also Condition C, now follows.

Proposition 6.4. *There exists a constant $C > 0$, depending only on n , m , α , and σ , such that $\|Q^{(\alpha)}\|_{L^2(\Omega)} \leq Cr^{l-|\alpha|}\|Q\|_{H^1(B)}$ for any $l \leq |\alpha| \leq m$, any $Q \in \mathcal{Q}_m$, any ball B of radius r , and any set Ω contained in $\sigma^{-1}B$ and star-shaped with respect to B .*

Proof. This follows from Corollary 6.2 and Lemma 6.3. $\square$

We now prove the following elementary lemma.

Lemma 6.5. *Assume the data defining S is fixed. Then for any $k \in \mathbb{Z}_+$ and any open sets $A_0 \Subset A_k \subset \Omega$, we can construct admissible open sets $B_1, \dots, B_{k-1}$ such that $A_0 =: B_0 \Subset B_1 \Subset B_2 \Subset \dots \Subset B_{k-1} \Subset B_k := A_k$ and $\hat{C} \text{dist}(\partial B_j, \partial B_{j-1}) \geq \theta/k$, where $\theta := \text{dist}(\partial A_0, \partial A_k)$, provided that $\hat{C}h < \theta$, where $\hat{C}$ depends on k and n only (in particular, $\hat{C}$ is independent of h).*

Proof. Take $k = 2$, for simplicity. The general result is proved similarly or by iterating this case. Let $\hat{C} = 4$. Let U be the union of all open sets ω_j at distance at most $\theta/4$ from A_0 . Let J be the set of indices j such that $\phi_j \neq 0$ on U and let G be the set where $\sum_{j \in J} \phi_j = 1$. We then define B_1 to be the interior of G . $\square$

6.1. Partition of Unity. We show in this subsection that, for a piecewise C^1 domain Ω , we can choose a sequence $\Sigma_\nu = \{\omega_j^\nu, \phi_j^\nu, \Psi_j^\nu, \omega_j^{*\nu}\}_{j=1}^{N_\nu}$, $\nu \in \mathbb{Z}_+$, of GFEM-data, $\cup_j \omega_j^\nu = \Omega$, satisfying the assumptions of Subsection 2.3. Other examples of sequences Σ_ν , possibly better suited for numerical implementation, will be included in the forthcoming paper [7], where we will also discuss the numerical implementation of the GFEM for boundary value problems with distributional data.

Theorem 6.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with piecewise C^1 -boundary and all angles > 0 . Then there exist structural constants A , C_j , σ , κ , λ , and m such that, for any small enough $h > 0$, we can construct a partition of unity $\{\phi_j\}$*

subordinated to the covering $\{\omega_j\}$ and satisfying all the assumptions of subsection (2.3), in particular, satisfying conditions $A(h)$, B , C , and D .

Proof. Let us first triangulate Ω with (possibly curvilinear) triangles T_j with diameters $\leq h/2$ and $\geq C_1 h$ and satisfying the angle condition. (Recall that this means that there exists an angle $\theta > 0$ such that all angles of the triangles are $\geq \theta$, with θ independent of h .) Only the sides on the boundary of Ω are allowed to be curvilinear. The interior sides are required to be straight segments. This is possible since Ω has a piecewise C^1 boundary and all angles are > 0 .

We can then choose $\sigma > 0$ small enough and independent of h such that each triangle will contain a ball of radius $\geq 2\sigma h$. Let x_j be the centers of these balls and let ω_j^* be the ball with radius σh and center x_j . Then let $\omega_j = B(x_j, h) \cap \Omega$, where $B(x, r)$ denotes the open ball with center x and radius r . Our construction shows that the balls $2\omega_j^* = B(x_j, 2\sigma h)$ are disjoint.

Then $\Omega = \cup_{j=1}^N \omega_j$. Let $\tilde{\phi}_j(x) = \eta(|x - x_j|/h)$, where η is a fixed, smooth function $\eta : [0, \infty) \rightarrow [0, 1]$ such that $\eta(t) = 1$ for $t \leq 1/2$ and $\eta(t) = 0$ for $t \geq 1$. Then $\|\partial^\alpha \phi_j\|_{L^\infty(\Omega)} \leq C_k / (\text{diam } \omega_j)^k$, for $k = |\alpha| \leq m$, and any $j = 1, \dots, N$. Moreover, $\tilde{\phi}_j = 1$ on ω_j and hence

$$\sum \tilde{\phi}_j \geq 1.$$

Let now $\psi_j(x) = \eta(|x - x_j|/(\sigma h))$, so that ψ_j has support in $2\omega_j^*$ and is equal to 1 on ω_j^* . Define $\hat{\psi}_j = 1 - \psi_j$ and let $\psi = \prod_{j=1}^N \psi_j$. Then $\psi = 0$ on $\cup \omega_j^*$ and $\|\partial^\alpha \phi_j\|_{L^\infty(\Omega)} \leq C_k / (\text{diam } \omega_j)^k$, for $k = |\alpha| \leq m$, and any $j = 1, \dots, N$. Let $\hat{\phi}_j := \psi \tilde{\phi}_j + \psi_j$. An easy verification shows that we still have $\sum_j \hat{\phi}_j \geq 1$, because

- (i) the balls $2\omega_j^*$ are disjoint;
- (ii) $\hat{\phi}_j = \tilde{\phi}_j$ outside $\cup 2\omega_j^*$,
- (iii) $\hat{\phi}_j = 1$ on $2\omega_j^*$, and
- (iv) $\hat{\phi}_j \geq 0$ everywhere.

Finally, we consider the Shephard functions

$$\phi_j := \left(\sum_k \hat{\phi}_k \right)^{-1} \hat{\phi}_j.$$

Our construction shows that ϕ_j is a $(\kappa, C_0, C_1, \dots, C_m)$ partition of unity. Moreover, $\phi_j = 1$ on ω_j^* , because $\phi_k(x) = \delta_{kj}$ for $x \in \omega_j^*$ (recall that $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ otherwise). $\square$

Some assumptions on the domain Ω in the above theorem are necessary, as shown by the following remark.

Remark 6.7. The non-Lipschitz domain

$$\Omega_c := \{(x, y), -x^2 \leq y \leq x^2, x^2 + y^2 \leq 1, x \geq 0\}$$

will have no covering $\{\omega_j\}$ satisfying the Condition A.

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