

PH.D. THESIS

Motion Control for Nonholonomic Systems on Matrix Lie Groups

by Herbert Karl Struemper
Advisor: P.S. Krishnaprasad

CDCSS Ph.D. 98-1
(ISR Ph.D. 98-1)



The Center for Dynamics and Control of Smart Structures (CDCSS) is a joint Harvard University, Boston University, University of Maryland center, supported by the Army Research Office under the ODDR&E MURI97 Program Grant No. DAAG55-97-1-0114 (through Harvard University). This document is a technical report in the CDCSS series originating at the University of Maryland.

Web site <http://www.isr.umd.edu/CDCSS/cdcss.html>

Report Documentation Page				Form Approved OMB No. 0704-0188	
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE 1998		2. REPORT TYPE		3. DATES COVERED -	
4. TITLE AND SUBTITLE Motion Control for Nonholonomic Systems on Matrix Lie Groups				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Army Research Office,PO Box 12211,Research Triangle Park,NC,27709				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited					
13. SUPPLEMENTARY NOTES The original document contains color images.					
14. ABSTRACT see report					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES 128	19a. NAME OF RESPONSIBLE PERSON
a. REPORT unclassified	b. ABSTRACT unclassified	c. THIS PAGE unclassified			

Abstract

Title of Dissertation: Motion Control for Nonholonomic Systems
on Matrix Lie Groups

Herbert Karl Struemper, Doctor of Philosophy, 1997

Dissertation directed by: Professor P. S. Krishnaprasad
Department of Electrical Engineering

In this dissertation we study the control of nonholonomic systems defined by invariant vector fields on matrix Lie groups. We make use of canonical constructions of coordinates and other mathematical tools provided by the Lie group setting. An approximate tracking control law is derived for so-called chained form systems which arise as local representations of systems on a certain nilpotent matrix group. After studying the technique of nilpotentization in the setting of systems on matrix Lie groups we show how motion control laws derived for nilpotent systems can be extended to nilpotentizable systems using feedback and state transformations. The proposed control laws exhibit highly oscillatory components both for tracking and feedback stabilization of local representations of nonholonomic systems on Lie groups. Applications to the control and analysis of the kinematics of mechanical systems are discussed and numerical simulations are presented.

Motion Control for Nonholonomic Systems on Matrix Lie Groups

by

Herbert Karl Struemper

Dissertation submitted to the Faculty of the Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1997

Advisory Committee:

Professor P. S. Krishnaprasad, Chairman/Advisor
Professor John J. Benedetto
Professor Steven I. Marcus
Professor Shihab A. Shamma
Professor Andre L. Tits

© Copyright by
Herbert Karl Struemper
1997

Dedication

To Andrea

Acknowledgements

In this dissertation we study the control of nonholonomic systems defined by invariant vector fields on matrix Lie groups. We make use of canonical constructions of coordinates and other mathematical tools provided by the Lie group setting. An approximate tracking control law is derived for so-called chained form systems which arise as local representations of systems on a certain nilpotent matrix group. After studying the technique of nilpotentization in the setting of systems on matrix Lie groups we show how motion control laws derived for nilpotent systems can be extended to nilpotentizable systems using feedback and state transformations. The proposed control laws exhibit highly oscillatory components both for tracking and feedback stabilization of local representations of nonholonomic systems on Lie groups. Applications to the control and analysis of the kinematics of mechanical systems are discussed and numerical simulations are presented.

In this dissertation we study the control of nonholonomic systems defined by invariant vector fields on matrix Lie groups. We make use of canonical constructions of coordinates and other mathematical tools provided by the Lie group setting. An approximate tracking control law is derived for so-called chained form systems which arise as local representations of systems on a certain nilpotent matrix group. After studying the technique of nilpotentization in the setting of systems on matrix Lie groups we show how motion control laws derived for nilpotent systems can be extended to nilpotentizable systems using feedback and state transformations. The proposed control laws exhibit highly oscillatory components both for tracking and feedback stabilization of local representations of nonholonomic systems on Lie groups. Applications to the control and analysis of the kinematics of mechanical systems are discussed and numerical simulations are presented.

In this dissertation we study the control of nonholonomic systems defined by invariant vector fields on matrix Lie groups. We make use of canonical constructions of coordinates and other mathematical tools provided by the Lie group setting. An approximate tracking control law is derived for so-called chained form systems which arise as local representations of systems on a certain nilpotent matrix group. After studying the technique of nilpotentization in the setting of systems on matrix Lie groups we show how motion control laws derived for nilpotent systems can be extended to nilpotentizable systems using feedback and state transformations. The proposed control laws ex-

hibit highly oscillatory components both for tracking and feedback stabilization of local representations of nonholonomic systems on Lie groups. Applications to the control and analysis of the kinematics of mechanical systems are discussed and numerical simulations are presented.

Table of Contents

<u>Section</u>	<u>Page</u>
1 Introduction	1
2 Preliminaries	9
2.1 Lie groups and Lie algebras	9
2.2 Matrix Lie groups	15
2.3 Systems on matrix Lie groups	19
2.4 Local representations of systems on Lie groups	21
2.4.1 Single exponential representation	22
2.4.2 Product of exponentials representation	25
2.5 Example Systems on Matrix Lie Groups	27
2.5.1 Rigid motions on $\mathbb{R}^n$	27
2.5.2 Unicycle	29
2.5.3 Spacecraft	31
2.5.4 Underwater Vehicle	33
3 Approximate Inversion	35
3.1 Problem Definition	35

3.2	Open-loop tracking for the nonholonomic integrator	38
3.3	Optimality	45
4	Approximate Inversion of Nilpotent Systems	50
4.1	Local Representations of Nilpotent Systems on Matrix Lie Groups	51
4.2	Approximate inversion of Chained-Form Systems	56
5	Nilpotentization of Invariant Systems on Matrix Lie Groups	68
5.1	Nilpotentization of Invariant Systems on Three-Dimensional Ma- trix Lie Groups	71
5.2	Nilpotentization for Higher-Dimensional Systems on Matrix Lie Groups	83
6	Motion Control For Nilpotentizable Systems	91
6.1	Approximate Tracking and Approximate Inversion	92
6.2	Approximate Inversion of Nilpotentizable Systems	95
6.3	Feedback Stabilization of Nilpotentizable Systems	100
7	Conclusion	109
	References	113

Motion Control for Nonholonomic Systems
on Matrix Lie Groups

Herbert Karl Struemper

November 4, 1997

This comment page is not part of the dissertation.

Typeset by $\text{\LaTeX}$ using the dissertation class by Pablo A. Straub, University of
Maryland.

Chapter 1

Introduction

Systems on matrix Lie groups, the main topic of this thesis, are an interesting subject of study both from a practical as well as from a control-theoretic point of view. Due to their unique combination of geometric and algebraic properties, Lie groups arise naturally as the models for the configuration space of mechanical systems which provide the major practical motivation for their study. For instance the position and orientation of a rigid body in Euclidean space can be completely characterized by the Special Euclidean Group $SE(3)$ or, more specifically by the matrix representation of the abstract Lie group $SE(3)$. Formulating the kinematics of such a mechanical system having a matrix Lie group G as a configuration space then leads to a matrix valued differential equation defined on G . We call such a differential equation a *system on a matrix Lie group* to emphasize our objective of motion control and interpret the system's configuration as an output while, in the case of a mechanical systems, viewing the velocities as inputs. System on matrix Lie groups thus find application in modeling and motion control of mechanical systems such robotic manipulators, wheeled robots, underwater vehicles and space-craft. Next to mechanical applications, Lie groups

also arise from physical conservation principles such as conservation of energy. For instance electrical networks used for power conversion can be modeled as evolving on the Special Orthogonal Group $SO(n)$ (Wood, 1974), and so-called multilevel systems used to model molecular bonds in the context of coherent control of quantum dynamics can naturally be represented as systems on the complex unitary group $U(n)$ (Dahleh *et al.*, 1996).

On the other hand invariant systems on Lie groups are of special interest in the theoretical context of nonlinear control theory since they form an important sub-class of nonlinear systems. Their structure naturally leads to simplifications which allows us to study the essence of various nonlinear control questions without the technicalities of more general formulations. For instance the Lie algebra of vector fields of invariant systems on finite-dimensional Lie groups is finite-dimensional and the corresponding distributions are regular. The study of problems like controllability can therefore be reduced to a study of the Lie algebra $\mathfrak{g}$ of the Lie group G underlying the given system. Thereby one can make use of the impressive geometric and algebraic machinery developed for Lie groups and Lie algebras in the course of the last century.

The global coordinate-free description of dynamical systems allows us to address certain qualitative questions with elegant geometric reasoning. For example the existence of a smooth, static feedback, globally asymptotically stabilizing the origin of a system on $SO(n)$ can be precluded immediately since Wilson (Wilson, 1967) showed that the domain of attraction of an asymptotically stable equilibrium point is diffeomorphic to $\mathbb{R}^n$, while it is well known that $SO(n)$ is not. In another instance the fact whether the underlying Lie group G is compact or not, turns out to be a crucial ingredient in establishing controllability of

systems with drift. Such geometric insight is either lost or concealed in a local representation of the given system, and arguments of the kind mentioned above might not be possible or as obvious if the given system is immediately expressed in local coordinates.

Even if we will venture into local coordinates eventually to obtain explicit control laws, it is of great importance which coordinate description we choose, emphasizing again the advantage of using the global description as a starting point when analyzing a dynamical system. In this thesis for instance two canonical constructions of coordinates and their properties will play a key role in the derivation of motion control results.

Systems on Lie groups were brought to the attention of the controls community by Roger Brockett (Brockett, 1972) in the early 70's and further analyzed by Jurdjevic and Sussmann (Jurdjevic & Sussmann, 1972). These and other early studies were mainly concerned with existence questions and established Lie algebraic criteria for problems such as controllability and observability. Constructive questions for systems on Lie groups such as deriving optimal controls for certain lower-dimensional system on Lie groups were taken up in (Krishnaprasad, 1993; Tsakiris, 1995). In (Leonard & Krishnaprasad, 1995) algorithms based on average theory are presented to steer generic, controllable, drift-free, invariant systems on matrix Lie groups approximately to a desired point in the configuration space.

We will focus in this thesis on nonholonomic (or under-actuated), drift-free, invariant systems on matrix Lie groups and consider mainly the open-loop tracking and feedback stabilization problem for such systems.

Nonholonomic systems for which the number m of controls is strictly smaller

than the dimension n of the underlying state space arise in practice through kinematic constraints, like the no-slip condition for wheeled vehicles, through conservation principles, such as the conservation of angular momentum for a floating bar linkage (Yang & Krishnaprasad, 1992), or simply through lack or failure of actuators, such as for a satellite with a failing gas jet. As opposed to holonomic systems with m inputs the motion of a m -input nonholonomic systems is not constrained to a m -dimensional submanifold of the state space, but can encompass the whole state space.

The problem of tracking a generic trajectory in state space with a nonholonomic system which is also labeled nonholonomic motion planning is obviously not exactly solvable due to the presence of the velocity constraints. But already in (Haynes & Hermes, 1970) it was shown that there exist sequences of controls for drift-free, nonholonomic systems such that the resulting trajectories uniformly approximate any given, sufficiently smooth, generic trajectory, although no concrete constructive procedure was given there.

The principle underlying such approximate tracking controls is that periodic controls with specific phase relations between certain inputs cause secular motions in the direction of the higher-order Lie brackets required for controllability. This corresponds to the familiar parallel parking maneuver, where a side-ways motion which is constrained by the no-slip condition of the wheels can be achieved secularly by switching between forwards, backwards and turning maneuvers in a suitable fashion. A tight parking space requires more of these elementary maneuvers, a fact which illustrates the need for high-frequency oscillatory controls if the accuracy of the trajectory approximation is to be improved.

The problem of actually constructing feasible paths between arbitrary config-

urations was first addressed in (Laumond, 1987) in the context of mobile wheeled robots. The realization that such wheeled vehicles can locally be transformed to the class of so-called nilpotent systems, so called chained form systems in particular, and that this class has properties facilitating the solution of motion control problems lead to a line of work documented in (Lafferriere & Sussmann, 1991; Brockett & Dai, 1992; Murray & Sastry, 1993). In (Sussmann & Liu, 1991) the problem of approximate tracking for drift-free, controllable, input-linear systems is tackled in a general setting, and highly oscillatory controls are presented such that resulting trajectories converge uniformly to a given generic smooth trajectory.

In parallel to the problem of steering of nonholonomic systems the problem of feedback stabilization of nonholonomic systems has received considerable attention in the last decade. In (Coron, 1992) the existence of time-varying stabilizing controls for drift-free, controllable, input-linear systems was shown, while in (Pomet, 1992) a method to explicitly construct such control laws for a restricted class of systems was presented. To improve the slow convergence rates of such smooth feedback laws (M'Closkey & Murray, 1995) introduced time-varying, non-smooth control laws based on the idea of homogeneous feedback leading to exponential convergence of trajectories. Making use of the constructions of (Sussmann & Liu, 1991) this idea was taken further in (Morin *et al.*, 1996) where explicit, homogeneous feedback laws for stabilization of generic, drift-free, homogeneous systems are presented.

As opposed to (Sussmann & Liu, 1991) where the problem of approximate tracking of drift-free system is tackled with a complex machinery in the most generic setting, we will follow a bottom-up strategy by first deriving relatively

simple control laws for a class of nilpotent systems. Analogous to the technique of feedback linearization this class of system is interpreted as a canonical form feedback equivalent to a wider class of systems, allowing us to extend the applicability of the derived control laws. The technique of feedback nilpotentization is then applied specifically to local representations of systems on matrix Lie groups to compute the required transformations, and the specifics arising from this set-up are studied. It is then shown how these transformations can be utilized to derive open-loop approximate tracking controls and stabilizing feedbacks for non-nilpotent systems on matrix Lie groups.

Chapter 2 reviews some basic notions concerning finite-dimensional Lie groups, Lie algebras and their concrete representations as matrix Lie groups and algebras. Systems evolving on matrix Lie groups are defined, categorized, and characterized in terms of their controllability properties. The choice of suitable coordinates for a given control problem turns out to be a crucial decision when studying systems on Lie groups. We present local representations of systems on Lie groups based on two canonical constructions of coordinates used in the course of the thesis and study their properties. Finally we describe in a series of examples how systems on matrix Lie groups arise as models of the kinematics of concrete mechanical systems and other motion control problems.

In Chapter 3 we describe the basic motion control problems for nonholonomic systems and define the notion of an approximate inverse system which was introduced in (Brockett, 1993). Using the example of Brockett's nonholonomic integrator it is demonstrated how oscillatory control components can be used to create secular motions in the direction of higher order Lie brackets and how these such motions can be made increasingly independent of each other by increasing

a frequency parameter in the control laws. Since an approximate tracking control law is defined only in terms of a limit process, i.e. the resulting trajectories converge to a desired trajectory in the high-frequency limit, solutions to this problem are not unique. We describe different ways to implement the motion generation by oscillatory control components leading to different requirements for the actuators of the underlying system. The chapter concludes with a discussion of the optimality of tracking controls with sinusoidal oscillatory components for the nonholonomic integrator.

It has been a common feature in recent work on motion control of nonholonomic systems to tackle control problem first in a nilpotent setting, for instance by restricting attention to systems in chained or in power form, to make use of simplifications arising from the corresponding Lie algebra structure. We follow a similar approach and start out in Chapter 4 by pointing out that chained form systems, power forms systems, as well as Brockett's nonholonomic integrator all originate from invariant systems on the same nilpotent matrix Lie group. In the main result of this chapter approximate inversion controls are presented for the single generator chained form systems with two or more inputs. A discussion of implementational issues related to this control laws ends the chapter.

Chapter 5 is concerned with the problem of nilpotentization of local representations of systems on matrix Lie groups. We show that for the most common invariant systems on three-dimensional matrix Lie groups the nilpotentizing transformations can be calculated in an explicit form directly from the corresponding Lie algebra structure. The so-called product of exponential coordinates plays a crucial role in this process, and the transformations basically fall out as byproducts of the computations for the corresponding local representation. This

explains the fact that nilpotentizing transformations derived in this way share their region of validity with the corresponding local representation itself. After reviewing the necessary and sufficient conditions for nilpotentization we explore the possibility of transforming system on higher-dimensional matrix Lie group into chained or other nilpotent forms in the second part of the chapter.

In the first section of Chapter 6 we use the results from Chapter 4 and Chapter 5 to construct approximate tracking controls for systems nilpotentizable to chained form. The resulting control laws which involve a feedback are then converted into open loop forms using a relatively simple feed-forward estimate of the state trajectories. In the last section of this chapter we show how exponentially stabilizing feedback laws can be derived via nilpotentization and a construction of feedback laws for nilpotent systems, which has been presented by Morin and co-workers (Morin *et al.*, 1996).

Numerical simulations are presented throughout the thesis wherever they enhance the understanding of the corresponding control laws.

Chapter 2

Preliminaries

This chapter introduces the mathematical concepts and tools related to differential equations evolving on finite-dimensional Lie groups which will be used throughout the thesis. After reviewing basic properties of Lie groups and Lie algebras, we define and characterize dynamical systems on Lie groups. Further, we introduce and compare local representations of these systems, in which most of the actual computations will be carried out later on. Hereby we follow the basic references (Varadarajan, 1974; Curtis, 1984; Marsden & Ratiu, 1994).

2.1 Lie groups and Lie algebras

Conceived in the last third of the 19th century by Marius Sophus Lie (1842–1899) as a tool for the solution of differential equations the theory of Lie groups and Lie algebras has since then developed into a discipline in its own right. By their very nature Lie groups bring together the mathematical disciplines of algebra and geometry to produce results which elegantly rely on the interaction of differential geometric and group-theoretic tools.

In systems theory Lie groups often arise as models for the configuration space

of mechanical systems. The fact that such systems allow at each point of the configuration space a continuous range of motions is accounted for by choosing a differentiable manifold as the underlying set of the configuration space. The algebraic character of the modeling Lie group, on the other hand, reflects that motions of such systems can be composed to obtain new motions and that any motion can be reversed.

Definition 2.1.1 (Lie group) *A Lie group G is a differentiable manifold which is also endowed with a group structure such that the group multiplication*

$$\mu : G \times G \rightarrow G; \quad (g, h) \mapsto gh \tag{2.1}$$

is a C^∞ map.

I.e., given a set of coordinates for the group manifold, the coordinates of the product of the two group elements have to be smooth functions of the coordinates of the factors. Using the implicit function theorem it can be shown that for a Lie group as defined above the inversion map

$$I : G \rightarrow G; \quad g \mapsto g^{-1}$$

is also a C^∞ function.

Given elements $g, h \in G$ we define the associated *left translation* $L_g : G \rightarrow G; \quad h \mapsto gh$, and the *right translation* $R_g : G \rightarrow G; \quad h \mapsto hg$, respectively. By the properties of Lie groups both these maps are diffeomorphisms. The map L_g applied to $h \in G$ induces a linear map $T_h L_g$ from the tangent space $T_h G$ of G to T_{gh} which turns out to be an isomorphism. Similarly, $T_h R_g$ is a tangent space isomorphism.

We can now construct a smooth vector field X on G by picking a vector $\xi = X(e)$ in $T_e G$ and transplanting it with $T_e L_g$ such that

$$X_\xi(g) = (T_e L_g)X(e), \forall g \in G. \quad (2.2)$$

In general, a vector field X satisfying

$$(T_h L_g)X(h) = X(gh), \forall g, h \in G \quad (2.3)$$

is called *left-invariant*. Similarly, we can define and construct *right-invariant* vector fields on G . Right-invariant vector fields Y_ξ are related to left-invariant vector fields X_ξ by

$$I_* X_\xi = -Y_\xi, \quad (2.4)$$

where I_* denotes the tangent space map associated with the inversion map I . Because of this correspondence we can focus without loss of generality on left-invariant notions, while keeping in mind that most constructions can be carried out analogously for the corresponding right-invariant notions.

Definition 2.1.2 (Lie Algebra) *A Lie algebra is a real vector space equipped with a bilinear operator $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called the Lie bracket, such that for all $x, y, z \in \mathfrak{g}$*

$$[x, y] = -[y, x] \quad (\text{skew-symmetry})$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad (\text{Jacobi identity})$$

Let Y_1 and Y_2 denote left-invariant vector fields on a Lie group G . Then, αY_1 for $\alpha \in \mathbb{R}$, $Y_1 + Y_2$, and $[Y_1, Y_2]$, where $[\cdot, \cdot]$ denotes the Jacobi-Lie bracket for vector fields, are also left-invariant vector fields on G . Thus, the set $\mathfrak{X}_L(G)$

of left-invariant vector fields on G forms a Lie algebra, actually a subalgebra of $\mathfrak{X}(G)$, the smooth vector fields on G .

As mentioned above the tangent space $T_e G$ at the identity and $\mathfrak{X}_L(G)$ are related by the transplantation map

$$t : T_e G \rightarrow \mathfrak{X}, \xi \mapsto X_\xi$$

and the evaluation map

$$e : \mathfrak{X}_L(G) \rightarrow T_e G, X \mapsto X(e)$$

and turn out to be isomorphic as vector spaces. Moreover, $T_e G$ inherits the Lie algebra structure of $\mathfrak{X}_L(G)$ by the induced bracket

$$[\xi, \eta] = [X_\xi, X_\eta](e).$$

and is called the *Lie algebra* $\mathfrak{g}$ of G .

This bijection between $T_e G$ and $\mathfrak{X}_L(G)$, which can also be extended to *one-parameter subgroups* $\phi : \mathbb{R} \rightarrow G; t \mapsto \phi(t)$ of G and *left-invariant $\mathbb{R}$ -actions* $\Phi : \mathbb{R} \times G \rightarrow G; (t, g) \mapsto g\phi(t)$ on G , is one of the main appeals of Lie's theory. Many properties of the Lie group G are reflected in the algebraic structure of the associated Lie algebra $\mathfrak{g}$. For connected Lie groups this correspondence goes as far as that the Lie group can be recovered up to an isomorphism from its Lie algebra. In our context the correspondence allows us to reduce problems involving invariant vector fields on a Lie group G to problems in the setting of Lie algebras $\mathfrak{g}$, thus reducing the inherent nonlinearity of invariant systems on Lie groups to the algebraic structure on a linear vector space.

Given two subspaces $\mathfrak{g}_1, \mathfrak{g}_2$ of a Lie algebra $\mathfrak{g}$ denote

$$[\mathfrak{g}_1, \mathfrak{g}_2] = \{[\xi, \eta] \in \mathfrak{g} \mid \xi \in \mathfrak{g}_1, \eta \in \mathfrak{g}_2\}.$$

A subspace of $\mathfrak{g}_1$ of $\mathfrak{g}$ is called a *subalgebra* if $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_1$. It is an *ideal* if $[\mathfrak{g}_1, \mathfrak{g}] \subseteq \mathfrak{g}_1$. The following nested sequences of ideals are used to characterize Lie algebras. The *derived series* of $\mathfrak{g}$ is defined by

$$\mathfrak{g} \subseteq \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}'' = [\mathfrak{g}', \mathfrak{g}'] \subseteq \cdots \subseteq \mathfrak{g}^n = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \subseteq \cdots$$

The *lower central series* is defined by

$$\mathfrak{g} \subseteq \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}^3 = [\mathfrak{g}^2, \mathfrak{g}] \subseteq \cdots \subseteq \mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}] \subseteq \cdots$$

We have

$$\mathfrak{g}^n \subseteq \mathfrak{g}^{(n+1)}, \quad n = 2, 3, \dots \quad (2.5)$$

The following characterizations are shared by Lie groups and Lie algebras but can be checked more easily in the linear setting underlying Lie algebras. Following the remarks above we will define them directly for Lie algebras.

Definition 2.1.3 (Abelian Lie Algebra) *A Lie algebra $\mathfrak{g}$ is called Abelian if $\mathfrak{g}' = \mathfrak{g}^2 = 0$.*

Definition 2.1.4 (Nilpotent Lie Algebra) *A Lie algebra $\mathfrak{g}$ is called nilpotent if the lower central series terminates after a finite number of steps, i.e. $\mathfrak{g}^k = 0$, for some integer k .*

Definition 2.1.5 (Solvable Lie Algebra) *A Lie algebra $\mathfrak{g}$ is called solvable if the derived series terminates after a finite number of steps, i.e. $\mathfrak{g}^{(k)} = 0$ for some integer k .*

Since $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^{n+1}$, $n = 1, 2, 3, \dots$ a solvable Lie algebra is also nilpotent.

Definition 2.1.6 (Simple Lie algebra) *A Lie algebra $\mathfrak{g}$ is called simple if it is non-Abelian and its only ideals are 0 and G*

Definition 2.1.7 (Semi-simple Lie algebra) *A Lie algebra $\mathfrak{g}$ is called semi-simple if its only Abelian ideal is 0.*

A simple Lie algebra is also semi-simple. For 3-dimensional Lie algebras $\mathfrak{g} = \mathfrak{g}'$ implies simplicity.

Next we will study a map relating a Lie algebra with the identity component of the corresponding Lie group which will be used later to obtain local coordinates for a neighborhood of the identity of G . Consider the integral curve $\phi_\xi : \mathbb{R} \rightarrow G$ of a left-invariant vector field X_ξ , $\xi \in \mathfrak{g}$ passing through the identity $e \in G$ at $t = 0$. The map ϕ_ξ turns out to be a homomorphism from $\mathbb{R}$ to G for all $\xi \in \mathfrak{g}$, i.e. it is a one-parameter subgroup of G .

Definition 2.1.8 (Exponential map) *The exponential map $\exp : \mathfrak{g} \rightarrow G$ is defined by setting*

$$\exp \xi = \phi_\xi(1).$$

It can be verified that $\exp(t\xi) = \phi_\xi(t)$, i.e. lines $t\xi$, $t \in \mathbb{R}$, $\xi \in \mathfrak{g}$ going through the origin of $\mathfrak{g}$, are mapped onto one-parameter subgroups of G . Conversely, every one-parameter subgroup of G can be expressed as $\exp(t\xi)$ for some $\xi \in \mathfrak{g}$.

Since the differential $d(\exp) : T_e G \rightarrow T_e G$ is the identity map on $T_e G$ we can conclude by the inverse function theorem and smoothness of $\exp$ that $\exp$ is a local diffeomorphism from a neighborhood U of the origin of $\mathfrak{g}$ onto a neighborhood V of the identity of G . We denote the inverse map of $\exp$ from V to U by $\log : G \rightarrow \mathfrak{g}$.

Even for connected Lie groups $\exp$ is in general neither one-to-one nor onto. However, there exist results exhibiting reasonably large neighborhoods of $0 \in \mathfrak{g}$

such that the exponential map is bijective (see pp. 110 of (Varadarajan, 1974)). So, for a given basis $\{\xi_1, \dots, \xi_n\}$ of $\mathfrak{g}$ there exists a reasonably large neighborhood U of the origin of $\mathbb{R}^n$ such that the map

$$\phi : U \rightarrow G; \quad (a_1, \dots, a_n) \mapsto \exp(a_1\xi_1 + \dots + a_n\xi_n)$$

is a smooth diffeomorphism onto an open subset $\phi(U) \subset G$ containing the identity. Then, the smooth functions $x_1, \dots, x_n$ on V satisfying

$$x_i(\exp(a_1\xi_1 + \dots + a_n\xi_n)) = a_i, \quad i = 1, \dots, n; \quad \forall a \in U$$

are called the *canonical coordinates of the first kind* or *single exponential coordinates* around $e \in G$ relative to the basis $\{\xi_1, \dots, \xi_n\}$.

Another set of canonical coordinates is based on the fact that there exists a $U \subset \mathbb{R}^n$ such that the map

$$\psi : U \rightarrow G; \quad (a_1, \dots, a_n) \mapsto \exp(a_1\xi_1) \cdots \exp(a_n\xi_n)$$

is a diffeomorphism onto a neighborhood $\psi(U)$ of the identity of G . Since in general every $g \in G$ can be written as a finite product $g_1 g_2 \cdots g_k$ of elements $g_1, \dots, g_k$ in an arbitrary neighborhood $V \subset G$ of the identity, ψ is surjective if we take $U = \mathbb{R}^n$. The smooth functions $x_1, \dots, x_n$ on $\psi(U)$ satisfying

$$x_i(\exp(a_1\xi_1) \cdots \exp(a_n\xi_n)) = a_i, \quad i = 1, \dots, n; \quad \forall a \in U$$

are called the *canonical coordinates of the second kind* or *product of exponentials coordinates* around $e \in G$ relative to the basis $\{\xi_1, \dots, \xi_n\}$.

2.2 Matrix Lie groups

Matrix Lie groups whose elements represent linear isomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$ provide us with numeric representations of abstract Lie groups. Since they al-

ready come along with a set of coordinates (the matrix elements) we can directly perform computations on matrix Lie groups and use the geometrical insight and the categorizations by canonical forms of linear algebra.

The most general matrix Lie group is the *general linear group* $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ of invertible $(n \times n)$ -matrices with real or complex entries, respectively. In the following we will focus on real matrix groups unless otherwise noted and denote $GL(n, \mathbb{R})$ simply with $GL(n)$. As the inverse image of $\mathbb{R} \setminus 0$ under the continuous map $X \mapsto \det(X)$, $GL(n)$ is an open subset of $M_{n,n} \cong \mathbb{R}^{n^2}$, and thus can be given the structure of a differentiable manifold. The group multiplication of $GL(n)$ is the regular matrix multiplication and the inversion map takes a matrix X in $GL(n)$ to its inverse X^{-1} , while $e \in GL(n)$ is assumed to be the identity matrix $I = \text{diag}(1, \dots, 1)$. Since $GL(n)$ is an open subset of $M_{n,n}$, the Lie algebra $\mathfrak{gl}(n)$ of $GL(n)$ turns out to be $M_{n,n}$ with the Lie bracket defined by the matrix commutator

$$[A, B] = AB - BA, \quad A, B \in \mathfrak{gl}(n).$$

All other real matrix Lie groups are subgroups of $GL(n)$ and the basic operations of $GL(n)$ and $\mathfrak{gl}(n)$ like the group multiplication or the Lie bracket restrict accordingly to the other matrix Lie groups.

The exponential map for a matrix Lie group G turns out to be just the matrix exponential, i.e. given an element $A \in G$ we write

$$\exp(A) = e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = I + A + \frac{1}{2}A^2 + \dots$$

For an element A in an n -dimensional matrix group whose entries are bounded above by c the entries of A^i are bounded above by $(cn)^i$. By relating this to the convergence of a scalar power series it can thus be established that the matrix

power series above converges for all $A \in G$. For a matrix Lie group G the exponential function is surjective whenever G is path-connected.

The matrix logarithm, the inverse of the matrix exponential, can accordingly be defined as a matrix power series

$$\log(X) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(X - I)^i}{i} = (X - I) - \frac{(X - I)^2}{2} + \frac{(X - I)^3}{3} \mp \dots$$

for X in a neighborhood of $e = I \in G$.

In the following we list for future reference the matrix Lie groups appearing throughout this thesis along with their properties.

Example 2.2.1 (Special Linear Group) Let $SL(n)$ denote the Special Linear group of $(n \times n)$ -matrices defined by

$$SL(n) = \{X \in GL(n) \mid \det(X) = 1\}.$$

The Lie algebra $\mathfrak{sl}(n)$ of $SL(n)$ is defined by

$$\mathfrak{sl}(n) = \{A \in \mathfrak{gl}(n) \mid \text{trace}(A) = 0\}$$

and can be shown to be simple. $SL(n)$ is of dimension $n^2 - 1$, non-compact, and connected.

Example 2.2.2 (Orthogonal Group) Let $O(n)$ denote the Orthogonal group of matrices defined by

$$O(n) = \{X \in GL(n) \mid X^T X = 1\}.$$

The Lie algebra $\mathfrak{o}(n)$ of $O(n)$ is defined by

$$\mathfrak{o}(n) = \{A \in \mathfrak{gl}(n) \mid A^T = -A\}.$$

Example 2.2.3 (Special Orthogonal Group) Let $SO(n)$ denote the Special Orthogonal group of matrices representing rotations in an n -dimensional Euclidean space defined by

$$SO(n) = \{X \in GL(n) \mid X \in O(n) \cap SL(n)\}.$$

Since $SO(n)$ is just the identity component of $O(n)$, the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ equals $\mathfrak{o}(n)$. $SO(n)$ is of dimension $\frac{n(n-1)}{2}$, simple, compact, and connected.

Example 2.2.4 (Special Euclidean Group) Let $SE(n)$ denote the Special Euclidean group of rigid motions in n -dimensional Euclidean space defined by

$$SE(n) = \left\{ \begin{pmatrix} X & p \\ 0^{1 \times n} & 1 \end{pmatrix} \in GL(n+1) \mid X \in SO(n), p \in \mathbb{R}^n \right\}.$$

The Lie algebra $\mathfrak{se}(n)$ of $SE(n)$ is defined by

$$\mathfrak{se}(n) = \left\{ \begin{pmatrix} A & p \\ 0^{1 \times n} & 0 \end{pmatrix} \in \mathfrak{gl}(n+1) \mid A \in \mathfrak{so}(n), p \in \mathbb{R}^n \right\}.$$

$SE(n)$ is of dimension $\frac{n(n+1)}{2}$, non-compact, and connected. $SE(2)$ is solvable.

Example 2.2.5 (Group of Unipotent Matrices) Let $UP(n)$ ¹ denote the group of upper-triangular $(n \times n)$ -matrices X with diagonal elements $X_{ii} = 1; i = 1, \dots, n$. The Lie algebra $\mathfrak{up}(n)$ of $UP(n)$ then consists of upper triangular matrices A with diagonal elements $A_{ii} = 0; i = 1, \dots, n$. It can be verified that $UP(n)$ is of dimension $\frac{n(n-1)}{2}$ and is nilpotent. Since we can use the elements $X_{ij}; i < j$ directly as global coordinate functions for $UP(n)$, the manifold underlying $UP(n)$ is diffeomorphic to $\mathbb{R}^{\frac{n(n-1)}{2}}$. Therefore $UP(n)$ is not compact, but simply connected.

¹This is non-standard notation.

2.3 Systems on matrix Lie groups

An affine control system on a n -dimensional Lie group G is defined by an affine combination of vector fields on G

$$\dot{g}(t) = \mathcal{X}_0(g(t)) + \sum_{i=1}^m u_i(t) \mathcal{X}_i(g(t)), \quad m \leq n, \quad u_i(t) \in R \quad (2.6)$$

where $g(t)$ is a curve on G , $\mathcal{X}_0$ is interpreted as the drift vector field and $\mathcal{X}_i$, $i = 1, \dots, m$ are interpreted as control vector fields. We focus in this thesis on the case where the vector fields $\mathcal{X}_i$, $i = 0, 1, \dots, m$ are left-invariant and also call the corresponding system (2.6) left-invariant. Following (2.2) a left-invariant control system on G can be written as

$$\dot{g}(t) = T_e L_{g(t)}(\xi_0 + \sum_{i=1}^m u_i(t) \xi_i) \quad (2.7)$$

where $\xi_0, \xi_1, \dots, \xi_m$ are fixed vectors in the Lie algebra $\mathfrak{g}$ of G . If $\xi_0 = 0$, the system (2.7) is called *drift-free*. Since the tangent lift $T_e L_X$, $X \in G$ of the left shift operation for matrix Lie groups is equal to L_X itself, a left-invariant system on a n -dimensional matrix Lie group can be expressed as

$$\dot{X}(t) = X(t)U(t) = X(t) \sum_i^m u_i(t) A_i, \quad m \leq n \quad (2.8)$$

where $X(t)$ is a curve in G , $U(t)$ is a curve in $\mathfrak{g}$ and $\{A_1, \dots, A_n\}$ is assumed to be a basis for $\mathfrak{g}$. We write

$$\dot{Y}(t) = U(t)Y(t) = (\sum_i^m u_i(t) A_i)Y \quad (2.9)$$

for the corresponding right-invariant system.

From (2.4) it follows that solution $Y(t)$ of (2.9) satisfies

$$\frac{d}{dt}(Y^{-1})(t) = Y^{-1}(t)(-U(t)). \quad (2.10)$$

Therefore, a right-invariant control system on a Lie group G can easily be converted to a left-invariant control system on G .

Turning to the question of controllability of (2.7) we define the set $\mathcal{U}$ of admissible controls as the class of locally bounded, measurable functions from $[0, \infty)$ to $\mathbb{R}^m$.

Definition 2.3.1 (Controllability) *The system (2.7) is called controllable if for any $g_0, g_1 \in G$ there exists a time $T > 0$ and an admissible control $u \in \mathcal{U}$ such that the corresponding solution of (2.8) satisfies $g(0) = g_0$ and $g(T) = g_1$.*

The following two theorems by Jurdjevic and Sussmann (Jurdjevic & Sussmann, 1972) demonstrate how controllability of (2.7) can be checked by studying the algebraic properties of $\mathfrak{g}$ and topological properties of the group manifold.

Theorem 2.3.2 (Controllability for Drift-Free System) *Let Σ denote a drift-free system of the form (2.7) on a connected Lie group G . Then Σ is controllable if and only if the Lie algebra generated by $\{\xi_1, \dots, \xi_m\}$ spans $\mathfrak{g}$.*

Theorem 2.3.3 (Controllability for System with Drift) *Let Σ_0 denote a system with drift of the form (2.7) on a compact, connected Lie group G . Then Σ_0 is controllable if and only if the Lie algebra generated by $\{\xi_0, \xi_1, \dots, \xi_m\}$ spans $\mathfrak{g}$.*

The corresponding controllability results for (2.8) can be obtained by replacing $g(0), g(T), \xi_i$ in Theorem (2.3.2) and Theorem (2.3.3) by $X(0), X(T), A_i$, respectively.

2.4 Local representations of systems on Lie groups

Even though important results for systems on matrix Lie groups, e.g. controllability, can be established directly from the algebraic and geometric properties of their global description (2.8), we need to resort to local representations of (2.8) when we compute the actual control laws.

Given a n -dimensional Lie group G we pick a diffeomorphism from a neighborhood U of $e \in G$ to neighborhood V of $0 \in \mathbb{R}^k, k \geq n$ which induces a local representation of (2.8) as a affine system

$$\dot{x} = f_0(x) + f_1(x)u_1 + \cdots + f_m(x)u_m, \quad x \in \mathbb{R}^n. \quad (2.11)$$

Note that distributions spanned by the $f_i, i = 0, 1, \dots, m$ are of constant dimension or *regular* on V since they originate from invariant vector fields on G .

Criteria for a desirable local representations include:

- coordinates have a large, well defined region of validity;
- representation is in terms of explicit functions;
- representation reflects structural properties of (2.8);
- representation has good numerical properties with respect to integration;
- coordinates have physical interpretation.

In the following we will present such local representations and discuss their advantages and disadvantages in terms of the criteria stated above.

2.4.1 Single exponential representation

Given a basis $\mathcal{B} = \{A_1, A_2, \dots, A_n\}$ for the Lie algebra $\mathfrak{g}$ of a n -dimensional matrix group G , the canonical coordinates of the first kind $z_1, z_2, \dots, z_n$ relative to $\mathcal{B}$ satisfy

$$X = e^Z = e^{z_1 A_1 + \dots + z_n A_n}$$

for X in V , a neighborhood of $e \in G$. Via the tangent lift of the coordinate map, the left-invariant control system $\dot{X}(t) = X(t) \sum_i^m u_i(t) A_i$ then induces a control system on $\mathbb{R}^n$. This local representation of (2.8) was characterized by Magnus (Magnus, 1954). Adhering to our convention we will present a *left*-invariant version of Magnus's result without proof.

Theorem 2.4.1 (Magnus) *Consider the left-invariant, drift-free system (2.8) on a matrix Lie group G and the single exponential representation $Z(t)$, $t \geq 0$ of its solution. Then, if certain unspecified conditions of convergence are satisfied $Z(t)$ can be written in the form*

$$\begin{aligned} \dot{Z}(t) &= \frac{ad_{Z(t)}}{1 - \exp(-ad_{Z(t)})} U(t) \\ &= \left(I + \frac{1}{2} ad_{Z(t)} + \sum_{p=1}^{\infty} \frac{\beta_{2p}}{(2p)!} ad_{Z(t)}^{2p} \right) U(t) \\ &= U(t) + \frac{1}{2} [Z(t), U(t)] + \frac{1}{12} [Z(t), [Z(t), U(t)]] - \frac{1}{720} [Z(t), [Z(t), [Z(t), U(t)]]] \pm \dots, \end{aligned} \tag{2.12}$$

where the β_{2p} are Bernoulli numbers.

We will call a local representation of the form above the *single exponential* or alternatively the *Magnus representation* of (2.8).

Example 2.4.2 Consider the left-invariant system (2.8) on the Special Euclidean group $SE(2)$ with a basis $\{A_1, A_2, A_3\}$ for $\mathfrak{g}$ chosen such that

$$[A_1, A_2] = A_3, \quad [A_1, A_3] = -A_2, \quad [A_2, A_3] = 0.$$

With $U(t) = u_1(t)A_1 + u_2(t)A_2$ and $Z(t) = z_1(t)A_1 + z_2(t)A_2 + z_3(t)A_3$ the single exponential representation of this system can be written as

$$\begin{aligned} \dot{Z}(t) &= U(t) + \frac{1}{2}[Z(t), U(t)] + \frac{1}{12}[Z(t), [Z(t), U(t)]] \pm \dots \\ &= u_1(t)A_1 + u_2(t)A_2 + \frac{1}{2}\left(z_1(t)u_2(t) - z_2(t)u_1(t)\right)A_3 \\ &\quad \frac{1}{12}\left(z_1(t)z_2(t)u_1(t) - z_1^2(t)u_2(t)\right)A_2 \pm \dots \end{aligned}$$

or due to the linear independence of the basis vectors A_i , $i = 1, 2, 3$ as

$$\begin{aligned} \dot{z}_1 &= u_1 \\ \dot{z}_2 &= u_2 + \frac{1}{12}\left(z_1z_2u_1 - z_1^2u_2\right) \pm \dots \\ \dot{z}_3 &= \frac{1}{2}\left(z_1u_2 - z_2u_1\right) \pm \dots \end{aligned}$$

For the system (2.8) on $SO(3)$ the single exponential coordinates turn out to be the Euler parameters for $SO(3)$.

Note that if $\mathfrak{g}$ is nilpotent of order k then the infinite sum terminates, and we need only consider the first $k + 1$ terms on the right hand side. Moreover, in the general situation the rapidly decreasing coefficients on the right hand side of (2.12) suggest that the original system can be approximated reasonably well by truncating the series expansion for $\dot{Z}(t)$ after a finite number of terms. In particular, if we start with a suitably ordered basis for $\mathfrak{g}$ one can determine a nilpotent approximation of a non-nilpotent system (2.8) by truncating (2.13) after the leading term for each $x_i, i = 1, 2, 3, \dots, n$. The resulting approximation has the same growth vector as the original system and is of relevance as

a nilpotent model system for nilpotentization or local exponential stabilization (see Chapters 5 and 6).

Another feature of single exponential representations is that the solution of the Magnus equation (2.12) can be written as a series of quadratures involving U using the so called *Fomenko-Chakon recursive expansion* (Fomenko & Chakon, 1991). It can be understood as a continuous version of the Baker–Campbell–Hausdorff formula and is presented here in its left-invariant version.

Theorem 2.4.3 (Fomenko-Chakon) *Let $\gamma = \{U(s), 0 \leq s \leq t\}$ be a Riemann integrable curve in Lie algebra $\mathcal{G}$ and assume that*

$$\int_0^t |U(s)| ds < \frac{\hat{b}}{M}.$$

where $\hat{b} < 2\pi$ and M is chosen such that $\|[X, Y]\| \leq M\|X\|\|Y\|, \forall X, Y \in \mathcal{G}$. Then (2.8) has a solution of the form $X(t) = e^{Z(t)}X(0)$ which satisfies (2.8) at each t for which $\lim_{h \rightarrow 0} \int_t^{t+h} U(s) = U(t)$, where

$$Z(t) = \sum_{i=1}^{\infty} (-1)^{i+1} Z_i(t) \quad (2.13)$$

converges and $\{Z_i(t)\}$ is uniquely defined by the recursion formulas

$$Z_1(t) = \int_0^t U(s) ds, \quad (2.14)$$

and for $n \geq 1$, with $T_0 = Z_1$

$$\begin{aligned} (i+1)Z_{i+1} &= T_i + \sum_{r=1}^i \left\{ \frac{1}{2} [Z_r, T_{i-r}] \right. \\ &\quad \left. + \sum_{p \geq 1, 2p \leq r} k_{2p} \sum_{m_i > 0, m_1 + \dots + m_{2p} = r} [Z_{m_1}, [\dots, [Z_{m_{2p}}, T_{n-r}] \dots]] \right\}, \end{aligned}$$

where $k_{2p}(2p)!$ are Bernoulli's numbers and for $k \geq 1$

$$T_k(t) = (-1)^k \int_{\tau_{k+1}=0}^t \int_{\tau_k=0}^{\tau_{k+1}} \dots \int_{\tau_1=0}^{\tau_2} [U(\tau_1), [U(\tau_2), \dots, [U(\tau_k), U(\tau_{k+1})] \dots]] d\tau_1 \dots d\tau_{k+1}$$

or equivalently

$$T_k(t) = \int_{\tau_1=0}^t \left[\cdots \left[\left[U(\tau_1), \int_{\tau_2=0}^{\tau_1} U(\tau_2) \right], \cdots \int_{\tau_k=0}^{\tau_{k-1}} U(\tau_k) \right], \int_{\tau_{k+1}}^{\tau_k} U(\tau_{k+1}) \right] d\tau_{k+1} \cdots d\tau_1.$$

2.4.2 Product of exponentials representation

Given a basis $\mathcal{B} = \{A_1, A_2, \dots, A_n\}$ for the Lie algebra $\mathfrak{g}$ of a n -dimensional matrix group G , the canonical coordinates of the second kind $x_1, x_2, \dots, x_n$ relative to $\mathcal{B}$ satisfy

$$X = e^{x_1 A_1} e^{x_2 A_2} \cdots e^{x_n A_n}$$

for X in V , a neighborhood of $e \in G$. Via the tangent lift of the coordinate map, the left-invariant control system $\dot{X}(t) = X(t) \sum_i^m u_i(t) A_i$ then induces a control system on $\mathbb{R}^n$, which was characterized by Wei and Norman (Wei & Norman, 1964). In the following theorem we use an extended control vector $u = (u_1, \dots, u_n)^T$ with $u_i = 0, i = m + 1, \dots, n$.

Theorem 2.4.4 (Wei–Norman) *Consider the system (2.8) and its solution $X(t), t \geq 0$. Then, in a neighborhood of $t = 0$ the solution may be expressed in the form*

$$X(t) = e^{x_1(t)A_1} e^{x_2(t)A_2} \cdots e^{x_n(t)A_n}. \quad (2.15)$$

The coordinate functions $x_i(t)$ evolve according to

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = M(x_1, \dots, x_n) \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \quad (2.16)$$

where M is analytic in the coordinates x_i and depends only on the structure of the Lie algebra $\mathcal{G}$.

Moreover, if $\mathcal{G}$ is solvable, then there exists a basis and an ordering of this basis for which the representation (2.15) is global and the x_i can be computed by quadratures.

Example 2.4.5 *The product of exponentials representation for the left-invariant system (2.8) on $SE(2)$ specified in Example 2.4.2 turns out to be*

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 + x_3 u_1 \\ \dot{x}_3 &= -x_2 u_1\end{aligned}$$

(see Chapter 5). The product of exponentials coordinates for $SO(3)$ are equivalent to the Euler angles.

For all systems considered in this dissertation the product of exponentials coordinates lead to simple local representations in terms of explicit functions. These coordinates have the additional advantage that we are guaranteed a global quadrature solutions for systems on solvable Lie groups. Since chained form systems are themselves in product of exponentials form this representation is especially suitable for feedback nilpotentization to chained form.

2.5 Example Systems on Matrix Lie Groups

In this section we demonstrate how systems on matrix Lie groups arise as models for the kinematics of mechanical systems and outline how certain assumptions on the actuators are reflected in the Lie group model.

Another source for systems on matrix Lie groups can be found in conservation laws governing physical processes such as in electronics or quantum mechanics. Other examples for systems on matrix Lie groups of interest but not included in the following list are switching circuits for power conversion leading to systems on $SO(n)$ (Wood, 1974), multi-level systems used to model molecular bonds in the context of coherent control of quantum dynamics leading to system on $SU(n)$ with drift (Dahleh *et al.*, 1996), a body-mass system used to model a satellite with flexible attachments leading to a system on $SO(3) \times \mathbb{R}^2$ (Yang, 1992), the model of a kinematic car leading to a non-invariant system on $SE(2) \times \mathbb{R}$ (Leonard, 1994), and so called G -Snakes modeling chains of rigid transformations (Tsakiris, 1995).

2.5.1 Rigid motions on $\mathbb{R}^n$

Let Σ_0 be an inertial frame for $\mathbb{R}^n$ and Σ_b be a frame fixed to a rigid body. Then a rigid motion of the body can be described by the motion of the body frame Σ_b relative to Σ_0 or, more directly, by the coordinate transformation relating the coordinates of a point with respect to the body frame Σ_b to its coordinates with respect to the inertial frame. In particular, the motion of a material point p of the body can be written as

$$p_s = X(t)p_b$$

where $X(t)$ is a transformation matrix and p_s and p_b are the coordinates of the point with respect to the inertial and body frame, respectively.

If the specific rigid motion involves translatory motion, p_s and p_b are written in homogeneous coordinates to obtain the coordinate transformation in matrix form.

The velocity of a point p with respect to Σ_0 is given by

$$v_s = \dot{X}(t)p_b = \dot{X}(t)X^{-1}(t)p_s,$$

and we call $\hat{V}_s = \dot{X}(t)X^{-1}(t)$ the *spatial velocity of the rigid motion* in matrix form.

By noting that the velocity v_s of a material point p expressed in body coordinates is

$$v_b = X^{-1}v_s = X^{-1}(t)\dot{X}(t)q_b$$

we define the *body velocity* of the rigid motion (in matrix form) to be $\hat{V}_b = X^{-1}(t)\dot{X}(t)$.

Depending on the type of actuation we write the kinematics of a mechanical system either in terms of V_s or V_b . For body-fixed actuation, as found for instance in the case of a satellite with body-fixed thrusters, it is convenient to model the kinematics in terms of V_b . The motion is then characterized by a left-invariant system

$$\dot{X}(t) = X(t)\hat{V}_b(t) \quad X \in G$$

where G is the matrix Lie groups corresponding to the respective rigid motion and $\hat{V}_b$ lies in $T_e G$ accordingly.

Similarly, in the case of spatially-fixed actuation it is convenient to write the

kinematics as the right-invariant system

$$\dot{X}(t) = \hat{V}_s(t)X(t), \quad X \in G.$$

We will view the velocities $V_b(t)$ and $V_s(t)$ as a matrix-valued input and denote them in the following simply by $U(t)$.

2.5.2 Unicycle

Let us consider a simplified model of a unicycle, where we just model the wheel which is assumed to roll without slipping on a plane with the wheel axis always parallel to the plane. Further we assume that we have control over the forward velocity as well as the steering velocity, which describes the angular velocity of the wheel around a line through the point of contact and perpendicular to the plane.

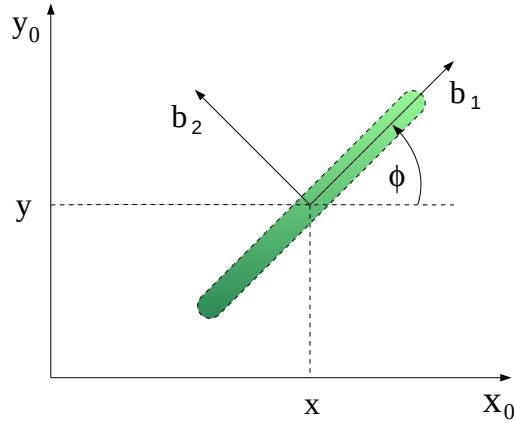


Figure 2.1: Overhead view of unicycle

Let Σ_0 be an inertial frame in $\mathbb{R}^2$ and Σ_b a frame attached to the wheel as shown in Figure 2.1. Then the matrix $X \in SE(2)$ relating the homogeneous

coordinates of a point in body coordinates to its coordinates with respect to the inertial frame Σ_0 is of the form

$$X = \begin{pmatrix} \cos \phi & -\sin \phi & x \\ \sin \phi & \cos \phi & y \\ 0 & 0 & 1 \end{pmatrix}.$$

while the body velocity $\hat{V}_b \in \mathfrak{se}(2)$ turns out to be of the form

$$\hat{V}_b = \begin{pmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{pmatrix}.$$

With a basis $\{A_1, A_2, A_3\}$ for $\mathfrak{se}(2)$ defined by

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

the kinematics of the unicycle can be written as

$$\dot{X} = X(u_1 A_1 + u_2 A_2), \tag{2.17}$$

where u_1 and u_2 are the steering and forward velocity, respectively.

Since $[A_1, A_2] = A_3$, i.e the input vectors A_1, A_2 generate the Lie algebra $\mathfrak{se}(2)$ (2.17) is controllable by Theorem 2.3.2.

Equation (2.17) describes generic nonholonomic motions on the plane and can for instance be used to represent the kinematics of a elementary **hovercraft** model if A_1 and A_2 are chosen to reflect the constellation of actuators.

2.5.3 Spacecraft

In the spacecraft attitude control problem we restrict our attention to the orientation of the satellite with respect to a reference frame Σ_0 and assume that the origin of the frame Σ_b coincides with the origin of Σ_0 . We assume that the actuators of the satellite, say thrusters or momentum wheels, are fixed to the body such that resulting angular velocity vectors are aligned with the orthonormal body frame Σ_b (see Figure 2.2). Further we make the idealizing assumption that we have direct control over the angular velocities resulting from the actuators.

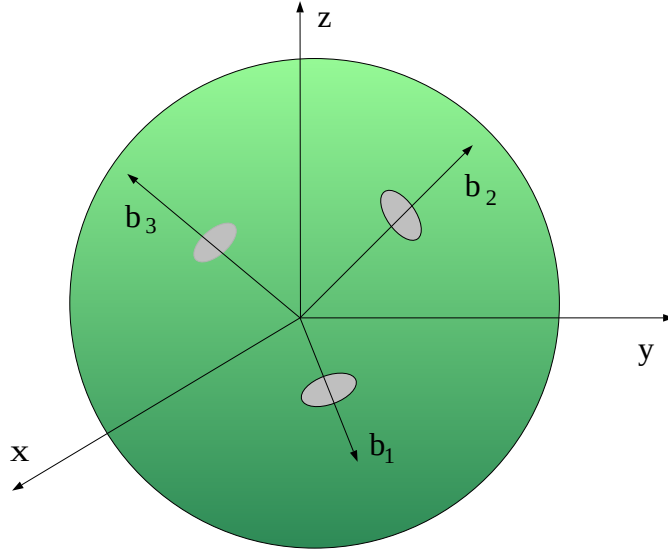


Figure 2.2: Satellite actuated by momentum wheels

The configuration of the spacecraft can thus be specified by the transformation $X \in SO(3)$ relating body coordinates to reference coordinates, while the

body angular velocity $\hat{V}_b = X^{-1}\dot{X} \in \mathfrak{so}(3)$ can be written as

$$\hat{V}_b = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where ω_i is the magnitude of the angular velocity around the axis b_i . Choosing a basis $\{A_1, A_2, A_3\}$ defined by

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $\mathfrak{so}(3)$ we define the corresponding left-invariant by control system on $SO(3)$ by

$$\dot{X}(t) = X(t)(u_1(t)A_1 + u_2(t)A_2 + u_3(t)A_3) \quad (2.18)$$

In this dissertation we are interested in the case where one of the controls u_i in (2.18) is identically equal to zero, due e.g. to a failure, which specifies a nonholonomic constraint for the evolution of $X(t)$ on $SO(3)$. Without loss of generality we assume $u_3 \equiv 0$ and study motion control of the corresponding system

$$\dot{X}(t) = X(t)(u_1(t)A_1 + u_2(t)A_2). \quad (2.19)$$

Since $[A_1, A_2] = A_3$, i.e the input vectors A_1, A_2 generate the Lie algebra $\mathfrak{so}(3)$, (2.19) is controllable by Theorem 2.3.2.

Equation (2.19) also serves as a model for the so called **ball and plate system** which was proposed as a vibratory actuator in (Leonard, 1994).

2.5.4 Underwater Vehicle

The configuration of an underwater vehicle is modeled as the position and orientation of a body-fixed $\Sigma_b = (b_1, b_2, b_3)$ with respect to an inertial frame $\Sigma_0 = (x, y, z)$ (see Figure 4.1). We assume that the individual actuators are configured such that the resulting angular and linear velocities are aligned with the body frame Σ_b . The configuration of the underwater vehicle can thus be specified by an element $X \in SE(3)$, and the corresponding body velocity $\hat{V}_b = X^{-1}\dot{X} \in \mathfrak{se}(3)$ can be written as

$$\hat{V}_b = \begin{pmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where ω_i , $i = 1, 2, 3$ is the magnitude of the angular velocity around the axis b_i and v_i , $i = 1, 2, 3$ represent the linear velocities.

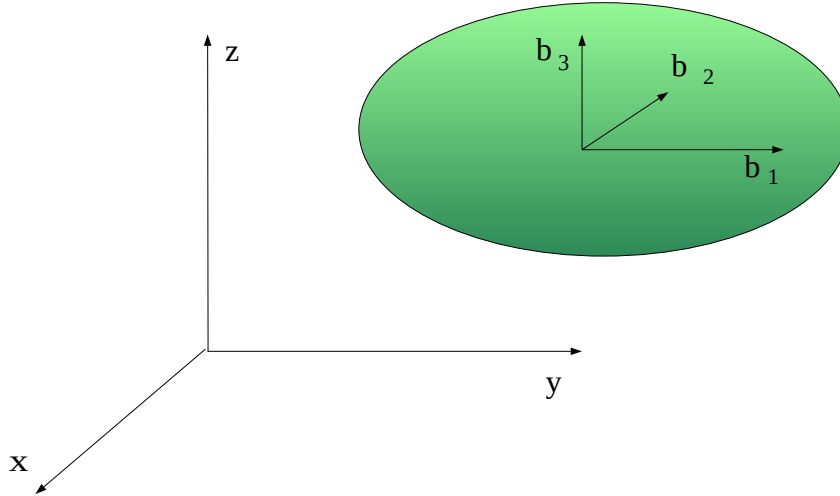


Figure 2.3: Underwater vehicle

Choosing a basis $\{A_1, \dots, A_6\}$ for $\mathfrak{se}(3)$ defined by

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we end up in the fully actuated case with the left-invariant control system on $SE(3)$ defined by

$$\dot{X}(t) = X(t)(u_1(t)A_1 + \dots + u_6(t)A_6).$$

The underwater vehicle is controllable with as few as two inputs (for details see Chapter 5).

Chapter 3

Approximate Inversion

This chapter serves to introduce the concept of approximate inversion of nonholonomic systems and to provide some motivation for the approaches taken in subsequent chapters. In the first section we define the problem of constructive controllability for nonholonomic systems and the notion of an approximate inverse dynamical system. Using the example of a simple three-dimensional system we study in the following sections different ways to formulate such approximate inversion control laws and related issues of optimality. Some of the ideas presented in this introductory chapter can already be found in (Brockett, 1993).

3.1 Problem Definition

It has been one of the achievements of linear systems theory to establish generic methods to construct explicit steering, tracking and stabilization control laws directly from the vector fields, i.e. the columns of the controllability Gramian used to establish controllability for a given linear system. No such methods exist for generic, controllable, nonlinear systems, but considerable progress has been

made for drift-free, input-linear systems of the type

$$\dot{x} = \sum_{i=1}^m f_i(x)u_i, \quad x \in \mathbb{R}^n. \quad (3.1)$$

Under the heading of *constructive controllability* two types of control problems for such systems are considered:

Point-to-Point Problem: Given any pair of points $x_0, x_1 \in \mathbb{R}^n$ find a control

$u(\cdot)$ steering the state of (3.1) from x_0 at time $t = 0$ to x_1 at some time $t = t_1 > 0$.

Approximate Tracking Problem Given a sufficiently smooth trajectory $\bar{x} :$

$[0, T] \rightarrow \mathbb{R}^n$ find controls $u(\cdot)$ steering the state $x(t), t \in [0, T]$ of (3.1) approximately (in a sense to be made precise below) along $\bar{x}$.

We will focus in this dissertation on the approximate tracking problem for systems (3.1) arising as local representations of invariant systems on matrix Lie groups or equivalently on approximately tracking a desired trajectory $\bar{g} : [0, T] \rightarrow G$ in a neighborhood of a point $g_0 \in G$.

In order to formalize the concept of approximate inversion we start by characterizing tracking controls for nonholonomic systems.

Definition 3.1.1 Let $\bar{x} : [0, T] \rightarrow \mathbb{R}^n$ be a sufficiently smooth desired trajectory for system (3.1) and assume that $x(0) = \bar{x}_0$. Let $u^{(\omega)}(t) = f(t, \omega, \bar{x}(t), \dot{\bar{x}}(t), x(t))$ be a control law parameterized by a parameter ω depending instantaneously on $\bar{x}, \dot{\bar{x}}$ and x at time t . If the sequence of trajectories $\{x^{(\omega)}\}$ of (3.1) resulting from the sequence of inputs $\{u^{(\omega)}\}$ satisfies

$$\lim_{\omega \rightarrow \infty} x^{(\omega)}(t) = \bar{x}(t), \quad \forall t \in [0, T]$$

where the above convergence is uniform with respect to $t \in [0, T]$, then we call $u^{(\omega)}$ an *approximate tracking control*. If moreover $u^{(\omega)}$ is independent of the state x of (3.1) we call $u^{(\omega)}$ an *open-loop approximate tracking control*.

The instantaneous dependency on the desired trajectory $\bar{x}$ is a crucial feature of this type of controls, since it allows on-line control of drift-free nonholonomic systems with an “on-the-fly” trajectory generator. If we are given $\bar{x}(t)$ and $x(t)$ as inputs we can therefore model the approximate tracking control law as an input-output map $\hat{\Sigma}^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^m; \bar{x}(t) \mapsto u(t)$ involving only the differentiation of $\bar{x}(t)$ and a subsequent nonlinear map with an explicit time dependence.

If this input-output map is independent of $x(t)$ then it can be interpreted as a dynamical system independent of (3.1). Following Brockett, we call such an input-output description of an open-loop approximate tracking control law the *approximate inverse* of the original system Σ . This terminology is motivated by the fact that the concatenation of the approximate inverse $\hat{\Sigma}^{-1}$ with the original system Σ yields the approximate identity operator in path space:

$$x(t) = \Sigma\left(\hat{\Sigma}_{\omega}^{-1}(\bar{x})\right) = \bar{x} + o(\omega^{\alpha}), \quad t \in [0, T], \quad \alpha < 0.$$

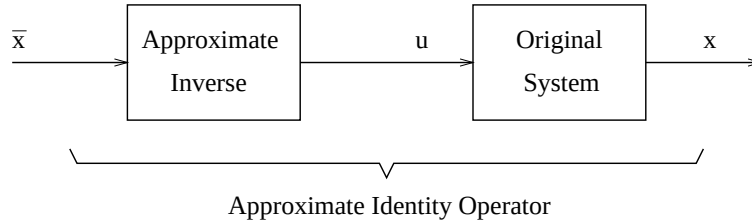


Figure 3.1: Interpretation of open-loop approximate tracking control law as approximate inverse operator in space of state trajectories

The interpretation of $\Sigma \circ \hat{\Sigma}^{-1}$ as an approximate identity operator has important consequences for the problem of asymptotic point stabilization of a drift-free nonholonomic system which will be explored in Chapter 6.

3.2 Open-loop tracking for the nonholonomic integrator

Consider the left-invariant drift-free system

$$\dot{X}(t) = X(t) (u_1(t)A_1 + u_2(t)A_2), \quad (3.2)$$

on the Heisenberg group $H(3)$, where $X \in H(3)$ is of the form

$$X = \begin{pmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and where we have fixed a basis

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $\mathfrak{h}(3)$. Since $[A_1, A_2] = A_3$, (3.2) is controllable by Theorem 2.3.2.

The single exponential representation of (3.2) is globally valid and turns out to be

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= \frac{1}{2}(x_1 u_2 - x_2 u_1). \end{aligned} \quad (3.3)$$

Despite its simplicity the system (3.3) already exhibits some of the fundamental features and problems encountered in motion control of nonholonomic systems. Following Brockett we call (3.3) the *nonholonomic integrator*.

While the states x_1 and x_2 of (3.3) can be controlled directly as the indefinite integrals of u_1 and u_2 , we can also assign a geometric meaning to the evolution of x_3 . Assuming $x_3(0) = 0$ and rewriting

$$x_3(t) = \frac{1}{2} \int_0^t x_1 u_2 - x_2 u_1 d\tau = \frac{1}{2} \int_0^t x_1 \dot{x}_2 - x_2 \dot{x}_1 d\tau \quad (3.4)$$

as a line integral

$$x_3(t) = \frac{1}{2} \int_{C_t} -x_2 dx_1 + x_1 dx_2 = \int_{C_t} \omega \quad (3.5)$$

we see that $x_3(t)$ is the integral of the solenoidal vector field with components $F_1(x_1, x_2) = -x_2$, $F_2(x_1, x_2) = x_1$ along the path $C_t = (x_1(\tau), x_2(\tau))$, $\tau \in [0, t]$ of the states x_1 and x_2 in the (x_1, x_2) -plane (see Figure 3.2).

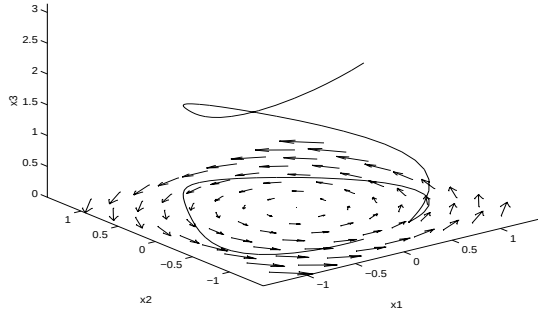


Figure 3.2: Interpretation of evolution of x_3 as line integral

Assuming that C is closed, i.e. $(x_1(T), x_2(T)) = (x_1(0), x_2(0))$ for some T and encloses the surface R in the (x_1, x_2) -plane positively, we can apply Green's

theorem to obtain

$$x_3(t) = \int_R d\omega = \int_R dx_1 dx_2 = a(R) \quad (3.6)$$

where $a(R)$ is the area of the surface R (see Figure 3.3). As pointed out in (Leonard, 1994), also for the generation of motion along higher-order brackets a geometric interpretation is possible.

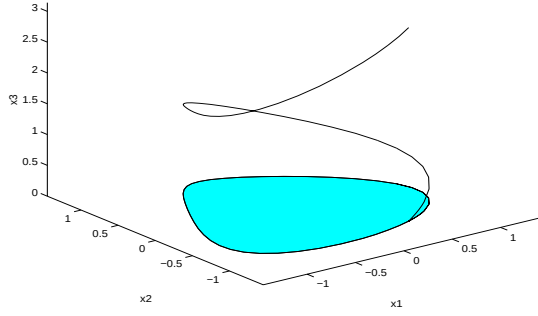


Figure 3.3: Interpretation of evolution of x_3 as surface integral

From (3.5) or Figure 3.2 we see that the only paths of x_1 and x_2 in the (x_1, x_2) -plane which leave x_3 constant are on lines through the origin of the (x_1, x_2) -plane. On the other hand, away from the origin in the (x_1, x_2) -plane any smooth trajectory $\bar{x}_3$ can be tracked exactly with x_3 by specifying a suitable trajectory in the (x_1, x_2) -plane.

Equation (3.6) shows how to achieve a secular motion for x_3 while staying close to an initial point in the (x_1, x_2) -plane. Using a T -periodic control $u(t) = (u_1(t), u_2(t))^T, t \geq 0$ the projection of the resulting trajectory onto the (x_1, x_2) -plane traverses successively a loop ¹ which encloses a surface with area A . Thus

¹For the sake of simplicity we assume simple loops here, i.e. there is no $t \in (0, T)$ such that $(x_1(t), x_2(t)) = (x_1(0), x_2(0))$.

by (3.6)

$$x_3(nT) = \pm nA, \quad n = 1, 2, \dots, \quad (3.7)$$

where the sign depends on the direction in which the loop is traversed. Note that while the trajectory of $x_3(t)$ changes for $t \neq nT$ if we vary $x_1(0)$ and $x_2(0)$, equation (3.7) is independent of the initial location in the (x_1, x_2) -plane, since the area enclosed by the loop depends only on the form of u but not on $x(0)$.

Moreover if we scale the amplitude and the time-dependence of control u such that $\hat{u}(t) = \alpha u(kt)$, $\alpha > 0, k > 0$ then we have for the resulting state $\hat{x}_3$ that

$$\hat{x}_3(nT) = \pm k\alpha^2 nA, \quad n = 1, 2, \dots \quad (3.8)$$

Letting $\alpha = \frac{1}{\sqrt{k}}$ for an integer $k > 1$ we have $\hat{x}_3(nT) = x_3(nT)$ while

$$\begin{aligned} \hat{x}_i(t) - \hat{x}_i(0) &= \int_0^t \sqrt{k} u(k\tau) d\tau = \frac{1}{\sqrt{k}} \int_0^t u(\tau) d\tau \\ &= \frac{1}{\sqrt{k}} (x_i(t) - x_i(0)), \quad i = 1, 2 \end{aligned}$$

where we have assumed $\hat{x}_i(0) = x_i(0)$. Thus, increasing the frequency of a periodic control by a factor k and scaling the amplitude as above yields the same secular motion for x_3 while reducing the deviation of $x_i(t), i = 1, 2$ from $x_i(0), i = 1, 2$ respectively by a factor $\frac{1}{\sqrt{k}}$.

This simple example reveals the principle underlying the use of *high-frequency controls* for trajectory tracking for drift-free nonholonomic systems: increasing the frequency of oscillatory controls which excite a certain Lie bracket of the system and scaling the amplitudes accordingly reduces the perturbations of the resulting motion in the direction of Lie brackets of a different order.

Brockett (Brockett, 1993) presented such open-loop approximate tracking controls for the nonholonomic integrator which we will state here in a slightly

modified version.

Proposition 3.2.1 *Let $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))^T$, $t \in [0, T]$ be a desired smooth trajectory for the system (3.3) and assume $x(0) = \bar{x}(0) = 0$. Define a sequence $\{u^{(\omega)}\}_{\omega=1}^{\infty}$ of controls*

$$\begin{aligned} u_1^{(\omega)}(t) &= \dot{\bar{x}}_1(t) - \sqrt{\omega} \sin(\omega t) \\ u_2^{(\omega)}(t) &= \dot{\bar{x}}_2(t) + \sqrt{\omega} m(t) \cos(\omega t) \end{aligned} \quad (3.9)$$

with $m(t) = 2\dot{\bar{x}}_3(t) - \bar{x}_1(t)\dot{\bar{x}}_2(t) + \bar{x}_2(t)\dot{\bar{x}}_1(t)$ and denote the solution of (3.3) with control $u^{(\omega)}$ by $x^{(\omega)}$. Then

$$\lim_{\omega \rightarrow \infty} x^{(\omega)}(t) = \bar{x}(t), \quad t \in [0, T], \quad (3.10)$$

where the above convergence is uniform with respect to t .

The proof of Proposition 3.2.1 is based on integration by parts and an application of the Riemann-Lebesgue Theorem and can be seen as a special case of the proof for Theorem 4.2.1 of Chapter 4.

We call the periodic functions appearing in (3.9) *carrier functions* since the role they play in the motion generation for nonholonomic systems is somewhat analogous to the role of carrier functions in tele-communications. To approximately track a continuous trajectory the carrier functions have to be modulated in order to specify the continuously varying velocity of the motion along the corresponding bracket direction. The nonholonomic system itself then acts as a de-modulator. In (3.9) amplitude modulation is used, but as will be shown below other types of modulations are possible.

To take the analogy with communications a little further, the problem of tracking a n -dimensional nonholonomic system with only $m < n$ controls can

be compared with the problem of transmitting n signals over only $m < n$ channels. Control laws of the type (3.9) can therefore be interpreted as frequency multiplexers since they distribute the motion information pertaining to different brackets of the system into distinct frequency bands. The nonholonomic system with m inputs (controls) and $n > m$ outputs (states) then plays the role of a de-multiplexer.

Approximate tracking controls are not unique since they are only required to satisfy a convergence condition. In the following we will discuss some possible variations.

The choice of sinusoidal carriers is motivated by their explicitness and smoothness, as well as by the fact that the circular trajectories in the (x_1, x_2) -plane resulting from sinusoidal carriers are extremals of a specific optimal tracking problem for (3.3) (see Section 3.3). Nevertheless, in the general case the controls (3.9) do not satisfy the necessary conditions for the standard optimal tracking problem. Other types of carriers might be used alternatively, for instance switching-type controls, depending on the properties of the available actuators.

In equation (3.9) the term specifying the velocity for x_3 is incorporated as an amplitude modulation of the sinusoidal carrier. It is also possible to use *frequency modulation* of the carrier for the generation of the motion in the direction of the first order Lie bracket as demonstrated in the following proposition.

Proposition 3.2.2 *Let $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))^T$, $t \in [0, T]$ be a desired smooth trajectory for the system (3.3) and assume $x(0) = \bar{x}(0) = 0$. Define a sequence $\{u^{(\omega)}\}_{\omega=1}^{\infty}$ of controls*

$$\begin{aligned} u_1^{(\omega)}(t) &= \dot{\bar{x}}_1(t) - \sqrt{\omega} \dot{m}_f(t) \sin(\omega m_f(t)) \\ u_2^{(\omega)}(t) &= \dot{\bar{x}}_2(t) + \sqrt{\omega} \dot{m}_f(t) \cos(\omega m_f(t)) \end{aligned} \quad (3.11)$$

with $m_f(t) = \bar{x}_3(t) - \int_0^t \bar{x}_1(\tau) \dot{\bar{x}}_2(\tau) + \bar{x}_2(\tau) \dot{\bar{x}}_1(\tau) d\tau$ and denote the solution of (3.3) with control $u^{(\omega)}$ by $x^{(\omega)}$. Then

$$\lim_{\omega \rightarrow \infty} x^{(\omega)}(t) = \bar{x}(t), \quad t \in [0, T], \quad (3.12)$$

where the above convergence is uniform with respect to t .

Proof: Integrating x_1 and x_2 of (3.3) with controls (3.11) yields

$$\begin{aligned} x_1(t) &= \bar{x}_1(t) + \omega^{-\frac{1}{2}}(\cos(\omega m_f(t)) - 1) \\ x_2(t) &= \bar{x}_2(t) + \omega^{-\frac{1}{2}} \sin(\omega m_f(t)), \end{aligned}$$

and it follows immediately that for $\omega \rightarrow \infty$, x_1 and x_2 converge uniformly on $[0, T]$ to $\bar{x}_1$ and $\bar{x}_2$ respectively. For x_3 we obtain

$$\begin{aligned} x_3(t) &= \\ &\frac{1}{2} \int_0^t \left(\bar{x}_1(\tau) + \omega^{-\frac{1}{2}}(\cos(\omega m_f(\tau)) - 1) \right) \left(\dot{\bar{x}}_2(\tau) + \sqrt{\omega} \dot{m}_f(\tau) \cos(\omega m_f(\tau)) \right) \\ &\quad - \left(\bar{x}_2(\tau) + \omega^{-\frac{1}{2}} \sin(\omega m_f(\tau)) \right) \left(\dot{\bar{x}}_1(\tau) - \sqrt{\omega} \dot{m}_f(\tau) \sin(\omega m_f(\tau)) \right) d\tau \\ &= \frac{1}{2} \int_0^t \left\{ \bar{x}_1(\tau) \dot{\bar{x}}_2(\tau) + \sqrt{\omega} \bar{x}_1(\tau) \dot{m}_f(\tau) \cos(\omega m_f(\tau)) + \omega^{-\frac{1}{2}} \dot{\bar{x}}_2(\tau) (\cos(\omega m_f(\tau)) - 1) \right. \\ &\quad \left. + \dot{m}_f(\tau) \underbrace{(\cos^2(\omega m_f(\tau)) + \sin^2(\omega m_f(\tau)))}_{=1} - \dot{m}_f(\tau) (\cos(\omega m_f(\tau))) \right. \\ &\quad \left. - \bar{x}_2(\tau) \dot{\bar{x}}_1(\tau) + \sqrt{\omega} \bar{x}_2(\tau) \dot{m}_f(\tau) \sin(\omega m_f(\tau)) + \sqrt{\omega} \dot{\bar{x}}_1(\tau) \dot{m}_f(\tau) \sin(\omega m_f(\tau)) \right\} d\tau \\ &= \bar{x}_3(t) + \frac{1}{2} \int_0^t \left\{ \sqrt{\omega} \bar{x}_1(\tau) \dot{m}_f(\tau) \cos(\omega m_f(\tau)) \right. \\ &\quad \left. + \omega^{-\frac{1}{2}} \dot{\bar{x}}_2(\tau) (\cos(\omega m_f(\tau)) - 1) - \dot{m}_f(\tau) (\cos(\omega m_f(\tau))) \right. \\ &\quad \left. + \sqrt{\omega} \bar{x}_2(\tau) \dot{m}_f(\tau) \sin(\omega m_f(\tau)) + \sqrt{\omega} \dot{\bar{x}}_1(\tau) \dot{m}_f(\tau) \sin(\omega m_f(\tau)) \right\} d\tau \\ &= \bar{x}_3(t) + o(\omega^{-\frac{1}{2}}) \end{aligned}$$

where the last equality follows from an application of the Riemann-Lebesgue Theorem (see proof of Theorem 4.2.1). Thus also x_3 converges uniformly on $[0, T]$ to $\bar{x}_3$ for $\omega \rightarrow \infty$.

□

Frequency modulation type controls are of advantage if for fixed ω the error in the (x_1, x_2) -plane is required not to exceed a certain limit independently of $m(t)$.

Completing the analogy with modulation techniques, also phase-shift type controls are possible for trajectory tracking. With such a technique the phase angle between the carrier functions is varied between $[-\frac{\pi}{2}, \frac{\pi}{2}]$ depending on $m(t)$ while keeping the frequency and amplitude of the carrier functions fixed. Even though phase-shift controls allow only a limited range of motions to be tracked, such controls would be of advantage if by the nature of the actuators a limited range phase shift of x_i , $i = 1, 2$ is easier to achieve than a variation of the frequency or amplitude of the loops in the (x_1, x_2) -plane.

Certain specific techniques and results with respect to asymptotic stabilization and tracking for higher-dimensional systems presented below are not compatible with frequency- and phase-modulation type controls. We will therefore for the remainder of the thesis focus on amplitude modulation type controls.

3.3 Optimality

In this section we will explore optimality issues for tracking control of drift-free nonholonomic systems using the example of the nonholonomic integrator. We will see that the inclusion of tracking performance in the objective function leads for some prescribed trajectories of the nonholonomic integrator to tractable necessary conditions for extremal functions. We will compare these necessary conditions to the explicit controls of type (3.9).

Significant progress in characterizing optimal controls has been made for the point-to-point optimal control problem, i.e steering a system from a given initial position at time T_0 to a desired final position at time T_f while minimizing an objective function $J = \int_{T_0}^{T_f} \sum_{i=1}^m u_i(t)^2 dt$. For a drift-free system $\dot{x} = \sum_{i=1}^m f_i(x)u_i$, $x \in \mathbb{R}^n$ it is well known (see for instance (Walsh *et al.*, 1994)) that the norm $\|u(t)\|$ of input functions satisfying the necessary conditions for the point-to-point optimal control problem is a constant of motion. Also, for invariant system on three-dimensional matrix Lie groups it has been shown by Poisson reduction in (Krishnaprasad, 1993) that the differential equations characterizing extremal controls of the corresponding Maximum Principle are integrable.

But even though the form of optimizing controls may be available explicitly, their parameters have to be determined numerically in most cases. Thus, even for the simpler point-to-point problem, optimizing controls can in most cases not be used to directly formulate explicit, open-loop control laws.

In the context of open-loop, approximate tracking we are not only interested in optimizing the control effort but in simultaneously minimizing the deviations from the desired trajectories. To keep the problem tractable we focus on the nonholonomic integrator and incorporate the deviations of x_i from $\bar{x}_i$ for $i = 1, 2$ in the objective function, i.e. the goal is to find $u_i(\cdot)$, $i = 1, 2$ which

$$\text{minimize} \quad \frac{1}{2} \int_0^1 \mu \left((x_1(t) - \bar{x}_1(t))^2 + (x_2(t) - \bar{x}_2(t))^2 \right) + u_1(t)^2 + u_2(t)^2 dt \quad (3.13)$$

subject to

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \end{aligned} \quad (3.14)$$

$$\dot{x}_3 = \frac{1}{2}(x_1 u_2 - x_2 u_1),$$

for given boundary conditions

$$\begin{aligned} x_1(0) &= x_{10}, \quad x_2(0) = x_{20}, \quad x_3(0) = x_{30} \\ x_1(1) &= x_{1f}, \quad x_2(1) = x_{2f}, \quad x_3(1) = x_{3f}. \end{aligned} \quad (3.15)$$

Proposition 3.3.1 *Given desired trajectories $\bar{x}_i(\cdot)$, $i = 1, 2$ assume that $x^*(\cdot) = (x_1^*(\cdot), x_2^*(\cdot), x_3^*(\cdot))^T$ is an optimal trajectory for the optimal control problem defined by (3.13), (3.14), (3.15). Then there exists a $\lambda \in \mathbb{R}$ such that $x_1^*(\cdot)$, $x_2^*(\cdot)$, $u_1^*(\cdot) = \dot{x}_1^*(\cdot)$, and $u_2^*(\cdot) = \dot{x}_2^*(\cdot)$ satisfy*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu & 0 & 0 & -\lambda \\ 0 & \mu & \lambda & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{pmatrix} - \mu \begin{pmatrix} 0 \\ 0 \\ \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} \quad (3.16)$$

Proof:

With the Lagrangian

$$L(x, u) = \frac{1}{2} \left\{ \mu \left((x_1(t) - \bar{x}_1(t))^2 + (x_2(t) - \bar{x}_2(t))^2 \right) + u_1(t)^2 + u_2(t)^2 \right\}$$

and the adjoint state $p = (p_1, p_2, p_3)^T$ define the Hamiltonian

$$\begin{aligned} H(x, p, u) &= p_0 L(x, u) + p^T f(x, u) \\ &= -L(x, u) + p_1 u_1 + p_2 u_2 + \frac{1}{2} p_3 (x_1 u_2 - x_2 u_1), \end{aligned}$$

where we have assumed without loss of generality $p_0 \equiv -1$ since we only consider regular extremals here. Given an optimal trajectory $x(\cdot)$ there exists according to Pontryagin's Maximum Principle a function $p(\cdot)$ satisfying the adjoint equations

$$\dot{p} = -\frac{\partial H(x, p, u)}{\partial x}$$

such that the optimal controls are characterized by

$$\frac{\partial H(x, p, u)}{\partial u_i} = 0, \quad i = 1, 2. \quad (3.17)$$

Writing out the adjoint equations we obtain

$$\begin{aligned} \dot{p}_1 &= -\frac{1}{2}p_3u_2 + \mu(x_1 - \bar{x}_1) \\ \dot{p}_2 &= \frac{1}{2}p_3u_1 + \mu(x_2 - \bar{x}_2) \\ \dot{p}_3 &= 0 \end{aligned} \quad (3.18)$$

and note that $p_3(t) \equiv p_{30}$, where we are free to choose p_{30} since we assumed fixed initial and final conditions on x_3 . It follows from (3.17) that

$$\begin{aligned} u_1 &= -\frac{1}{2}p_{30}x_2 + p_1 \\ u_2 &= \frac{1}{2}p_{30}x_1 + p_2, \end{aligned}$$

which yields after differentiating with respect to time

$$\begin{aligned} \dot{u}_1 &= -\frac{1}{2}p_{30}\dot{x}_2 + \dot{p}_1 \\ &= -p_{30}u_2 + \mu(x_1 - \bar{x}_1) \\ \dot{u}_2 &= \frac{1}{2}p_{30}\dot{x}_1 + \dot{p}_2 \\ &= p_{30}u_1 + \mu(x_2 - \bar{x}_2). \end{aligned}$$

Thus, setting $\lambda = p_{30}$ we have proven the result.

□

Note, that since (3.16) is a linear time-invariant differential equation we can explicitly obtain the form of controls satisfying the Maximum Principle for arbitrary $\bar{x}_1$ and $\bar{x}_2$.

It turns out that the sinusoidal carrier functions of (3.9) satisfy the necessary conditions of the optimal control problem (3.13), (3.14), (3.15) only in a very

special case, namely if $\bar{x}_1 = \bar{x}_2 \equiv 0$ and $x_1(0), x_1(1), x_2(0), x_2(1)$ are assumed to be free. Then it can be verified that the trajectories

$$\begin{aligned} x_1(t) &= m \cos(\omega t); & u_1(t) &= -m\omega \sin(\omega t) \\ x_2(t) &= m \sin(\omega t); & u_1(t) &= m\omega \cos(\omega t) \end{aligned} \quad (3.19)$$

with $m \in \mathbb{R}$ and $\omega \in \mathbb{R}$ satisfy (3.16) with $\lambda = \omega + \frac{\mu}{\omega}$. To achieve $\Delta x_3 \triangleq x_{3f} - x_{30}$ we have to have $m = \sqrt{\frac{\Delta x_3}{\omega}}$ in (3.19)². To obtain the optimal frequency we then plug (3.19) in to the objective function J and differentiate to obtain

$$\frac{\partial J}{\partial \omega} = \frac{\Delta x_3}{2} \left(-\frac{\mu}{\omega^2} + 1 \right).$$

It follows that the optimal frequency $\omega = \sqrt{\mu}$ is independent of Δx_3 . Thus the controls (3.9) (using amplitude modulation) are preferable to the controls (3.11) (using frequency modulation) in this case and given $x_1(0) = \frac{\sqrt{\Delta x_3}}{\omega}, x_2(0) = 0$ indeed satisfy the Maximum Principle for the above optimal control problem.

The dependency of the optimal frequency on μ characterizes the trade-off between control effort and tracking accuracy under the above assumptions but might also be used as a suggestion of how to adjust the frequency parameter in the general case.

²It then follows from (3.19) that $x_3(t) = \Delta x_3 * t$ and that the above controls also satisfy an optimal control problem which includes a trajectory error term $(x_3 - \Delta x_3 * t)^2$ in (3.13) with otherwise unchanged assumptions.

Chapter 4

Approximate Inversion of Nilpotent Systems

The focus of this chapter will be the definition, significance and control of nilpotent systems. We call an invariant system on a matrix Lie group G a *nilpotent system* if G or equivalently the Lie corresponding Lie algebra $\mathfrak{g}$ is nilpotent. Nilpotent systems play a special role in control of nonholonomic systems since their Lie algebra structure leads to significant simplifications in their treatment. On the other hand nilpotent systems have been shown to be locally feedback equivalent, for instance, to the kinematics of wheeled robots, which play an important role in robotic applications and have been a major motivation for the study of drift-free nonholonomic systems.

Looking at the Fomenko Chakon expansion (2.13) for the solution of a nilpotent system in the coordinates of the first kind the advantages of $\mathfrak{g}$ being nilpotent, say of order k , become immediately apparent. Since the series expansion terminates after the term Z_k , (i) one does not need to worry about components of the trajectories coming from higher order Lie brackets and (ii) the system can be integrated by a finite series of quadratures.

Restricting the input to a linear combination of explicitly integrable functions, property (ii) implies that forward integration of the nilpotent system can be replaced by a fixed map from the coefficients of the input function to a set of coefficients parameterizing an explicit given output function. This allows us to reduce the point-to-point problem to the inversion of a polynomial map and to cast the problem of optimal tracking control as a finite dimensional static optimization problem.

In our context of open-loop tracking property (ii) will allow us to construct simple control laws and prove the corresponding convergence result for open-loop tracking by means of elementary mathematical tools.

In Section 4.1 we show how equivalent forms of nilpotent systems extensively studied in research on nonholonomic path planning do in fact arise as local representations of invariant systems on a single nilpotent matrix group. The main result of this chapter concerning approximate inversion of chained-form systems is proven in Section 4.2.

4.1 Local Representations of Nilpotent Systems on Matrix Lie Groups

Recall that a $p \times p$ -matrix A is called *nilpotent* if $A^k = 0$ for some integer k and the smallest such k is called the *index of nilpotency*. Nilpotent matrices are of interest also in linear systems theory as a matrix in Jordan form can be decomposed into a diagonal matrix and a nilpotent matrix.

A $p \times p$ -matrix $X = (X_{ij})$ satisfying $X_{ii} = 1$, $i = 1, \dots, p$ and $X_{ij} = 0$ whenever $i > j$, i.e. a upper triangular matrix with ones on the diagonal, is

called *unipotent*. The unipotent matrices $UP(n)$ form a Lie group with the Lie algebra $\mathfrak{up}(n)$ consisting of matrices $A = (A_{ij})$ satisfying $A_{ij} = 0$, whenever $i \leq j$. It can be verified that the elements of the Lie algebra of $p \times p$ unipotent matrices are nilpotent with index k and moreover that the Lie algebra itself is nilpotent of order k , where $k \leq p$.

Consider a n -dimensional subgroup G of the unipotent matrices consisting of elements X of the form

$$X = \begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_n \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 & \cdots & \frac{1}{(n-2)!}x_1^{n-2} \\ & & 1 & x_1 & \frac{1}{2}x_1^2 & \ddots & \vdots \\ & & \ddots & 1 & x_1 & \ddots & \frac{1}{6}x_1^3 \\ \vdots & & & & 1 & \ddots & \frac{1}{2}x_1^2 \\ & & & & & \ddots & x_1 \\ 0 & & \cdots & & 0 & & 1 \end{pmatrix}, \quad x = (x_1, \dots, x_n)^T \in \mathbb{R}^n, \quad (4.1)$$

which we call $SUP(n)$ for future reference. ¹ Note, that for $n = 3$ $SUP(n)$ is isomorphic to the real Heisenberg group $H(3)$.

Fix the following basis for the Lie algebra of $SUP(n)$:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & & 0 \end{pmatrix}, \quad (4.2)$$

¹We are not aware of a standard nomenclature for the above matrix Lie group.

$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & & \cdots & & & 0 \\ & & & & & \\ \vdots & & \ddots & & & \vdots \\ 0 & & \cdots & & & 0 \end{pmatrix}, \dots, A_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ \vdots & \ddots & & \\ & & & \vdots \\ 0 & \cdots & 0 & \end{pmatrix}.$$

This choice of a basis results in the following non-zero Lie brackets

$$\begin{aligned} ad_{A_1} A_2 &= -A_3 \\ ad_{A_1}^2 A_2 &= A_4 \\ &\vdots \\ ad_{A_1}^{n-2} A_2 &= (-1)^{n-2} A_n, \end{aligned} \tag{4.3}$$

while all other Lie brackets vanish. Thus, $SUP(n)$ is nilpotent of order $n - 2$.

We start by considering the following two-input, drift-free, *left*-invariant system on $SUP(n)$

$$\dot{X} = X(u_1 A_1 + u_2 A_2), \quad X \in SUP(n) \tag{4.4}$$

with A_1, A_2 as defined above. It follows from (4.3) and Theorem 2.3.2 that (4.4) is controllable, more specifically, that (4.4) is a depth- $(n - 2)$ system.

The entries $x_1, \dots, x_n$ of $X \in SUP(n)$ as described in (4.1) provide us naturally with global coordinates for $SUP(n)$. Thus, by equating the entries on either side of equation (4.4) we obtain the following globally valid representation of (4.4) on $\mathbb{R}^n$:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \end{aligned}$$

$$\dot{x}_3 = x_2 u_1 \quad (4.5)$$

$$\dot{x}_4 = x_3 u_1$$

$$\vdots$$

$$\dot{x}_n = x_{n-1} u_1.$$

Note that $x_1, \dots, x_n$ coincide with the canonical coordinates of the second kind for $SUP(n)$ and that (4.5) therefore represents the product of exponentials representation of (4.4). We say that system (4.5) is in *chained form*. Due to its lower triangular structure we can integrate (4.5) by quadratures and the solution for the i^{th} , $i = 3, \dots, n$ state can be written as

$$x_i(t) = \int_0^t u_1(\tau_1) \int_0^{\tau_1} u_1(\tau_2) \dots \int_0^{\tau_{i-3}} u_1(\tau_{i-2}) \int_0^{\tau_{i-2}} u_2(\tau_{i-1}) d\tau_{i-1} \dots d\tau_1,$$

assuming $x(0) = 0$.

Next consider the *right*-invariant version

$$\dot{Y} = (u_1 A_1 + u_2 A_2) Y, \quad X \in SUP(n) \quad (4.6)$$

of system (4.4). Using again the canonical coordinates $y_1, \dots, y_n$ of the second kind for $SUP(n)$ the product of exponentials representation of (4.6) turns out to be

$$\dot{y}_1 = u_1$$

$$\dot{y}_2 = u_2$$

$$\dot{y}_3 = y_1 u_2 \quad (4.7)$$

$$\dot{y}_4 = \frac{1}{2} y_1^2 u_2$$

$$\vdots$$

$$\dot{y}_n = \frac{1}{(n-2)!} y_1^{n-2} u_2.$$

Systems of the type (4.7) are called systems in *power form*.

Rewriting (4.4) and (4.5) as

$$\dot{\bar{X}} = \bar{X}(-u_1 A_1 - u_2 A_2), \quad X \in SUP(n) \quad (4.8)$$

with

$$\bar{X} = \begin{pmatrix} 1 & -x_2 & x_3 & -x_4 & x_5 & \cdots & (-1)^{n-1}x_n \\ 0 & 1 & -x_1 & \frac{1}{2}x_1^2 & -\frac{1}{6}x_1^3 & \cdots & (-1)^{n-2}\frac{1}{(n-2)!}x_1^{n-2} \\ & & 1 & -x_1 & \frac{1}{2}x_1^2 & \ddots & \vdots \\ & & \ddots & 1 & -x_1 & \ddots & -\frac{1}{6}x_1^3 \\ \vdots & & & & 1 & \ddots & \frac{1}{2}x_1^2 \\ & & & & & \ddots & -x_1 \\ 0 & & \cdots & & 0 & & 1 \end{pmatrix}$$

we see from the relationship (2.10) between left-invariant and right invariant systems on matrix Lie groups that if $\bar{X}$ is a solution of (4.8) then $\bar{X}^{-1}$ satisfies (4.6). The coordinate transformation T mapping the coordinates of the chained form system to the coordinates of the power form system can therefore be obtained by $Y = T(X) = (\bar{X})^{-1}$ and component-wise we have

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= -x_3 + x_1 x_2 \\ y_4 &= x_4 - x_1 x_3 + \frac{1}{2} x_1 x_2 \\ &\vdots \end{aligned} \quad (4.9)$$

$$y_n = (-1)^x_n + \sum_{i=2}^{n-1} (-1)^i \frac{1}{(n-1)!} x_1^{n-i} x_i. \quad (4.10)$$

In the case $n = 3$ the group $SUP(n)$ coincides with the Heisenberg group $H(3)$ and the single exponential representation of (4.6) yields Brockett's nonholonomic integrator.

A generalization of the Lie algebra structure (4.3) to systems with $m + 1$ inputs leads to so called $(m + 1)$ -input, (m) -chain, single generator chained form systems which are described in the following section.

Chained and power forms system play an important role in the study of the kinematics of various types of wheeled robots. Kinematic models of a simplified unicycle, a car, or a tractor with trailers, for instance, have a Lie algebra structure compatible with the Lie algebra structure of chained form systems and can be locally converted via a feedback and coordinate transformation into chained form. Control issues such as open-loop point-to-point steering, as well as point and trajectory stabilization for chained form systems were studied for instance in (Murray & Sastry, 1993; Samson, 1995).

While chained form systems reflect more directly how motion is propagated in the kinematic description of a tractor and trailer system, power form systems resolve this recursive dependency and describe a state $y_i, i \geq 3$ solely in terms of y_1 and u_2 . This property makes the power form description especially suited for the derivation and proof of stabilization results.

4.2 Approximate inversion of Chained-Form Systems

In this section we will present open-loop tracking controls or, in the terminology established in Chapter 3, an approximate inverse for (n) -dimensional

chained form systems extending the result of Brockett (Brockett, 1993) for a three-dimensional system. To create independent motions in the direction of the $l = n - 2$ higher order Lie brackets required for controllability we can make use of the particularly simple Lie algebra structure of (4.5).

As opposed to the general method of Liu and Sussmann (Sussmann & Liu, 1991) which requires l sets of so-called mutually independent minimally cancelling frequencies the following results use only one set of l integrally related frequencies.

This is achieved by specifying the desired velocity terms for the states x_i , $i = 3, \dots, n$ in u_2^ω as amplitude modulations of integrally related sinusoids. The cosine term in u_1^ω shifts the frequency components by the successive multiplications in the chain of (4.5) such that for each state x_i , $i = 3, \dots, n$ the velocity term corresponding to $\bar{x}_i$ is in resonance with the cosine term.

Theorem 4.2.1 *Given a n -dimensional chained-form system (4.5), let $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \dots, \bar{x}_n(t))^T$, $t \in [0, T]$ denote the desired trajectory. Assume that $\bar{x}$ is twice differentiable on $[0, T]$ and that $x(0) = \bar{x}(0) = \dot{\bar{x}}(0) = 0$. Define a sequence $\{u^{(\omega)}\}_{\omega=1}^\infty$ of controls*

$$\begin{aligned} u_1^{(\omega)} &= \dot{\bar{x}}_1(t) + 2\omega^{\frac{l}{l+1}} \cos(\omega t) \\ u_2^{(\omega)} &= \dot{\bar{x}}_2(t) + \sum_{m=1}^l \alpha_m(t) \omega^{-\frac{ml}{l+1}} \frac{m!}{m^m} \frac{d^m}{dt^m}(\cos(m\omega t)), \quad l = n - 2 \end{aligned} \quad (4.11)$$

where

$$\alpha_m(t) = \dot{\bar{x}}_{m+2}(t) - \dot{\bar{x}}_1(t)\bar{x}_{m+1}(t).$$

Let $x^{(\omega)}(t)$, $0 \leq t \leq T$ be the solution of (4.5) with $u^{(\omega)}$ as input. Then

$$\lim_{\omega \rightarrow \infty} x^{(\omega)}(t) = \bar{x}(t), \quad \forall t \in [0, T] \quad (4.12)$$

where the convergence is uniform with respect to t .

Proof: We start by proving a lemma which enables us to discard terms in the solution of (4.5) which vanish in the high-frequency limit.

Lemma 4.2.2 *Let f be of bounded variation on $[a, b]$ and let $\phi \in [0, 2\pi]$. Then as $\omega \rightarrow \infty$*

$$\int_a^b f(t) \cos(\omega t + \phi) dt = o(1/\omega). \quad (4.13)$$

Proof: (Lemma 4.2.2) By the Jordan Decomposition Theorem we can write a function f of bounded variation as the difference of two non-decreasing functions, and therefore it suffices to show the lemma for non-decreasing functions.

Now, assuming f to be non-negative and non-decreasing and g continuous, it follows from a Bonnet form of the Second Mean Value Theorem that there exists a $\xi \in [a, b]$ such that

$$\int_a^b f(t)g(t)dt = f(b) \int_{\xi}^b g(t)dt.$$

Hence, there exists a $\xi \in [a, b]$ such that

$$\begin{aligned} \left| \int_a^b f(t) \cos(\omega t + \phi) dt \right| &= \left| f(b) \int_{\xi}^b \cos(\omega t + \phi) dt \right| \\ &\leq \frac{2f(b)}{\omega}, \end{aligned}$$

and (4.13) follows from the boundedness of f on $[a, b]$. □

Note that we can readily apply Lemma 4.2.2 in our context since the smoothness assumption on $\bar{x}$ implies that the $\bar{x}_i$, $i = 1, 2, \dots, n$ are of bounded variation.

We proceed to show convergence $x_i^{(\omega)}(t) \rightarrow \bar{x}_i(t)$, $1 \leq i \leq n$ for $\omega \rightarrow \infty$, which implies the convergence $x^{(\omega)}(t) \rightarrow \bar{x}(t)$ with respect to the standard norm on $\mathbb{R}^n$. Writing out the solution for the first state

$$x_1^{(\omega)}(t) = x_1(0) + \int_0^t u_1^{(\omega)}(\tau) d\tau$$

$$\begin{aligned}
&= \int_0^t \dot{\bar{x}}_1(\tau) + 2\omega^{\frac{l}{l+1}} \cos(\omega\tau) d\tau \\
&= \bar{x}_1(t) + 2\omega^{-\frac{1}{l+1}} \sin(\omega t),
\end{aligned}$$

it follows that

$$\lim_{\omega \rightarrow \infty} x_1^{(\omega)}(t) = \bar{x}_1(t), \quad \forall t \in [0, T], \quad (4.14)$$

where the convergence is uniform with respect to t .

Using integration by parts we have for the second state

$$\begin{aligned}
x_2^{(\omega)}(t) &= x_2(0) + \int_0^t u_2^{(\omega)}(\tau) d\tau \\
&= \int_0^t \dot{\bar{x}}_2(\tau) + \sum_{m=1}^l \alpha_m(\tau) \omega^{-\frac{ml}{l+1}} \frac{m!}{m^m} \frac{d^m}{d\tau^m}(\cos(m\omega\tau)) d\tau \\
&= \bar{x}_2(t) + \sum_{m=1}^l \alpha_m(t) \omega^{-\frac{ml}{l+1}} \frac{m!}{m^m} \frac{d^{m-1}}{dt^{m-1}}(\cos(m\omega t)) \\
&\quad - \sum_{m=1}^l \int_0^t \dot{\alpha}_m(t) \omega^{-\frac{ml}{l+1}} \frac{m!}{m^m} \frac{d^{m-1}}{dt^{m-1}}(\cos(m\omega t)),
\end{aligned}$$

where the first sum is of order $\omega^{-\frac{1}{l+1}}$ and the second sum is of order $\omega^{-(1+\frac{1}{l+1})}$ by Lemma 4.2.2. Thus

$$\lim_{\omega \rightarrow \infty} x_2^{(\omega)}(t) = \bar{x}_2(t), \quad \forall t \in [0, T], \quad (4.15)$$

where the convergence is uniform with respect to t .

In writing down the solution for the third state it will become clear how the successive multiplications with u_1 and subsequent integrations required to solve (4.5) affect the limit behavior of the involved terms. Plugging in for $\alpha_1(\tau)$, using the identities

$$\cos(\omega t) \cos(n\omega t) = \frac{1}{2} \left(\cos((n-1)\omega t) + \cos((n+1)\omega t) \right), \quad (4.16)$$

$$2 \cos(\omega t) \sum_m \frac{m!}{m^m} \frac{d^{m-1}}{d\tau^{m-1}}(\cos(m\omega\tau)) = \sum_m \frac{(m-1)!}{(m-1)^{m-1}} \frac{d^{m-1}}{d\tau^{m-1}} \left(\cos((m-1)\omega\tau) + \cos((m+1)\omega\tau) \right)$$

and integrating by parts we obtain:

$$\begin{aligned}
& x_3^{(\omega)}(t) \\
&= x_3(0) + \int_0^t u_1^{(\omega)}(\tau) x_2^{(\omega)}(\tau) d\tau \\
&= \int_0^t \left(\dot{x}_1(\tau) + 2\omega^{\frac{l}{l+1}} \cos(\omega\tau) \right) \left(\bar{x}_2(\tau) + \sum_{m=1}^l \alpha_m(\tau) \omega^{-\frac{ml}{l+1}} \frac{m!}{m^m} \frac{d^{m-1}}{d\tau^{m-1}}(\cos(m\omega\tau)) \right. \\
&\quad \left. + o(\omega^{-(1+\frac{1}{l+1})}) \right) d\tau \\
&= \int_0^t \left\{ \dot{x}_1(\tau) \bar{x}_2(\tau) + \dot{x}_1(\tau) \alpha_1(\tau) \omega^{-\frac{l}{l+1}} \cos(\omega\tau) \right. \\
&\quad + 2\omega^{\frac{l}{l+1}} \bar{x}_2(\tau) \cos(\omega\tau) + \alpha_1(\tau) + \alpha_1(\tau) \cos(2\omega\tau) \\
&\quad + \left(\dot{x}_1(\tau) + 2\omega^{\frac{l}{l+1}} \cos(\omega\tau) \right) \left(\sum_{m=2}^l \alpha_m(\tau) \omega^{-\frac{ml}{l+1}} \frac{m!}{m^m} \frac{d^{m-1}}{d\tau^{m-1}}(\cos(m\omega\tau)) \right. \\
&\quad \left. \left. + o(\omega^{-(1+\frac{1}{l+1})}) \right) \right\} d\tau \\
&= \int_0^t \left\{ \dot{x}_3(\tau) + \dot{x}_1(\tau) \alpha_1(\tau) \omega^{-\frac{l}{l+1}} \cos(\omega\tau) + 2\omega^{\frac{l}{l+1}} \bar{x}_2(\tau) \cos(\omega\tau) + \alpha_1(\tau) \cos(2\omega\tau) \right. \\
&\quad + \sum_{m=2}^l \alpha_m(\tau) \omega^{-\frac{ml}{l+1}} \frac{(m-1)!}{(m-1)^{m-1}} \frac{d^{m-1}}{d\tau^{m-1}}(\cos((m-1)\omega\tau) + \cos((m+1)\omega\tau)) \\
&\quad + 2\omega^{\frac{l}{l+1}} \cos(\omega\tau) o(\omega^{-(1+\frac{1}{l+1})}) \\
&\quad \left. + \dot{x}_1(\tau) \left(\sum_{m=1}^l \alpha_m(\tau) \omega^{-\frac{ml}{l+1}} \frac{m!}{m^m} \frac{d^{m-1}}{d\tau^{m-1}}(\cos(m\omega\tau)) + o(\omega^{-(1+\frac{1}{l+1})}) \right) \right\} d\tau \\
&= \bar{x}_3(t) + 2\omega^{-\frac{1}{l+1}} \bar{x}_2(t) \sin(\omega t) \\
&\quad + \sum_{m=2}^l \alpha_m(t) \omega^{-\frac{ml}{l+1}} \frac{(m-1)!}{(m-1)^{m-1}} \frac{d^{m-2}}{dt^{m-2}}(\cos((m-1)\omega t) + \cos((m+1)\omega t)) \\
&\quad - \int_0^t \left\{ \sum_{m=2}^l \dot{\alpha}_m(\tau) \omega^{-\frac{ml}{l+1}} \frac{(m-1)!}{(m-1)^{m-1}} \frac{d^{m-2}}{d\tau^{m-2}}(\cos((m-1)\omega\tau) + \cos((m+1)\omega\tau)) \right\} d\tau \\
&\quad + o(\omega^{-1}) \\
&= \bar{x}_3(t) + o(\omega^{-\frac{1}{l+1}}).
\end{aligned}$$

Again as a consequence of Lemma 4.2.2 all terms in the expression for $x_3(t)$

except for $\bar{x}_3(t)$ are of negative order with respect to ω and therefore

$$\lim_{\omega \rightarrow \infty} x_3^{(\omega)}(t) = \bar{x}_3(t), \quad \forall t \in [0, T], \quad (4.17)$$

where convergence is again uniform with respect to t .

To establish an induction argument assume for the states $x_j, j = 4, \dots, n$ recall that for $i = 1, \dots, n - 2$

$$x_{i+2}(t) = \int_0^t u_1(\tau_1) \int_0^{\tau_1} u_1(\tau_2) \dots \int_0^{\tau_{i-1}} u_1(\tau_i) \int_0^{\tau_i} u_2(\tau_{i+1}) d\tau_{i+1} \dots d\tau_1, \quad (4.18)$$

i.e. x_{i+2} is obtained by applying an iteration to u_2 consisting of integration and multiplication with u_1 . According to (4.16) the terms of u_2 are iteratively frequency-shifted by $\pm\omega$ for each multiplication with the $2\omega^{\frac{l}{l+1}} \cos(\omega t)$ -terms of u_1 . We write

$$\begin{aligned} x_{i+2}^{(\omega)}(t) &= \bar{x}_{i+2} + \sum_{m=i+1}^l \left\{ \alpha_m(t) \omega^{\frac{-(m-i)l}{l+1}} \frac{(m-i)!}{(m-i)^{m-i}} \frac{d^{m-i+1}}{dt^{m-i+1}} (\cos((m-i)\omega t)) \right\} \\ &\quad + 2\omega^{-\frac{1}{l+1}} \bar{x}_{i+1}(t) \sin(\omega t) + \rho_{i+2} + o(\omega^{-1}), \end{aligned} \quad (4.19)$$

where the summation in (4.19) is comprised of the terms whose frequencies have been shifted by $-\omega$ for all previous multiplications with $2\omega^{\frac{l}{l+1}} \cos(\omega t)$. The terms whose frequencies have been shifted at least once by $+\omega$ due to multiplication with $2\omega^{\frac{l}{l+1}} \cos(\omega t)$ are subsumed under ρ_{i+2} . The terms in ρ_{i+2} are of lower order in ω as compared to terms in the summation with the same frequency. It can be verified that therefore the contributions of ρ_{i+2} to the states $x_j, j = i + 3, \dots, n$ vanish in the high-frequency limit. We assume further that $\rho_{i+2} = o(\omega^{\frac{-1}{l+1}})$ such that $\lim_{\omega \rightarrow \infty} x_{i+2}^{(\omega)}(t) = \bar{x}_{i+2}(t), \forall t \in [0, T]$. Note that $x_3(t)$ is of the form (4.19).

The following state can be written as

$$x_{i+3}^{(\omega)}(t)$$

$$\begin{aligned}
&= \int_0^t u_1(\tau) x_{i+2}(\tau) d\tau \\
&= \int_0^t \left(\dot{x}_1(\tau) + 2\omega^{\frac{l}{l+1}} \cos(\omega\tau) \right) \left(\bar{x}_{i+2}(\tau) + 2\omega^{-\frac{1}{l+1}} \bar{x}_{i+1}(\tau) \sin(\omega\tau) \right. \\
&\quad \left. + \sum_{m=i+1}^l \alpha_m(\tau) \omega^{\frac{-(m-i)l}{l+1}} \frac{(m-i)!}{(m-i)^{m-i}} \frac{d^{m-i+1}}{d\tau^{m-i+1}} (\cos((m-i)\omega\tau)) + \rho_{i+2} + o(\omega^{-1}) \right) d\tau \\
&= \int_0^t \left\{ \dot{x}_1(\tau) \bar{x}_{i+2}(\tau) + \dot{x}_1(\tau) (x_{i+2}(\tau) - \bar{x}_{i+2}(\tau)) \right. \\
&\quad \left. + 2\omega^{\frac{l}{l+1}} \bar{x}_{i+2}(\tau) \cos(\omega\tau) + 2\omega^{\frac{l-1}{l+1}} \bar{x}_{i+1}(\tau) \sin(\omega\tau) + \alpha_{i+1}(\tau) + \alpha_{i+1}(\tau) \cos(2\omega\tau) \right. \\
&\quad \left. + \sum_{m=i+2}^l \alpha_m(\tau) \omega^{\frac{-(m-i-1)l}{l+1}} \frac{(m-i-1)!}{(m-i-1)^{m-i-1}} \frac{d^{m-i+1}}{d\tau^{m-i+1}} (\cos((m-i-1)\omega\tau) + \cos((m-i+1)\omega\tau)) \right. \\
&\quad \left. + 2\omega^{\frac{l}{l+1}} \cos(\omega\tau) (\rho_{i+2} + o(\omega^{-1})) \right\} d\tau \\
&= \bar{x}_{i+3}(t) + 2\omega^{\frac{-1}{l+1}} \bar{x}_{i+2}(t) \sin(\omega t) \\
&\quad + \sum_{m=i+2}^l \alpha_m(t) \omega^{\frac{-(m-i-1)l}{l+1}} \frac{(m-i-1)!}{(m-i-1)^{m-i-1}} \frac{d^{m-i}}{dt^{m-i}} (\cos((m-i-1)\omega t) + \cos((m-i+1)\omega t)) \\
&\quad + \int_0^t \left\{ \dot{x}_1(\tau) (x_{i+2}(\tau) - \bar{x}_{i+2}(\tau)) \right. \\
&\quad \left. - \sum_{m=i+2}^l \dot{\alpha}_m(\tau) \omega^{\frac{-(m-i-1)l}{l+1}} \frac{(m-i-1)!}{(m-i-1)^{m-i-1}} \frac{d^{m-i}}{d\tau^{m-i}} (\cos((m-i-1)\omega\tau) + \cos((m-i+1)\omega\tau)) \right. \\
&\quad \left. + 2\omega^{\frac{l}{l+1}} \cos(\omega\tau) (\rho_{i+2} + o(\omega^{-1})) \right\} d\tau \\
&= \bar{x}_{i+3}(t) + \sum_{m=i+2}^l \alpha_m(t) \omega^{\frac{-(m-i-1)l}{l+1}} \frac{(m-i-1)!}{(m-i-1)^{m-i-1}} \frac{d^{m-i}}{dt^{m-i}} (\cos((m-i-1)\omega t)) \\
&\quad + 2\omega^{\frac{-1}{l+1}} \bar{x}_{i+2}(t) \sin(\omega t) + \rho_{i+3} + o(\omega^{-1}).
\end{aligned}$$

Note that also here the contributions of ρ_{i+3} and the $o(\omega^{-1})$ to $x_j, j = i+4, n$ vanish in the high-frequency limit. Again, we have

$$\begin{aligned}
\lim_{\omega \rightarrow \infty} x_{i+3}^{(\omega)}(t) &= \lim_{\omega \rightarrow \infty} \left(\bar{x}_{i+3}(t) + o(\omega^{-\frac{1}{l+1}}) \right) \\
&= \bar{x}_{i+3}(t), \quad \forall t \in [0, T]
\end{aligned}$$

where the convergence is uniform in t . Our claim follows by induction on i .

□

Theorem 4.2.1 can be straightforwardly extended to so called $(m+1)$ -input, m -chain, single generator systems of the form

$$\begin{aligned}
\dot{x}_0 &= u_0 & \dot{x}_{1,0} &= u_1 & \cdots & \dot{x}_{m,0} &= u_m \\
& & \dot{x}_{1,1} &= x_{1,0} u_1 & \cdots & \dot{x}_{m,1} &= x_{m,0} u_m \\
& & \vdots & & \ddots & \vdots & \\
& & \dot{x}_{1,n_1} &= x_{1,n_1-1} u_1 & \cdots & \dot{x}_{m,n_m} &= x_{m,n_m-1} u_m
\end{aligned} \tag{4.20}$$

whose state vector $x = (x_0, x_{1,0}, \dots, x_{1,n_1}, \dots, x_{m,0}, \dots, x_{m,n_m})$ is of dimension $1 + \sum_{i=1}^m n_i$. These systems were studied in (Bushnell *et al.*, 1993) where it is also shown that the kinematic description of a fire-truck is locally feedback equivalent to a three-input, two-chain, single generator system.

Corollary 4.2.3 *Given a $(m+1)$ -input chained-form system of the form (4.20), let $\bar{x}(t)$, $t \in [0, T]$ denote a twice differentiable desired trajectory satisfying $x(0) = \bar{x}(0) = \dot{x}(0) = 0$. Define a sequence $\{u^{(\omega)}\}_{\omega=1}^{\infty}$ of controls*

$$\begin{aligned}
u_1^{(\omega)} &= \dot{x}_0(t) + 2\omega^{\frac{\nu}{\nu+1}} \cos(\omega t) \\
u_i^{(\omega)} &= \dot{x}_{i,0}(t) + \sum_{j=1}^{n_i} \alpha_{i,j}(t) \omega^{-\frac{j\nu}{\nu+1}} \frac{j!}{j^j} \frac{d^j}{dt^j}(\cos(j\omega t)), \quad i = 1, \dots, m
\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}
\nu &= \max_{i \in \{1, \dots, m\}} n_i \\
\alpha_{i,j}(t) &= \dot{x}_{i,j+2}(t) - \dot{x}_0(t) \bar{x}_{i,j+1}(t).
\end{aligned}$$

Let $x^{(\omega)}(t)$, $0 \leq t \leq T$ be the solution of (4.5) with $u^{(\omega)}$ as input. Then

$$\lim_{\omega \rightarrow \infty} x^{(\omega)}(t) = \bar{x}(t), \quad \forall t \in [0, T] \tag{4.22}$$

where the convergence is uniform with respect to t .

Proof: The result of Corollary 4.2.3 follows from applying the proof of Theorem 4.2.1 to each of the m chains of (4.20) separately.

□

The following series of remarks concern implementational issues of the above control laws and a discussion of the resulting error terms.

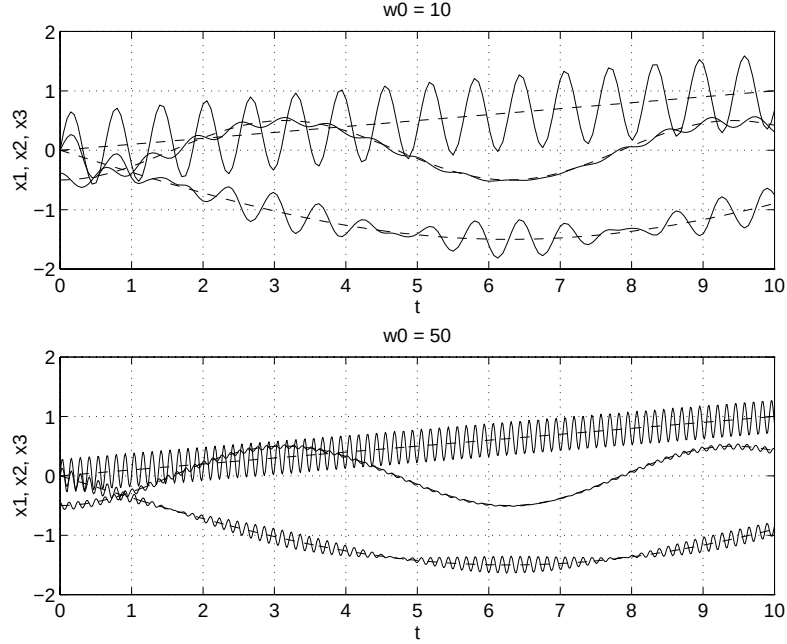


Figure 4.1: Trajectory tracking for three-dimensional chained form system with carrier frequencies $\omega = 10$ and $\omega = 50$

Remark 4.2.4 Since the convergence in Theorem 4.2.1 and Corollary 4.2.3 is uniform in t , also the tracking error defined as $E = \int_0^T \|\bar{x}(\tau) - x^{(\omega)}(\tau)\|_p d\tau$, with $\|\cdot\|$ a l_p -norm on $\mathbb{R}^n$, goes to zero with $\omega \rightarrow \infty$.

Remark 4.2.5 The trajectory errors $x - \bar{x}$ in Theorem 4.2.1 and Corollary 4.2.3 are of order $-\frac{1}{n-1}$ and $-\frac{1}{\nu+1}$ in ω , respectively. Hence the convergence properties

with respect to the frequency parameter ω worsen with the maximal depth of the Lie brackets required for controllability of the system at hand. This type of phenomenon, namely that control becomes increasingly difficult from a practical point of view with increasing depth of the nonholonomic system, manifests itself also in other motion control problems such as feedback stabilization (Gurvits & Lie, 1992).

Remark 4.2.6 We have assumed $x(0) = \bar{x}(0) = \dot{\bar{x}}(0) = 0$ in Theorem 4.2.1 and Corollary 4.2.3 so that we can discard the initial conditions and evaluations of the lower limit of any definite integral in the proof. Nevertheless, the above results hold whenever $x(0) = \bar{x}(0)$ and $\dot{\bar{x}}(0) = 0$. The control laws of Theorem 4.2.1 and Corollary 4.2.3 share the drawbacks common to all open-loop control laws: an error for the initial condition $x(0)$ or any other perturbation of the state on the interval $[0, T]$ is not compensated for. In fact, since the chained form systems (4.5), (4.20) are local representations of an invariant system on a matrix Lie group the initial error $\tilde{x}(0)$ corresponding to $\tilde{X}(0) = \tilde{X}_0 \in G$ causes an error $\tilde{x}(t)$, $t \in [0, T]$ corresponding to a translation of the desired trajectory $\bar{X}$ on the group by $\tilde{X}_0$.

Remark 4.2.7 The control law in Theorem 4.2.1 contains several degrees of freedom under which the convergence to the desired trajectory is preserved. For instance other shapes of the carrier functions would be possible. Sinusoids are chosen here due to their smoothness and since they facilitate explicit computations. Also, the amplitudes of the oscillatory terms in (4.11) can be modified in the following way

$$u_1^{(\omega)} = \dot{\bar{x}}_1(t) + 2\rho\omega^{\frac{l}{l+1}}\cos(\omega t)$$

$$u_2^{(\omega)} = \dot{\bar{x}}_2(t) + \sum_{m=1}^l \rho^m \alpha_m(t) \omega^{-\frac{ml}{l+1}} \frac{m!}{m^m} \frac{d^m}{dt^m} (\cos(m\omega t)), \quad (4.23)$$

with $\rho > 0$, allowing to adapt the amplitude of the oscillatory component of $u_1^{(\omega)}$ to the given desired trajectory.

We conclude this chapter with a lemma presenting an estimate $\hat{x}(t)$ for the state $x(t)$ of the chained form system (4.5) with controls (4.11) with $\hat{x}(t)$ depending explicitly on the desired trajectory $\bar{x}$ and $\dot{\bar{x}}$. This estimate will be used in Chapter 5 for converting feedback control laws into open-loop form. Since the estimate becomes rather complex with increasing dimension n we do not present a general, explicit form of the estimate. An example of the precise form of $\hat{x}$ for $n = 3$ also can be found in Chapter 5.

Lemma 4.2.8 *Consider the n -dimensional chained-form system (4.5), a twice differentiable desired trajectory $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \dots, \bar{x}_n(t))^T$, $t \in [0, T]$ satisfying $x(0) = \bar{x}(0) = \dot{\bar{x}}(0) = 0$, and approximate inversion controls of the form (4.11). Let α_j, β_j be rationals satisfying $\alpha_j, \beta_j \leq \frac{1}{n-1}$, f_j, g_j be monomials in the components of $\bar{x}$ and $\dot{\bar{x}}$, and m_j, n_j be positive integers. Then, there exist estimates*

$$\hat{x}_i(t) = \bar{x}_i(t) + \gamma_i(t, \omega, \bar{x}, \dot{\bar{x}}) \quad (4.24)$$

for the states $x_i(t)$, $i = 1, 2, \dots, n$ of (4.5) with controls (4.11), where $\gamma_i(\cdot)$ is a linear combination of terms of the form

$$\begin{aligned} & \omega^{\alpha_j} f_j(\bar{x}, \dot{\bar{x}}) \sin(m_j \omega t) \\ & \omega^{\beta_j} g_j(\bar{x}, \dot{\bar{x}}) \cos(n_j \omega t), \end{aligned} \quad (4.25)$$

such that

$$x_i(t) - \bar{x}_i(t) = o(\omega^{-1}), \quad t \in [0, T] \quad (4.26)$$

Proof: Since x_1 of (4.5) with control (4.11) is explicitly integrable the estimate for x_1 is

$$\hat{x}_1(t) = x_1(t) = \bar{x}_1(t) + 2\omega^{-\frac{1}{l+1}}.$$

As can be seen from the proof of Theorem 4.2.1 or from applying controls (4.11) with terms α_i accordingly modified to a power form system (4.7), the states x_i , $i = 2, 3, \dots, n$ can be written as

$$x_i(t) = \int_0^t \nu_{0i}(\bar{x}(\tau), \dot{\bar{x}}(\tau)) + \nu_i(\tau, \omega, \bar{x}(\tau), \dot{\bar{x}}(\tau)) d\tau,$$

where under the given assumptions the terms subsumed under $\nu_{0i}(\cdot)$ add up to give $\nu_{0i}(\bar{x}(\tau), \dot{\bar{x}}(\tau)) = \dot{\bar{x}}_1(\tau)$ and $\nu_i(\cdot)$ is a linear combination of terms of the form (4.25). Applying integration by parts results in

$$x_i(t) = \bar{x}_i(t) + \gamma_i(t, \omega, \bar{x}, \dot{\bar{x}}) - \int_0^t \eta_i(\tau, \omega, \bar{x}, \dot{\bar{x}}) d\tau.$$

The terms $\gamma_i(\cdot)$ and $\eta_i(\cdot)$ are again a linear combination of term of the form (4.25) where, for instance, a term $\omega^{\beta_j} g_j(\bar{x}, \dot{\bar{x}}) \cos(n_j \omega \tau)$ taken from $\nu_i(\cdot)$ enters $\gamma_i(\cdot)$ as $\omega^{\beta_j-1} g_j(\bar{x}, \dot{\bar{x}}) \sin(n_j \omega \tau)$ and $\eta_i(\cdot)$ as $\omega^{\beta_j-1} \frac{d}{d\tau} (g_j(\bar{x}, \dot{\bar{x}})) \sin(n_j \omega \tau)$. It follows from the proof of Theorem 4.2.1 that both $\gamma_i(\cdot)$ and $\eta_i(\cdot)$ are of order $-\frac{1}{n-1}$ in ω and that therefore by Lemma 4.2.2

$$\int_0^t \eta_i(\tau, \omega, \bar{x}, \dot{\bar{x}}) d\tau = o(\omega^{-1-\frac{1}{n-1}}).$$

Setting $\hat{x}_i(t) = \bar{x}_i(t) + \gamma_i(t, \omega, \bar{x}, \dot{\bar{x}})$ therefore yields

$$x_i(t) - \hat{x}_i(t) = o(\omega^{-1}), \quad i = 2, 3, \dots, n.$$

□

Chapter 5

Nilpotentization of Invariant Systems on Matrix Lie Groups

As pointed out above, the study of certain control problems is greatly facilitated for nilpotent systems since they are integrable by quadratures and due to their relatively simple Lie algebra structure.

In particular, since as shown in Theorem 4.2.1 the approximate tracking problem can be solved with relatively simple controls for chained form systems the question arises as to what class of systems and how one can construct transformations bringing non-nilpotent systems into chained form or, more generally, making these systems nilpotent.

This process which we henceforth call nilpotentization is comparable to the technique of feedback linearization in that here, as there, a certain class of nonlinear systems characterized by conditions on their associated Lie algebra of vector fields is transformed by a feedback transformation to a canonical form. For feedback linearization this canonical form is the class of controllable, linear systems for which an abundance of control results exist. Nilpotent systems, and in particular chained form systems, can therefore be seen as providing such a canonical

form for another class of nonlinear systems which can be made nilpotent by a state-feedback.

Given an input-affine system of the type

$$\Sigma : \quad \dot{x} = f_0(x) + f_1(x)u_1 + \cdots + f_m(x)u_m, \quad x \in M \quad (5.1)$$

defined on a n -dimensional manifold M with $m \leq n$, assume that the real-analytic vector fields f_i , $i = 0, 1, \dots, m$ are linearly independent around a point $p \in M$, span the distribution $\Delta = \text{span}\{f_0, f_1, \dots, f_m\}$, and generate a Lie algebra $L(f_0, f_1, \dots, f_m)(x)$ of vector fields of dimension n around p . Nilpotentization of (5.1) then amounts to finding a nilpotent basis for the input distribution Δ . Since the analysis in this chapter is of local nature, we will assume for simplicity of notation that Σ is already defined on $\mathbb{R}^n$ as a local representation of a system on a manifold M .

The availability of a nilpotent basis for Δ has different consequences depending on whether or not a drift vector field f_0 is present. For drift-free systems Σ the existence of nilpotent basis for Δ implies that there exists a locally invertible $(m \times m)$ -matrix $H(x)$ such that using the feedback $u = H(x)v$ the system Σ is transformed to

$$\Sigma_{nil} : \quad \dot{x} = F(x)H(x)v \triangleq G(x)v \quad (5.2)$$

where the columns $g_1(x), \dots, g_m(x)$ of $G(x)$ are again real-analytic vector fields forming a nilpotent basis for Δ . The nilpotent system then is *locally trajectory equivalent* to Σ in the sense that any trajectory $x^*(\cdot)$ of Σ can be achieved by Σ_{nil} with control $v^* = H^{-1}(x^*)u^*$ as long $x(\cdot)$ remains within the region for which $H(x)$ is invertible.

To transform Σ in particular to a chained form system the Lie algebra

$L(g_1, \dots, g_m)$ has to be isomorphic to the Lie algebra generated by the corresponding chained form vector fields and, in general, a coordinate transformation T has to be applied to Σ_{nil} since the notion of a chained form system is coordinate dependent.

As shown in (Hermes *et al.*, 1984) the existence of a nilpotent basis for Δ for a system with drift implies only the weaker notion of *orbit equivalence*; i.e. using an affine feedback transformation $u = h(x) + H(x)u$ the resulting system Σ_{nil} can locally trace the same orbits in state space as Σ although in general not with the same time parameterization as Σ .

The dual description of Σ in terms of a set of independent one-forms representing the corresponding nonholonomic constraints allows us to pose the question of nilpotentization equivalently in the cotangent setting. Namely, given a set of independent one-forms $\{\alpha_1(x), \dots, \alpha_{n-m}(x)$ annihilating Δ we are looking for a change of coordinates S such that $\alpha_i, i = 1, \dots, n - m$ assume the form of the one-forms $\beta_i, i = 1, \dots, n - m$ characterizing a compatible nilpotent system.

In particular, for nilpotentization to a chained form systems it has been pointed out by Murray (Murray, 1994) that the one-forms characterizing chained form systems are in Goursat normal form allowing the use of the machinery and the results of the theory of exterior differential systems (see (Bryant *et al.*, 1991)). The sufficiency conditions for the existence of a nilpotentizing transformation to chained form systems have been established in this setting.

Nevertheless, in both the tangent and the cotangent space setting the determination of the coordinate transform T requires in general the solution of a partial differential equation. Moreover, the transformation to a certain nilpotent form is non-unique, since there always exist a multitude of transformation $H(x)$

and $T(x)$ achieving this transition.

We will show in Section 5.1 that for a wide class of three-dimensional systems on matrix Lie groups the transformation T can be computed directly from the structure of the Lie algebra. This allows in principle to solve the nilpotentization problem for this class of systems with an algorithm implementable on a symbolic computing package. After a review of the necessary and sufficient condition for nilpotentization we study in Section 5.2 the problem of nilpotentization for higher dimensional matrix Lie groups paying special attention to Lie groups arising from the kinematics of mechanical devices.

5.1 Nilpotentization of Invariant Systems on Three-Dimensional Matrix Lie Groups

Three-dimensional matrix Lie groups are of theoretical interest since they allow a classification of their algebraic structure into a tractable set of equivalence classes of Lie algebras and therefore a complete treatment of nilpotentization for the corresponding nonholonomic systems. They are also of practical interest since the rigid motions on the plane and rigid reorientation in three-space can be modeled by the three-dimensional Lie groups $SE(2)$ and $SO(3)$ respectively.

Classification of nonholonomic systems on three-dimensional Lie groups requires the classification of three-dimensional Lie algebras as well as within each isomorphism class of Lie algebras the classification of two-dimensional subspaces which are not subalgebras. This is achieved in the following classification listing the three-dimensional Lie algebras along with their commutation relations with respect to a basis $\{A_1, A_2, A_3\}$ and Proposition 5.1.1 both taken from (Vershik

& Gershkovich, 1994).

Classification of Three-Dimensional Lie Algebras:

1. Abelian Lie Algebra $\mathfrak{t}(3)$
2. Nilpotent Heisenberg Lie algebra $\mathfrak{h}(3)$

$$[A_1, A_2] = A_3, \quad [A_1, A_3] = [A_2, A_3] = 0$$

3. Solvable Lie algebras:

$$[A_1, A_2] = a_{11}A_1 + a_{12}A_3, \quad [A_2, A_3] = a_{21}A_1 + a_{22}A_3, \quad [A_1, A_3] = 0$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2, \mathbb{R})$$

Depending on the eigenvalues of A we distinguish between the following subclasses:

- (a) A is diagonal

$$a_{11} = a_{22} = 1, \quad a_{12} = a_{21} = 0$$

- (b) A has different real eigenvalues $\lambda_1 \neq \lambda_2$

$$a_{11} = \lambda_1, \quad a_{22} = \lambda_2, \quad a_{12} = a_{21} = 0$$

- (c) A is conjugate to a rotation ($\phi = \frac{\pi}{2} \rightarrow \mathfrak{se}(2)$)

$$a_{11} = \cos \phi, \quad a_{12} = \sin \phi, \quad a_{21} = -\sin \phi, \quad a_{22} = \cos \phi$$

- (d) A is conjugate to the Jordan matrix

$$a_{11} = a_{22} = a_{12} = 1, \quad a_{21} = 0$$

4. Semi-simple Lie algebras

(a) Special orthogonal Lie algebra $\mathfrak{so}(3)$

$$[A_1, A_2] = A_3, \quad [A_1, A_3] = -A_2, \quad [A_2, A_3] = A_1$$

(b) Special linear algebra $\mathfrak{sl}(2)$

$$[A_1, A_2] = A_3, \quad [A_1, A_3] = -2A_1, \quad [A_2, A_3] = 2A_2$$

Proposition 5.1.1 *There are no nonholonomic left-invariant distributions either on the Abelian group $T(3)$, or on the solvable group of type $3(a)$.*

For the groups $H(3)$, the solvable groups of type $3(b)$, $3(c)$, $3(d)$, and the group $SO(3)$ all nonholonomic left-invariant distributions lie on the same orbit of the group of automorphisms of the corresponding Lie algebra.

The set of nonholonomic left-invariant distribution on $SL(2)$ splits into two orbits represented by the subspaces $V_1 = \text{span}\{A_1, A_2\}$ and $V_2 = \text{span}\{A_1 + A_2, A_3\}$.

In the following Lemma we show that the product of exponentials representation of a nonholonomic system has under a mild condition on the Lie algebra structure a special form which turns out to be of advantage for nilpotentization.

Lemma 5.1.2 *Consider a left-invariant system*

$$\dot{X} = X(A_1 u_1 + A_2 u_2 + A_3 u_3), \quad X \in G, \quad A_i \in \mathfrak{g}, i = 1, 2, 3 \quad (5.3)$$

on a three-dimensional matrix Lie group G and let the associated Lie algebra $\mathfrak{g} = \text{span}\{A_1, A_2, A_3\}$ be such that

$$[A_1, A_2] = A_3 \quad (5.4)$$

$$\Gamma_{i3}^3 = 0 \quad i = 1, 2. \quad (5.5)$$

Then the local product of exponentials representation of (5.3) has the form

$$\dot{x} = \begin{pmatrix} f_{11}(x) & f_{12}(x) & 0 \\ f_{21}(x) & f_{22}(x) & 0 \\ f_{31}(x) & f_{32}(x) & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad (5.6)$$

and there exists a diffeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}; x_2 \mapsto h(x_2)$ with $h(0) = 0$ such that

$$f_{1i} h(x_2) = -f_{3i} \quad i = 1, 2. \quad (5.7)$$

Proof: Given a $X \in G$ in a sufficiently small neighborhood of the identity of G we can write $X = e^{x_1 A_1} e^{x_2 A_2} e^{x_3 A_3}$, where the x_i , $i = 1, 2, 3$ are the single exponential coordinates for G . Differentiating and using $Ae^A = e^A A$ as well as

$$e^A B e^{-A} = e^{ad(A)} B = B + \sum_{k=1}^{\infty} \frac{1}{k!} ad_A^k B \quad A, B \in \mathfrak{g}$$

we obtain:

$$\begin{aligned} \frac{dX}{dt} &= \frac{d}{dt} (e^{x_1 A_1} e^{x_2 A_2} e^{x_3 A_3}) \\ &= \dot{x}_1 e^{A_1 x_1} A_1 e^{A_2 x_2} e^{A_3 x_3} + \dot{x}_2 e^{A_1 x_1} e^{A_2 x_2} A_2 e^{A_3 x_3} + \dot{x}_3 e^{A_1 x_1} e^{A_2 x_2} e^{A_3 x_3} A_3 \\ &= e^{x_1 A_1} e^{x_2 A_2} e^{x_3 A_3} (\dot{x}_1 e^{-x_3 A_3} e^{-x_2 A_2} A_1 e^{A_2 x_2} e^{A_3 x_3} + \dot{x}_2 e^{-x_3 A_3} A_2 e^{A_3 x_3} + \dot{x}_3 A_3) \\ &= X (\dot{x}_1 e^{ad(-x_3 A_3)} e^{ad(-x_2 A_2)} A_1 + \dot{x}_2 e^{ad(-x_3 A_3)} A_2 + \dot{x}_3 A_3). \end{aligned}$$

Setting this equal to the right hand side of (5.3) yields

$$A_1 u_1 + A_2 u_2 + A_3 u_3 = \dot{x}_1 e^{ad(-x_3 A_3)} e^{ad(-x_2 A_2)} A_1 + \dot{x}_2 e^{ad(-x_3 A_3)} A_2 + \dot{x}_3 A_3. \quad (5.8)$$

Let $\{A_1^b, \dots, A_n^b\}$ be a basis for the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ such that

$$A_i^b(A_j) = \delta_i^j, i, j = 1, \dots, n,$$

i.e $A_i^\flat$ projects vectors in $\mathfrak{g}$ to their component in the direction associated with A_i . Then it follows by (5.5) that

$$A_3^\flat(e^{ad(-x_3 A_3)} A_2) = 0 \quad \forall x_3 \in \mathbb{R} \quad (5.9)$$

$$A_3^\flat(e^{ad(-x_3 A_3)} e^{ad(-x_2 A_2)} A_1) = A_3^\flat(e^{ad(-x_2 A_2)} A_1), \quad \forall x_3 \in \mathbb{R}. \quad (5.10)$$

Introducing the notation $\eta_1 = e^{ad(-x_3 A_3)} e^{ad(-x_2 A_2)} A_1$, $\eta_2 = e^{ad(-x_3 A_3)} A_2$, and $\eta_3 = e^{ad(-x_2 A_2)} A_1$ and using (5.9), (5.10), as well as the fact that $\{A_1, A_2, A_3\}$ forms a basis for $\mathfrak{g}$ we can rewrite (5.8) as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} A_1^\flat(\eta_1) & A_1^\flat(\eta_2) & 0 \\ A_2^\flat(\eta_1) & A_2^\flat(\eta_2) & 0 \\ A_3^\flat(\eta_3) & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \\ \triangleq \hat{M}(x)\dot{x}.$$

The product of exponentials representation $\dot{x} = M(x)u$ of (5.3) can then be obtained by setting $M(x) = \hat{M}^{-1}$. As a consequence of the form of $\hat{M}$ it turns out that

$$M = \hat{M}^{-1} = \frac{1}{A_1^\flat(\eta_1)A_2^\flat(\eta_2) - A_2^\flat(\eta_1)A_1^\flat(\eta_2)} \begin{pmatrix} A_2^\flat(\eta_2) & -A_1^\flat(\eta_2) & 0 \\ -A_2^\flat(\eta_1) & A_1^\flat(\eta_1) & 0 \\ -A_3^\flat(\eta_3)A_2^\flat(\eta_2) & A_3^\flat(\eta_3)A_1^\flat(\eta_2) & 1 \end{pmatrix} \quad (5.11)$$

from where it follows that $h = A_3^\flat(\eta_3)$. Since with (5.15)

$$\begin{aligned} \eta_3 = e^{ad(-x_2 A_2)} A_1 &= A_1 + \sum_{k=1} (-1)^k \frac{x_2^k}{k!} ad_{A_2}^k A_1 \\ &= (1 + o(x_2^2))A_1 + o(x_2^2)A_2 + \underbrace{(x_2 + o(x_2^2))}_{h(x_2)} A_3 \end{aligned}$$

the function h is indeed a local diffeomorphism depending on x_2 only and $h(0) = 0$.

□

Note that the familiar matrix Lie groups $H(3)$, $SE(2)$, $SL(2)$, and $SO(3)$ which form a “nearly” exhaustive set of representatives for three-dimensional Lie groups exhibiting nonholonomy (leaving aside only the solvable cousins of $SE(2)$) all satisfy the conditions (5.4) and (5.5).

For drift-free, three-dimensional nonholonomic systems of form

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2, \quad x \in \mathbb{R}^n$$

the problem of nilpotentization is best approached in the cotangent formulation and can be shown to always have a solution (see Theorem 5.2.2). Let the one-form

$$\alpha(x) = \alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dx_3$$

be such that

$$\alpha(f_i)(x) \equiv 0, \quad i = 1, 2$$

and define

$$\beta(y) = -y_2 dy_1 + dy_3$$

as the one-form annihilating the vector fields

$$\begin{aligned} g_1(y) &= \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_3} \\ g_2(y) &= \frac{\partial}{\partial y_2} \end{aligned}$$

of a three-dimensional chained form system. The problem of nilpotentization is then reduced to finding a coordinate transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfying

$$\alpha(x) = \beta(T(x)) \left(\frac{\partial T}{\partial x} \right). \quad (5.12)$$

The following Theorem shows how for a wide class of nonholonomic systems on three-dimensional matrix groups T and $H(x)$ can be directly determined from the Lie algebra structure as a “byproduct” of the computations for the product of exponentials representation. It is this choice of coordinates which simplifies nilpotentization greatly which is not surprising since we saw in Chapter 4 that the chained form system is itself in product of exponential coordinates. The especially simple form of T , i.e. $T_1(x) = x_1, T_2(x) = h(x_2), T_3(x) = x_3$, obtained with this algorithm will be of advantage for motion control applications.

Theorem 5.1.3 Consider a left-invariant, nonholonomic system

$$\dot{X} = X(A_1 u_1 + A_2 u_2), \quad X \in G, \quad A_1, A_2 \in \mathfrak{g}, \quad (5.13)$$

on a three-dimensional matrix Lie group G and its local product of exponentials representation

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2 \quad (5.14)$$

relative to the basis $\{A_1, A_2, A_3\}$ for $\mathfrak{g}$. Assume that

$$[A_1, A_2] = A_3 \quad (5.15)$$

$$\Gamma_{i3}^3 = 0 \quad i = 1, 2. \quad (5.16)$$

Then the one-form α annihilating $\Delta = \text{span}\{f_1(x), f_2(x)\}$ can be specified as $\alpha = A_3^\flat(e^{ad(-x_2 A_2)} A_1) dx_1 + dx_3$ and is related to the one-form $\beta = -y_2 dy_1 + dy_3$ by the change of coordinates

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ -A_3^\flat(e^{ad(-x_2 A_2)} A_1) \\ x_3 \end{pmatrix} \quad (5.17)$$

which uniquely determines the feedback transformation $H(x)$ making (5.14) nilpotent.

Proof: It follows directly from Lemma 5.1.2 that the transformation equation $\alpha(x) = \beta(T(x)) \left(\frac{\partial T}{\partial x} \right)$ is satisfied by α and T as specified above.

The nilpotent basis $\{\tilde{g}_1, \tilde{g}_2\}$ for Δ on the other hand is determined from the vector fields $g_1 = \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_3}$, $g_2 = \frac{\partial}{\partial y_2}$ of the chained form system by the induced tangent map

$$\tilde{g}_i(x) = \frac{\partial S}{\partial x} g_i(S^{-1}(x)) = \left(\frac{\partial T}{\partial x} \right)^{-1} g_i(T(x)),$$

of $S = T^{-1}$, i.e. with $\tilde{G} = (\tilde{g}_1 \tilde{g}_2)$ we obtain

$$\tilde{G} = \begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{\partial T_2}{\partial x_2} \right)^{-1} \\ T_2(x) & 0 \end{pmatrix}$$

with $T_2(x) = -A_3^b(e^{ad(-x_2 A_2)} A_1)$.

Letting $e_3 = (0 \ 0 \ 1)^T$, $F_{ext} = (f_1(x) \ f_2(x) \ e_3)$, $\tilde{G}_{ext} = (\tilde{g}_1 \ \tilde{g}_2 \ e_3)$, the desired feedback transformation $H(x)$ is uniquely defined by

$$\begin{pmatrix} H(x)_{2 \times 2} & 0 \\ 0 & 1 \end{pmatrix} = F_{ext}^{-1} \tilde{G}_{ext}.$$

Since $F_{ext}^{-1} = \hat{M}$ (see proof of previous Lemma) it follows that $H(x)$ is completely determined by the terms required for the derivation of the product of exponential representation:

$$H(x) = \begin{pmatrix} A_1^b(\eta_1) & -A_1^b(\eta_2) \frac{\partial A_3^b(\eta_3)}{\partial x_2} \\ A_2^b(\eta_1) & -A_2^b(\eta_2) \frac{\partial A_3^b(\eta_3)}{\partial x_2} \end{pmatrix}. \quad (5.18)$$

□

In the following examples we apply the procedure presented in Theorem 5.1.3 to nonholonomic systems on the non-nilpotent matrix Lie groups $SE(2)$, $SL(2)$, and $SO(3)$. Their corresponding Lie algebras satisfy the conditions (5.15), (5.16) and the feedback and state transformation required for nilpotentization of these systems can be obtain directly from the $A_i^b(\eta_j)$ -terms required for the derivation of the product of exponentials representation.

Example 5.1.4 Consider the system (5.13) on $G = SE(2)$ with a basis for $\mathfrak{g}$ spanned by

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the resulting bracket structure

$$[A_1, A_2] = A_3, \quad [A_1, A_3] = -A_2, \quad [A_2, A_3] = 0$$

we obtain

$$\begin{aligned} e^{ad(-x_2 A_2)} A_1 &= A_1 + x_2 A_3 = \eta_3 \\ e^{ad(-x_3 A_3)} e^{ad(-x_2 A_2)} A_1 &= A_1 - x_3 A_2 + x_2 A_3 = \eta_1 \\ e^{ad(-x_3 A_3)} A_2 &= A_2 = \eta_2 \end{aligned}$$

and the product of exponentials representation turns out to be

$$\Sigma : \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x_3 & 1 \\ -x_2 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

With

$$H(x) = \begin{pmatrix} 1 & 0 \\ -x_3 & -1 \end{pmatrix}$$

the feedback law $u = H(x)v$ transforms Σ to the trajectory equivalent nilpotent system

$$\Sigma_{nil} : \quad \dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -x_2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (5.19)$$

which is state-equivalent to Σ_{cfs} via the change of coordinates

$$y = T(x) = \begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix}. \quad (5.20)$$

Example 5.1.5 Consider the system (5.13) on $G = SL(2)$ with a basis for $\mathfrak{g}$ spanned by

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From the resulting bracket structure

$$[A_1, A_2] = A_3, \quad [A_1, A_3] = -2A_1, \quad [A_2, A_3] = 2A_2$$

we obtain

$$\begin{aligned} e^{ad(-x_2 A_2)} A_1 &= A_1 - x_2^2 A_2 + x_2 A_3 = \eta_3 \\ e^{ad(-x_3 A_3)} e^{ad(-x_2 A_2)} A_1 &= e^{-2x_3} A_1 - x_2^2 e^{2x_3} A_2 + x_2 A_3 = \eta_1 \\ e^{ad(-x_3 A_3)} A_2 &= e^{2x_3} A_2 = \eta_2 \end{aligned}$$

and the product of exponentials representation turns out to be

$$\Sigma : \quad \dot{x} = \begin{pmatrix} e^{-2x_3} & 0 \\ -x_2^2 e^{2x_3} & e^{2x_3} \\ -x_2 e^{2x_3} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

With

$$H(x) = \begin{pmatrix} e^{-2x_3} & 0 \\ -x_2^2 e^{2x_3} & -e^{2x_3} \end{pmatrix}$$

the feedback law $u = H(x)v$ transforms Σ to the trajectory equivalent nilpotent system

$$\Sigma_{nil} : \quad \dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -x_2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (5.21)$$

which is state-equivalent to Σ_{cfs} via the change of coordinates

$$y = T(x) = \begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix}. \quad (5.22)$$

Example 5.1.6 ($SO(3)$) Consider the system (5.13) on $G = SO(3)$ with a basis for $\mathfrak{g}$ spanned by

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the resulting bracket structure

$$[A_1, A_2] = A_3, \quad [A_1, A_3] = -A_2, \quad [A_2, A_3] = A_1$$

we obtain

$$\begin{aligned}
e^{ad(-x_2 A_2)} A_1 &= \cos x_2 A_1 + \sin x_2 A_3 = \eta_3 \\
e^{ad(-x_3 A_3)} e^{ad(-x_2 A_2)} A_1 &= \cos x_2 \cos x_3 A_1 - \cos x_2 \sin x_3 A_2 + \sin x_2 A_3 = \eta_1 \\
e^{ad(-x_3 A_3)} A_2 &= \sin x_3 A_1 + \cos x_3 A_2 = \eta_2,
\end{aligned}$$

and the product of exponentials representation turns out to be

$$\Sigma : \quad \dot{x} = \begin{pmatrix} \sec x_2 \cos x_3 & -\sec x_2 \sin x_3 \\ \sin x_3 & \cos x_3 \\ -\tan x_2 \cos x_3 & \tan x_2 \sin x_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

With

$$H(x) = \begin{pmatrix} \cos x_2 \cos x_3 & -\sec x_2 \sin x_3 \\ -\cos x_2 \sin x_3 & -\sec x_2 \cos x_3 \end{pmatrix}$$

the feedback law $u = H(x)v$ transforms Σ to the trajectory equivalent nilpotent system

$$\Sigma_{nil} : \quad \dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -\sec x_2 \\ -\sin x_2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (5.23)$$

which is state-equivalent to Σ_{cfs} via the change of coordinates

$$y = T(x) = \begin{pmatrix} x_1 \\ -\sin x_2 \\ x_3 \end{pmatrix}. \quad (5.24)$$

For motion control applications it is of interest to have the transformations $T(x)$ and $H(x)$ defined on a neighborhood $U \subset \mathbb{R}^n$ of the origin as large as possible. For Examples 5.1.4 and 5.1.5 the change of coordinates $T(x)$ is a

diffeomorphism and $H(x)$ is a well defined invertible feedback transformation for all finite $x \in R^n$. For Example 5.1.6 $T(x)$ and $H(x)$ have the desired properties for $x \in U = \{x = (x_1, x_2, x_3) \in R^n | x_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$. Thus, in all three cases T and H are valid on the whole domain for which the local representation of the invariant system itself is valid, and nilpotentization therefore does not restrict the domain allowed for x any further.

5.2 Nilpotentization for Higher-Dimensional Systems on Matrix Lie Groups

In this section we study the problem of nilpotentization for systems on higher dimensional matrix Lie groups paying special attention to systems arising as models for the kinematics of mechanical systems or from other applications.

A necessary condition for nilpotentization to a specific nilpotent system can be deduced from the following invariance property of distributions under a change of basis.

Proposition 5.2.1 ((Hermes *et al.*, 1984)) *Consider a locally regular distribution $\Delta(x) = \text{span}\{f_1(x), \dots, f_m(x)\}$ let $\{g_1, \dots, g_m\}$ be another basis for Δ . Given $\mathcal{F}^0 = \{f_1(x), \dots, f_m(x)\}$ define $\mathcal{F}^i$ inductively as the set of j^{th} -order brackets of vector fields taken from $\mathcal{F}^0$ with $j \leq i$ and define $\mathcal{G}^i$ analogously with $\mathcal{G}^0 = \{g_1, \dots, g_m\}$.*

Then, locally we have

$$\dim(\text{span } \mathcal{F}^i) = \dim(\text{span } \mathcal{G}^i), \quad i = 0, 1, 2, \dots \quad (5.25)$$

Applying an invertible feedback transformation $u = H(x)v$ to a drift-free system $\dot{x} = F(x)u$ then implies that distributions $\text{span}(\mathcal{G}^i)$ of the resulting system $\dot{x} = F(x)H(x)v = G(x)v$ have the same dimensionality as the corresponding distributions $\text{span}(\mathcal{F}^i)$ of the original system. The nilpotent model system to which the original system is supposed to be transformed therefore has to be chosen to satisfy (5.25).

For the case of a two-input, single chain, single generator chained form system as the nilpotent model system, Murray has derived sufficient conditions for nilpotentization using tools of exterior algebra and a result related to Goursat Normal Forms.

The condition is formulated in terms of the filtrations $\{E_i\}$ and $\{F_i\}$ derived from $\Delta(x) = \text{span}\{f_1(x), \dots, f_m(x)\}$ as follows:

$$\begin{array}{ll}
E_0 &= \Delta & F_0 &= \Delta \\
E_1 &= E_0 + [E_0, E_0] & F_1 &= F_0 + [F_0, F_0] \\
E_2 &= E_1 + [E_1, E_1] & F_2 &= F_1 + [F_1, F_0] \\
&\vdots & &\vdots \\
E_{i+1} &= E_i + [E_i, E_i] & F_{i+1} &= F_i + [F_i, F_0].
\end{array} \tag{5.26}$$

We assume henceforth that the distributions E_i and F_i are regular which is naturally the case for systems arising as local representation of systems on matrix Lie groups. In fact, for systems on matrix Lie groups we can directly consider the Lie algebra filtrations, i.e. nested sequences of subspaces of $\mathfrak{g}$, corresponding to $\{E_i\}$ and $\{F_i\}$.

Theorem 5.2.2 ((Murray, 1994)) *A feedback transformation which puts a system $\dot{x} = f_1(x)u_1 + f_2(x)u_2$, $x \in \mathbb{R}^n$ into chained form (4.5) exists if and*

only if

$$\dim E_i = \dim F_i = i + 2, i = 0, \dots, n - 2. \quad (5.27)$$

It follows from a count of dimensions that controllable two-input systems $\dot{x} = f_1(x)u_1 + f_2(x)u_2$ with $x \in \mathbb{R}^3$ and $x \in \mathbb{R}^4$ satisfy the growth vector condition (5.27) and therefore always can be transformed to chained form.

Revisiting an example given by Murray (Murray, 1994) we present a nilpotentizing feedback for a four-dimensional system and show that if the original system is written in product of exponentials coordinates the computations and resulting transformations are very simple.

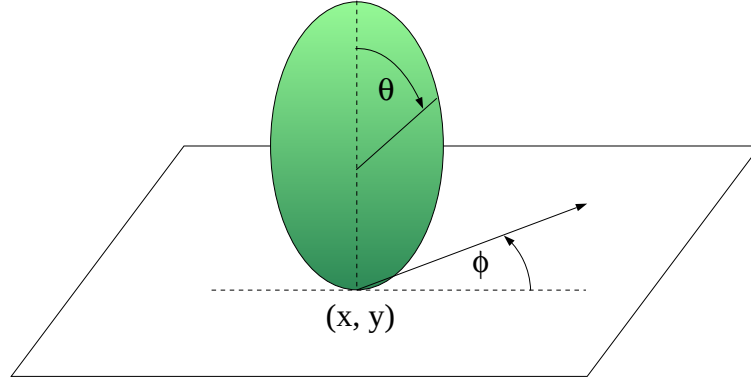


Figure 5.1: Rolling penny

Example 5.2.3 (Rolling Penny) Consider the kinematic model of a disk rolling without slipping on a plane as depicted in Figure 5.1. We assume that we have control over the heading angle and the angular velocity which is proportional to the forward velocity due to the no-slip constraint. The configuration of the disk can be described by an element of $SE(2)$ describing the location of the contact point of the penny on the plane and by an element of $SO(2)$ describing the angular displacement of a fixed line on the penny with respect to the vertical. An

element of the configuration space $G = SE(2) \times SO(2)$ can be written

$$X = \begin{pmatrix} \cos \phi & -\sin \phi & x & 0 & 0 \\ \sin \phi & \cos \phi & y & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}.$$

while a basis for $\mathfrak{g}$ is given by

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra is not nilpotent since

$$ad_{A_1} A_2 = -ad_{A_1}^3 A_2.$$

The corresponding left-invariant system

$$\dot{X} = X(A_1 u_1 + A_2 u_2), \quad X \in G \quad (5.28)$$

is depth-two controllable since

$$[A_1, A_2] = -A_3, \quad [A_1, [A_1, A_2]] = A_4.$$

Using the product of exponential coordinates the local representation of (5.28) turns out to be

$$\dot{x} = \begin{pmatrix} 1 \\ 0 \\ x_2 - x_4 \\ x_3 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u_2 \quad (5.29)$$

with the one-forms $\alpha_i(x)$, $i = 1, 2$ annihilating $\Delta(x) = \text{span}\{f_1(x), f_2(x)\}$ given as

$$\begin{aligned} \alpha_1(x) &= -(x_2 - x_4)dx_1 + dx_3 \\ \alpha_2(x) &= -x_3dx_1 + dx_4. \end{aligned}$$

It follows immediately that the change of coordinates relating $\alpha_i(x)$, $i = 1, 2$ to the one-forms $\beta_1(y) = -y_2dy_1 + dy_3$, $\beta_2(y) = -y_3dy_1 + dy_4$ characterizing the corresponding chained form system is given by

$$T : y_1 = x_1; \ y_2 = (x_2 - x_4); \ y_3 = x_3; \ y_4 = x_4$$

while the feedback transformation

$$H(x) = \begin{pmatrix} 1 & 0 \\ x_3 & 1 \end{pmatrix}$$

makes (5.29) nilpotent.

For a state space of dimension five or higher growth vectors of E_i and F_i can occur which rule out nilpotentization to a two-input, single chain, single generator chained form system.

Example 5.2.4 (Body-Mass System) Consider the system (5.13) on $G = SO(3)$ with a basis for $\mathfrak{g}$ spanned by

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It turns out that the Body-Mass System is controllable using the brackets

$$[A_1, A_2] = A_3, \quad [A_1, A_3] = [A_1, [A_1, A_2]] = -A_4, \quad [A_2, A_3] = [A_2, [A_1, A_2]] = A_5,$$

but since

$$\{\dim F_i\}_{i=0}^2 = \{2, 3, 5\}$$

does not satisfy (5.27) it is not nilpotentizable to chained form. However, this does not preclude that the Body-Mass System can be transformed to a nilpotent system with a matched growth vector satisfying the necessary condition of Proposition 5.2.1.

Example 5.2.5 (Underwater Vehicle) We consider the kinematic model of a underwater vehicle as detailed in Chapter 2 and distinguish different cases depending on the numbers of actuators present. Defining two controls associated

with elements $B_1, B_2 \in \mathfrak{se}(3)$, such that each control generates a translational velocity along and a rotational velocity about the same axis yields a controllable system on $SE(3)$. Taking for instance $B_1 = A_1 + A_4$ and $B_2 = A_2 + A_5$ results in a depth-3 system since

$$B_1, B_2, [B_1, B_2], [B_1, [B_1, B_2]], [B_2, [B_1, B_2]], [B_1, [B_1, [B_1, B_2]]]$$

are linearly independent elements of $\mathfrak{se}(3)$. But the resulting growth vector $\{\dim F_i\}_{i=0}^3 = \{2, 3, 5, 6\}$ does not satisfy (5.27) and the corresponding system therefore is not nilpotentizable to chained form. As in the previous example, the *rotational* components associated with u_1 and u_2 induce a relative growth of two for $\dim F_i$ at $i = 2$ thus violating the necessary condition for nilpotentization. Since on the other hand a rotational component in each of B_1 and B_2 is necessary to achieve controllability, we conjecture that there is in fact no two-dimensional subspace of $\mathfrak{se}(3)$ which generates $\mathfrak{se}(3)$ and satisfies the growth condition (5.27).

For three-input and four-input systems on $SE(3)$ the situation is inconclusive. Since $\mathfrak{se}(3)$ does not have a five-dimensional subalgebra the sufficient conditions given by Bushnell (Bushnell *et al.*, 1993) for nilpotentization to a m -input, $(m - 1)$ -chain, single generator system are not satisfied and the corresponding constructive procedure can not be used. But due to the lack of sufficiently strong necessary conditions, as for instance in Theorem 5.2.2, nilpotentization to a m -input chained form systems can not be precluded either.

For systems with five inputs the situation becomes conclusive again. Using the rank condition based on Darboux's Theorem given in Theorem 3 of (Hermes *et al.*, 1984) it can be shown, for instance, that for the left-invariant distribution corresponding to the subspace $\text{span}\{A_1, \dots, A_5\}$ a nilpotent basis can be found.

Example 5.2.6 ($SO(4)$) For two-input systems on $SO(4)$ a situation similar to the one of two-input systems on $SE(3)$ is encountered. In (Jurdjevic & Sussmann, 1972) a two-dimensional subspace is presented which generates the six-dimensional Lie algebra $\mathfrak{so}(4)$ but also fails to satisfy (5.27) having a growth vector $\{\dim F_i\}_{i=0}^3 = \{2, 3, 5, 6\}$.

In the examples studied we have seen that due to the special Lie algebra structure of two-input, single generator chained form systems and the corresponding restrictiveness of the growth vector condition (5.27) nilpotentization to this form is possible only in very special cases. These include three and four-dimensional nonholonomic system and those where the underlying physical phenomena reflects the chained structure as illustrated by the tractor and trailer system.

More work needs to be done to clarify the situation for systems with codimension larger than two. It would be desirable to obtain selection criteria for a suitable nilpotent model system based on the Lie algebra structure of the system under study and derive constructive procedures to obtain the corresponding feedback and state transformations.

Chapter 6

Motion Control For Nilpotentizable Systems

This chapter demonstrates how motion control problems for non-nilpotent systems can be reduced by feedback nilpotentization to motion control problems for nilpotent systems which often turn out to be much simpler.

For instance the point-to-point steering problem for a controllable, drift-free nilpotent system can be reduced to the inversion of a polynomial map. The steering laws presented in (Leonard & Krishnaprasad, 1995), which steer generic, drift-free, invariant systems on matrix Lie groups into a $o(\epsilon^\alpha)$ neighborhood of the target point (where α depends on the order of averaging employed), can therefore be made precise for the case of a nilpotentizable system.

We start by extending the approximate inversion control law obtained in Chapter 4 to drift-free systems which are nilpotentizable to chained form. Due to the feedback terms in the nilpotentizing input transformation the resulting control law will not be open-loop anymore. Section 6.2 shows how these control can be converted to open-loop form again using an estimate of the state such that the resulting control law can also be interpreted as an approximate

inverse system. Even though approximate tracking via feedback nilpotentization is less general than for instance the method of Liu and Sussmann (Sussmann & Liu, 1991), it has advantages steering the system along trajectories of feedback equivalent nilpotent systems. For instance, if one needs to compute the exact trajectory resulting from a specific control, say for example in an iterative motion planning scheme, an inversion scheme based on nilpotentization has the advantage that efficient quadrature algorithms can be used to carry out the forward integration of the system.

We conclude this chapter by giving feedback laws which exponentially stabilize equilibria of local representations of invariant systems on three-dimensional matrix Lie groups drawing on Morin's construction procedure.(Morin *et al.*, 1996).

6.1 Approximate Tracking and Approximate Inversion

Let

$$\Sigma : \quad \dot{x} = F(x)u = f_1(x)u_1 + f_2(x)u_2, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^2 \quad (6.1)$$

be a input-linear system which is nilpotentizable to two-input, single generator, single chain form. Thus, locally there exists an invertible linear feedback transformation $H(x)$ and a change of coordinates T such that the system

$$\Sigma_{nil} : \quad \dot{x} = F(x)H(x)v = \tilde{G}(x)v, \quad x \in \mathbb{R}^n, v \in \mathbb{R}^2 \quad (6.2)$$

is state-equivalent to the two-input, single generator, single chain system

$$\Sigma_{cfs} : \quad \dot{y} = G(y)v, \quad x \in \mathbb{R}^n, v \in \mathbb{R}^2, \quad (6.3)$$

i.e. we have $G(y) = \frac{\partial T}{\partial x}(T^{-1}(y))\tilde{G}(T^{-1}(y))$.

Then, given a sufficiently smooth desired trajectory $\bar{x}(t)$, $t \in [0, T]$ for Σ satisfying $x(0) = \dot{x}(0) = 0$, let $\bar{y}(t) = T(\bar{x}(t))$, $t \in [0, T]$ be the corresponding desired trajectory for the state-equivalent system Σ_{cfs} . Let $v^{(\omega)} = \Sigma_{cfs}^{-\dagger}(\bar{y})$ denote the approximate inversion controls for Σ_{cfs} specified in Theorem 4.2.1 and $y^{(\omega)}$ the trajectory of Σ_{cfs} with controls $v^{(\omega)}$. From Theorem 4.2.1 it follows that

$$\lim_{\omega \rightarrow \infty} y^{(\omega)}(t) = \bar{y}(t), \quad t \in [0, T]$$

where convergence is uniform with respect to t . Now, using $v^{(\omega)}$ as input for Σ_{nil} and denoting the resulting trajectory by $x^{(\omega)}$ it follows since T is assumed to be a diffeomorphism that

$$\lim_{\omega \rightarrow \infty} x^{(\omega)}(t) = \lim_{\omega \rightarrow \infty} T^{-1}(y^{(\omega)}(t)) = T^{-1}(\bar{y}(t)) = \bar{x}(t), \quad t \in [0, T]. \quad (6.4)$$

Finally, we obtain the approximate tracking controls for Σ as

$$u = H(x)v^{(\omega)} = H(x)\Sigma_{cfs}^{-\dagger}(\bar{y}).$$

To summarize the preceding development:

Proposition 6.1.1 *Consider an input-linear system $\Sigma : \dot{x} = f_1(x)u_1 + f_2(x)u_2, x \in \mathbb{R}^n$, nilpotentizable to a two-input single-generator chained form system (6.3), and a sufficiently smooth desired trajectory $\bar{x}(t)$, $t \in [0, T]$ for Σ satisfying $x(0) = \dot{x}(0) = 0$. Let $x^{(\omega)}$ denote the trajectory of Σ with controls*

$$u = H(x)\Sigma_{cfs}^{-\dagger}(T(\bar{x})) \quad (6.5)$$

where $H(x)$, $\Sigma_{cfs}^{-\dagger}$, T are as specified above. Then

$$\lim_{\omega \rightarrow \infty} x^{(\omega)}(t) = \bar{x}(t), \quad t \in [0, T]$$

where convergence is uniform with respect to t .

Using Corollary 4.2.3 instead of Theorem 4.2.1, Proposition 6.1.1 can be straightforwardly extended to include m -input, input-linear systems Σ which are nilpotentizable to an m -input, single generator, chained form system. One can obtain approximate tracking controls for systems nilpotentizable to systems other than single generator chained form systems equally well, but the case considered above has special significance since the approximate inversion controls of Theorem 4.2.1 have an especially simple form.

The above approach to tracking for nonholonomic systems is illustrated in Figure 6.1 and the following example.

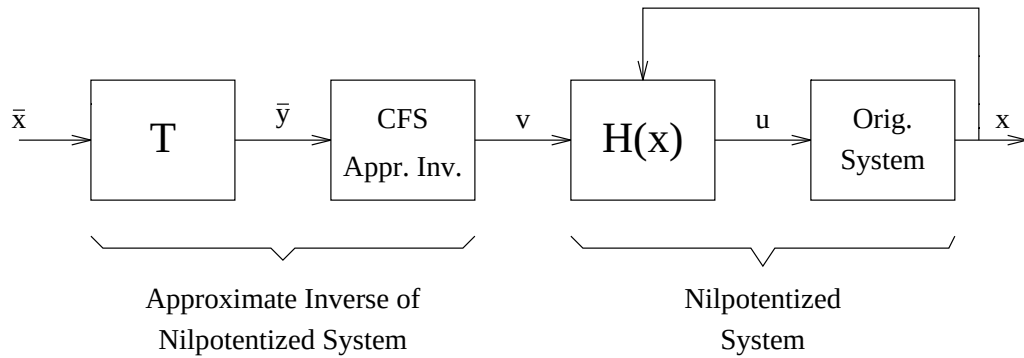


Figure 6.1: Approximate tracking for nilpotentizable systems

Example 6.1.2 ($SO(3)$) Consider the product of exponentials representation of a two-input, drift-free, left-invariant system on $SO(3)$ presented in Example 5.1.6. Then, with the nilpotentizing transformations T and $H(x)$ derived there

the approximate tracking controls (6.5) turn out to be

$$\begin{aligned}
u &= H(x) \Sigma_{cfs}^{-\dagger} (T(\bar{x})) \\
&= H(x) \begin{pmatrix} \frac{d}{dt}(T(\bar{x}_1)) + 2\omega^{\frac{1}{2}} \cos \omega t \\ \frac{d}{dt}(T(\bar{x}_2)) - \omega^{-\frac{1}{2}} \alpha_1 \sin \omega t \end{pmatrix} \\
&\quad \begin{pmatrix} \cos x_2 \cos x_3 & -\sec x_2 \sin x_3 \\ -\cos x_2 \sin x_3 & -\sec x_2 \cos x_3 \end{pmatrix} \begin{pmatrix} \dot{\bar{x}}_1 + 2\omega^{\frac{1}{2}} \cos \omega t \\ -\dot{\bar{x}}_2 \cos \bar{x}_2 - \omega^{-\frac{1}{2}} \alpha_1 \sin \omega t \end{pmatrix}
\end{aligned} \tag{6.6}$$

where

$$\alpha_1 = \frac{d}{dt}(T(\bar{x}_3)) - \frac{d}{dt}(T(\bar{x}_1))T(\bar{x}_2) = \dot{\bar{x}}_3 + \dot{\bar{x}}_1 \sin \bar{x}_2.$$

Simulations for approximate tracking controls derived by nilpotentization turn out to be identical to the simulations presented in Chapter 4 if numerical errors are neglected.

Note that due to the feedback occurring in $H(x)$ the control law (6.5) cannot be interpreted as providing an approximate inverse system independent of the original system Σ .

6.2 Approximate Inversion of Nilpotentizable Systems

In certain practical applications it might be costly or even impossible to measure the current state of the system. Also the nature of the controls at our disposal might preclude an on-line modification of the controls (see e.g. (Dahleh *et al.*, 1996)). For these cases it would be desirable to convert the approximate tracking controls derived above into an open-loop form.

Even though $\lim_{\omega \rightarrow \infty} x^{(\omega)} = \bar{x}$, replacing the current state x in the approximate tracking control law $u = H(x) \Sigma_{cfs}^{-\dagger}(T(\bar{x})) = H(x)v^{(\omega)}$ simply by $\bar{x}$ does not guarantee that the resulting open-loop control $u = H(\bar{x})v^{(\omega)}$ achieves approximate tracking. This is due to the fact that the control $v^{(\omega)}$ for Σ_{nil} depends itself on the frequency parameter ω , for instance using the approximate inversion control given in Theorem 4.2.1 we have $v^{(\omega)} = o(\omega^{\frac{n-2}{n-1}})$. Thus, since in this case $x^{(\omega)} - \bar{x} = o(-\frac{1}{n-1})$, the convergence of $H(x^{(\omega)})$ to $H(\bar{x})$ is not sufficiently fast, and we have

$$\lim_{\omega \rightarrow \infty} (H(\bar{x}) - H(x^{(\omega)}))v^{(\omega)} \neq 0$$

in general.

But since the state of x of $\Sigma : \dot{x} = F(x)u$ can also be interpreted as the state of $\Sigma_{nil} : \dot{x} = F(x)H(x)v$, which is state-equivalent to the chained form system (6.3) and can therefore be integrated by quadratures, we can obtain a better open-loop approximation to $H(x)$. The first possibility is to directly use the quadrature solution of Σ_{nil} as a state-estimate. But using Lemma 4.2.8 we can also obtain a sufficiently good estimate $\hat{x}_i$ for x_i , $i = 1, 2, \dots, n$ based on elementary functions in the components of $\bar{x}$ and $\dot{\bar{x}}$.

Given the states $y_i^{(\omega)}$, $i = 1, 2, \dots, n$ of Σ_{cfs} with approximate inversion controls $v^{(\omega)}$, let $\hat{y}_i$ be an estimate for $y_i^{(\omega)}$ as specified in Lemma 4.2.8 satisfying $y_i^{(\omega)} - \hat{y}_i = o(\omega^{-1})$. Then, since T is assumed to be a local diffeomorphism and with $T^{-1}(\cdot) = (T_1^{-1}, T_2^{-1}, \dots, T_n^{-1})$, $\hat{x}_i = T_i^{-1}(y^{(\omega)})$ provides an estimate for the state $x_i^{(\omega)}$ of Σ_{nil} with controls $v^{(\omega)}$ such that

$$x_i^{(\omega)} - \hat{x}_i = o(\omega^{-1}).$$

Example 6.2.1 Consider the three-dimensional chained form system

$$\begin{aligned}
\dot{y}_1 &= v_1 \\
\Sigma_{cfs} : \quad \dot{y}_2 &= v_2 \\
\dot{y}_3 &= y_2 v_1
\end{aligned}$$

with $y(0) = 0$ and approximate inversion controls

$$\begin{aligned}
v_1^{(\omega)}(t) &= \dot{\bar{y}}_1 + 2\omega^{\frac{1}{2}} \cos(\omega t) \\
v_2^{(\omega)}(t) &= \dot{\bar{y}}_2 - \omega^{-\frac{1}{2}} \underbrace{(\dot{\bar{y}}_3 - \bar{y}_2 \dot{\bar{y}}_1)}_{\alpha(t)} \sin(\omega t),
\end{aligned}$$

and a smooth desired trajectory satisfying $\bar{y} = \dot{\bar{y}} = 0$. Then the estimate for the resulting state $y_1^{(\omega)}(t)$ is directly

$$\hat{y}_1(t) = y_1^{(\omega)}(t) = \bar{y}_1(t) + 2\omega^{-\frac{1}{2}} \sin(\omega t).$$

For the state

$$y_2^{(\omega)}(t) = \bar{y}_2(t) + \omega^{-\frac{1}{2}} \alpha(t) \cos(\omega t) - \omega^{-\frac{1}{2}} \underbrace{\int_0^t \frac{d}{d\tau}(\alpha(\tau)) \cos(\omega \tau) d\tau}_{o(\omega^{-1})}$$

we choose the estimate

$$\hat{y}_2(t) = \bar{y}_2(t) + \omega^{-\frac{1}{2}} \alpha(t) \cos(\omega t).$$

For the state

$$\begin{aligned}
y_3^{(\omega)}(t) &= \int_0^t \left(\bar{y}_2(\tau) + \omega^{-\frac{1}{2}} \alpha(\tau) \cos(\omega \tau) - \omega^{-\frac{1}{2}} \int_0^\tau \frac{d}{d\sigma}(\alpha(\sigma)) \cos(\omega \sigma) d\sigma \right) \\
&\quad \times \left(\dot{\bar{y}}_1(\tau) + 2\omega^{\frac{1}{2}} \cos(\omega \tau) \right) d\tau \\
&= \bar{y}_3(t) + 2\omega^{-\frac{1}{2}} \bar{y}_2(t) \sin(\omega t) + \frac{1}{2} \omega^{-1} \alpha(t) \sin(2\omega t) \\
&\quad + \omega^{-\frac{3}{2}} \alpha(t) \dot{\bar{y}}_1(t) \sin(\omega t) + o(\omega^{-\frac{3}{2}})
\end{aligned}$$

we pick

$$\hat{y}_3(t) = \bar{y}_3(t) + 2\omega^{-\frac{1}{2}}\bar{y}_2(t) \sin(\omega t).$$

It follows that

$$y_i^{(\omega)}(t) - \hat{y}_i(t) = o(\omega^{-1}), \quad i = 1, 2, 3.$$

Thus, if $x^{(\omega)}$ is the state of a three-dimensional system, two-input system Σ_{nil} which is equivalent with Σ_{cfs} via the smooth change of coordinates $y = T(x)$, the estimates $\hat{x}_i = T^{-1}(\hat{y})$ for $x^{(\omega)}$ also satisfy

$$x_i^{(\omega)}(t) - \hat{x}_i(t) = o(\omega^{-1}), \quad i = 1, 2, 3.$$

Such estimates turn out to be sufficient for converting the approximate tracking control law (6.5) into open-loop form and achieving convergence of the resulting trajectory to the desired trajectory $\bar{x}$ in the high-frequency limit. Since the resulting control law is independent of the current state x of Σ it can also be interpreted as providing an approximate inverse system $\Sigma^\dagger$ for the non-nilpotent system Σ .

Theorem 6.2.2 *Consider a input-linear system $\Sigma : \dot{x} = f_1(x)u_1 + f_2(x)u_2, x \in \mathbb{R}^n$, nilpotentizable to a two-input single-generator chained form system (6.3), and a sufficiently smooth desired trajectory $\bar{x}(t)$, $t \in [0, T]$ for Σ satisfying $x(0) = \dot{x}(0) = 0$. Let $x^{(\omega)}$ denote the trajectory of Σ with open-loop controls*

$$u = H(\hat{x})\Sigma_{cfs}^{-\dagger}(T(\bar{x})) \tag{6.7}$$

where $H(x)$, $\Sigma_{cfs}^{-\dagger}$, T , and $\hat{x}$ are as specified above. Then

$$\lim_{\omega \rightarrow \infty} x^{(\omega)}(t) = \bar{x}(t), \quad t \in [0, T]$$

where convergence is uniform with respect to t .

Proof: We will show that the solution $x^{(\omega)}$ of Σ with open-loop controls $u = H(\hat{x})\Sigma_{cfs}^{-\dagger}(T(\bar{x})) = H(\hat{x})v^{(\omega)}$ will converge to the solution $\tilde{x}^{(\omega)}$ of Σ with feedback controls $u^{\tilde{(\omega)}} = H(x)v^{(\omega)}$ in the high-frequency limit. Then, since by Proposition 6.1.1 $\tilde{x}^{(\omega)}$ converges to the desired trajectory $\bar{x}$, also $x^{(\omega)}$ converges to $\bar{x}$ in the high-frequency limit.

Now, for $\lim_{\omega \rightarrow \infty} \|x^{(\omega)} - \tilde{x}^{(\omega)}\| = 0$ it is sufficient to have $\lim_{\omega \rightarrow \infty} \|u - \tilde{u}\| = 0$. But, since by Lemma 4.2.8 and the argument above there exists an estimate $\hat{x}$ satisfying $\hat{x} - x^{(\omega)} = o(\omega^{-1})$ and since the elements of $H(x)$ are continuously differentiable around the origin, we also have $\|H(\hat{x}) - H(x^{(\omega)})\| = o(\omega^{-1})$ where $\|\cdot\|$ is an induced matrix norm, say the maximum column sum associated with the l_1 vector norm. Since the approximate inversion controls $v^{(\omega)}$ satisfy $v^{(\omega)} = o(\omega^{\frac{n-2}{n-1}})$ it follows that

$$\begin{aligned} \|u(t) - \tilde{u}(t)\| &= \|(H(\hat{x}(t)) - H(x^{(\omega)}(t)))v\| \\ &\leq \|H(\hat{x}(t)) - H(x^{(\omega)}(t))\| \|v\| \\ &= o(\omega^{-\frac{1}{n-1}}), \quad \forall t \geq 0, \end{aligned}$$

and therefore $\lim_{\omega \rightarrow \infty} \|u(t) - \tilde{u}(t)\| = 0, \forall t \geq 0$.

□

Example 6.2.3 Continuing Example 6.1.2 and using the estimates $\hat{x}_i, i = 1, 2, 3$ from Example 6.2.1, the approximate inversion controls for the product of exponentials representation of a two-input, drift-free, left-invariant system on $SO(3)$ can be written as

$$u = H(\hat{x})\Sigma_{cfs}^{-\dagger}(T(\bar{x}))$$

with

$$\hat{x}_1 = \bar{x}_1(t) + 2\omega^{-\frac{1}{2}}\sin(\omega t)$$

$$\hat{x}_2 = -\arcsin(\sin(\bar{x}_2(t)) + \omega^{-\frac{1}{2}}\alpha(t) \cos(\omega t))$$

$$\hat{x}_3 = \bar{y}_3(t) - 2\omega^{-\frac{1}{2}}\sin(\bar{x}_2(t)) \sin(\omega t).$$

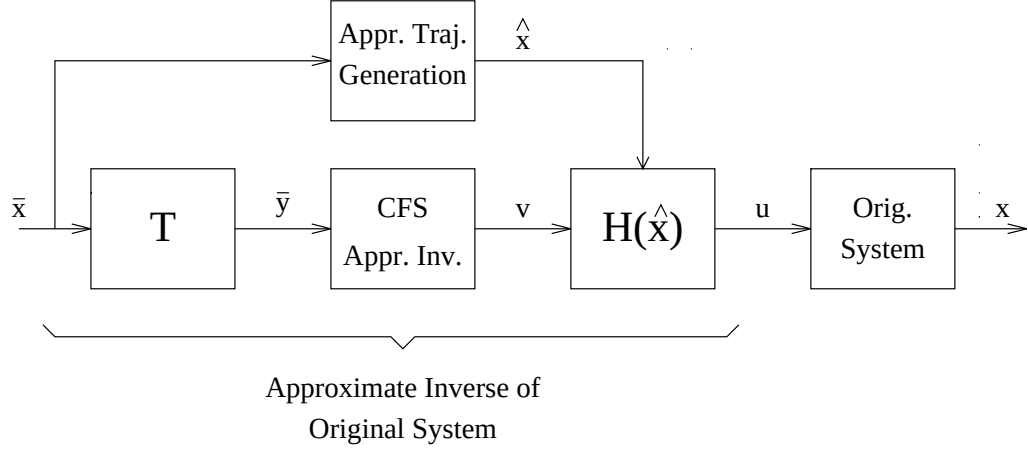


Figure 6.2: Approximate inversion for nilpotentizable systems

6.3 Feedback Stabilization of Nilpotentizable Systems

According to Brockett's necessary condition (Brockett, 1983) the origin of a regular, drift-free systems of the type

$$\dot{x} = f_1(x)u_1 + \dots + f_m(x)u_m = F(x)u, \quad x \in \mathbb{R}^n \quad (6.8)$$

with $m < n$ cannot be asymptotically stabilized with a smooth, time-invariant feedback law, since there does not exist a neighborhood Ω of the origin such that the mapping $\gamma : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^n; (x, u) \mapsto \dot{x}$ is onto an open set containing the origin. Coron (Coron, 1992) has shown, though, that there exists a smooth,

time-varying feedback globally asymptotically stabilizing the origin of (6.8) as long as (6.8) is controllable, although no explicit construction procedure is given there.

Drawing on the concept of approximate inversion one can make a simple heuristic argument for why time-varying controls achieve asymptotic stabilization. The system (6.8) together with an approximate tracking control yields an approximate identity operator in path space which can be robustly stabilized by a negative feedback and an integrator (see Figure 6.3) given that the frequency parameter in the approximate inversion control law is chosen sufficiently large. The feedback law thus can be conceptually decomposed into a steering component requiring the periodic oscillations to excite higher order Lie brackets and a robust stabilization component. Although most time-varying feedback laws basically follow this concept technical difficulties obstruct a stability proof based directly on this idea.

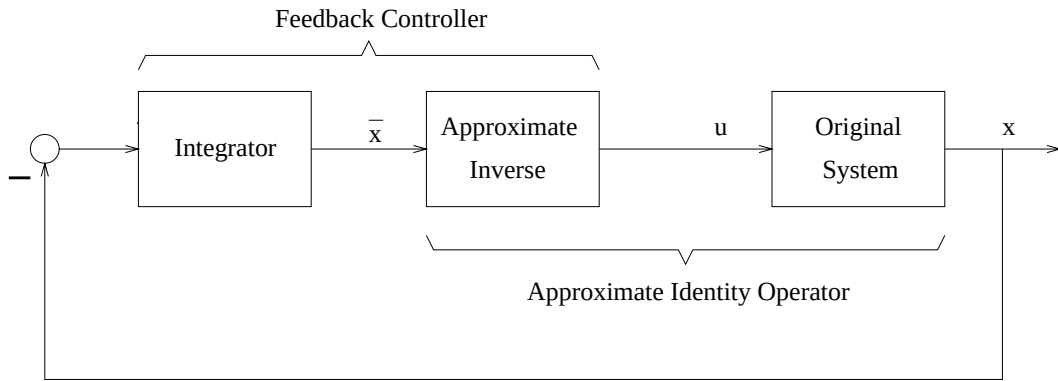


Figure 6.3: Decomposition of time-varying stabilizing controller into steering and robust stabilization component

By restricting the attention to chained form systems or a class of systems having similar properties the construction of smooth explicit control laws for

asymptotic stabilization was achieved (Pomet, 1992; Teel *et al.*, 1992). These smooth control laws suffered from slow convergence rates and it was pointed out (M'Closkey & Murray, 1995) that achieving exponential convergence actually requires feedback which is non-smooth at the origin.

In (Struemper & Krishnaprasad, 1997) we showed that a non-smooth feedback based on the approximate inversion achieves global exponential convergence for the nonholonomic integrator given the frequency parameter of the control is sufficiently large. Since the nonholonomic integrator can be viewed as a nilpotent approximation of the single exponential representation of non-nilpotent controllable systems on three-dimensional matrix groups it could also be shown that the same control law locally stabilizes this class of systems. But this control has the drawback that the subset of the state space where the feedback is non-Lipschitz can be reached in finite time which causes loss of uniqueness of trajectories and numerical difficulties for the simulation of the corresponding trajectories.

Based on techniques in (Sussmann & Liu, 1991; Morin *et al.*, 1996) presents a method to construct exponentially stabilizing controls for drift-free homogeneous systems. The problem of non-uniqueness is resolved there by making use of homogeneous feedback which is non-smooth *only* at the origin. We will use this method to construct a control law for a three-dimensional chained form system and show how the applicability of this control law can be extended to non-nilpotent systems by using feedback nilpotentization. To do so let us recall some basic notion concerning homogeneous vector fields (for details see (Hermes, 1991; M'Closkey & Murray, 1995)).

Let $x = (x_1, \dots, x_n)$ be local coordinates for $\mathbb{R}^n$. A *dilation* $\delta_\lambda^r : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is a map parameterized by a *weight vector* $r = (r_1, \dots, r_n)$ of rationals

satisfying $r_1 = 1 \leq r_2 \leq \dots \leq r_n$ such that

$$\delta_\lambda^r x = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n), \quad \lambda \in \mathbb{R}, \lambda > 0.$$

A continuous function $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *homogeneous* of degree σ with respect to the dilation δ_λ^r if

$$f(t, \delta_\lambda^r x) = \lambda^\sigma f(t, x).$$

We call a continuous vector field $\mathcal{X} = \sum_{i=1}^n a_i(t, x) \frac{\partial}{\partial x_i}$ on $\mathbb{R} \times \mathbb{R}^n$ *homogeneous* of degree m with respect to the dilation δ_λ^r if a_i is homogeneous of degree $r_i - m$ with respect to δ_λ^r for $i = 1, 2, \dots, n$. A so-called *homogeneous norm* is a continuous map $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying $\rho(x) \geq 0$, $\rho(x) = 0 \Leftrightarrow x = 0$ and $\rho(\delta_\lambda^r x) = \lambda \rho(x)$, $\forall \lambda > 0$. Thus the three-dimensional chained form system

$$\dot{x} = \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_2 = f_1(x)u_1 + f_2(x)u_2 \quad (6.9)$$

has degree one vector fields with respect to the dilation δ_λ^r with $r = (1, 1, 2)$, while a homogeneous norm for this case can be chosen to be $\rho(x) = (x_1^4 + x_2^4 + x_3^2)^{\frac{1}{4}}$. To make optimal use of this setting in stability theory the usual definition of exponential stability is generalized to account for the modified measure of distance induced by the norm $\rho(\cdot)$ homogeneous with respect to δ_λ^r .

Definition 6.3.1 *The equilibrium point $x = 0$ is locally exponentially stable with respect to the homogeneous norm $\rho(\cdot)$ if there exist two constants $\alpha, \beta > 0$ and a neighborhood U of the origin such that*

$$\rho(\phi(t, t_0, x_0)) = \beta \rho(x_0) e^{-\alpha(t-t_0)}, \quad \forall t \geq 0, \forall x_0 \in U. \quad (6.10)$$

This notion of stability is often called ρ -*exponential stability*, but we will also refer to it simply by exponential stability in the context of systems with homogeneous vector fields. An appealing consequence of homogeneity is the fact that for homogeneous degree zero vector fields local asymptotic stability of the origin is equivalent to global exponential stability.

Using the methods described in (Morin *et al.*, 1996) a exponentially stabilizing feedback law can be given as

$$\begin{aligned} u_1(t, x) &= -x_1 + h_1(x) + \rho \omega^{\frac{1}{2}} \cos(\omega t) \\ u_2(t, x) &= -x_2 + h_2(x) - 2 \rho^{-1} \omega^{\frac{1}{2}} \alpha_1(x) \sin(\omega t) \end{aligned} \tag{6.11}$$

with

$$\begin{aligned} h_1(x) &= \rho^{-4} \alpha_1(x) (2x_1^3 + x_2x_3) \\ h_2(x) &= 2(\rho^{-4} x_2^3 \alpha_1(x) - x_1) \\ \alpha_1(x) &= -x_3 + x_1x_2. \end{aligned}$$

Note that this control can be interpreted as originating from the open-loop control law 4.11 by replacing $\dot{x}_i$, $i = 1, 2, 3$ by $-x_i$, introducing the norm $\rho(\cdot)$, and determining the functions $h_i(\cdot)$, $i = 1, 2$ such that

$$[2\rho f_1(x), \rho^{-1} \alpha_1(x) f_2(x)] = h_1(x)f_1(x) + h_2(x)f_2(x) + [f_1(x), f_2(x)].$$

Asymptotic stability of control laws of this kind for ω sufficiently large is shown in (Morin *et al.*, 1996) by choosing a time-varying Lyapunov function and decomposing the differentiable operator corresponding to the closed loop vector field suitably with a process similar to integration by parts. Since the feedback law $u(t, x)$ is homogeneous of degree one the closed loop vector field is

homogeneous of degree zero with respect to δ_λ^r and global exponential stability for (6.9) follows.

Since the construction procedure in (Morin *et al.*, 1996) is restricted to systems with homogeneous vector fields it cannot be directly applied to local representation of invariant systems on non-nilpotent matrix groups.

A first strategy to extend these stabilizing feedbacks to nonholonomic systems on non-nilpotent matrix groups is to compute homogeneous approximations for their local representations. This can be done conveniently by reading of the leading terms of the single exponential representation (2.12) in the coordinates relative to a basis $\{A_1, \dots, A_m\}$ which is adapted to the growth vector of the system. Applying the construction procedure of (Morin *et al.*, 1996) to the resulting homogeneous approximation yields feedback laws locally exponentially stabilizing the original systems.

On the other hand we can achieve exponentially stabilizing feedback laws with a reasonably large guaranteed domain of stability by converting the original system with a nilpotentizing feedback into homogeneous form and applying the construction procedure to the transformed system. Thus the resulting region of stability is only limited by the region for which the nilpotentizing transformation is defined. The relationship of nilpotent approximation and (exact) feedback nilpotentization to the point stabilization problem is therefore comparable to the situation encountered when using exact feedback linearization as opposed to ordinary (Jacobian) linearization in the construction of stabilizing feedback laws.

Example 6.3.2 For each of the systems on $SE(2)$, $SL(2)$, and $SO(3)$ for which we derived in Chapter 5 the transformations $H(x)$ and $T(x)$ converting them

to the form (6.8) we can thus derive a exponentially stabilizing feedback law. Letting $u_{cfs}(t, x) = (u_1(t, x), u_2(t, x))^T$ with u_i , $i = 1, 2$ taken from (6.11) we thus obtain in the case of $SO(3)$ the exponentially stabilizing control law

$$\begin{aligned} u_{SO(3)} &= H(x)u_{cfs}(t, T(x)) \\ &= H(x) = \begin{pmatrix} \cos x_2 \cos x_3 & -\sec x_2 \sin x_3 \\ -\cos x_2 \sin x_3 & -\sec x_2 \cos x_3 \end{pmatrix} \end{aligned} \tag{6.12}$$

given ω is chosen sufficiently large. Figures 6.4 and 6.5 show simulations of trajectories resulting from this control law using frequencies $\omega = 1$ and $\omega = 10$, respectively. Simulations with different initial conditions suggest that $\omega = 1$ is indeed already sufficiently large for exponential stabilization.

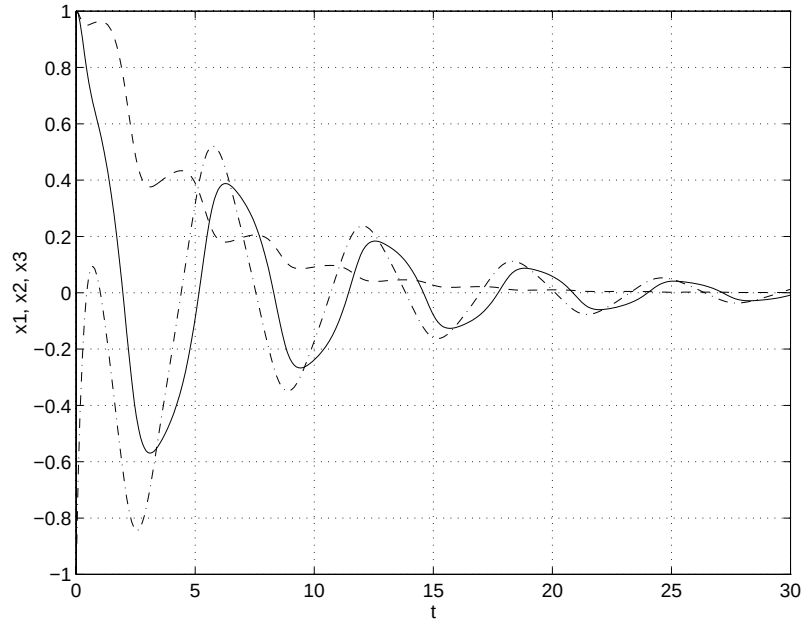


Figure 6.4: Stabilization of local representation of system on $SO(3)$ using feedback nilpotentization and homogeneous feedback ($\omega = 1$, $x_1(\cdot) —$, $x_2(\cdot) -.-$, $x_3(\cdot) - -$)

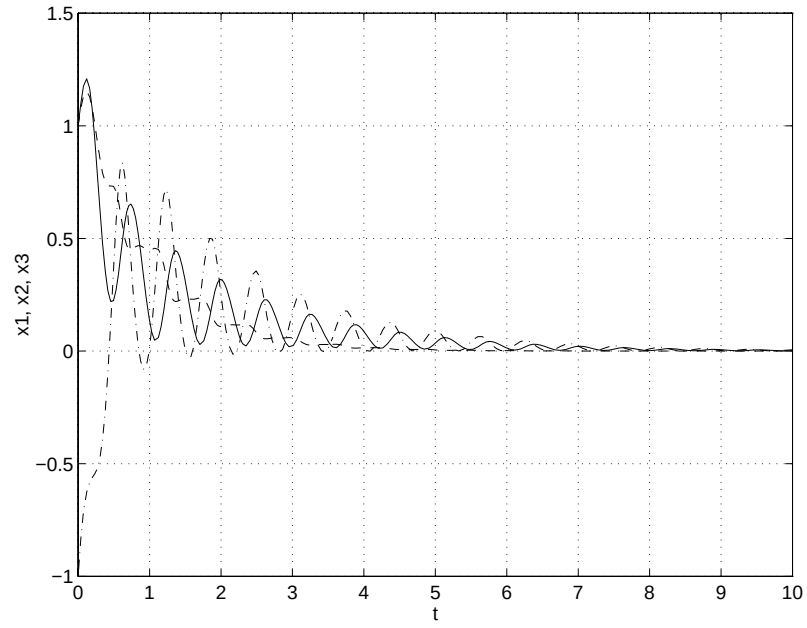


Figure 6.5: Stabilization of local representation of system on $SO(3)$ using feedback nilpotentization and homogeneous feedback ($\omega = 10$, $x_1(\cdot) —$, $x_2(\cdot) -.-$, $x_3(\cdot) - -$)

Chapter 7

Conclusion

In this dissertation we investigated the problem of motion control for nonholonomic systems on finite-dimensional matrix Lie groups. This work contributes to the larger project in nonlinear control theory of deriving control laws methodically from the Lie algebra of vector fields associated with a given system and characterizing classes of systems for which this is possible. Systems on nilpotent matrix Lie groups and their local representations proved to be an especially suitable setting to address motion control problems. Comparable to the technique of feedback linearization the approach taken here was to interpret nilpotent system as a canonical form for certain nonlinear systems. Characterizing the systems reducible to this form and identifying the corresponding transformations we extended the relatively simple control laws for nilpotent system to a wider class of non-nilpotent systems.

In Chapter 2 we reviewed basic notions concerning Lie groups, Lie algebras and invariant vector fields defined on Lie groups. We characterized systems on matrix Lie groups and studied their local representations. To illustrate the intimate relationship between Lie groups and mechanical systems of practical

interest we presented and analyzed several examples of the kinematics of mechanical systems.

Chapter 3 defined basic problems of motion control and introduced the interpretation of an open-loop tracking control law as an approximate inverse system of a given system. Using the example of Brockett's nonholonomic integrator we illustrated the basic principles of motion generation via high-frequency controls and presented different forms of approximate inversion control laws. With the goal of characterizing the inherent trade-off between tracking accuracy and control effort we studied the optimal control problem for the nonholonomic integrator with a control objective which also included trajectory error terms.

In Chapter 4 we started out by relating the nonholonomic integrator and the familiar chained and power form systems to invariant systems on the same nilpotent matrix Lie group. The main result of this chapter was an open-loop approximate tracking control law for chained form systems which could be cast in a relatively simple form due to the convenient Lie algebra structure of chained form systems.

In Chapter 5 we introduced the technique of nilpotentization and studied the question to which systems on matrix Lie groups this technique could be applied. In the case of systems on three-dimensional matrix Lie groups we succeeded in deriving a systematic procedure to construct the nilpotentizing transformations directly from the Lie algebra structure of the underlying matrix group. This is remarkable since the problem of finding such transformations generally requires the solution of a partial differential equation. We also investigated the possibility of transforming systems on higher-dimensional matrix Lie groups into chained or other nilpotent forms.

Chapter 6 addressed the problem of motion control for nilpotentizable systems. We started out by extending the approximate tracking control law derived in Chapter 4 to non-nilpotent systems. We then succeeded in converting these control laws, which involved a state feedback due to the nilpotentizing transformation, back into open-loop form. Again using feedback nilpotentization, we constructed exponentially stabilizing control laws for non-nilpotent systems by extending the region of attraction of otherwise only locally valid control laws.

As an extension of this work we will investigate the application of the methods derived above to systems with drift. Here, one needs to consider the case of invariant drift vector fields as encountered in coherent control of quantum dynamics (Dahleh *et al.*, 1996) as well as the case of the full dynamic formulation of a mechanical system on the tangent or cotangent bundle of the group manifold.

We have seen in Chapter 5 that for higher-dimensional nonholonomic systems only systems modeling a physical phenomenon with an inherent chained structure can actually be transformed into chained form. However, there exist nilpotent model systems which do exhibit the same growth vectors as, for instance, invariant systems on $SE(3)$. The question whether and how such systems which violate the necessary conditions for transformation into chained form can be brought into other nilpotent forms needs further investigation.

Finally, considering the control cost of moving in the direction of higher-order Lie brackets it is desirable from a practical point of view to integrate the trajectory planner with the approximate tracking control law. On the one hand the carrier frequencies of the control law should be adapted to the accuracy required in a certain phase of a tracking task, while the trajectory planner itself should construct the desired paths based on the notion that shorter paths might

be inefficient depending on the degree to which they violate the nonholonomic constraints.

References

- Brockett, R.W. 1972. Systems Theory on Group Manifolds and Coset Spaces. *SIAM Journal of Control*, **10**(2), 265–284.
- Brockett, R.W. 1983. Asymptotic Stability and Feedback Stabilization. *In*: Brockett, R.W., Millman, R.S., & Sussmann, H.J. (eds), *Differential Geometric Control Theory*. Basel: Birkhäuser.
- Brockett, R.W. 1993. Pattern Generation and Feedback Control of Nonholonomic Systems. *In*: *Conf. on Decision and Contr.: Workshop on Mechanics, Holonomy, and Control*.
- Brockett, R.W., & Dai, L. 1992. Nonholonomic kinematics and the Role of Elliptic Functions in Constructive Controllability. *In*: *Progress in Nonholonomic Motion Planning*. New York: Kluwer.
- Bryant, R.L., Chern, S.S., R.B. Gardner, H.L. Goldschmidt, & Griffiths, P.A. 1991. *Exterior Differential Systems*. New York: Springer.
- Bushnell, L., Tilbury, D., & Sastry, S. 1993. Steering Three-Input Chained Form Nonholonomic Systems Using Sinusoids: The Fire Truck Example. *In*: *Proceedings of the European Control Conference*.

- Coron, J.-M. 1992. Global Asymptotic Stabilization for Controllable Linear Systems without Drift. *Mathematics of Control, Systems, and Signals*, **5**, 295–312.
- Curtis, M. L. 1984. *Matrix Groups*. 2nd edn. New York: Springer-Verlag.
- Dahleh, M., Peirce, A., Rabitz, H.A., & Ramakrishna, V. 1996. Control of Molecular Motion. *Proc. of the IEEE*, **84**(1), 7–15.
- Fomenko, A.T., & Chakon, R.V. 1991. Recursion Formulas for the Lie Integral. *Advances in Mathematics*, **88**, 200–257.
- Gurvits, L., & Lie, Z. X. 1992. *Smooth Time-Periodic Feedback Solutions for Nonholonomic Motion Planning*. Tech. rept. Courant Institute of Mathematical Sciences.
- Haynes, G.W., & Hermes, H. 1970. Nonlinear Controllability via Lie Theory. *SIAM Journal of Control*, **8**(4), 450–460.
- Hermes, H. 1991. Nilpotent and High-Order Approximations of Vector Field Systems. *SIAM Review*, **33**(2), 238–264.
- Hermes, H., Lundell, A., & Sullivan, D. 1984. Nilpotent Bases for Distributions and Control Systems. *J. of Diff. Equations*, **55**, 385–400.
- Jurdjevic, V., & Sussmann, H. J. 1972. Control Systems on Lie Groups. *Journal of Differential Equations*, **12**, 313–329.
- Krishnaprasad, P.S. 1993. *Optimal Control and Poisson Reduction*. Tech. rept. Institute for Systems Research Technical Report 93-87.

- Lafferriere, G., & Sussmann, H.J. 1991. Motion planning for Controllable Systems without Drift. *Pages 1148–1153 of: IEEE International Conference on Robotics and Automation.*
- Laumond, J.-P. 1987. Feasible Trajectories for Mobile Robots with Kinematic and Environment Constraints. *In: Intelligent Autonomous Systems.* Amsterdam: North Holland.
- Leonard, N. E., & Krishnaprasad, P. S. 1995. Motion Control of Drift-Free, Left-Invariant Systems on Lie Groups. *IEEE Trans. on Automat. Contr.*, **40**(9), 1539–1554.
- Leonard, N.E. 1994. *Averaging and motion control of systems on Lie groups.* Ph.D. thesis, Univ. Maryland, College Park, MD.
- Magnus, W. 1954. On the exponential solution of differential equations for a linear operator. *Comm. on Pure and Appl. Math.*, **7**, 649–673.
- Marsden, J.E., & Ratiu, T.S. 1994. *Introduction to Mechanics and Symmetry.* New York: Springer-Verlag.
- M'Closkey, R.T., & Murray, R. M. 1995. *Exponential Stabilization of Driftless Nonlinear Control Systems Using Homogenous Feedback.* Tech. rept. CDS95-012. California Institute of Technology.
- Morin, P., Pomet, J.-B., & Samson, C. 1996. *Design of Homogeneous Time-Varying Stabilizing Control Laws for Driftless Controllable Systems via Oscillatory Approximation of Lie Brackets in Closed Loop.* Tech. rept. 3077. INRIA Sophia Antipolis.

- Murray, R.M. 1994. Nilpotent Bases for a Class of Nonintegrable Distributions with Application to Trajectory Generation for Nonholonomic Systems. *Math. of Contr., Signals, and Syst.*, **7**, 58–75.
- Murray, R.M., & Sastry, S.S. 1993. Nonholonomic motion planning: Steering using sinusoids. *IEEE Trans. on Automat. Contr.*, **38**(5), 700–716.
- Pomet, J.-B. 1992. Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *System and Control Letters*, **18**, 147–158.
- Samson, C. 1995. Control of Chained Systems: Application to Path Following and Time-Varying Point-Stabilization of Mobile Robots. *IEEE Transactions on Automatic Control*, **40**(1), 64–76.
- Struemper, H.K., & Krishnaprasad, P.S. 1997. Approximate inversion and feedback stabilization for systems on matrix Lie groups. *In: Proceedings of the 1997 American Control Conference*.
- Sussmann, H.J., & Liu, W. 1991. Limits of highly oscillatory controls and the approximation of general paths by admissible trajectories. *Pages 437–442 of: Proc. of the 30th IEEE Conf. on Decision and Contr.*
- Teel, A., Murray, R., & Walsh, G. 1992. Nonholonomic Control Systems: From Steering to Stabilization with Sinusoids. *Pages 1603–1609 of: Proceedings of the IEEE Conference on Decision and Control*.
- Tsakiris, D.P. 1995. *Motion Control and Planning for Nonholonomic Kinematic Chains*. Ph.D. thesis, Univ. Maryland, College Park, MD.

- Varadarajan, V. S. 1974. *Lie groups, Lie algebras, and their representations*. New York: Springer.
- Vershik, A.M., & Gershkovich, V.Y. 1994. *Dynamical Systems VII, Encyclopedia of Mathematical Sciences*. Vol. 16. New York: Springer Verlag. Chap. Non-holonomic Dynamical Systems, Geometry of Distributions and Variational Problems.
- Walsh, G., Montgomery, R., & Sastry, S. 1994. Optimal Path Planning on Matrix Lie Groups. *In: Proceedings of the 1994 IEEE Conference on Decision and Control*.
- Wei, J., & Norman, E. 1964. On global representations of the solution of linear differential equations as a product of exponentials. *Proc. of the Amer. Math. Soc.*, **15**, 327–334.
- Wilson, F. W. 1967. The Structure of the Level Surfaces of a Lyapunov Function. *Journal of Differential Equations*, **3**, 323–329.
- Wood, J. R. 1974. *Power conversion in electrical networks*. Tech. rept. Rep. No. CR-120830. NASA. (also PhD thesis, Harvard University).
- Yang, R. 1992. *Nonholonomic Geometry, Mechanics and Control*. Ph.D. thesis, University of Maryland.
- Yang, R., & Krishnaprasad, P.S. 1992. *On the Geometry and Dynamics of Floating Four-Bar Linkages*. Tech. rept. TR 92–9. Institute for Systems Research.