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Efficient Implementation of Controllers for Large Scale Linear Systems via Wavelet Packet Transforms

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Efficient Implementation of Controllers for Large Scale Linear Systems via Wavelet Packet Transforms

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Abstract

In this paper we present a method of efficiently implementing controllers for linear systems with large numbers of sensors and actuators. It is well known that singular value decomposition can be used to diagonalize any real matrix. Here, we use orthogonal transforms from the wavelet packet to "approximate" SVD of the plant matrix. This yields alternate bases for the input and output vector which allow for feedback control using local information. This fact allows for the efficient computation of a feedback control law in the alternate bases. Since the wavelet packet transforms are also computationally efficient, this method provides a good alternative to direct implementation of a controller matrix for large systems.

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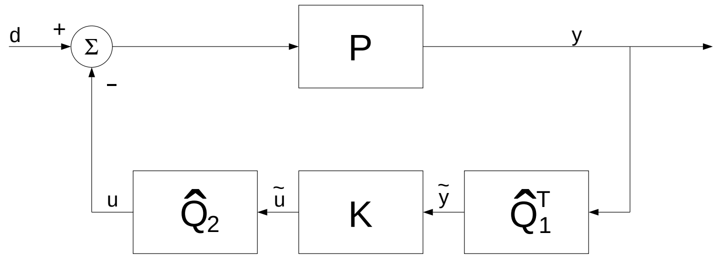

Figure 1: Block diagram of control law implementation.

1 Introduction

This document outlines a novel method of implementing controllers for dynamic systems with many (on the order of one thousand) inputs and outputs. The idea presented here is to perform a fast orthogonal transform over the output space of the system, changing the basis of the output signal from the Euclidean basis (the standard basis for $\mathbb{R}^n$) to the basis associated with the transform. The hope is that this new basis will allow the implementation of the desired control law with fewer computations and less communication than would be required to implement the equivalent control law in the original basis. Chou, Guthart, and Flamm [1] have proposed the use of the fast wavelet transform for this purpose, however these authors do not suggest a systematic way of finding a suitable wavelet basis.

Instead of limiting ourselves to the wavelet basis, our approach uses an orthogonal transform selected from the large collection of such transforms which constitute the wavelet packet. Further, we borrow an image processing technique developed by Wickerhauser [4] to select the "best" basis from the packet.

Once the basis is selected, the associated transform can be used toward the efficient implementation of a given controller. The output vector, y, is transformed into the new basis to yield the transformed output vector, $\tilde{y}$. The transformed input vector, $\tilde{u}$ is then computed based on $\tilde{y}$. Finally, the actual control vector, u, is found by performing the inverse transform on $\tilde{u}$. This implementation is depicted in Figure 1, where the forward and inverse transforms are represented by $\widehat{Q}_1^T$ and $\widehat{Q}_2$, respectively.

The forward and inverse transforms from the wavelet packet are fast; an N dimensional vector can be transformed in $O(N\log N)$ operations [3]. In this paper, we will show that it is possible to find a basis such that the transformed control vector, $\tilde{u} = K\tilde{y}$, can be computed in $O(N)$ operations. As a result, the controller implementation depicted in Figure 1 requires $O(N\log N) + O(N) + O(N\log N)$ operations. This compares favorably with the $O(N^2)$ operations required to implement a controller directly.

In Section 2, we introduce the plant model which is used in this paper. Section 3 describes the notion of plant diagonalization. It contains a review of the well known matrix factorization technique of singular value decomposition (SVD) along with a discussion of how to approximate SVD using a wavelet packet transform. The design of a simple controller which takes advantage of the approximately diagonalized plant matrix is discussed in Section 4. Finally, techniques presented in Sections 3 and 4 are illustrated with an example in Section 5.

2 Plant Structure

For this discussion, we restrict ourselves to plants of the form

$$y_c = P_c u_c \tag{1}$$

where $y_c \in \mathbb{C}^p$, $u_c \in \mathbb{C}^m$, and $P_c \in \mathbb{C}^{p\times m}$. Such a system can be used to model any linear time invariant system whose inputs are all sinusoids of a fixed frequency, ω. Each element of the input and output vectors is a complex number denoting the amplitude and phase of the sinusoid. The complex valued plant matrix can be thought of as a transfer function matrix evaluated at the fixed frequency ω.

The approximate diagonalization algorithm discussed in Section 3 works only for systems with real valued inputs, outputs, and plant matrix elements. For this reason, it is necessary to convert the complex valued system of Equation 1 to an equivalent real valued representation. This can be done via the following steps:

1. Replace the kth element of u_c with the column vector $[\text{real}(u_c(k)) \quad \text{imag}(u_c(k))]'$, for $k = 1, \ldots, m$.

2. Replace the kth element of y_c with the column vector $[\text{real}(y_c(k)) \quad \text{imag}(y_c(k))]'$, for $k = 1, \ldots, p$.

3. Replace the (h, k)th element of P_c with the 2×2 matrix

$$\begin{bmatrix} \text{real}(P_c(h,k)) & -\text{imag}(P_c(h,k)) \\ \text{imag}(P_c(h,k)) & \text{real}(P_c(h,k)) \end{bmatrix}.$$

Clearly, this transformation is invertible. Inspection shows that complex multiplication is equivalent to matrix multiplication under this transformation. Hence, this transformation yields an equivalent real valued system of the form

$$y = Pu \tag{2}$$

where $y \in \mathbb{R}^{2p}$, $u \in \mathbb{R}^{2m}$, and $P \in \mathbb{R}^{2p \times 2m}$.

Control design for plants of this form leads naturally to a "frequency banded" approach to developing controllers for more general plants. In such an approach, the spectrum to be controlled is divided up into frequency bands so within each band the plant can be represented as a complex valued matrix. Individual controllers are then developed for each band. These individual controllers can then be added together (with the proper filtering) to create a controller which works well over the whole spectrum. Hence, the results presented in this paper can be applied to a range of plants which is much broader than LTI systems with a fixed disturbance frequency.

3 Plant Matrix Diagonalization

Recall that we wish to find a fast transform so that, in the transformed basis, the control $\widetilde{u}$ can be efficiently computed from the output vector $\widetilde{y}$. Singular value decomposition (SVD) can be used to generate a good basis for our purposes. SVD can be used to diagonalize any real matrix [2]. Consider the real valued $p \times m$ matrix P. SVD states that P can be factored such that

$$P = Q_1 \Sigma Q_2^T, \tag{3}$$

where

1. The columns of $Q_1 \in \mathbb{R}^{p \times p}$ are the eigenvectors of PP^T.
2. The columns of $Q_2 \in \mathbb{R}^{m \times m}$ are the eigenvectors of $P^T P$.
3. The $p \times m$ matrix Σ is diagonal, and values on the diagonal are the square roots of the eigenvalues of PP^T and $P^T P$.

The matrices Q_1 and Q_2 are orthogonal, i.e. $Q_1 Q_1^T = \mathbf{I}_p$ and $Q_2 Q_2^T = \mathbf{I}_m$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Hence, multiplying P by Q_1^T on the left and Q_2 on the right yields

$$Q_1^T P Q_2 = \Sigma. \tag{4}$$

In the present context, P is a plant matrix of an m-input, p-output system. In other words,

$$y = Pu, \tag{5}$$

where $u \in \mathbb{R}^m$ is the input vector and $y \in \mathbb{R}^p$ is the output vector.

Now let $\widetilde{y} = Q_1^T y$ and $\widetilde{u} = Q_2^T u$. Substitution into the plant equation (Equation 5) yields

$$\begin{aligned} \widetilde{y} &= Q_1^T P Q_2 \widetilde{u} \\ &= \Sigma \widetilde{u}. \end{aligned} \tag{6}$$

The ith element of the transformed input vector $\widetilde{u}$ affects only the ith element of the transformed output vector $\widetilde{y}$ for $i = 1, 2, \ldots, \min(p, m)$. The entire system can be controlled using local feedback, i.e. each input is determined based solely on the value of its corresponding output. In the transformed basis, the control $\widetilde{u}$ can be computed from $\widetilde{y}$ in $O(\min(p, m))$ operations.

Unfortunately, the SVD generated transforms are computationally intensive; it takes $O(N^2)$ operations to transform an N dimensional vector. This obstacle can be overcome by using a fast wavelet packet transform (FWPT) which approximates the Karhunen-Loève transform. As will be shown later in this section, the Karhunen-Loève transform can be used to perform SVD. An algorithm to find such a transform has been presented by Wickerhauser [4]. The FWPT is computationally efficient; it takes $O(N \log N)$ operations to transform an N dimensional vector.

We know that the Karhunen-Loève representation of a given ensemble of vectors has a lower entropy than any other orthonormal representation. Wickerhauser's algorithm recursively searches all of the orthogonal bases in the wavelet packet to find the basis in which the vector ensemble has the lowest entropy representation. This basis is then "closer" to Karhunen-Loève than any other basis in the packet. Hence, the wavelet packet transform to this basis will be used as our "approximate" Karhunen-Loève transform.

If we consider an ensemble of vectors composed of the columns of P, the Karhunen-Loève transform for this ensemble is identical to the SVD factor Q_1^T. Similarly, if we let the ensemble be the rows of P, the resulting Karhunen-Loève transform is identical to the SVD factor Q_2^T. Thus, we can use the Karhunen-Loève transform to diagonalize the plant matrix P. Or we can use an "approximate" Karhunen-Loève transform to "approximately" diagonalize P.

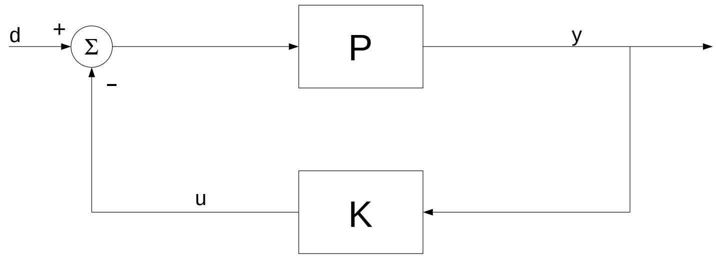

Figure 2: Block diagram of full feedback system.

4 Controller Design

Before we discuss the development of the feedback control law for this method, we review the centralized case so we have something to which we can compare the results. Consider the feedback system depicted in Figure 2. Here, d is the disturbance input, u is the control input, and y is the output. The object is to find a control matrix K such that the effect of the disturbance on the output it minimized. Consider the case where P is square (i.e. $p = m$) and invertible. Then letting $K = \gamma P^{-1}$ yields

$$\begin{aligned} y &= P(d - \gamma P^{-1} y) \\ &= Pd - \gamma y \\ &= \frac{P}{1+\gamma} d. \end{aligned}$$

This control law reduces the transmission from d to y by a factor of $1 + \gamma$.

Now consider the system shown in Figure 1. If we let $\widehat{\Sigma}$ be the approximately diagonalized plant matrix, then simple matrix manipulation shows that $\widehat{Q_2}\gamma\widehat{\Sigma}^{-1}\widehat{Q_1}^T = \gamma P^{-1}$. As a result, letting $K = \gamma\widehat{\Sigma}^{-1}$ accomplishes a disturbance transmission reduction by a factor of $1 + \gamma$.

Unfortunately, $\widehat{\Sigma}$ is only "approximately" diagonal, so $\gamma\widehat{\Sigma}^{-1}$ will not be a diagonal matrix. This fact ruins the locality of the feedback law in the transformed space. This locality is the feature which allows the efficient computation of $\widetilde{u}$. Hence, we need to find a way to implement something close to $\gamma\widehat{\Sigma}^{-1}$ while preserving as much locality as possible.

The first idea that comes to mind is to simply ignore the off-diagonal elements of $\widehat{\Sigma}$. In other words, define $\widetilde{\Sigma}$ to be equal $\widehat{\Sigma}$ with all off-diagonal elements replaced by zeros. Now the feedback law $\gamma\widetilde{\Sigma}^{-1}$ is diagonal and can be implemented locally. In sequel, we will denote this feedback law as the "pure local controller".

The pure local feedback law will work well if the transforms $\widehat{Q_1}^T$ and $\widehat{Q_2}$ are good approximations of Q_1^T and Q_2, but this is not always the case. If the approximate transforms are not sufficiently close to the desired Karhunen-Loève transforms, then $\widehat{\Sigma}$ will have off-diagonal elements which cannot simply be ignored. In this case, we will be forced to violate the locality of the feedback by allowing a small number of the transformed inputs have access to the information from a small number of the transformed outputs.

The first step in doing this is to set all of the off-diagonal elements of $\widehat{\Sigma}$ whose absolute values are below a certain threshold to zero. Let n be the number of non-zero off-diagonal elements which remain. Next, the transformed inputs and outputs are reordered so that the resulting matrix takes the block diagonal form

$$\widetilde{\Sigma} = \begin{bmatrix} M_1 & & & 0 \\ & \ddots & & \\ & & M_k & \\ 0 & & & S \end{bmatrix}, \tag{7}$$

where M_i, $i = 1, \ldots, k$, $k \leq n$ are square and S is $(m-n) \times (m-n)$ and diagonal. The inverse of $\widetilde{\Sigma}$ is then

$$\widetilde{\Sigma}^{-1} = \begin{bmatrix} M_1^{-1} & & & 0 \\ & \ddots & & \\ & & M_k^{-1} & \\ 0 & & & S^{-1} \end{bmatrix}. \tag{8}$$

As a result, to implement the control law $K = \gamma\widehat{\Sigma}^{-1}$, a few of the inputs require information from a few of outputs but the rest of the inputs can be calculated locally. We will denote this feedback law as the "local+n controller".

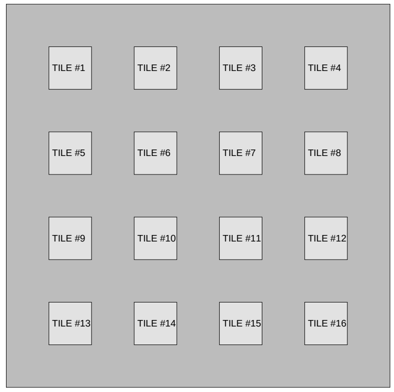

Figure 3: Diagram of 4×4 rectangular array of CSM Tiles.

The hope is that feedback of the pure local or local+n (with n small) form will perform comparably with the full feedback law. For this comparison, we look to the example in the next section.

5 Example

The Composite Smart Material (CSM) Tile [5] provides the motivation for the following example. The CSM tile is a self contained unit containing underwater acoustic sensors and actuators along with a power supply and signal processing hardware. In practice, thousands of these "smart tiles" will be mounted on the outside of the hull of a submarine, covering the entire hull. This massive sensor/actuator array will then be used to actively reduce acoustic radiation, cancel enemy sonar pulses, and perform acoustic sensing.

Here we consider the example of a rectangular 4×4 array of the CSM Tiles. We assume that the disturbance signal is a sinusoid of a fixed frequency, $d = \sin(2\pi f)$, so that we can model that interaction between the tiles as a complex valued plant matrix. The medium between the tiles is homogeneous so that the wavelength of the disturbance signal, λ, is fixed. The magnitude of the coupling between any two tiles is $\frac{1}{d^2}$, where d is the distance between them. The phase shift between any two tiles is $2\pi \ (d \mod \lambda)$. These matrices are shown in the top half of Figure 4 where the real valued elements of the matrix are replaced by pixels of varying intensity.

The first step is to transform the complex valued plant matrix to its real matrix equivalent as described in Section 2. This yields a 32×32 real valued matrix. The absolute values of the elements of this matrix are shown in the bottom half of Figure 4. Now we can write a model of the system in the form of Equation 2. The input vector u and the output vector y are both in $\mathbb{R}^{32}$. The real and imaginary parts of the sensor output of the kth tile become $y(2k-1)$ and $y(2k)$, respectively. Likewise, the real and imaginary parts of the input to the actuator on the kth tile become $u(2k-1)$ and $u(2k)$, respectively. This means something important to the local+n controller design: since each sensor effectively measures two outputs (real and imaginary part) and each actuator effectively takes two control signals, adding one connection between two tiles effectively brings the data from 2 inputs and 2 outputs together. This means that the local+n controller will include the 2×2 blocks with the largest norm in the approximately diagonalized matrix.

The next step is to find the approximate forward and inverse transforms to diagonalize the plant matrix P. This was done and the transforms were used to compute the approximately diagonalized plant matrix, $\widehat{\Sigma} = \widehat{Q_1}^T P \widehat{Q_2}$. The absolute values of the elements of this matrix are shown in Figure 5.

Controllers for this system were developed as outlined in Section 4. The parameter γ was set to 9 so that the full feedback control case attenuates the disturbance by a factor of 10. The disturbance transmission matrices for the local+5 controller are shown in Figure 6. For this controller, Figure 5 was used to determine which off-diagonal blocks to include. As the figures show, the performance of the local+5 controller is very close to that of the full feedback control.

Intuition tells us that as n gets larger, the performance of the local+n controller should get better. To check this, we introduce a measure of closeness of the performance of the local+n controller to the performance of the full feedback controller. Let P_{full} and P_n be the closed loop disturbance transmission matrix under full feedback control and local+n control, respectively. We now define the error of the local+n controller, E_n, as

$$E_n = \|P_{full} - P_n\|_2 \tag{9}$$

where $\|\cdot\|_2$ represents the matrix 2-norm, or largest singular value. A plot of E_n vs. n is shown in Figure 7. The plot matches our intuition, but it also shows something that is not obvious. In a 16-input, 16 output system, there

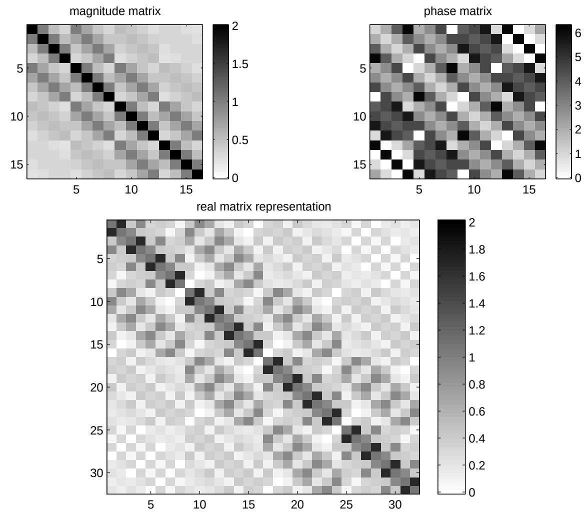

Figure 4: Plant matrices for 4×4 rectangular CSM Tile array.

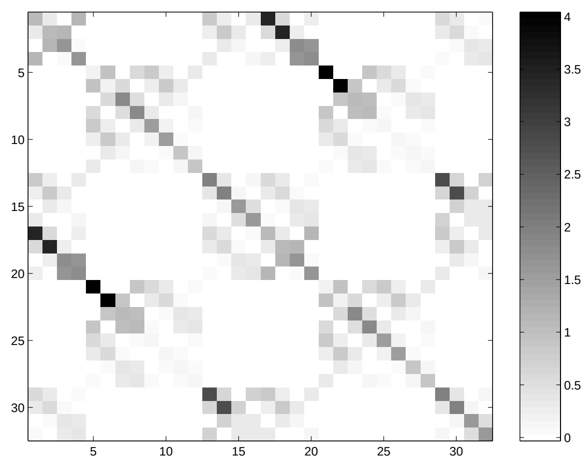

Figure 5: Approximately diagonalized plant matrix, $\widehat{\Sigma} = \widehat{Q_1}^T P \widehat{Q_2}$.

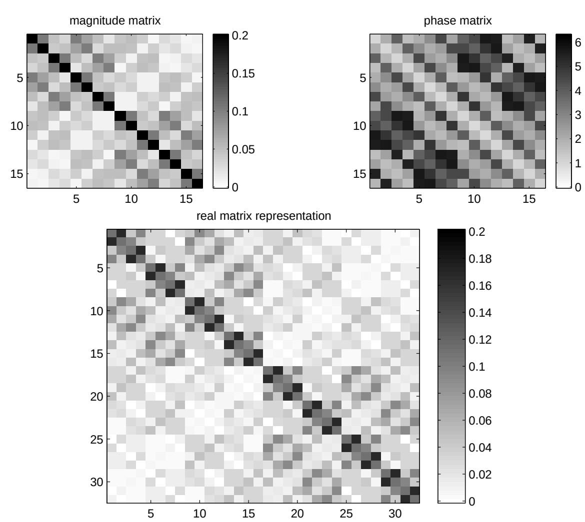

Figure 6: Disturbance transmission matrices for local+5 controller.

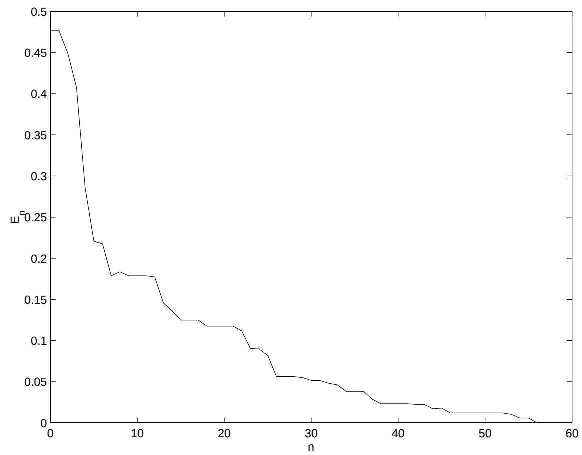

Figure 7: Error of local+n controller, $E_n = \|P_{full} - P_n\|_2$.

are $(16 \times 16 - 16)/2 = 120$ possible off-diagonal connections to be made. In other words, the local+120 controller is equivalent to the full feedback controller. But as Figure 7 shows, the error of the local+n controller is zero for all $n \geq 56$. This means that for this example, it is possible to exactly reproduce the performance of the full feedback controller using less than half of the possible connections.

6 Conclusion

Here we have presented a computationally efficient method of implementing controllers for large scale systems. In this method, the output vector is transformed into a basis which allows full feedback control using only local information. This can be done because the basis transform is chosen to approximately diagonalize the plant matrix. The resulting locality provides advantages in both design and implementation. These results should prove useful for the growing number of applications which require large numbers of sensors and actuators.

References

[1] K. Chou, G. Guthart, and D. Flamm. A multiscale approach to the control of smart materials. In *Proceedings of the SPIE Conference on Smart Structures and Materials*, volume 2447, pages 249–263, 1995.

[2] G. Strang. *Linear Algebra and Its Applications.* Saunders HBJ, 1988.

[3] G. Strang and T. Nguyen. *Wavelets and Filter Banks.* Wellesley-Cambridge Press, 1996.

[4] M.V. Wickerhauser. Large-rank approximate principal component analysis with wavelets for signal feature discrimination and the inversion of complicated maps. *J. Chem. Inf. Comput. Sci.*, 34:1036–1046, 1993.

[5] S.R. Winzer. Composite smart material for defense and dual use applications. In *Smart Materials and Structures: DARPA Technology Interchange Meeting #4*, pages 113–133, August 1997.