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Modeling and Control of Mixed and Flexible Structures

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MODELING AND CONTROL OF MIXED AND FLEXIBLE STRUCTURES

by

Thomas Alfred Posbergh

Dissertation submitted to the Faculty of the Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1988

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ABSTRACT

Title of Dissertation: Modeling and Control of Mixed and Flexible Structures

Thomas Alfred Posbergh, Doctor of Philosophy, 1988

Dissertation directed by: P. S. Krishnaprasad, Professor, Department of Electrical Engineering

The design of control systems for flexible spacecraft continues to be an important problem in current and future space missions. Crucial to successful controller design is accurate modeling of the underlying distributed parameter system. Current techniques frequently fail to capture the nonlinear features of the dynamic behavior of flexible spacecraft. From a practical point of view a closely related issue is the fidelity of approximations in preserving the essential characteristics of the underlying distributed parameter system.

This dissertation is concerned with distributed parameter models and rigorous approximations of the same as the basis for control system analysis and design. Specifically, we examine the generic case of a rigid spacecraft to which a flexible appendage is attached. The flexible appendage is modeled using geometrically exact rod theory. Equilibria for stationary and rotating configurations are computed and used as the basis of a subsequent linearization which preserves the Hamiltonian structure of the underlying system. These linearized models are the basis of the construction of the corresponding transfer functions. The associated transfer functions relate tip position and acceleration of the appendage to rigid body torques. In addition, stability of these equilibria is investigated using the Energy-Casimir method.

Using the transfer functions of the linearized model, modern frequency domain methods can be employed to do compensator design. In addition, we show that a rigid n-body chain is a natural approximation to a limiting case of the geometrically exact beam. Such an approximation provides the basis for finite dimensional compensator design for our infinite dimensional system. The design, implementation, and actual performance of such a compensator for an existing laboratory test fixture is discussed.

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CHAPTER ONE

INTRODUCTION

The control of mixed and flexible structures is a problem which is being confronted more than ever before in the design of lightweight, high performance systems. By a mixed structure we have in mind a multibody system, some elements of which are modeled as flexible, while rigid body modeling suffices for the rest. Such models are basic to the control of large, flexible spacecraft, high precision pointing and tracking, and many robotic applications (see for example Johnson [1983]).

Flexibility becomes significant when the performance demands render inadequate the modeling of a structure as one or more interconnected rigid bodies. The need to acknowledge and account for the flexibility of a structure is a consequence of either increasing size of the structure, or more stringent performance demands. In either case, the flexibility introduces a spatially dependent component into the model. With the introduction of this spatially dependent component we have moved into the realm of distributed parameter models which are described by systems of partial differential equations.

In the engineering analysis of these models current practice usually resorts to finite element modeling. Finite element methods replace the partial differential equations modeling the distributed parameter structure with a finite, although often very large, number of ordinary differential equations. In effect, one approximates the continuous structure by a finite number of interconnected elements with well-defined structural dynamics. Consequently, the infinite number of modes associated with the distributed parameter model will be replaced by a finite, although possibly very large, number of modes. The purpose of this analysis is to compute the frequencies of the modes, and the mode shapes, which approximate the partial differential equation. Current methods of finite element analysis are described in many books (see for example the recent textbook of Hughes [1987]).

The large, finite element model used to describe the flexible spacecraft must generally be reduced in size before it can reasonably be used as a model for the purpose of control system design. This further approximation is generally accomplished by retaining only a small number of the modes. The task of choosing which modes to retain and which modes to ignore is nontrivial, and the wrong choice will produce a bad design. As a consequence, a great deal of research has been devoted to the problem of model reduction (see for example the references in Balas [1982]). We note that this problem is fundamental to the design of finite dimensional controllers for infinite dimensional systems. For discussion of these issues in the case of as they relate to spacecraft modeling and control see also the thesis of B. Wie [1981].

Recent efforts that seek to perform the dynamic analysis of flexible spacecraft from other than the perspective of modal decomposition include Baillieul and Levi [1983] in which an examination is made of the exact equilibrium solutions of two specific distributed parameter systems, and Krishnaprasad and Marsden [1987] in which the model for a rigid body with an attached flexible appendage is formulated in the context of Hamiltonian mechanics. See also the recent thesis of Sreenath [1987] for a discussion of multi-body systems from the Hamiltonian point of view.

For general mechanical systems the equations which model the dynamics are nonlinear. When the system is rotating, Coriolis and centrifugal forces can significantly change the dynamics from what they were when the system was at rest. If we desire to obtain a linear model, then we need to account for these additional forces and torques. That such effects are significant has been recognized (see for example Kane, Ryan, and Banerjee [1986], or Simo and VuQuoc [1987]).

An example we explore in depth in this dissertation is that of a rigid body to which a flexible appendage is attached. Our model for the appendage is that of a *geometrically exact rod.* Such a model captures the important properties of the appendage, in particular, for large deformations it is geometrically exact in allowing finite strain. In addition, if the model is linearized about an equilibrium, it captures the important physical effects such as stiffening. This is in contrast to more *ad hoc* methods which can be found in the literature. One of the important themes of this dissertation is that *good models are necessary for successful control system design* .

Stability of flexible space structures has been studied in the past. Indeed, the issue of attitude stability has been an important area of research since the very first artificial earth satellites were placed into orbit. An early effort in the analysis of the stability of flexible spacecraft was that of Meirovitch [1979], who attempted to use the Hamiltonian as a Lyapunov function in testing the stability of a damped flexible spacecraft. More recently Krishnaprasad and Marsden [1987] discussed an energy-Casimir method for investigating the stability of a rigid body-flexible appendage configuration. This method can be used to obtain qualitative information about the stability of a dynamical system.

Chapter two of this thesis is concerned with the modeling of flexible structures, in particular a rod which is attached to a rigid body. The model we develop is based on the concept of a geometrically exact rod. In contrast to the classical Euler-Bernoulli model, a geometrically exact rod can be used to accurately model large deformations. This is done by allowing finite strain in the rod model. Recent interest in these models originates with Erickson and Truesdell [1958]. The notation we employ is that of Simo [1985] which is based on an earlier, planar model of Reissner [1973]. For a very clear exposition of the basic models of a rod see Antman [1972,1976].

In chapter three we linearize these models for a rigid body with an attached flexible appendage. The rotating system can be formally expressed in the form of a linear control system on an appropriate Hilbert space. (see for example Slemrod [1987], or Curtain [1987]). For the case of a rotating configuration the associated operator can be written as the operator of the nonrotating configuration perturbed by an operator arising from the rotation. For these systems we can compute transfer functions.

Chapter four addresses the issue of stability of the rigid body and rod configurations. In this chapter we apply the energy-Casimir method to establish Lyapunov stability of a rigid body with attached extensible shear beam. We then introduce the method of energy-momentum and use it to reproduce these results for the same model.

Chapter five addresses the control of the rigid body with an attached appendage modeled as an inextensible, nonshearable rod. For this problem we can employ geometric ideas to perform exact input-output linearization of the N-body approximation. In addition, we examine the application of the L_∞ techniques of Curtain and Glover [1986] to our model.

CHAPTER TWO

ANALYTICAL MODELING

In this chapter we lay the foundation for the remainder of this thesis. Our main objective is the derivation of *mathematically rigorous* models which describe the deformation of a thin rod, especially in the case when the thin rod is attached to a rigid body. Unlike an Euler-Bernoulli model, this model accurately describes large angle, geometrically nonlinear deformations.

The theory of the bending of elastic rods has it's origin in the investigations of J. Bernoulli in 1691. The investigations of L. Euler into the *elastica* were done with the encouragement of D. Bernoulli and were published as an appendix to Euler's famous book on the calculus of variations [1744]. Over the next century and a half investigation continued and the theory was further developed by among others St. Venant, Cauchy, Poisson, Kirchhoff, Clebsch, and Love.

In the last decade of the nineteenth century Duhem [1898] proposed the idea of describing a body as not only a collection of points but also of directions associated with the points (this idea of an *oriented body* is fundamentally geometric.) Shortly thereafter there appeared the work of the brothers Cosserat [1909] in which this idea was used in the representation of the twisting and bending of rods and shells.

Modern interest in rod theory (the special Cosserat theory) begins with the paper of Ericksen and Truesdell [1958] who presented a modern, generalized version of the Cosserat theory and developed nonlinear theories of strain for oriented curves and surfaces. This was the start of a series of investigations. Notable contributions among these are those of Cohen [1966], Green and Laws [1966], Green, Naghdi and Wainwright [1967], Whitman and DeSilva [1969], Antman [1970]. For a complete account of the theory up to that time see Antman [1972, 1976].

More recent work has addressed the issue of numerical simulation and given rise to explicit formulations which are more natural in this regard. The formulation originally developed by Reissner [1974] for the static case was generalized and extended to the fully three dimensional case by Simo [1985] and Simo and VuQuoc [1986]. In this work a major motivation is the accurate modeling by computer simulation of large deformations of rods, plates, and shells. The model we discuss in this chapter is the model used by Simo and his coworkers.

We should note that there exists with the theory of rods a fundamental dichotomy. One can develop the theory as a true, one dimensional theory in the tradition of the Cosserat brothers. Alternatively, one can view the theory as a special case of three dimensional elasticity. For the relationship between these two perspectives we refer to Antman [1976].

We begin our discussion with a general rod model, valid in three dimensions, and derive a special case, that of a nonshearable, inextensible rod. This model can be obtained in several ways, as a special case of a general theory, directly, or as the limit of an N-body chain of rigid bodies. We restrict ourselves to the planar case in this model. Futhermore, we consider such a model when the boundary condition is time varying and described by a system of ordinary differential equations. Such a model arises in describing a flexible appendage attached to a rigid body. In the limiting case, when the masses and inertias associated to the rigid body go to zero we recover equations which represent a generalization of Euler's elastica. The linearization of this model in a nonrotating frame is exactly the Euler-Bernoulli Beam equation with rotatory inertia (Love, [1944]). The proper linearization of this model is fundamental to successful control system design, for certain space structures.

In the last two sections of this chapter we turn our attention to the Hamiltonian formulation of the rod model. Formulating the model in the Hamiltonian setting enables us to employ powerful, modern geometric methods. Specifically, we will address the issue of stability in the reduced phase space by these methods.

2.1. Rod Theory and Geometrically Exact Models

In this section we introduce the rod model to be used in this paper. In particular, we are concerned with the dynamical model of a geometrically exact finite strain rod. The model we use is based on the classical Kirchhoff-Love model as modified by Reissner to account for shear deformation (Reissner, [1972]). The notation we use is based mainly on the work of Simo [1985]. In our analysis we attempt to adhere to the modern geometric perspective as described in Marsden and Hughes [1983].

2.1.1. A Geometrically Exact, Dynamic Rod Model

A rod is a special case of an elastic continuum which physically occupies a region of space. The *reference configuration* $\mathcal{B}$ of the rod is a manifold. We denote points in this manifold as $\mathbf{X}$. We assume that $\mathcal{B}$ is diffeomorphic to a domain of Euclidean space. *Ambient Space* $\mathcal{S}$ is the region within which the body deforms. Points in ambient space are denoted by $\mathbf{x}$. A *configuration* $\mathbf{\Phi}$ is a mapping $\mathbf{\Phi}: \mathcal{B} \longrightarrow \mathcal{S}$ which carries the point $\mathbf{X}$ in the body to the point $\mathbf{x}$ in ambient space.

The configurations of a rod in the ambient space $\mathbb{R}^3$ can be specified by a curve which we call the *line of centroids* and the orientation of a *cross section* associated with each point S on the curve. These are represented respectively by, (i) the mapping $\phi: [0, L] \longrightarrow \mathbb{R}^3$ which takes the line of centroids in the reference configuration into the corresponding curve of the deformed rod in ambient space and; (ii) the orientations of the cross sections as defined by the family of orthogonal transformations $\mathbf{\Lambda} : [0, L] \longrightarrow SO(3)$.

We will use two bases in our subsequent development. The basis $\{\boldsymbol{E}_I\}$, which is fixed in $\mathbb{R}^3$,and is referred to as the *material* or *inertial frame*. The moving basis, $\{\boldsymbol{t}_I\}$ is fixed in each cross section. Its orientation can be related to the material frame through the mapping $\mathbf{\Lambda}$, according to the relation: $\boldsymbol{t}_I(S) = \mathbf{\Lambda}(S)\mathbf{E}_I$.

We assume the rod has a finite cross section given by the compact set $A \subset \mathbb{R}^2$. Any point in the reference configuration is mapped into ambient space by the configuration mapping, $\mathbf{\Phi}: A \times [0, L] \longrightarrow \mathbb{R}^3$ where

$$\mathbf{\Phi}(S) = \phi(S) + \xi_1 \mathbf{\Lambda}(S)\boldsymbol{E}_1 + \xi_2 \mathbf{\Lambda}(S)\boldsymbol{E}_2, \tag{2.1}$$

and $(\xi_1, \xi_2, S) \in A \times [0, L]$. This is illustrated in figure 1.

We call the pair $(\phi(S), \mathbf{\Lambda}(S))$ a configuration. In what follows we assume the rod does not intersect itself, and undergoes no deformation of the cross sections. We associate with any configuration the *arc length* defined by

$$s \triangleq \int_0^S \| \frac{\partial}{\partial \zeta} \phi(\zeta) \| \, d\zeta. \tag{2.2}$$

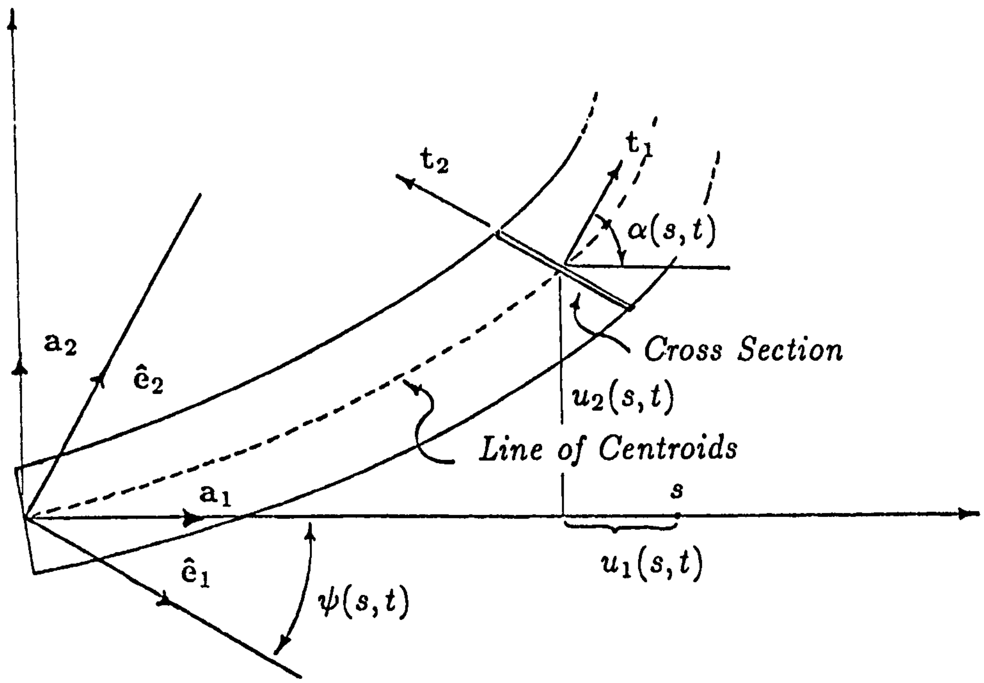

Figure 2.1. The Geometry of the Rod Model.

The arc length is used in the spatial description of the dynamics of the rod.

As part of the kinematic description of the deformed rod we need to determine the evolution of the basis $\{\boldsymbol{t}_I(S)\}$ as we move along the line of centroids. In this case we define the skew symmetric tensors fields, $\hat{\omega}: [0, L] \longrightarrow so(3)$ and $\hat{\boldsymbol{\Omega}}: [0, L] \longrightarrow so(3)$ by

$$\hat{\boldsymbol{\Omega}}(S) = \boldsymbol{\Lambda}^T(S)\frac{d\boldsymbol{\Lambda}(S)}{dS} \; ; \qquad\qquad \hat{\omega}(S) = \frac{d\boldsymbol{\Lambda}(S)}{dS}\boldsymbol{\Lambda}^T(S). \tag{2.3}$$

Here, we use the standard Lie algebra isomorphism $\hat{}: R^3 \longrightarrow so(3)$, to denote the skew symmetric tensor $\hat{\omega} \in so(3)$ associated with the axial vector $\omega \in R^3$. One has $\omega = \boldsymbol{\Lambda}\boldsymbol{\Omega}$. The evolution of the basis $\{\boldsymbol{t}_I(S)\}$ is now described as

$$\frac{d}{dS}\boldsymbol{t}_I(S) = \hat{\omega}(S)\boldsymbol{t}_I(S). \tag{2.4}$$

A motion of the rod is a curve in the configuration space assigning to each $(S, t) \in [0, L] \times R_+$ the pair $(\phi(S, t), \boldsymbol{\Lambda}(S, t))$. Associated with a motion we have velocities and accelerations. We define the *material velocity field* as

$$\mathbf{V}(S, t) \triangleq \Big(\frac{\partial \phi(S, t)}{\partial t}, \frac{\partial \boldsymbol{\Lambda}(S, t)}{\partial t}\Big) \tag{2.5}$$

We can use this to associate with any point of the rod a material velocity vector. Thus, from expression (2.1),

$$\frac{\partial \mathbf{\Phi}(S,t)}{\partial t} = \frac{\partial \boldsymbol{\phi}(S,t)}{\partial t} + \xi_1 \frac{\partial \mathbf{\Lambda}(S,t)}{\partial t} \boldsymbol{E}_1 + \xi_2 \frac{\partial \mathbf{\Lambda}(S,t)}{\partial t} \boldsymbol{E}_2 \tag{2.6}$$

Since $\mathbf{\Lambda}(S,t) \in SO(3)$ we define $\hat{\boldsymbol{w}}: [0,L] \times I\!R \longrightarrow so(3)$ and $\hat{\boldsymbol{W}}: [0,L] \times I\!R \longrightarrow so(3)$ by the expressions

$$\hat{\boldsymbol{W}}(S,t) = \mathbf{\Lambda}^T(S,t) \frac{\partial \mathbf{\Lambda}(S,t)}{\partial t}\ ; \qquad \hat{\boldsymbol{w}}(S,t) = \frac{\partial \mathbf{\Lambda}(S,t)}{\partial t} \mathbf{\Lambda}^T(S,t). \tag{2.7}$$

We call the skew symmetric linear map $\hat{\boldsymbol{w}}$ the *spin* of the moving basis $\{\boldsymbol{t}_I(S,t)\}$. Its axial vector $\boldsymbol{w}$ is the vorticity associated with the moving basis. Observe that $\boldsymbol{w} = \mathbf{\Lambda} \boldsymbol{W}$. We also define the *material acceleration field* as

$$\mathbf{A}(S,t) \triangleq \Big(\frac{\partial^2 \phi_t(S)}{\partial t^2}, \frac{\partial^2 \mathbf{\Lambda}_t(S)}{\partial t^2} \Big), \tag{2.8}$$

thus

$$\frac{\partial^2 \mathbf{\Phi}(S,t)}{\partial t^2} = \frac{\partial^2 \boldsymbol{\phi}(S,t)}{\partial t^2} + \xi_1 \frac{\partial^2 \mathbf{\Lambda}(S,t)}{\partial t^2} \boldsymbol{E}_1 + \xi_2 \frac{\partial^2 \mathbf{\Lambda}(S,t)}{\partial t^2} \boldsymbol{E}_2. \tag{2.9}$$

Observe that from the point of view of the three dimensional theory, the *deformation gradient* (a two-point tensor) is defined by

$$\mathbf{F} \triangleq \left[\frac{\partial \boldsymbol{\phi}(S,t)}{\partial S} - \boldsymbol{t}_3 + \boldsymbol{w} \times (\mathbf{x} - \boldsymbol{\phi}) \right] \otimes \boldsymbol{E}_3 + \mathbf{\Lambda}. \tag{2.10}$$

Next we proceed to introduce the equations of motion.

We define the two spatial quantities, the *resultant contact force* per unit of reference length, $\boldsymbol{n}(S,t)$, and the *resultant torque* per unit of reference length, $\boldsymbol{m}(S,t)$.

Making use of the balance of momentum equation of the three dimensional theory (see Marsden and Hughes [1983]), one can derive the equations of balance of linear and angular momentum for the rod. In the spatial representation one has

$$\rho_A \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \boldsymbol{n}}{\partial S} + \bar{\boldsymbol{n}} \tag{2.11}$$

$$\mathbf{I}_\rho \frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w} \times \mathbf{I}_\rho \boldsymbol{w} = \frac{\partial \boldsymbol{m}}{\partial S} + \frac{\partial \boldsymbol{\phi}}{\partial S} \times \boldsymbol{n} + \bar{\boldsymbol{m}} \tag{2.12}$$

In the above, ρ_A is the mass density of a cross section, and $\mathbf{I}_\rho$ is the time dependent inertia dyadic cross sections. One has the relations

$$\mathbf{J}_\rho = \int_0^L \rho \xi_\alpha \xi_\beta dS \big[\mathbf{1} \delta_{\alpha\beta} - \boldsymbol{E}_\alpha \otimes \boldsymbol{E}_\beta \big]\ ; \qquad \mathbf{I}_\rho = \mathbf{\Lambda} \mathbf{J}_\rho \mathbf{\Lambda}^T \tag{2.13}$$

where $\mathbf{J}_\rho$ is the time independent inertia dyadic in the convected (body) representation. In addition, $\bar{\boldsymbol{n}}$ is the vector field of external forces, and $\bar{\boldsymbol{m}}$ is the vector field of external torques. The quantities $\boldsymbol{n}$ and $\boldsymbol{m}$ are internal forces and moments arising from the potential energy function and we refer to them as *stress resultants.*

By considering a virtual work type argument we can compute the strain measures conjugate to the stress resultants and stress couples. One can show that the stress power given by

$$\int_{A\times[0,L]} \mathbf{P}:\dot{\mathbf{F}}\, d\Gamma dS = \int_{[0,L]} \mathbf{N}\cdot\dot{\mathbf{\Gamma}} + \mathbf{M}\cdot\dot{\mathbf{\Omega}}\, dS \tag{2.14}$$

where $\mathbf{P}$ is the Piola-Kirchhoff stress tensor, $\mathbf{N}$, $\mathbf{M}$ are convected stress resultants defined as

$$\mathbf{N} = \mathbf{\Lambda}^T\boldsymbol{n} \qquad \text{and} \qquad \mathbf{M} = \mathbf{\Lambda}^T\boldsymbol{m}, \tag{2.15}$$

and $\mathbf{\Gamma}$, $\mathbf{\Omega}$ are convected strain measures defined by

$$\mathbf{\Gamma} = \mathbf{\Lambda}^T\frac{\partial\phi}{\partial S} - \boldsymbol{E}_3 \qquad \text{and} \qquad \mathbf{\Omega} = \mathbf{\Lambda}^T\boldsymbol{\omega}. \tag{2.16}$$

We next consider global constitutive equations for the elastic case and the pure mechanical theory. We assume the existence of a free energy function $\psi(S,\gamma,\boldsymbol{\omega},\mathbf{\Lambda})$ where $\gamma \overset{\triangle}{=} \frac{\partial}{\partial S}\phi$, such that

$$\boldsymbol{n} = \frac{\partial\psi}{\partial\gamma} \qquad \text{and} \qquad \boldsymbol{m} = \frac{\partial\psi}{\partial\boldsymbol{\omega}}. \tag{2.17}$$

By postulating that these equations are frame indifferent (as they must be); i.e. invariant under the left action of the Euclidean group of spatial isometries, we obtain the representation $\psi = \Psi(\mathbf{\Lambda}^T\gamma, \mathbf{\Lambda}^T\boldsymbol{\omega})$. Hence it follows that

$$\mathbf{N} = \frac{\partial\Psi}{\partial\mathbf{\Gamma}} \qquad \text{and} \qquad \mathbf{M} = \frac{\partial\Psi}{\partial\mathbf{\Omega}}. \tag{2.18}$$

Linear constitutive equations consistent with the above invariance requirements are furnished by the *uncoupled linear systems*

$$\mathbf{N} = \mathbf{C}_N(\mathbf{\Gamma} - \mathbf{\Gamma}_0) \qquad \text{and} \qquad \mathbf{M} = \mathbf{C}_M(\mathbf{\Omega} - \mathbf{\Omega}_0) \tag{2.19}$$

Here $\mathbf{C}_N$, and $\mathbf{C}_M$ are symmetric, positive definite matrices. One typically assumes that $\mathbf{C}_N$, and $\mathbf{C}_M$ are diagonal with constant coefficients. The assumption of an uncoupled quadratic expression for the material stored energy function $\Psi(S,\mathbf{\Gamma},\mathbf{\Omega})$ is the counterpart of the Saint Venant – Kirchhoff model of three dimensional elasticity.

2.1.2. Admissible Variations

The configuration space of a rod is a differentiable manifold. Subsequently, we will need to take variations of the configuration at which time the nonlinear nature of the manifold will determine the admissible variations.

From an abstract point of view, according to the preceding discussion, the configuration space $\mathcal{C}$ is the set of mappings Φ such that

$$\mathcal{C} \stackrel{\triangle}{=} \{\Phi = (\phi, \Lambda) \,|\, [0, L] \longrightarrow I\!R^3 \times SO(3)\}. \tag{2.20}$$

Thus the configurations take values on the differentiable manifold $I\!R^3 \times SO(3)$. Let $so(3)$ be the Lie algebra associated with the tangent space to $SO(3)$ at the identity. Elements of $so(3)$ are the skew-symmetric matrices with the standard matrix commutator as Lie bracket. Recall that we can associate with any element $\delta\hat{\boldsymbol{\theta}} \in so(3)$ a vector $\delta\boldsymbol{\theta} \in I\!R^3$ such that for any $\mathbf{h} \in I\!R^3$ we have $\delta\hat{\boldsymbol{\theta}}\,\mathbf{h} = \delta\boldsymbol{\theta} \times \mathbf{h}$. Since the tangent space at $\boldsymbol{\Lambda} \in SO(3)$ is given by $\mathbf{T}_{\boldsymbol{\Lambda}} SO(3) \stackrel{\triangle}{=} \{\delta\Theta\Lambda \,|\, \delta\Theta \in so(3)\}$, the space of admissible variations at the configuration $\boldsymbol{\Phi} = (\phi, \boldsymbol{\Lambda})$ is given by

$$\mathbf{T}_{\boldsymbol{\Phi}}\mathcal{C} = \left\{(\delta\phi, \delta\hat{\theta}\Lambda) \quad | \quad (\delta\phi, \delta\hat{\theta}): \quad [0, L] \longrightarrow I\!R^3 \times so(3)\right\}. \tag{2.21}$$

We call $\delta\phi$, and $\delta\hat{\boldsymbol{\theta}}$ *admissible variations*. We define a *perturbed configuration*, $\boldsymbol{\Phi}_\epsilon = (\phi_\epsilon, \boldsymbol{\Lambda}_\epsilon)$, for $\epsilon > 0$ by using the exponential map and letting

$$\phi_\epsilon(S,t) = \phi(S,t) + \epsilon\delta\phi(S,t), \qquad \Lambda_\epsilon(S,t) = \exp[\epsilon\delta\hat{\theta}]\Lambda(S,t). \tag{2.22}$$

The above are used in computing the variational derivatives associated with the system. A more complete discussion of these matters is found in Simo & Vu-Quoc [1986].

2.2. A Special Case — The Shear Beam Model

An illutrative for the previous development is provided by the case of a rigid body to which a linear extensible shear beam is attached. Our model for the appendage is sometimes referred to as a string model. Such a model is discussed in Krishnaprasad & Marsden [1986]. Here we show how the dynamics can be derived using the rod theory discussed in the previous section.

For the linear extensible shear beam we assume that the stored energy function is of the form,

$$\Psi(\mathbf{\Gamma}, \mathbf{\Omega}) = \tfrac{1}{2}\mathbf{K}\frac{\partial \mathbf{r}}{\partial S} \cdot \frac{\partial \mathbf{r}}{\partial S} \tag{2.23}$$

where $\mathbf{r}$ is the configuration variable in the convected representation, $\mathbf{r} = \mathbf{\Lambda}^T \boldsymbol{\phi}$. Note that in the case of the linear extensible shear beam $\mathbf{\Lambda}(S,t) = \mathbf{\Lambda}(t)$, since there is no local freedom of rotation for the cross sections. The symmetric, positive definite operator $\mathbf{K}$ corresponds to the matrix of coefficients of elasticity in the linear theory. In this case we assume $\mathbf{K} = \mathrm{diag}(EA_1,\ EA_2,\ GA)$.

In addition we assume that there is a rigid body fixed to the base of the appendage with convected inertia dyadic $\mathbf{J}$. We will assume that the mass of the appendage is small with respect to the mass of the rigid body, consequently we implicitly assume that the center of mass of the system coincides with the center of mass of the rigid body. We will assume that the entire configuration is rotating with convected angular velocity $\mathbf{W} = \mathbf{J}^{-1}\mathbf{p}$, here $\mathbf{p}$ is the convected angular momentum vector of the rigid body.

From the definition of $\mathbf{r}$, we have upon differentiating

$$\dot{\boldsymbol{\phi}} = \mathbf{\Lambda}(\mathbf{W} \times \mathbf{r} + \dot{\mathbf{r}}) \tag{2.24}$$

where $\mathbf{W}$ is as defined above. If we define $\mathbf{m} = \rho_A \mathbf{\Lambda}^T \dot{\boldsymbol{\phi}}$ as the convected momentum density then the time derivative of $\mathbf{r}$ in the convected frame is

$$\dot{\mathbf{r}} = \rho_A^{-1}\mathbf{m} + \mathbf{r} \times \mathbf{J}^{-1}\mathbf{p}. \tag{2.25}$$

Similarly, if we differentiate the the convected momentum density we get

$$\ddot{\boldsymbol{\phi}} = \mathbf{\Lambda}\rho_A^{-1}(\mathbf{J}^{-1}\mathbf{p} \times \mathbf{m} + \dot{\mathbf{m}}), \tag{2.26}$$

while from our definition of the stored energy function Ψ we get

$$\frac{\partial \Psi}{\partial \gamma} = \mathbf{\Lambda}\mathbf{K}\frac{\partial \mathbf{r}}{\partial S}. \tag{2.27}$$

Thus, substitution into (2.11) yields

$$\mathbf{\Lambda}(\dot{\mathbf{m}} - \mathbf{m} \times \mathbf{J}^{-1}\mathbf{p}) = \mathbf{\Lambda}\mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial S^2}, \tag{2.28}$$

or

$$\dot{\mathbf{m}} = \mathbf{m} \times \mathbf{J}^{-1}\mathbf{p} + \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial S^2} \tag{2.29}$$

in the convected representation.

Finally, we have from (2.12) an expression for the moment about the origin. In this expression we have $\mathbf{m} = 0$ from our definition of Ψ. Thus, for the right hand side of (2.12) we compute

$$\frac{\partial \mathbf{m}}{\partial S} + \frac{\partial \phi}{\partial S} \times \mathbf{n} = -\Lambda(\mathbf{r} \times \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial S^2}) \tag{2.30}$$

To include the rigid body attached to the base we integrate (2.12) from 0 to L, picking up the rigid body dynamics as a boundary term. Thus in convected coordinates one finds

$$\dot{\mathbf{p}} - \mathbf{p} \times \mathbf{J}^{-1}\mathbf{p} = -\int_0^L (\mathbf{r} \times \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial S^2})\, dS \tag{2.31}$$

where we have neglected the effect of the inertia of the appendage.

Thus we conclude that the equations of motion for the rigid body plus linear extensible shear beam are, in convected coordinates

$$\dot{\mathbf{p}} = \mathbf{p} \times \mathbf{J}^{-1}\mathbf{p} - \int_0^L (\mathbf{r} \times \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial S^2})\, dS \tag{2.32}$$

$$\dot{\mathbf{r}} = \rho_A^{-1}\mathbf{m} + \mathbf{r} \times \mathbf{J}^{-1}\mathbf{p} \tag{2.33}$$

$$\dot{\mathbf{m}} = \mathbf{m} \times \mathbf{J}^{-1}\mathbf{p} + \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial S^2} \tag{2.34}$$

We will return to this example again when we discuss Hamiltonian methods.

2.3. A Special Case — A Nonshearable, Inextensible Rod

In this section we examine the limiting case of the rod model when the shear and axial deformation are prohibited. Physically we can think of a material in which the deformation arising from shear and axial forces is negligible, the material is inelastic with regard to those components of stress. We will show that such a rod represents a limiting case of a chain of rigid bodies in ambient space.

There are at least three ways in which we can arrive at the dynamical equations for this rod model. The first method is to simply prohibit any deformation due to shear or axial force. In this case we postulate the strain arising from this stress as zero and then proceed to derive the associated dynamical equations from the balance of momentum principle.

The second method can be employed when the stress and strain are linearly related by constant coefficients. The constitutive model in this case is related to the Saint Venant–Kirchoff model of three dimensional elasticity and is typically restricted to small strains. In this case the coefficients, sometimes called elasticity coefficients, can be treated as parameters and those associated with the shear and axial stress allowed to go to infinity. The resulting equations will reflect the inelasticity in the shear and axial directions. The equations of motion are then determined from the balance of momentum equations.

The third method which could be employed involves treating the material of the rod as a directed media or a Cosserat continuum. We have not used this approach explicitly in our derivation, but the interested reader is directed to Antman [1972], for an account of these types of material.

In the remainder of this section we derive the equations of motion of a rod when the shear and axial stresses produce no deformations in the rod. We conclude with the special case of a rod which is restricted to deform in the plane.

2.3.1. Derivation of the Equations of Motion

In this section we derive the equations of motion for the rod model when there is no axial or shear deformation permitted. In an earlier section we introduced the expressions for the strain. Our constraint on strain therefore corresponds to $\mathbf{\Gamma}(S,t) = 0$.

Recall the expression for the strain, $\mathbf{\Gamma}$ in (2.16). The postulated strain condition allows us to relate the orientation of the $\{\mathbf{t}_I\}$ frame fixed in the cross section to the line of centroids $\boldsymbol{\phi}$. Thus,

$$\frac{\partial \boldsymbol{\phi}}{\partial S} = \mathbf{\Lambda}(S,t)\boldsymbol{E}_3. \tag{2.35}$$

This equation introduces a constraint which effectively reduces the number of independent quantities needed to describe the current configuration.

From the three dimensional material form of the balance of linear and angular momentum we can obtain two equations in terms of the resultant force $\boldsymbol{n}(S,t)$ and the resultant moment $\boldsymbol{m}(S,t)$. With no external forces or torques, these take the form (in the spatial representation)

$$\rho_A \ddot{\boldsymbol{\phi}} = \frac{\partial}{\partial S}\boldsymbol{n}, \tag{2.36}$$

$$\mathbf{I}_\rho \dot{\boldsymbol{w}} + \boldsymbol{w} \times \mathbf{I}_\rho \boldsymbol{w} = \frac{\partial}{\partial S}\boldsymbol{m} + \frac{\partial \boldsymbol{\phi}}{\partial S} \times \boldsymbol{n}. \tag{2.37}$$

These correspond to equations (2.11) and (2.12) in the previous section. We can obtain an expression for $\ddot{\boldsymbol{\phi}}$ in (2.36) by integrating (2.35) and then differentiating with respect to time twice. Thus

$$\boldsymbol{\phi}(S,t) = \int_0^S \boldsymbol{\Lambda}(\sigma_1, t)\boldsymbol{E}_3\, d\sigma_1, \tag{2.38}$$

and differentiating

$$\frac{\partial \boldsymbol{\phi}}{\partial t} = \int_0^S \frac{\partial}{\partial t}\boldsymbol{\Lambda}(\sigma_1, t)\boldsymbol{E}_3\, d\sigma_1, \tag{2.39}$$

$$\frac{\partial^2 \boldsymbol{\phi}}{\partial t^2} = \int_0^S \frac{\partial^2}{\partial t^2}\boldsymbol{\Lambda}(\sigma_1, t)\boldsymbol{E}_3\, d\sigma_1, \tag{2.40}$$

from which, upon substitution into (2.36), we get

$$\frac{\partial \boldsymbol{n}}{\partial S} = \rho_A \int_0^S \frac{\partial^2}{\partial t^2}\boldsymbol{\Lambda}(\sigma_1, t)\boldsymbol{E}_3\, d\sigma_1. \tag{2.41}$$

Integrating this from 0 to S and substituting into (2.37) along with (2.35) we have for the angular momentum

$$\mathbf{I}_\rho \dot{\boldsymbol{w}} + \boldsymbol{w} \times \mathbf{I}_\rho \boldsymbol{w} = \frac{\partial \boldsymbol{m}}{\partial S} + \boldsymbol{\Lambda}(S,t)\boldsymbol{E}_3 \times (\boldsymbol{n}(0) + \int_0^S \rho_A \int_0^{\sigma_1} \frac{\partial^2}{\partial t^2}\boldsymbol{\Lambda}(\sigma_2, t)\boldsymbol{E}_3\, d\sigma_2 d\sigma_1), \tag{2.42}$$

where we have used the initial condition on $\boldsymbol{n}$.

Equation (2.42) describes the dynamics of the rod in terms of the orientations of the cross sections. (Recall $[\dot{\boldsymbol{\Lambda}}(S,t)\boldsymbol{\Lambda}(S,t)]\boldsymbol{w}(S,t) = 0$). We were able to eliminate the equation for the line of centroids by the algebraic constraint (2.35). Physically, what we have done is remove some of the degrees of freedom which the cross sections had with respect to the line of tangent to the line of centroids. We note that in this case the convected basis corresponds to the moving basis of Simo [1985]. In fact, the normal to the cross sections is now in the same direction as the tangent to the line of centroids.

2.3.2. An Example, the Plane Case

We can now specialize the results of the previous section to the case where the rod is restricted to lay in a plane. In this case there are two linear displacements, $u_1(S,t)$, the displacement from rest along $\boldsymbol{E}_3$, and $u_2(S,t)$, the displacement along $\boldsymbol{E}_1$ and one angular variable $\alpha(S,t)$, the tangent to the line of centroids. The line of centroids is described by

$$\phi(S,t) \triangleq (S - u_1(S,t))\boldsymbol{E}_3 + u_2(S,t)\boldsymbol{E}_1, \tag{2.43}$$

while the orientation of a cross section is described by

$$\Lambda(S,t) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}. \tag{2.44}$$

Using the expressions (2.16) for the strain measures we find,

$$\Gamma_1 = -(1 + \frac{\partial u_1}{\partial S})\sin\alpha + \frac{\partial u_2}{\partial S}\cos\alpha, \tag{2.45}$$

$$\Gamma_2 = (1 + \frac{\partial u_1}{\partial S})\cos\alpha + \frac{\partial u_2}{\partial S}\sin\alpha, \tag{2.46}$$

and with $\Gamma_1 = 0$, $\Gamma_3 = 0$ we have

$$\cos\alpha = (1 + \frac{\partial u_1}{\partial S}), \qquad \sin\alpha = \frac{\partial u_2}{\partial S}. \tag{2.47}$$

We will assume $\boldsymbol{m}(S,t) = EI\frac{\partial\alpha}{\partial S}$, so the resultant moment is proportional to the curvature of the rod. Note that E is the modulus of elasticity and I is the cross sectional inertia. In addition we assume $\mathbf{J}_\rho$ is diagonal with the second diagonal element $J_{\rho,22}$. Since the deformation is in a plane the only nonzero element of $\boldsymbol{w}$ is the second and this is exactly $\ddot{\alpha}$. Combining this all into equation (2.42) we find that the dynamics of the rod about the axis along $\boldsymbol{E}_2$ is described by

$$\begin{aligned} I\frac{\partial^2\alpha}{\partial S^2} &+ \sin\alpha\int_0^S \rho_A \int_0^{\sigma_1} (\cos\alpha\left(\frac{\partial\alpha}{\partial t}\right)^2 - \sin\alpha\frac{\partial^2\alpha}{\partial t^2})\, d\sigma_2 d\sigma_1 \\ &+ \cos\alpha\int_0^S \rho_A \int_0^{\sigma_1} (-\sin\alpha\left(\frac{\partial\alpha}{\partial t}\right)^2 + \cos\alpha\frac{\partial^2\alpha}{\partial t^2})\, d\sigma_2 d\sigma_1 = J_{\rho,22}\frac{\partial^2\alpha}{\partial t^2}. \end{aligned} \tag{2.48}$$

If we linearize this equation we can recover the classical Euler-Bernoulli beam equation with rotatory inertia. Note that in this case $\cos\alpha = 1$, $\sin\alpha = \alpha$. Thus, the above becomes

$$\int_0^S \int_0^{\sigma_1} \rho_A \frac{\partial^2\alpha}{\partial t^2}\, d\sigma_2\, d\sigma_1 + EI\frac{\partial^2\alpha}{\partial s^2} = J_{\rho,22}\frac{\partial^2\alpha}{\partial t^2}. \tag{2.49}$$

Next we use the linear approximation $\alpha = \frac{\partial u_2}{\partial s}$, and differentiate once with respect to S. Note that the inner integral disappears when we integrate the substituted term while we get rid of the outer integral when we differentiate. The result is then

$$\rho_A \frac{\partial^2 u_2}{\partial t^2} + EI \frac{\partial^4 u_2}{\partial S^4} = J_{\rho,22} \frac{\partial^4 u_2}{\partial t^2 \partial S^2}, \tag{2.50}$$

which is exactly the Euler-Bernoulli beam equation with rotatory inertia. [1]

2.4. An Alternative Derivation

In this section we discuss in detail an alternative derivation of the equations of motion for a planar, inextensible, nonshearable rod, including the case when the base of the rod is attached to a rigid body. The approach is an application of Hamilton's principle, a variational principle based on minumizing an integral of the Lagrangian. For the case we consider, the appropriate kinetic and potential energy will be

$$\begin{aligned} K &= \tfrac{1}{2}\mathbf{I}_M \dot{\alpha}^2(0,t) + \tfrac{1}{2}M(\dot{x}^2(0,t) + \dot{y}^2(0,t)) \\ &\qquad + \int_0^L \tfrac{1}{2}\mathbf{I}_\rho \dot{\alpha}^2(S,t) + \tfrac{1}{2}\rho_A(\dot{x}^2(S,t) + \dot{y}^2(S,t))\,dS, \\ U &= \int_0^L \tfrac{1}{2} EI\alpha''(S,t)\,dS. \end{aligned}$$

Here I_M is the inertia of the rigid body, and M is its mass. We assume its center of mass coincides with the base point of the rod. For the rod we assume $\mathbf{I}_\rho$ is the rotatory inertia while ρ_A is the mass per unit length. A point on the line of centroids of the rod is located with respect to an inertial frame by the coordinates $x(S,t)$, and $y(S,t)$. The tangent to this point makes an angle $\alpha(S,t)$ with respect to the inertial coordinates. Thus

$$\begin{aligned} x(S,t) &= x(0,t) + \int_0^S \cos(\alpha(\sigma,t))\,d\sigma, \\ y(S,t) &= y(0,t) + \int_0^S \sin(\alpha(\sigma,t))\,d\sigma. \end{aligned}$$

[1] This is exactly equation (5), p. 430 of Love [1944]

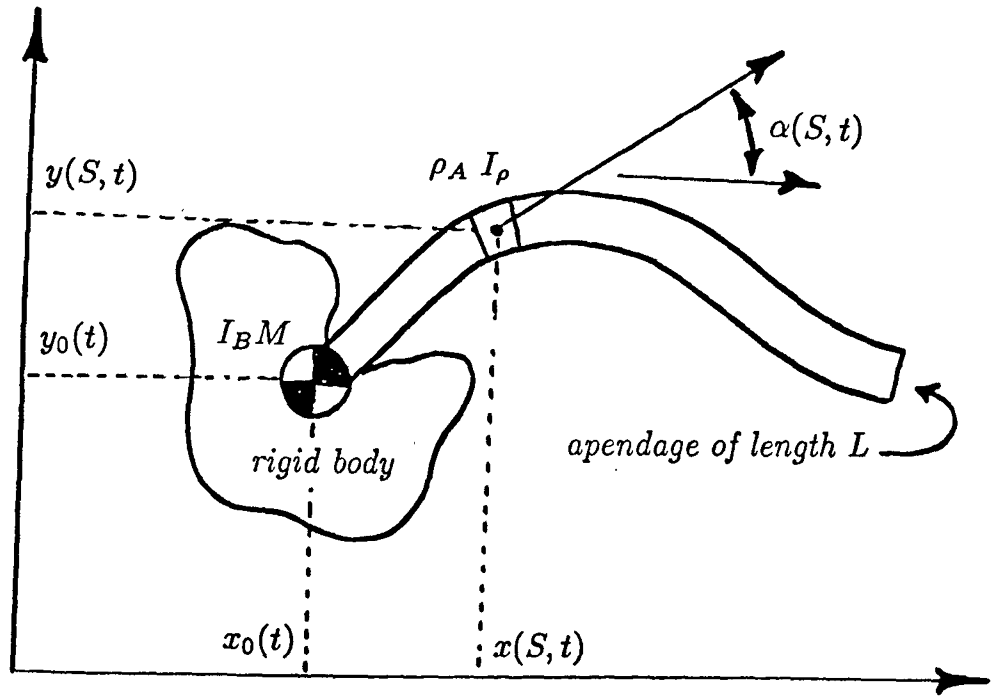

Figure 2.2. A Planar Rigid Body and Rod Configuration.

2.4.1. A Free, Planar Rod in Space

We assume the rod satisfies the conditions of the first section. The equations of motion are obtained by Hamilton's principle . We begin with the integral of the Lagrangian as in the first section.

$$I = \int_{t_0}^{t_1} \left\{ \int_0^L \tfrac{1}{2} I_\rho \dot{\alpha}^2 + \tfrac{1}{2}\rho_A(\dot{x}(0,t) - \int_0^S \sin(\alpha)\dot{\alpha}\, d\sigma_1)^2 \right.$$
$$\left. + \tfrac{1}{2}\rho_A(\dot{y}(0,t) + \int_0^S \cos(\alpha)\dot{\alpha}\, d\sigma_1)^2 - \tfrac{1}{2}EI\,(\alpha')^2\, dS \right\} d\tau, \qquad (2.51)$$

and consider test functions of the form $\alpha(S,t) + \epsilon\eta(S,t)$, and subject to the boundary conditions that arise from Hamilton's principle, i.e. $\eta(S,t_0) = \eta(S,t_1) = 0$. We also consider the boundary conditions at $S = 0$, or $S = L$. We note that for a free end, $\eta(S,t)\Big|_{S=0,L}$ is arbitrary, as are $x(S,t)\Big|_{S=0,L}$, and $y(S,t)\Big|_{S=0,L}$. For a pinned end $x(S,t)\Big|_{S=0,L} = 0$, and $y(S,t)\Big|_{S=0,L} = 0$ with $\eta(S,t)\Big|_{S=0,L}$ again arbitrary. Finally, for a clamped end $x(S,t)\Big|_{S=0,L} = 0$, $y(S,t)\Big|_{S=0,L} = 0$, $\eta(S,t)\Big|_{S=0,L} = 0$

For the first term in the expression for I we can take the first variation and then perform and integration by parts in t

$$\delta I_1 = \int_{t_0}^{t_1} \int_0^L I_\rho \dot{\alpha}\dot{\eta}\, dS\, d\tau$$
$$= \int_0^L \dot{\alpha}\eta \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} I_\rho \ddot{\alpha}\eta\, dS\, d\tau$$
$$= -\int_{t_0}^{t_1} \int_0^L I_\rho \ddot{\alpha}\eta\, dS\, d\tau \tag{2.52}$$

where we have used the boundary conditions $\eta(S, t_0) = 0$, and $\eta(S, t_1) = 0$. We have also interchanged the order of integration twice.

For the fourth term we again take the first variation and this time perform integration by parts in space.

$$\delta I_4 = -\int_{t_0}^{t_1} \int_0^L EI\, \alpha' \eta'\, dS\, d\tau$$
$$= \int_{t_0}^{t_1} -EI\, \alpha' \eta \Big|_0^L + \int_0^L EI\, \alpha'' \eta\, dS\, d\tau \tag{2.53}$$

In this case we observe that $\eta(0, t)$, and $\eta(L, t)$ maybe arbitrary and consequently $-EI\, \alpha' \eta$ need not be zero at the boundary.

For the remaining two terms we proceed with a somewhat more tedious calculation.

$$\delta I_2 = \int_{t_0}^{t_1} \int_0^L \rho_A(\dot{x}(0,\tau) - \int_0^S \sin(\alpha)\dot{\alpha}\, d\sigma_1)(\int_0^S -\cos(\alpha)\eta\dot{\alpha} - \sin(\alpha)\dot{\eta}\, d\sigma_1)\, dS\, d\tau$$
$$= \int_{t_0}^{t_1} \int_0^L \rho_A \Big(\dot{x}(0,\tau) - \int_0^S \sin(\alpha)\dot{\alpha}\, d\sigma_1\Big)(\int_0^S -\frac{\partial}{\partial t}(\sin(\alpha)\eta)\, d\sigma_1)\, dS\, d\tau.$$

Integrating by parts in t;

$$= \int_0^L \rho_A \Big(\dot{x}(0,\tau) - \int_0^S \sin(\alpha)\dot{\alpha}\, d\sigma_1\Big)(\int_0^S -\sin(\alpha)\eta\, d\sigma_1)\Big|_{t_0}^{t_1}$$
$$- \int_0^L \int_{t_0}^{t_1} \rho_A \Big(\ddot{x}(0,\tau) - \int_0^S (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_1\Big)(\int_0^S -\sin(\alpha)\eta\, d\sigma_1)\, dS\, d\tau,$$
$$= \int_0^L \int_{t_0}^{t_1} \rho_A \Big(\ddot{x}(0,\tau) - \int_0^S (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_1\Big)(\int_0^S \sin(\alpha)\eta\, d\sigma_1)\, dS\, d\tau,$$

where we have used $\eta(0, t_0) = \eta(0, t_1) = 0$. Integrating by parts in S

$$= \int_{t_0}^{t_1} \int_0^S \rho_A \Big(\ddot{x}(0,\tau) - \int_0^{\sigma_1} (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_2 \Big)\, d\sigma_1 (\int_0^S \sin(\alpha)\eta\, d\sigma_1) \Big|_0^L$$
$$- \int_0^L \left\{ \int_0^S \rho_A(\ddot{x}(0,\tau) + \int_0^{\sigma_1} (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_2)\, \sigma_1 \right\} \sin(\alpha)\eta\, dS\, d\tau. \tag{2.54}$$

We can proceed in an analogous fashion to compute the variation associated with the third term

$$\delta I_3 = \int_{t_0}^{t_1} \int_0^S \rho_A \Big(\ddot{y}(0,\tau) + \int_0^{\sigma_1} (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_2 \Big)\, d\sigma_1 (\int_0^S \cos(\alpha)\eta\, d\sigma_1) \Big|_0^L$$
$$+ \int_0^L \left\{ \int_0^S \rho_A \Big(\ddot{y}(0,\tau) + \int_0^{\sigma_1} (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_2 \Big)\, \sigma_1 \right\} \cos(\alpha)\eta\, dS\, d\tau. \tag{2.55}$$

Since the first variation of I must be zero, combining the above we find

$$0 = \int_{t_0}^{t_1} \Big\{ (\int_0^S \rho_A \Big(\ddot{x}(0,\tau) + \int_0^{\sigma_1} (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_2 \Big)\, d\sigma_1 (\int_0^S \sin(\alpha)\eta\, d\sigma_1)$$
$$- (\int_0^S \rho_A \Big(\ddot{y}(0,\tau) + \int_0^{\sigma_1} (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_2 \Big)\, d\sigma_1)$$
$$\cdot (\int_0^S (\cos(\alpha)\eta\, d\sigma_1) \Big\} - EI\,\alpha'\eta \Big|_0^L$$
$$+ \int_0^L \Big\{ -I_\rho \ddot{\alpha} - \sin(\alpha) \left(\int_0^S \rho_A(\ddot{x}(0,\tau) - \int_0^{\sigma_1} (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_2)\, \sigma_1 \right)$$
$$+ \cos(\alpha) \left(\int_0^S \rho_A(\ddot{y}(0,\tau) + \int_0^{\sigma_1} (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_2)\, \sigma_1 \right) + EI\,\alpha'' \Big\} \eta\, dS\, d\tau. \tag{2.56}$$

/smallskip The requirement that (2.56) must be equal to zero along with properties of η and the assumptions of continuity can be used to find the dynamics of the rod along with the end conditions. We first note that since η is arbitrary for the above to hold and the conditions of Hamilton's principle to be satisfied we require

$$0 = -\sin(\alpha) \left(\int_0^L \rho_A(\ddot{x}(0,t) - \int_0^{\sigma_1} (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_2)\, d\sigma_1 \right)$$
$$- \cos(\alpha) \left(\int_0^L \rho_A(\ddot{y}(0,t) + \int_0^{\sigma_1} (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_2)\, d\sigma_1 \right),$$
$$0 \le S \le L, \tag{2.57}$$

$$EI\,\alpha'(0,t)\eta(0,t) = 0, \tag{2.58}$$

$$EI\,\alpha'(L,t)\eta(L,t) = 0, \tag{2.59}$$

$$\begin{aligned} 0 = -I_\rho\ddot{\alpha} - \sin(\alpha)&\left(\int_0^S \rho_A(\ddot{x}(0,t) - \int_0^{\sigma_1}(\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\,d\sigma_2)\,d\sigma_1\right) \\ + \cos(\alpha)&\left(\int_0^S \rho_A(\ddot{y}(0,t) + \int_0^{\sigma_1}(-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\,d\sigma_2)\,d\sigma_1\right) + EI\,\alpha''. \\ &0 \le S \le L \end{aligned} \tag{2.60}$$

If we set $S = L$ in (2.60) and use (2.57) we find

$$0 = -I_\rho\ddot{\alpha}(L,t) + EI\,\alpha''(L,t), \tag{2.61}$$

this equation describes dynamics of the rod at the tip. What remains is to specify the boundary conditions, there are two cases:

2.4.2. A Fixed-Free, Planar Rod

In this case we demand $x(0,t) = 0$, $y(0,t) = 0$, and $\alpha(0,t) = 0$ at the base of the rod. This represents the base being fixed in space and not being allowed to rotate.

At the tip of the rod there remain two conditions to specify. The first is found by recalling that for the free end, η is arbitrary. Thus from the condition at $S = L$ (2.58), we require $\alpha'(L,t) = 0$ to satisfy Hamilton's principle. The second condition is (2.61).

Physically this equation corresponds to the balance of force and torque at the tip of the rod.

2.4.3. Hinged-Free Rod

Again we demand that $x(0,t) = 0$, and $y(0,t) = 0$ as in the fixed-free case. In this case however the base of the rod is free to rotate, hence $\eta(0,t)$ is arbitrary. As a consequence, in order to satisfy (2.59) we require $\alpha_{,s}(0,t) = 0$. The free end is the same as in the fixed-free case.

2.4.4. Rigid Body and Rod

In the preceding section it was assumed that $x(0,t)$, and $y(0,t)$ were fixed. In this section we will remove that restriction and assume that a rigid body is attached to the point $(x(0,t), y(0,t))$. In what follows the rigid body is assumed to have mass M, and moment of inertia I_M. We will also put a torque, $u(t)$, about an axis through the center of the rigid body.

The integral of the Lagrangian for this problem is

$$I = \int_{t_0}^{t_1} \tfrac{1}{2} I_M \dot{\alpha}^2(0,\tau) + \tfrac{1}{2} M(\dot{x}^2(0,\tau) + \dot{y}^2(0,\tau) - \alpha(0,\tau)u(\tau)\, d\tau$$
$$+ \int_{t_0}^{t_1} \Big\{ \int_0^L \tfrac{1}{2} I_\rho \dot{\alpha}^2 + \tfrac{1}{2}\rho_A(\dot{x}(0,\tau) - \int_0^S \sin(\alpha)\dot{\alpha}\, d\sigma_1)^2$$
$$+ \tfrac{1}{2}\rho_A(\dot{y}(0,\tau) + \int_0^S \cos(\alpha)\dot{\alpha}\, d\sigma_1)^2 - \tfrac{1}{2} EI\,(\alpha')^2\, dS \Big\}\, d\tau. \qquad (2.62)$$

We consider test functions of the form $\alpha(S,t) + \epsilon\eta(S,t)$, as before and in addition $x(0,t) + \epsilon\eta^{(x)}(t)$, and $y(0,t) + \epsilon\eta^{(y)}(t)$.

The variation associated with the rigid body is computed,

$$\delta I = \int_{t_0}^{t_1} I_M \dot{\alpha}(0,\tau)\dot{\eta}(0,t) + M\dot{x}(0,\tau)\dot{\eta}^{(x)} + M\dot{y}(0,\tau)\dot{\eta}^{(y)} - u(\tau)\eta(0,\tau)\, d\tau,$$
$$= \int_{t_0}^{t_1} -I_M \ddot{\alpha}(0,\tau)\eta(0,\tau) - M\ddot{x}(0,\tau)\eta^{(x)} - M\ddot{y}(0,\tau)\eta^{(y)} - u(\tau)\eta(0,\tau)\, d\tau, \qquad (2.63)$$

where we have preformed integration by parts and used the boundary conditions.

The expressions for δI_1, and δI_4 remain the same. The expressions for δI_2, and δI_3 now have additional components associated with $x(0,t)$, and $y(0,t)$. In particular,

$$\delta I_2^{(x)} = \int_{t_0}^{t_1} \int_0^L \rho_A\big(\dot{x}(0,\tau) - \int_0^S \sin(\alpha)\dot{\alpha}\, d\sigma_1\big)\dot{\eta}^{(x)}\, dS\, d\tau$$
$$= -\int_{t_0}^{t_1} \int_0^L \rho_A\big(\ddot{x}(0,\tau) - \int_0^S (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_1\big)\eta^{(x)}\, dS\, d\tau. \quad (2.64)$$

Thus we get for the variation of the second term

$$\delta I_2 = -\int_{t_0}^{t_1} \int_0^L \rho_A\big(\ddot{x}(0,\tau) - \int_0^S (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_1\big)\eta^{(x)}\, dS\, d\tau$$
$$+ \int_{t_0}^{t_1} \int_0^S \rho_A\Big(\ddot{x}(0,\tau) - \int_0^{\sigma_1} (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_2\Big)\, d\sigma_1 \big(\int_0^S \sin(\alpha)\eta\, d\sigma_1\big)\Big|_0^L$$
$$- \int_0^L \Big\{ \int_0^S \rho_A(\ddot{x}(0,\tau) + \int_0^{\sigma_1} (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_2)\, \sigma_1 \Big\} \sin(\alpha)\eta\, dS\, d\tau.$$
$$(2.65)$$

We can proceed in an analogous fashion to compute the variation associated with the third term

$$
\begin{aligned}
\delta I_3^{(y)} &= \int_{t_0}^{t_1} \int_0^L \rho_A\big(\dot{y}(0,\tau) + \int_0^S \cos(\alpha)\dot{\alpha}\, d\sigma_1\big)\dot{\eta}^{(y)}\, dS\, d\tau \\
&= -\int_{t_0}^{t_1} \int_0^L \rho_A\big(\ddot{y}(0,\tau) + \int_0^S (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_1\big)\eta^{(y)}\, dS\, d\tau. \quad (2.66)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\delta I_3 = &-\int_{t_0}^{t_1} \int_0^L \rho_A\big(\ddot{y}(0,\tau) + \int_0^S (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_1\big)\eta^{(y)}\, dS\, d\tau \\
&+ \int_{t_0}^{t_1} \int_0^S \rho_A\bigg(\ddot{y}(0,\tau) + \int_0^{\sigma_1} (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_2\bigg)\, d\sigma_1 (\int_0^S \cos(\alpha)\eta\, d\sigma_1)\bigg|_0^L \\
&\quad + \int_0^L \bigg\{\int_0^S \rho_A\bigg(\ddot{y}(0,\tau) + \int_0^{\sigma_1} (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_2\bigg)\, \sigma_1\bigg\} \cos(\alpha)\eta\, dS\, d\tau.
\end{aligned}
$$

(2.67)

From the above, with $\eta^{(x)}(t)$, $\eta^{(y)}(t)$, and $\eta(S,t)$ arbitrary we have the following conditions for Hamilton's principle to be satisfied. Associated with the variations in $x(0,t)$, and $y(0,t)$ we have

$$
0 = M\ddot{x}(0,t) + \int_0^L \rho_A\big(\ddot{x}(0,t) - \int_0^S (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_1\big)\, dS, \qquad (2.68)
$$

$$
0 = M\ddot{y}(0,t) + \int_0^L \rho_A\big(\ddot{y}(0,t) + \int_0^S (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_1\big)\, dS. \qquad (2.69)
$$

We can recognize this as the sum of reaction forces in a system with no external forces acting on it. As before, again we have

$$
\begin{aligned}
0 = &-\sin(\alpha)\left(\int_0^L \rho_A(\ddot{x}(0,t) - \int_0^{\sigma_1} (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_2)\, d\sigma_1\right) \\
&- \cos(\alpha)\left(\int_0^L \rho_A(\ddot{y}(0,t) + \int_0^{\sigma_1} (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_2)\, d\sigma_1\right), \\
&\qquad 0 \le S \le L, \qquad (2.70)
\end{aligned}
$$

condition (2.58) now becomes

$$
EI\,\alpha'(0,t) + I_M\ddot{\alpha}(0,t) = u(t), \qquad (2.71)
$$

while (2.59) remains the same

$$
EI\,\alpha'(L,t) = 0. \qquad (2.72)
$$

Again we have the rod dynamics

$$
\begin{aligned}
0 = -I_\rho \ddot{\alpha} - \sin(\alpha)\left(\int_0^S \rho_A(\ddot{x}(0,t) - \int_0^{\sigma_1} (\cos(\alpha)\dot{\alpha}^2 + \sin(\alpha)\ddot{\alpha})\, d\sigma_2)\, d\sigma_1\right) \\
+ \cos(\alpha)\left(\int_0^S \rho_A(\ddot{y}(0,t) + \int_0^{\sigma_1} (-\sin(\alpha)\dot{\alpha}^2 + \cos(\alpha)\ddot{\alpha})\, d\sigma_2)\, d\sigma_1\right) + EI\,\alpha'', \\
0 \leq S \leq L. \qquad (2.73)
\end{aligned}
$$

If we divide the conditions containing the mass M by the mass then let $M \to \infty$ we find $\ddot{x}(0,t) = 0$, and $y(0,t) = 0$. However we let I_M remain fixed. Our equations now correspond to a rod with a rigid body at one end pinned in space. If we let $I_M \to 0$ we recover the equations for a hinged rod. If we let $I_M \to \infty$ we recover the equations for a clamped rod.

2.5. Continuum Models as Limits of N-Body Chains

The planar, inextensible, nonshearable rod dynamics we have considered can also be thought of as the limiting case of a planar chain of rigid bodies. In this section we will consider a planar chain of rigid bodies and show that it has in the limit the continuum model of the previous two sections. Consequently, in approximating the continuum model there is a natural approximation which can be physically interpreted as a chain of N-rigid bodies.

2.5.1. Formulation of the Equations of Motion

For a chain of N rigid bodies we assume that the distance between the two hinge points at opposite ends of the i^{th} rigid body has length r_i. We define the *centerline* of a rigid body as the line connecting the two hinge points. We assume the center of mass of the i^{th} rigid body lays on the centerline at a point $\epsilon_i r_i$ from the lower hinge. This rigid body is characterized by its mass m_i and the moment of inertia about it's center of mass, I_i.

The N rigid bodies are connected together to form a chain (see figure 2.3).

We call the first rigid body the *base link*. The i^{th} hinge is the hinge which connects the i^{th} rigid body to the $(i+1)^{st}$ rigid body. The elastic hinges produce a torque linearly proportional to their angular displacement by a constant k_i. We assume the

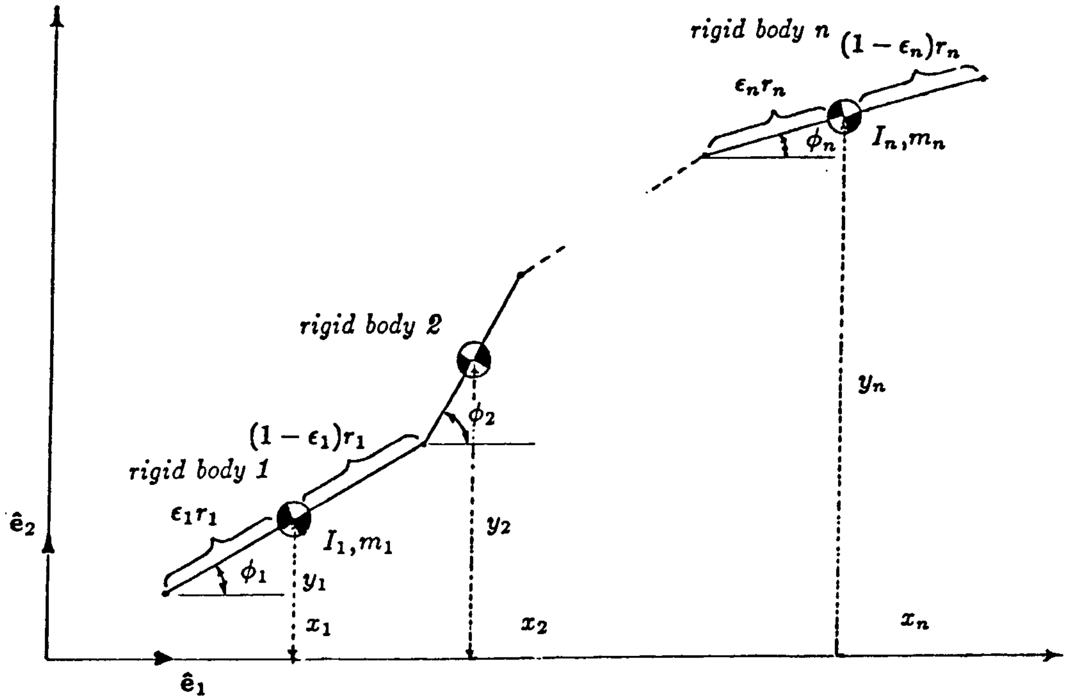

Figure 2.3. A Chain of N Rigid Bodies.

total length of the chain is constant, we denote it by L. Similarly, we assume that total mass of the chain is constant and we denote it by m_{tot}.

The configurations of the chain generally lays in R^3. However, by restricting our attention to the planer case our configurations lay in R^2. If $\{ \mathbf{E}_I \}$ denotes an inertial reference frame then the center of mass of the i^{th} rigid body will be located at $x_i\mathbf{E}_1 + y_i\mathbf{E}_2 + z_i\mathbf{E}_3$. By a suitable choice of $\{ \mathbf{E}_I \}$ we have $z_i = 0,\ i = 1, ..., N$. In addition each rigid body is characterized by an angle ϕ_i which is the angle the centerline of the i^{th} rigid body makes with the vector $\mathbf{E}_2$.

The relationship between the positions of the center's of mass of the rigid bodies in the chain will be described by the two constraint equations for $1 \leq j < i \leq N$

$$x_i = f_{ij}(\phi_i, ..., \phi_j) + x_j \qquad (2.74),$$

$$y_i = g_{ij}(\phi_i, ..., \phi_j) + y_j \qquad (2.75),$$

where we have $f_{ij} = 0,\ i \leq j$ and $g_{ij} = 0,\ i \leq j$. If we denote $f_i = f_{ij}|_{j=1}$ and $g_i = g_{ij}|_{j=1}$ then the velocity and acceleration of the center of mass of the i^{th} rigid body can be expressed in terms of that of the 1^{st} rigid body and the angles $\phi_1, ..., \phi_i$;

$$\dot{x}_i = \sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j} \dot{\phi}_j + \dot{x}_1, \tag{2.76}$$

$$\ddot{x}_i = \sum_{j=1}^{i} \sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell} \dot{\phi}_j \dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j} \ddot{\phi}_j + \ddot{x}_1, \tag{2.77}$$

$$\dot{y}_i = \sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j} \dot{\phi}_j + \dot{y}_1, \tag{2.78}$$

$$\ddot{y}_i = \sum_{j=1}^{i} \sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell} \dot{\phi}_j \dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j} \ddot{\phi}_j + \ddot{y}_1. \tag{2.79}$$

The total kinetic energy of the chain is the sum of kinetic energy arising from the linear and angular motion of each rigid body

$$T = \sum_{i=1}^{N} \tfrac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2) + \sum_{i=1}^{N} \tfrac{1}{2} I_i \dot{\phi}_i^2. \tag{2.80}$$

The potential energy in the chain arises from the deformation at the hinges. The total potential energy in the chain will be

$$U = \sum_{i=1}^{N} \tfrac{1}{2} k_i (\phi_{i+1} - \phi_i)^2, \tag{2.81}$$

where we require $k_N = 0$ and leave ϕ_{N+1} unspecified. Combining the above in the Lagrangian $\mathcal{L} = T - U$ and using the expressions for $\dot{x}$ and $\dot{y}$ we have

$$\mathcal{L} = \sum_{i=1}^{N} \tfrac{1}{2} m_i \left\{ (\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j} \dot{\phi}_j + \dot{x}_1)^2 + (\sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j} \dot{\phi}_j + \dot{y}_1)^2 \right\}$$
$$+ \sum_{i=1}^{N} \tfrac{1}{2} I_i \dot{\phi}_i^2 - \sum_{i=1}^{N} \tfrac{1}{2} k_i (\phi_{i+1} - \phi_i)^2, \tag{2.82}$$

Next we use Lagrange's method to obtain the equations of motion. There are $N + 2$ independent variables needed to specify the equations of motion. These are $x_1, y_1, \phi_1, ..., \phi_N$. These arise as follows: From the three variables, x_i, y_i, ϕ_i associated with each of the rigid bodies, We can eliminate $2N - 2$ variables by the constraints (2.74)-(2.75) for $j = 1$. Thus the $N + 2$ dynamical equations of motion are given by:

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_k}\right) - \frac{\partial \mathcal{L}}{\partial \phi_k}; \qquad k = 1, ..., N, \tag{2.83}$$

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1}, \tag{2.84}$$

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}_1}\right) - \frac{\partial \mathcal{L}}{\partial y_1}. \tag{2.85}$$

The first equation describes the dynamics of each rigid body in the chain with respect to the base link while the remaining two describe the position of the center of mass of the base link with respect to inertial space. Note that in our formulation the configuration of the chain is not related to the center of mass of the chain but instead to the position of the base link center of mass.

We now proceed to compute the first equation,

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} = \sum_{i=1}^{N} m_i (\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\dot{\phi}_j + \dot{x}_1)\frac{\partial f_i}{\partial \phi_j}$$
$$+ \sum_{i=1}^{N} m_i (\sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\dot{\phi}_j + \dot{y}_1)\frac{\partial g_i}{\partial \phi_j} + \sum_{i=1}^{N} I_i \dot{\phi}_i. \tag{2.86}$$

Next we differentiate this quantity with respect to time, note that each ϕ_i depends on time, thus

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} = \sum_{i=1}^{N} m_i \{ (\sum_{j=1}^{i}\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_j\dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\ddot{\phi}_j + \ddot{x}_1)\frac{\partial f_i}{\partial \phi_k}$$
$$+ (\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\dot{\phi}_j + \dot{x}_1)\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell \}$$
$$+ \sum_{i=1}^{N} m_i \{ (\sum_{j=1}^{i}\sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_j\dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\ddot{\phi}_j + \ddot{y}_1)\frac{\partial g_i}{\partial \phi_k}$$
$$+ (\sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\dot{\phi}_j + \dot{y}_1)\sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell \} + I_k \ddot{\phi}_k. \tag{2.87}$$

Next we differentiate with respect to the ϕ_i,

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} = \sum_{i=1}^{N} m_i (\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\dot{\phi}_j + \dot{x}_1)\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell$$
$$+ \sum_{i=1}^{N} m_i (\sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\dot{\phi}_j + \dot{y}_1)\sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell$$
$$+ k_{k+1}(\phi_{k+1} - \phi_k) - k_k(\phi_k - \phi_{k-1}). \tag{2.88}$$

Combining the above to form Lagrange's equations we obtain N equations of motion.

$$0 = \sum_{i=1}^{N} m_i \{ (\sum_{j=1}^{i}\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_j\dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\ddot{\phi}_j + \ddot{x}_1)\frac{\partial f_i}{\partial \phi_k}$$

$$
\begin{aligned}
&\quad + (\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\dot{\phi}_j + \dot{x}_1)\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell \,\} \\
&+ \sum_{i=1}^{N} m_i \{ \, (\sum_{j=1}^{i}\sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_j\dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\ddot{\phi}_j + \ddot{y}_1)\frac{\partial g_i}{\partial \phi_k} \\
&\quad + (\sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\dot{\phi}_j + \dot{y}_1)\sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell \,\} + I_k\ddot{\phi}_k \\
&+ \sum_{i=1}^{N} m_i(\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\dot{\phi}_j + \dot{x}_1)\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell + \sum_{i=1}^{N} m_i(\sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\dot{\phi}_j + \dot{y}_1)\sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell \\
&\quad + k_{k+1}(\phi_{k+1} - \phi_k) - k_k(\phi_k - \phi_{k-1}). \qquad (2.89)
\end{aligned}
$$

Note that we can cancel several terms in the above to get

$$
\begin{aligned}
0 = &\sum_{i=1}^{N} m_i \{ \, (\sum_{j=1}^{i}\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_j\dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\ddot{\phi}_j + \ddot{x}_1)\frac{\partial f_i}{\partial \phi_k} \\
&\quad + (\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\dot{\phi}_j + \dot{x}_1)\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell \,\} \\
&+ \sum_{i=1}^{N} m_i \{ \, (\sum_{j=1}^{i}\sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_j\dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\ddot{\phi}_j + \ddot{y}_1)\frac{\partial g_i}{\partial \phi_k} \\
&\quad + (\sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\dot{\phi}_j + \dot{y}_1)\sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell \,\} + I_k\ddot{\phi}_k \\
&+ \sum_{i=1}^{N} m_i(\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j}\dot{\phi}_j + \dot{x}_1)\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell + \sum_{i=1}^{N} m_i(\sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j}\dot{\phi}_j + \dot{y}_1)\sum_{\ell=1}^{i} \frac{\partial^2 g_i}{\partial \phi_j \partial \phi_\ell}\dot{\phi}_\ell \\
&\quad + k_{k+1}(\phi_{k+1} - \phi_k) - k_k(\phi_k - \phi_{k-1}). \qquad (2.90)
\end{aligned}
$$

We can rewrite these in the form

$$
\begin{aligned}
0 = &\sum_{i=1}^{N} m_i \left\{ \sum_{j=1}^{i} (\frac{\partial f_i}{\partial \phi_j}\frac{\partial f_i}{\partial \phi_k} + \frac{\partial g_i}{\partial \phi_j}\frac{\partial g_i}{\partial \phi_k})\ddot{\phi}_j \right\} \\
&+ \sum_{i=1}^{N} m_i \left\{ \sum_{j=1}^{i}\sum_{\ell=1}^{i} (\frac{\partial^2 f_i}{\partial \phi_j \phi_\ell}\frac{\partial f_i}{\partial \phi_k} + \frac{\partial^2 g_i}{\partial \phi_j \phi_\ell}\frac{\partial g_i}{\partial \phi_k})\dot{\phi}_j\dot{\phi}_\ell \right\} \\
&+ I_k\ddot{\phi}_k - k_k(\phi_{k+1} - \phi_k) + k_{k-1}(\phi_k - \phi_{k-1}) \\
&+ \sum_{i=1}^{N} m_i \left\{ \ddot{x}_1 \frac{\partial f_i}{\partial \phi_k} + \ddot{y}_1 \frac{\partial g_i}{\partial \phi_k} \right\}. \qquad (2.91)
\end{aligned}
$$

In addition we have the two equations which describe the position of the base link.

For the dynamics resolved along the x axis,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \sum_{i=1}^{N} m_i (\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j} \dot{\phi}_j + \dot{x}_1) \tag{2.92}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \sum_{i=1}^{N} m_i (\sum_{j=1}^{i}\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell} \dot{\phi}_j \dot{\phi}_\ell + (\sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j} \ddot{\phi}_j + \ddot{x}_1). \tag{2.93}$$

Since there is no potential energy due to the linear displacements we have $\frac{\partial L}{\partial x_1} = 0$ and the last expression above describes the dynamics. If we proceed in a similar fashion along the y axis we find from the above

$$m_{tot}\ddot{x}_1 = -\sum_{i=1}^{N} m_i \sum_{j=1}^{i}\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell} \dot{\phi}_j \dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial f_i}{\partial \phi_j} \ddot{\phi}_j, \tag{2.94}$$

$$m_{tot}\ddot{y}_1 = -\sum_{i=1}^{N} m_i \sum_{j=1}^{i}\sum_{\ell=1}^{i} \frac{\partial^2 f_i}{\partial \phi_j \partial \phi_\ell} \dot{\phi}_j \dot{\phi}_\ell + \sum_{j=1}^{i} \frac{\partial g_i}{\partial \phi_j} \ddot{\phi}_j. \tag{2.95}$$

For the particular case which we consider

$$f_i(\phi_i, ..., \phi_1) = \sum_{j=1}^{i} r_j \epsilon_j \cos(\phi_j) + r_{j+1}(1-\epsilon_{j-1})\cos(\phi_{j-1}), \tag{2.96}$$

$$g_i(\phi_i, ..., \phi_1) = \sum_{j=1}^{i} r_j \epsilon_j \sin(\phi_j) + r_{j+1}(1-\epsilon_{j-1})\sin(\phi_{j-1}). \tag{2.97}$$

From which we find

$$\frac{\partial f_i}{\partial \phi_k} = \begin{cases} 0 & k > i \\ -r_k \epsilon_k \sin(\phi_k) & k = i \\ -r_k \sin(\phi_k) & i > k > 1 \\ -r_1(1-\epsilon_1)\sin(\phi_1) & k = 1 \end{cases}, \quad \frac{\partial g_i}{\partial \phi_k} = \begin{cases} 0 & k > i \\ r_k \epsilon_k \cos(\phi_k) & k = i \\ r_k \cos(\phi_k) & i > k > 1 \\ r_1(1-\epsilon_1)\cos(\phi_1) & k = 1 \end{cases}$$

and for the second partial derivatives

$$\frac{\partial^2 f_i}{\partial \phi_k \partial \phi_\ell} = \begin{cases} 0 & k \neq \ell, k > i \\ -r_k \epsilon_k \cos(\phi_k) & k = \ell, k = i \\ -r_k \cos(\phi_k) & k = \ell, i > k > 1 \\ -r_1(1-\epsilon_1)\cos(\phi_1) & k = \ell, k = 1 \end{cases}$$

$$\frac{\partial^2 g_i}{\partial \phi_k \partial \phi_\ell} = \begin{cases} 0 & k \neq \ell, k > i \\ -r_k \epsilon_k \sin(\phi_k) & k = \ell, k = i \\ -r_k \sin(\phi_k) & k = \ell, i > k > 1 \\ -r_1(1-\epsilon_1)\sin(\phi_1) & k = \ell, k = 1 \end{cases}$$

The dynamics for our N-body planar chain are described by equations (2.91),(2.94), and (2.95) with the appropriate partial derivatives substituted from the above.

2.5.2. Limiting Case of the N-Body Chain

In this section we will take the limit as $N \to \infty$ of the chain of rigid bodies when the total mass and length remain fixed. In addition, we require that the potential energy in the chain remain independent of N. Additional assumptions will be needed to assure this is well behaved.

The starting point in our development is (2.91). We will consider this equation as $N \to \infty$ with $\sum_i m_i = m_{tot}$, and $\sum_i r_i = L$. We also assume that $k_i r_i \to \mu$, and $m_i / r_i \to \rho$.

We will construct a partition of the interval $[0, L]$ by the points $s_i = \sum_j r_j$, thus $0 \leq s_1, \leq \ldots \leq s_{N-1} \leq N$. For any partition we let $R = \max_i\{r_i\}$. We require that $R \to 0$, and $N \to \infty$. We assume that the angles between the rigid bodies go to the limit in a uniform fashion. First we will recast (2.91) in a slightly different form. We note that

$$\begin{aligned}
0 = &\sum_{i=k}^{N} m_i \left\{ \sum_{j=1}^{i} \left(\frac{\partial f_i}{\partial \phi_j} \frac{\partial f_i}{\partial \phi_k} + \frac{\partial g_i}{\partial \phi_j} \frac{\partial g_i}{\partial \phi_k} \right) \ddot{\phi}_j \right\} \\
&+ \sum_{i=k}^{N} m_i \left\{ \sum_{j=1}^{i} \sum_{\ell=1}^{i} \left(\frac{\partial^2 f_i}{\partial \phi_j \phi_\ell} \frac{\partial f_i}{\partial \phi_k} + \frac{\partial^2 g_i}{\partial \phi_j \phi_\ell} \frac{\partial g_i}{\partial \phi_k} \right) \dot{\phi}_j \dot{\phi}_\ell \right\} \\
&+ I_k \ddot{\phi}_k - k_k(\phi_{k+1} - \phi_k) + k_{k-1}(\phi_k - \phi_{k-1}) \\
&+ \sum_{i=k}^{N} m_i \left\{ \ddot{x}_1 \frac{\partial f_i}{\partial \phi_k} + \ddot{y}_1 \frac{\partial g_i}{\partial \phi_k} \right\},
\end{aligned} \tag{2.98}$$

where we have used the fact that $\frac{\partial f_i}{\partial \phi_k} = 0$, $\frac{\partial g_i}{\partial \phi_k} = 0$ for $k > i$. This can be rewritten as

$$\begin{aligned}
0 = &\sum_{i=k}^{N} m_i \frac{\partial f_i}{\partial \phi_k} \frac{d}{dt} \left\{ \sum_{j=1}^{i} f_j + x_1 \right\} + \sum_{i=k}^{N} m_i \frac{\partial g_i}{\partial \phi_k} \frac{d}{dt} \left\{ \sum_{j=1}^{i} g_j + y_1 \right\} \\
&+ I_k \ddot{\phi}_k - k_k(\phi_{k+1} - \phi_k) + k_{k-1}(\phi_k - \phi_{k-1}) \\
= &\sum_{i=k}^{N} m_i \frac{\partial f_i}{\partial \phi_k} \ddot{x}_i + \sum_{i=k}^{N} m_i \frac{\partial g_i}{\partial \phi_k} \ddot{y}_i \\
&+ I_k \ddot{\phi}_k - k_k(\phi_{k+1} - \phi_k) + k_{k-1}(\phi_k - \phi_{k-1}).
\end{aligned} \tag{2.99}$$

If we recognize that we have no external forces and appeal to conservation of linear momentum.

$$\sum_{i=k+1}^{N} m_i \ddot{x}_i = - \sum_{i=1}^{k} m_i \ddot{x}_i.$$

Using this result we can change the limits of summation on the sums which are from $i = k+1$ to N in equation (2.91). Doing this, dividing through by r_k, and rearranging we can rewrite Lagrange's equations for the k^{th} hinge in the form

$$\sin(\phi_k)\sum_{i=1}^{k} m_i\{\sum_{j=1}^{i-1}\frac{\partial^2}{\partial t^2}(r_{j+1}\epsilon_{j+1}\cos(\phi_{j+1}) + r_j(1-\epsilon_j)\cos(\phi_j)) + \ddot{x}_1\}$$
$$+\cos(\phi_k)\sum_{i=1}^{k} m_i\{\sum_{j=1}^{i-1}\frac{\partial^2}{\partial t^2}(r_{j+1}\epsilon_{j+1}\sin(\phi_{j+1}) + r_j(1-\epsilon_j)\sin(\phi_j)) + \ddot{y}_1\}$$
$$+\frac{I_k\ddot{\phi}_k}{r_k} - \frac{k_{k+1}(\phi_{k+1}-\phi_k) - k_k(\phi_k-\phi_{k-1})}{r_k} = 0. \quad (2.100)$$

We are now in a position to take the limit of the chain of N-rigid bodies as $N \to \infty$ while the total length L, the mass M, and the potential energy of the chain remain fixed. For any particular N the joints partition the length into N intervals $r_1, ..., r_N$. Note that each interval r_i is a function of N. As we refine the partitions we have $\lim_{N\to\infty} r_i = 0$.

Under suitable assumptions the limits of the first two terms in (2.100) can be shown to be Reimann-Stieljes integrals integrated along the curve defined in the limit of the n-body chain. These two terms are

$$\lim_{N\to\infty}\sin(\phi_k)\sum_{i=1}^{k} m_i\{\sum_{j=1}^{i-1}\frac{\partial^2}{\partial t^2}(r_{j+1}\epsilon_{j+1}\cos(\phi_{j+1}) + r_j(1-\epsilon_j)\cos(\phi_j)) + \ddot{x}_1\}$$
$$= \sin(\phi_S)\int_0^S \rho_A(\sigma)\left\{\int_0^\sigma \frac{\partial^2}{\partial t^2}\cos(\phi_\tau)\,d\tau + \ddot{x}(0)\right\}d\sigma, \quad (2.101)$$
$$\lim_{N\to\infty}\cos(\phi_k)\sum_{i=1}^{k} m_i\{\sum_{j=1}^{i-1}\frac{\partial^2}{\partial t^2}(r_{j+1}\epsilon_{j+1}\sin(\phi_{j+1}) + r_j(1-\epsilon_j)\sin(\phi_j)) + \ddot{y}_1\}$$
$$= \cos(\phi_S)\int_0^S \rho_A(\sigma)\left\{\int_0^\sigma \frac{\partial^2}{\partial t^2}\sin(\phi_\tau)\,d\tau + \ddot{y}(0)\right\}d\sigma. \quad (2.102)$$

Note that the right hand side of the above are written in a form suggestive of the underlying physical system.

The remaining two terms arise for the rotational and kinetic energy of the individual rigid bodies and the potential energy in the joints. Both k_k and I_k depend on N. For these two terms we compute

$$\lim_{N\to\infty}\frac{I_k}{r_k} = I_\rho(S), \quad (2.103)$$

and

$$\lim_{N\to\infty}\frac{k_{k+1}(\phi_{k+1}-\phi_k) - k_k(\phi_k-\phi_{k-1})}{r_k} = \mu\frac{\partial^2\phi}{\partial S^2}. \quad (2.104)$$

We can now combine these results. If we do this we have the following; the limit of the N-body chain as $N \to \infty$, under the assumption that the potential and kinetic energy remain constant will be

$$\begin{aligned}
\sin(\phi(S)) \int_0^S \rho_A(S) \left\{ \int_0^\sigma \frac{\partial^2}{\partial t^2} \cos(\phi(\tau))\, d\tau + \ddot{x}(0) \right\} d\sigma \\
+ \cos(\phi(S)) \int_0^S \rho_A(S) \left\{ \int_0^\sigma \frac{\partial^2}{\partial t^2} \sin(\phi(\tau))\, d\tau + \ddot{y}(0) \right\} d\sigma \\
+ \mu \frac{\partial^2 \phi(S)}{\partial S^2} - I_\rho \ddot{\phi}(S) = 0, \qquad 0 \leq S \leq L. \qquad (2.105)
\end{aligned}$$

Finally we show that (2.105) corresponds to the expression for the limiting case of the continuous model in the second section. Let $\phi(S) = \alpha(S) + \psi$ where ψ is an angle between a rotating coordinate frame and the inertial frame and $\alpha(S)$ angle of a tangent to the chain at the point S with with respect to the rotating coordinate frame. The the previous expression can be rewritten

$$\begin{aligned}
- \sin(\alpha(S)) \int_0^S \{ \rho(\sigma) \int_0^\sigma \left\{ \cos(\alpha(\tau))(\dot{\alpha}^2(\tau) + 2\dot{\alpha}(\tau)\dot{\psi} + \dot{\psi}^2) + \sin(\alpha(\tau))(\ddot{\alpha} + \ddot{\psi}) \right\} d\tau \\
+ \cos(\psi)\ddot{x}(0) - \sin(\psi)\ddot{y}(0) \} \, d\sigma \\
+ \cos(\alpha(S)) \int_0^S \{ \rho(\sigma) \int_0^\sigma \left\{ \cos(\alpha(\tau))(\ddot{\alpha} + \ddot{\psi}) - \sin(\alpha(\tau))(\dot{\alpha}^2(\tau) + 2\dot{\alpha}(\tau)\dot{\psi} + \dot{\psi}^2) \right\} d\tau \\
+ \sin(\psi)\ddot{x}(0) + \cos(\psi)\ddot{y}(0) \} \, d\sigma \\
+ \mu \frac{\partial^2 \alpha_S}{\partial S^2} = I_\rho(\ddot{\alpha}_\tau + \ddot{\psi}), \qquad 0 \leq S \leq L. \qquad (2.106)
\end{aligned}$$

which agrees with the continuous case when $\ddot{x}(0) = 0$ and $\ddot{y}(0) = 0$.

2.6. Dissipative Mechanisms

Damping mechanisms are frequently modeled in a less than satisfactory manner in the engineering literature. In part this reflects complicated and poorly understood physical mechanisms which may cause these effects. In solids, the nature of the internal forces which produce dissipative effects is generally very complex, and varies considerably between different types of materials. However, some dissipative mechanisms are able to be incorporated in the continuum constitutive equations of elasticity under reasonable assumptions. Two of these are square root damping and rate damping.

Square root damping has been employed by Balas [1979], and Gibson [1979] in studies of large space structures. This type of damping in a simple Euler-Bernoulli beam has the form

$$EI\frac{\partial^4 u}{\partial S^4} + c\frac{\partial^3 u}{\partial S^2 \partial t} + \rho_A \frac{\partial^2 u}{\partial t^2} = 0.$$

Such a model may however produce negative energy dissipation, a physically unrealistic situation. (For an example, see Wie [1981]).

Rate damping is one of the more commonly employed damping models. Rate damping has it's origin in the work of Voigt (Voigt, [1892]) and is frequently referred to as the Kelvin-Voigt model. This damping model employed with an Euler-Bernoulli type beam is of the form

$$EI\frac{\partial^4 u}{\partial S^4} + c\frac{\partial^5 u}{\partial S^4 \partial t} + \rho_A \frac{\partial^2 u}{\partial t^2} = 0.$$

In the following section we discuss how this model is incorporated in the constitutive equations.

2.6.1. Constitutive Equations with Rate Damping

Incorporated in the stored energy function, the conditions under which the mechanism of rate damping generates a contractive semigroup is described in Marsden & Hughes [1983], p.357. The conditions are satisfied if, for example, the first partials of Φ with respect to the strains are strongly elliptic.

An example of a material stored energy function incorporating rate damping is

$$\tilde{\Psi} = \Psi(\Lambda^T \gamma, \Lambda^T \omega) + \mathbf{B}_1 \dot{\mathbf{\Gamma}} + \mathbf{B}_2 \dot{\mathbf{\Omega}} \tag{2.107}$$

When used in our rod model such a constitutive equation gives the linear and rotational equations of motion

$$\rho_A \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial}{\partial S}(\mathbf{n} + \mathbf{B}_1 \dot{\mathbf{n}}) + \bar{\mathbf{n}} \tag{2.108}$$

$$\mathbf{I}_\rho \frac{\partial \mathbf{w}}{\partial t} + \mathbf{w} \times \mathbf{I}_\rho \mathbf{w} = \frac{\partial}{\partial S}(\mathbf{m} + \mathbf{B}_2 \dot{\mathbf{m}}) + \frac{\partial \phi}{\partial S} \times (\mathbf{n} + \mathbf{B}_1 \dot{\mathbf{n}}) + \bar{\mathbf{m}} \tag{2.109}$$

If we proceed as before then we set $\mathbf{\Gamma} = 0$, and obtain

$$\frac{\partial}{\partial S}(\mathbf{m} + \mathbf{B}_2 \dot{\mathbf{m}}) + \Lambda(S,t)\mathbf{E}_3 \times \int_0^S \rho_A \int_0^{\sigma_1} \frac{\partial^2}{\partial t^2} \Lambda(\sigma_2, t)\mathbf{E}_3 \, d\sigma_2 d\sigma_1 = \mathbf{I}_\rho \dot{\mathbf{w}} + \mathbf{w} \times \mathbf{I}_\rho \mathbf{w} \tag{2.110}$$

For the example of the plane case this yields

$$EI\frac{\partial^2\alpha}{\partial S^2} + B_2{}^{(y)}\frac{\partial^3\alpha}{\partial S^2\partial t} + \sin\alpha\int_0^S \rho_A\int_0^{\sigma_1}(\cos\alpha\left(\frac{\partial\alpha}{\partial t}\right)^2 - \sin\alpha\frac{\partial^2\alpha}{\partial t^2})\,d\sigma_2 d\sigma_1$$
$$+\cos\alpha\int_0^S \rho_A\int_0^{\sigma_1}(-\sin\alpha\left(\frac{\partial\alpha}{\partial t}\right)^2 + \cos\alpha\frac{\partial^2\alpha}{\partial t^2})\,d\sigma_2 d\sigma_1 = I_\rho^{(y)}\frac{\partial^2\alpha}{\partial t^2} \tag{2.111}$$

and upon linearization this yields

$$\rho_A\frac{\partial^2 u_2}{\partial t^2} + EI\frac{\partial^4 u_2}{\partial S^4} = B_2{}^{(y)}\frac{\partial^5 u_2}{\partial S^4\partial t} + I_\rho^{(y)}\frac{\partial^4 u_2}{\partial t^2\partial S^2} \tag{2.112}$$

2.7. The Hamiltonian Formulation

In this section we present very brief outline of the Hamiltonian formulation of the rod model. The Hamiltonian formulation provides a framework for the systematic procedure of reduction, the exploitation of symmetries in our problem which enables us to eliminate variables. The reduced model provides the basis for the analysis of stability of relative equilibria in the next section.

In the remainder of this section we introduce some key ideas for recasting the rod theory into the Hamiltonian framework. The material on Poisson manifolds is found in Weinstein [1983], a complete discussion of the geometric viewpoint will be found in Arnold [1978], or Abraham & Marsden [1978]. The geometric viewpoint as regards elasticity can be found in Marsden & Hughes [1983], while the specific case of rod theory is outlined in Simo, Marsden, & Krishnaprasad [1987]

2.7.1. The Rod as a Hamiltonian System

The configuration of the rod takes values on a Poisson manifold. If we denote by $C^\infty(M)$ the set of infinitely differentiable real functions defined on a manifold M then we have the following;

Definition: *Let M be a manifold of dimension n. A* Poisson structure *on M is a Lie algebra structure $\{\cdot,\cdot\}: C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ which satisfies the Leibnitz identity, $\{fg,h\} = f\{g,h\} + \{f,h\}g$. The manifold M with such a structure is called a* Poisson manifold.

The Leibnitz identity guarantees that the bracket is a derivation in each entry. Consequently, for $H \in C^\infty(M)$ there is a vector field ξ_H such that $\xi_H \cdot f = \{f, H\}$ for all f. We call ξ_H the *Hamiltonian vector field* generated by H. Note that symplectic manifolds are a special case of Poisson manifolds, since we can always construct a natural Poisson structure on a symplectic manifold by setting $\{f,g\} = \omega(\xi_f, \xi_g)$, where ω is the given symplectic 2-form on the manifold. Thus, since the cotangent bundle T^*G of a Lie group G is a symplectic manifold, it is also a Poisson manifold.

We can also use the bracket operation in the Lie algebra $\mathcal{G}$ of a group G to obtain a natural Poisson structure on the dual of the Lie algebra. This Poisson structure on $\mathcal{G}^*$ is called the Lie-Poisson structure and makes $\mathcal{G}^*$ into a Poisson manifold. Let $[\cdot,\cdot]$ denote the Lie algebra operation in $\mathcal{G}$, the Lie algebra of the Lie group G. If $\mu \in \mathcal{G}^*$ we can define the bracket operation

$$\{f,g\}(\mu) = -\langle \mu, [\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}]\rangle \tag{2.113}$$

where $\delta f/\delta\mu$ is the variational or Frechét derivative of f and $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathcal{G}$ and $\mathcal{G}^*$. Recall that $\delta f/\delta\mu \colon \mathcal{G} \to \mathcal{G}^*$ as

$$DF(\mu)\cdot\delta\mu = \langle \delta F/\delta\mu, \delta\mu\rangle \tag{2.114}$$

Sometimes $\mathcal{G}^*$ with this structure is denoted by $\mathcal{G}^*_-$.

Subsequently we will need the notion of a Poisson mapping. Such a mapping preserves the structure of the underlying Poisson manifold. More precisely, we have the following definition

Definition. *A* Poisson mapping $\phi \colon M_1 \to M_2$ *between two Poisson manifolds is a map such that*

$$\{f\circ\phi, g\circ\phi\}_1 = \{f,g\}_2 \circ \phi \tag{2.115}$$

where $\{\cdot,\cdot\}_1$ *is the Poisson bracket of* M_1 *and* $\{\cdot,\cdot\}_2$ *is the Poisson bracket of* M_2.

We next recall the definition of a Hamiltonian system,

Definition. *A* Hamiltonian system *is a Poisson manifold* M*, together with a Poisson bracket and function* $H \in C^\infty(M)$*. The field* ξ_H *determined by the condition*

$$\{\xi_H, Y\} = dH\cdot Y \tag{2.116}$$

is called the Hamiltonian vector field. *Integral curves of the Hamiltonian vector field* ξ_H *describe the time evolution of the Hamiltonian system* $(M,\{,\},\xi_H)$.

From this definition we have, for $x \in M$, $\dot{x} = \xi_H(x)$. We call the Poisson manifold the *phase space* of the system and the scalar function H the *Hamiltonian* of the system. The dynamics are made more explicit by observing that if F_t is the flow of the Hamiltonian vector field ξ_H, then we have for a real function g defined on M

$$\frac{d}{dt}(g \circ F_t) = \{g \circ F_t, H\} \tag{2.117}$$

Thus, one sees that the Hamiltonian and the associated bracket determine the dynamics on M of the system.

We will refer to functions $f \in C^\infty(M)$ for which $\{f,g\} = 0$ for all $g \in C^\infty(M)$ as Casimir functions. Casimirs represent quantities which are constants of motion, for any Hamiltonian system on M. The Hamiltonian is one example of a Casimir function.

We now turn our attention to the geometrically exact rod model, outlined in an earlier section. The manifold for the geometrically exact rod is the cotangent bundle of the configuration, $T^*\mathcal{C}$, of the configuration space $\mathcal{C}$ defined in (2.20). This is a symplectic manifold. We can take as a bracket the canonical 2-form associated with it. The Hamiltonian for the geometrically exact rod model can be written in the spatial description as

$$H = \tfrac{1}{2}\int_0^L (\rho_A^{-1}\|\boldsymbol{p}\|^2 + \boldsymbol{\pi}\cdot I_\rho^{-1}\boldsymbol{\pi})\,dS + \tfrac{1}{2}\int_0^L \Psi(\boldsymbol{\Lambda}^T\boldsymbol{\gamma}, \boldsymbol{\Lambda}^T\boldsymbol{\omega})\,dS \tag{2.118}$$

where the the first integral corresponds to the inertial component and the second integral corresponds to the potential energy. In the above $\boldsymbol{p} \triangleq \rho_A\dot{\boldsymbol{\phi}}$, and $\boldsymbol{\pi} \triangleq \mathbf{I}_\rho\mathbf{w}$ denote the linear and angular momenta of the rod, and $\boldsymbol{\Gamma} = \boldsymbol{\Lambda}^T\boldsymbol{\gamma}$, $\boldsymbol{\Omega} = \boldsymbol{\Lambda}^T\boldsymbol{\omega}$ are the convected strain measures. The total linear and angular momentum of the rod is given by

$$\boldsymbol{\ell} = \int_0^L \boldsymbol{p}\,dS, \qquad \boldsymbol{\alpha} = \int_0^L (\boldsymbol{\pi} + \boldsymbol{\phi}\times\boldsymbol{p})\,dS. \tag{2.119}$$

A direct calculation (Simo and Vu-Quoc [1986a]) shows that the total angular momentum of the rod is a conserved quantity; i.e.:

$$\frac{d\boldsymbol{\alpha}(t)}{dt} = \mathbf{0} \tag{2.120}$$

Note that the total angular momentum is a mapping, $\boldsymbol{\alpha}: \mathcal{C} \to I\!R^3$, where $I\!R^3$ is equipped with the standard cross product identified with $so(3)^*$ — the dual of $so(3)$ — via the Lie-algebra isomorphism $\hat{}: so(3) \to I\!R^3$; i.e.:

$$[\, \hat{\Theta}_1 \, , \, \hat{\Theta}_2 \,] = (\, \hat{\Theta}_1 \times \hat{\Theta}_2 \,)\hat{} \tag{2.121}$$

where $[\, \hat{\Theta}_1 \, , \, \hat{\Theta}_2 \,] = \hat{\Theta}_1 \hat{\Theta}_2 - \hat{\Theta}_2 \hat{\Theta}_1$, is the matrix commutator.

2.8. Symmetries and Model Reduction

The general notion of reduction on a Poisson manifold and the procedure for computing the reduced space is found in Marsden & Ratiu [1986]. In Marsden & Weinstein [1974], a procedure for the reduction of symplectic manifolds with symmetry is described. This procedure exploits the symplectic structure of such a manifold (see also Abraham & Marsden [1978]). In particular, a momentum map is used to partition the manifold into equivalence classes $J^{-1}(\mu)$, each corresponding to a fixed value of momentum μ. We then quotient this manifold by an isotropy group G_μ constructed from the coadjoint action, the resulting manifold corresponding to our reduced phase space $M_\mu = J^{-1}(\mu)/G_\mu$.

An alternative method of reduction which avoids the explicit computation of T^*G/G is the basis of the reduction described in Krishnaprasad & Marsden [1986]. Under suitable assumptions we can identify $\mathcal{G}^*$ with the quotient space T^*G/G, avoiding explicit computation of T^*G/G and working instead with $\mathcal{G}^*$.

2.8.1. Reduction of Symplectic Manifolds with Symmetry

The reduction process used by Marsden & Weinstein [1974] to construct the reduced phase space is described in this section. This is a special case of the more general notion of Poisson reduction (see Marsden & Ratiu [1986]). In this case we exploit the underlying symplectic structure.

The first part of the procedure uses the momentum of the system to partition the manifold. This is based on the idea of a momentum map, $J: M \to \mathcal{G}^*$. The momentum map is defined as follows.

Definition. *Let (M, ω) be a connected symplectic manifold and $\phi: G \times M \to M$ a symplectic action of the Lie group G on M. We say that a mapping $J: M \to \mathcal{G}^*$ is a* *momentum mapping for the action if*

$$\langle T_x J \cdot v \rangle = \omega_x(\xi_X(x), v), \qquad \textit{for} \quad \xi \in G, v \in T_x M, \tag{2.122}$$

where ξ_X is the infinitesimal generator of the action corresponding to ξ.

In other words, the vector field corresponding to the infinitesimal generators has the dual map $\hat{J}$, $\hat{J} = \langle J, \xi \rangle$ corresponding to a hamiltonian function. In fact J is an integral of the action. This abstracts the physical notion of a momentum.

An important group action is the adjoint representation of a Lie group G, defined as

$$\phi_g : x \to gxg^{-1}. \tag{2.123}$$

The linearized map, at the identity is called the adjoint representation $Ad: \mathcal{G} \to \mathcal{G}$. The coadjoint action is the induced mapping $Ad^*: \mathcal{G} \to \mathcal{G}$. For a fixed $\mu \in \mathcal{G}^*$ we can define the orbits $\mathcal{O}(\mu)$ of the coadjoint representation as

$$\mathcal{O}(\mu) = \{Ad^*(g)\mu | g \in G\}. \tag{2.124}$$

Recall that a map $\tau: M \to \tilde{M}$ is equivariant for any two group actions $\phi_g: M \to M$ and $\tilde{\phi}_g: \tilde{M} \to \tilde{M}$ if $\tau\phi_g = \tilde{\phi}_g\tau$. If ϕ_g is an exact symplectic group action then the moment map $J: M \to \mathcal{G}$ is equivariant with respect to ϕ_g and the coadjoint representation of $\mathcal{G}$. Consequently, the image of $\{\phi_g(x) | g \in G\}$ for fixed $x \in M$ under J is given for $\mu = J(x)$ by

$$\{Ad^*(g)\mu | g \in G\}. \tag{2.125}$$

We define the isotropy group G_μ of μ,

$$G_\mu = \{g \in G | Ad^*(g)\mu = \mu\}. \tag{2.126}$$

Now we consider, for a fixed $\mu \in \mathcal{G}^*$ the set

$$J^{-1}(\mu) = \{x \in M | J(x) = \mu\}. \tag{2.127}$$

This partitions M into regions corresponding to fixed values of the momentum.

The reduced phase space is now defined to be

$$M_\mu = J^{-1}(\mu)/G_\mu. \tag{2.128}$$

This reduced manifold M_μ has a natural symplectic structure ω_μ such that $i^*\omega = \pi^*\omega_\mu$ where $i: j^{-1}(\mu) \to M$ is the inclusion map and $\pi: J^{-1}(\mu) \to M_\mu$ is the natural projection.

We now return to the original problem of reducing a phase space T^*G by quotienting out by the group G and outline an alternative method of reduction. Since T^*G is a symplectic manifold we know it is also a Possion manifold. In addition the dual of the Lie-algebra $\mathcal{G}^*$ is a Poisson manifold with the Lie-Poisson structure. Since the Lie group G acts on T^*G freely and properly and the action is a Poisson map, then T^*G/G is a Poisson manifold. If we consider the group action of left translation, $L_g: G \times G \to G$, then it turns out that the map $\phi: T^*G \to \mathcal{G}^*$ sending $\alpha_g \in T_g^*G$ to $TL_g^* \cdot \alpha_g \in \mathcal{G}^*$ is a Poisson map which induces a Poisson diffeomorphism of T^*G/G with $\mathcal{G}^*$ (see Marsden, et. al. [1983] for details). This identifies T^*G/G with $\mathcal{G}^*$.

A simple example of the above is furnished by the free rigid body rotating about a fixed point. The configuration of the rigid body is given by elements of $SO(3)$, the phase space is $T^*SO(3)$, typically consisting of the Euler angles and the components of the spatial angular momentum vector. The Hamiltonian is given by $H = \frac{1}{2}\mathbf{p} \cdot \mathbf{J}^{-1}\boldsymbol{p}$ where $\boldsymbol{p}$ is the momentum vector and $\mathbf{J}$ is the inertia operator.

In this case if we use the expression for the Lie-Poisson structure where the Frechét derivatives now correspond to ordinary gradients, $\{,\}$ is the cross product, and $\langle , \rangle$ is the dot product in $\mathbb{R}^3$ we have, for $f \in C^\infty(M)$

$$\dot{f} = \{f, H\}, \tag{2.129}$$

from which

$$\nabla f \cdot \dot{\boldsymbol{p}} = \boldsymbol{p} \cdot (\nabla f \times \nabla H), \qquad \text{or} \qquad \dot{\boldsymbol{p}} = \boldsymbol{p} \times \mathbf{J}^{-1}\boldsymbol{p}. \tag{2.130}$$

which are simply Euler's equations of motion.

2.8.2. Rigid Body with Linear, Extensible Shear Beam

In Krishnaprasad & Marsden [1987] a rigid body with an attached appendage is reduced to a system which eliminates the variables associated with the rotation of the configuration.

In this example we consider a rigid body to which a flexible appendage is attached. The phase space associated with this coupled system is $T^*G \times P$ where P is the phase space of the flexible appendage. Since the appendage is attached to the rigid body we assume G acts on both the rigid body and appendage simultaneously.

We define a mapping $\phi: T^*G \times P \to \mathcal{G}^* \times P$ by

$$\phi(\alpha_g, x) = (TL_g^* \cdot \alpha_g, g^{-1} \cdot x). \tag{2.131}$$

This mapping transforms the conjugate momentum α_g and $x \in P$ to body representation. Furthermore, this mapping identifies the quotient manifold $(T^*G \times P)/G$ with $\mathcal{G}^* \times P$. A specific example is furnished by the example used in Krishnaprasad & Marsden. If we endow $\mathcal{G}^* \times P$ with the bracket

$$\{f, g\} = -\langle \mu, [\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}] \rangle + \{f, g\}_P - d_x f \cdot \left(\frac{\delta g}{\delta \mu}\right)_P + d_x g \cdot \left(\frac{\delta f}{\delta \mu}\right)_P. \tag{2.132}$$

Then $\mathcal{G}^* \times P$ is a Poisson manifold and ϕ a Poisson map. In addition, ϕ is G invariant and induces a Poisson diffeomorphism of $(T^*G \times P)/G$ with $\mathcal{G}^* \times P$.

In the case of a rigid body plus appendage, we can again define a Poisson structure on $T^*G \times P$. Here T^*G is the phase space associated with the rigid body and P is the phase space associated with the flexible appendage.

In this case Krishnaprasad & Marsden [1986] show that the appropriate bracket on the reduced space is

$$\begin{aligned}
\{f, g\} = -p \cdot (\nabla_m f \times \nabla_m g) &+ \int_0^L \left(\frac{\delta f}{\delta r} \cdot \frac{\delta g}{\delta m} - \frac{\delta f}{\delta m} \cdot \frac{\delta g}{\delta r}\right) ds \\
&+ \int_0^L \left[\frac{\delta g}{\delta r} \cdot \left(\nabla_m f \times r\right) + \frac{\delta g}{\delta m} \cdot \left(\nabla_m f \times m\right)\right] ds \\
&- \int_0^L \left[\frac{\delta f}{\delta r} \cdot \left(\nabla_m g \times r\right) + \frac{\delta f}{\delta m} \cdot \left(\nabla_m g \times m\right)\right] ds.
\end{aligned}$$

The reduced Hamiltonian, with the mass of the appendage neglected is

$$\mathbf{H} = \tfrac{1}{2}\mathbf{J}^{-1}\boldsymbol{p} \cdot \boldsymbol{p} + \tfrac{1}{2} \int_0^L \frac{\|\mathbf{m}(s)\|^2}{\rho_0} ds + \tfrac{1}{2} \int_0^L \mathbf{K}\frac{\partial \mathbf{r}}{\partial s} \cdot \frac{\partial \mathbf{r}}{\partial s} ds \tag{2.133}$$

from which we can compute the dynamics on the reduced phase space to be

$$\dot{\boldsymbol{p}} = \boldsymbol{p} \times J^{-1}\boldsymbol{p} - \int_0^L \mathbf{r} \times \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial s^2} ds \tag{2.134}$$

$$\dot{\mathbf{r}} = \rho_A \mathbf{m} + \mathbf{r} \times J^{-1}\boldsymbol{p} \tag{2.135}$$

$$\dot{\mathbf{m}} = \frac{\partial^2 \mathbf{r}}{\partial s^2} + \mathbf{m} \times J^{-1}\boldsymbol{p} \tag{2.136}$$

For the system examined by Krishnaprasad & Marsden a family of Casimir functions on the reduced space is given by

$$C = \tfrac{1}{2}\phi(\|\boldsymbol{p} + \int_0^L \mathbf{r} \times \mathbf{m}\, dS\|^2) \tag{2.137}$$

where $\phi(\cdot)$ is any differentiable function. Physically this corresponds to a function of the magnitude of the momentum vector of the system. Note that if we take the time derivative of C along a solution to the dynamics we find $\dot{C} = 0$, confirming that C is indeed a Casimir.

CHAPTER THREE

LINEARIZATION AND TRANSFER FUNTIONS

In this chapter we will develop linearized versions of some specific cases of the nonlinear models of the previous chapter. Linearized models are an important class, there being an extensive theory for the control of linear models which we can exploit. In particular, for linear models we can use Laplace transform techniques for the analysis and synthesis tasks in the frequency domain.

The linearization of a dynamic model takes place in a region local to an equilibrium of the system. In this region we consider the linear component of the vector field. In the case of continuum models our configurations take values on a suitable space of functions.

3.1. Equilibria and Relative Equalibria

An equilibrium point of a dynamical system corresponds to a fixed point in the phase space.

Definition. *Let ξ be a C^1 vector field on an n-dimensional manifold M. A point x_0 is called an* equilibrium point *of ξ if $\xi(x_0) = 0$. A* closed orbit *is an orbit $\gamma(t)$ for the vector field ξ when $\gamma(t)$ is not a fixed point and there is a $\tau > 0$ such that $\gamma(t+\tau) = \gamma(t)$ for all t.*

Sometimes equilibrium points are referred to as critical points or singular points. Closed orbits are also called limit cycles. It should be clear from the definition that if x_0 is an equilibrium point, then the flow F_t, leaves x_0 fixed, i.e. $F_t x_0 = x_0$. For a closed orbit, if $x_0 \in \gamma(t)$, then $F_t x_0 = F_{t+\tau} x_0$.

For the case of a rod this configuration space is the set of mappings taking points in $[0, L]$ into points on the differentable manifold $\mathbb{R}^3 \times SO(3)$. Thus, a point in $\mathcal{C}$ corresponds to a pair (ϕ, Λ). The tangent space to the rod at a particular point $T_{(\phi,\Lambda)}\mathcal{C}$ is the space of pairs $(\delta\phi, \delta\hat{\theta})$. These are functions taking $[0, L]$ into $\mathbb{R}^3 \times so(3)$.

We consider the cotangent bundle $T^*\mathcal{C}$ of the configuration which is a symplectic manifold. A point on this manifold corresponds to a configuration variable and a comomentum variable. Both, functions are defined on the interval $[0, L]$. If ξ_H is a Hamiltonian vector field on $T^*\mathcal{C}$ then if $H \neq 0$ the critical elements are closed orbits.

These notions can be generalized to that of an invariant set. In this case we let N be a submanifold of M. Then N is an *invariant set* if the flow of a vector field ξ leaves N invariant, i.e. if $x_0 \in N$ then $F_t(x_0) \in N$ for all t. Geometrically this is the case if and only if ξ is tangent to N.

We next consider the notion of an equilibrium on a reduced manifold. We have the following

Definition. *A point $x \in M$ is called* relative equilibrium *if $\pi_\mu(x) \in M_\mu$ is a fixed point for the induced Hamiltonian system ξ_{H_μ} on M_μ where $\mu = J(x)$.*

In other words $dH_\mu(\pi_\mu(x)) = 0$.

A useful criteria involving the Hamiltonian and a momentum mapping on M is the following (due to Souriau-Smale-Robbin);

Proposition. *Let the conditions of of the previous definition hold. Then $x \in J^{-1}(\mu)$ is a relative equilibrium if and only if x is a critical point of $H \times J: M \to I\!R \times \mathcal{G}^*$.*

Proof: See Abraham & Marsden [1978], p.307

3.2. Linearization of Continuum Models

Linearized systems are an important class of models for which a well developed theory exists. To correctly linearize a model we must first determine the associated equilibria about which to perform the linearization.

Given an equilibrium point x_0 and vector field ξ we have the following

Definition. *The* Linearization *of ξ at an* equilibrium point x_0 *is the linear map*

$$\xi'(x_0): T_{x_0}M \to T_{x_0}M$$

defined by

$$\xi'(x_0) \cdot v = \frac{d}{dt}(TF_t(x_0) \cdot v)\bigg|_{t=0}$$

where F_t is the flow of ξ at x_0.

For a finite dimensional system this corresponds to taking the linear component in a Taylor series.

In the case of a finite dimensional system the definition corresponds to the standard notion of linearization. For the infinite dimensional case we need the notion of a variational or Frechét derivative (see Marsden & Hughes [1983], p.183), a generalization of the ordinary derivative.

Let $\mathcal{X}$ be a Banach space and consider a function $f: \mathcal{X} \to \mathbb{R}$. The function f is said to be *Fréchet differentiable at a point* x, if for every $h \in \mathcal{X}$

$$\lim_{\epsilon \to 0} \frac{f(x + \epsilon h) - f(x)}{\epsilon} = \delta f(x; h) \tag{3.1}$$

exists and defines a linear bounded transformation (in h) mapping $\mathcal{X}$ into $\mathbb{R}$. Equation (3.1) defines the differential of f which we denote as $\delta f(x; h)$. Since the differential is a linear bounded transformation in h we can write

$$\langle \frac{\delta f}{\delta x}, h \rangle = \delta f(x; h) \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ is the norm defined on $\mathcal{X}$. We call $\frac{\delta f}{\delta x}$ the *Frechét derivative of f at the point $x \in \mathcal{X}$.*

3.3. Examples

In this section we will consider equilibria for the linear extensible shear beam and the nonshearable, inextensible rod model as well as linearized versions of these.

3.3.1. The Linear Extensible Shear Beam

Equations (2.135)-(2.137) describe the dynamics of the reduced model for the linear extensible shear-beam attached at the base to a rigid body. In these equations we assume that $\mathbf{J}$ is the inertia matrix of the rigid body and that ρ_A is the uniform mass per unit length of the attached appendage of length L. The configuration at any times is described by $\boldsymbol{p}$, the momentum vector of the rigid body; $\boldsymbol{r}(S)$, the displacement of the shear beam at a point S, $0 \leq S \leq L$; and $\boldsymbol{m}(S)$ the momentum density of shear beam at the point S. The equation at an equilibrium is

$$0 = \boldsymbol{p} \times \mathbf{J}^{-1}\boldsymbol{p} + \mathbf{a} \times \left.\frac{\partial \boldsymbol{r}}{\partial S}\right|_{S=0} - \left.\boldsymbol{r}\right|_{S=L} \times \mathbf{K}\mathbf{e}_2 + \int_0^L \frac{\partial \boldsymbol{r}}{\partial S} \times \mathbf{K}\frac{\partial \boldsymbol{r}}{\partial S}\, dS, \tag{3.3}$$

$$0 = \rho_A{}^{-1}\boldsymbol{m} + \boldsymbol{r} \times \mathbf{J}^{-1}\boldsymbol{p}, \tag{3.4}$$

$$0 = \mathrm{K}\frac{\partial^2 \boldsymbol{r}}{\partial S^2} + \boldsymbol{m} \times \mathbf{J}^{-1}\boldsymbol{p}. \tag{3.5}$$

Two boundary values are associated with these equations,

$$\left.\frac{\partial \boldsymbol{r}}{\partial S}\right|_{S=L} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2, \qquad \text{and} \qquad \left.\boldsymbol{r}\right|_{S=0} = \begin{bmatrix} 0 \\ a_2 \\ 0 \end{bmatrix} = \mathbf{a}. \tag{3.6}$$

For convenience we will let $\boldsymbol{\omega} = \mathbf{J}^{-1}\boldsymbol{r}$ and assume $\mathrm{K} = \mathrm{diag}(k_x\,,\, k_y\,,\, k_z)$. Physically $\boldsymbol{\omega}$ corresponds to the rotation rate vector of the rigid body. For the above equations we note that for any equilibrium a given $\boldsymbol{\omega}$ will uniquely specify the values of $\boldsymbol{r}$ and $\boldsymbol{m}$. This is easily seen by noting that for a fixed $\boldsymbol{p}$, equation (3.4) can be substituted into (3.5) to yield a single second order differential equation in $\boldsymbol{r}$ with the two boundary conditions specified. Thus

$$\mathrm{K}\frac{\partial^2 \boldsymbol{r}}{\partial S^2} = -\rho_A(\boldsymbol{r} \times \boldsymbol{\omega}) \times \boldsymbol{\omega}. \tag{3.7}$$

Solving the differential equation for $\boldsymbol{r}$ we get

$$\boldsymbol{r}(s) = \phi(S, \boldsymbol{\omega})\left.\frac{\partial \boldsymbol{r}}{\partial S}\right|_{S=0} + \left.\frac{\partial \phi(S, \boldsymbol{\omega})}{\partial S}\boldsymbol{r}\right|_{S=0}, \tag{3.8}$$

where $\phi(S, \boldsymbol{\omega})$ is computed from the solution of the matrix differential equation for $\boldsymbol{r}(S)$. Differentiating once

$$\frac{\partial \boldsymbol{r}(S)}{\partial S} = \frac{\partial \phi(S, \boldsymbol{\omega})}{\partial S}\left.\frac{\partial \boldsymbol{r}}{\partial S}\right|_{S=0} + \left.\frac{\partial^2 \phi(S, \boldsymbol{\omega})}{\partial S^2}\boldsymbol{r}\right|_{S=0}. \tag{3.9}$$

Evaluating at $S = L$, and using the boundary conditions we can solve for the value of the first derivative of $\boldsymbol{r}(S)$ evaluated at $S = 0$.

$$\left.\frac{\partial \boldsymbol{r}}{\partial S}\right|_{s=0} = \left[\left.\frac{\partial \phi(S, \boldsymbol{\omega})}{\partial S}\right|_{S=L}\right]^{-1}\left[\mathbf{e}_2 - \left.\frac{\partial^2 \phi(s, \boldsymbol{\omega})}{\partial S^2}\right|_{S=L}\mathbf{a}\right]. \tag{3.10}$$

We now have both the initial conditions, thus from (3.8) both $\boldsymbol{r}(S)$ and $\frac{\partial \boldsymbol{r}(S)}{\partial S}$ are completely specified. If we substitute these expressions into (3.3) we obtain an equation in $\boldsymbol{\omega}$. Solutions of this equation for $\boldsymbol{\omega}$ give the steady state rotation rates for the configuration. These can be used in equation (3.5) to compute $\boldsymbol{r}(S)$, and then $\boldsymbol{m}(S)$ via equation (3.4).

Our computation of equilibria has been based on solving a wave equation satisfied by $\boldsymbol{r}$, assuming that $\boldsymbol{\omega}$ is fixed, and then substituting the solution into (3.3) to obtain an expression for $\boldsymbol{\omega}$. In fact this is a special case of the eigenvalue problem associated with this configuration that we have solved.

We can recast our problem into the explicit form of an eigenvalue problem as follows. We have for the momentum

$$\boldsymbol{\alpha} = \boldsymbol{p} + \int_0^L \boldsymbol{r} \times \boldsymbol{m} \, dS \tag{3.11}$$

Using (3.3), $\boldsymbol{\omega} = \mathbf{J}\boldsymbol{p}$, and the vector identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ one finds

$$\boldsymbol{\alpha} = \mathbf{J}\boldsymbol{\omega} - \int_0^L \rho_A \boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\omega}) \, dS \tag{3.12}$$

$$= \mathbf{J}\boldsymbol{\omega} + \int_0^L \rho_A (\mathbf{1}\|\boldsymbol{r}\|^2 - \boldsymbol{r} \otimes \boldsymbol{r}) \, dS \boldsymbol{\omega} \tag{3.13}$$

$$= \mathbf{J}_\infty \boldsymbol{\omega} \tag{3.14}$$

where $\mathbf{J}_\infty = \int_0^L \rho_A (\mathbf{1}\|\boldsymbol{r}\|^2 - \boldsymbol{r} \otimes \boldsymbol{r}) \, dS$ is the augmented inertia matrix corresponding to the total inertia of the rigid body plus that of the deformed appendage.

Similarly, we can show

$$0 = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} - \int_0^L \boldsymbol{r} \times \frac{\partial^2 \boldsymbol{r}}{\partial S^2} \, dS \tag{3.15}$$

$$= \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \int_0^L \rho_A \boldsymbol{r} \times (\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) \, dS \tag{3.16}$$

$$= \mathbf{J}_\infty \boldsymbol{\omega} \times \boldsymbol{\omega} \tag{3.17}$$

From this last equation we conclude that $\mathbf{J}_\infty \boldsymbol{\omega} = \lambda \boldsymbol{\omega}$, where λ is a scalar number. But this is simply an eigenvalue problem

$$(\mathbf{J}_\infty - \lambda \mathbf{1})\boldsymbol{\omega} = 0 \tag{3.18}$$

Since the computation of $\mathbf{J}_\infty$ depends on a solution $\boldsymbol{r}$ of the wave equation,

$$\mathbf{K} \frac{\partial^2 \boldsymbol{r}}{\partial S^2} = \rho_A (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})) \tag{3.19}$$

in general the equation which we will need to solve will consist of three, coupled multinomial equations in ω_1, ω_2, and ω_3. There are however some fairly obvious cases where we can compute the equilibrium. These cases correspond to rotations about principal axes of inertia.

Assume that the linear extensible shear beam lies along the second principal axis. From geometric considerations the position of the shear beam will cause the principal axes of the rigid-body-shear-beam configuration to lie in the same directions as those of the rigid body. In this case the addition of the shear beam will have the effect of increasing the moments of inertia about the first and the third principal axes. Because the linear extensible shear beam cannot deflect laterally the principal axes of the of the configuration remain fixed for any longitudinal extension of the shear beam. Thus, for this configuration there are three axes about which the equilibria can exist. These axes will correspond to the three principal axes of the rigid body.

The simplest case is the first one to be considered. Here we assume that rotation takes place about the axis along which the linear extensible shear beam lies. This assumption is satisfied when $\omega_2 > 0$ and $\omega_1 = \omega_3 = 0$, from which $||\boldsymbol{\omega}|| = \omega_2$. In addition we are given the boundary conditions. We compute

$$\frac{\partial \boldsymbol{\phi}(S,\boldsymbol{\omega})}{\partial S} = \begin{bmatrix} \cos(\sqrt{\frac{\rho A}{k_x}}\omega_2 S) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(\sqrt{\frac{\rho A}{k_x}}\omega_2 S) \end{bmatrix} \tag{3.20}$$

and,

$$\frac{\partial^2 \boldsymbol{\phi}(S,\boldsymbol{\omega})}{\partial S^2} = \begin{bmatrix} -\sqrt{\frac{\rho A}{k_x}}\omega_2 \sin(\sqrt{\frac{\rho A}{k_x}}\omega_2 S) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{\rho A}{k_x}}\omega_2 \sin(\sqrt{\frac{\rho A}{k_x}}\omega_2 S) \end{bmatrix}. \tag{3.21}$$

Substituting these into the expression for the first partial of $\boldsymbol{r}$ at the boundary we obtain

$$\left.\frac{\partial \boldsymbol{r}}{\partial S}\right|_{S=0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{3.22}$$

from which it is readily apparent that

$$\boldsymbol{r}(S) = \begin{bmatrix} 0 \\ a_2 + S \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{m}(S) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.23}$$

Physically this corresponds to the linear extensible shear beam being unstretched by the rotation. We note that the total angular momentum is associated with the rigid body and determines the value of ω_2 by the relationship

$$\|p\| = j_{22}\omega_2. \tag{3.24}$$

For the second case, we consider rotations of the rigid-body-shear-beam configuration about the first or third principal axes of inertia. We consider the case when the rotation is about the first principal axis of inertia, and note that rotations about the third axis are similar.

Assume that $\omega_1 > 0$ and $\omega_2 = \omega_3 = 0$, then

$$\frac{\partial \phi(s,\omega)}{\partial S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\sqrt{\frac{\rho A}{k_y}}\omega_1 s) & 0 \\ 0 & 0 & \cos(\sqrt{\frac{\rho A}{k_z}}\omega_1 s) \end{bmatrix}. \tag{3.25}$$

Differentiating,

$$\frac{\partial^2 \phi(s,\omega)}{\partial s^2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{\frac{\rho A}{k_y}}\omega_1 \sin(\sqrt{\frac{\rho A}{k_y}}\omega_1 s) & 0 \\ 0 & 0 & -\sqrt{\frac{\rho A}{k_z}}\omega_1 \sin(\sqrt{\frac{\rho A}{k_z}}\omega_1 s) \end{bmatrix}. \tag{3.26}$$

Using these in the expression for the second partial we get,

$$\left.\frac{\partial r}{\partial S}\right|_{S=0} = \begin{bmatrix} 0 \\ \frac{1+a_2\sqrt{\frac{\rho A}{k_y}}\omega_1 \sin(\sqrt{\frac{\rho A}{k_y}}\omega_1 L)}{\cos(\sqrt{\frac{\rho A}{k_y}}\omega_1 L)} \\ 0 \end{bmatrix}. \tag{3.27}$$

We can now use equation (3.8) to compute $r(s)$,

$$r(S) = \begin{bmatrix} \frac{\sin(\sqrt{\frac{\rho A}{k_y}}\omega_1 S)}{\sqrt{\frac{\rho A}{k_y}}\omega_1} \begin{bmatrix} 0 \\ \frac{1+a_2\sqrt{\frac{\rho A}{k_y}}\omega_1 \sin(\sqrt{\frac{\rho A}{k_y}}\omega_2 L)}{\cos(\sqrt{\frac{\rho A}{k_y}}\omega_1 L)} \\ 0 \end{bmatrix} + a_2 \cos(\sqrt{\frac{\rho A}{k_y}}\omega_1 S) \end{bmatrix}. \tag{3.28}$$

This equation holds over the interval $0 \leq s \leq L$ and satisfies the given boundary conditions.

Using the value for the first derivative of r evaluated at $S = 0$ and computed above we find,

$$\mathbf{a} \times \left.\frac{\partial r(S)}{\partial S}\right|_{S=0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.29}$$

In the present case we have $\omega = [\omega_1 \quad 0 \quad 0]^T$ and it only remains to compute the vector $\boldsymbol{m}(S)$. From equation (3.4),

$$\boldsymbol{m}(S) = \begin{bmatrix} 0 \\ 0 \\ \rho_0\omega_1 \frac{\sin(\sqrt{\frac{\rho A}{k_y}}\omega_1 s)}{\sqrt{\frac{\rho A}{k_y}}\omega_1} \left[\frac{1+a_2\sqrt{\frac{\rho A}{k_y}}\omega_1 \sin(\sqrt{\frac{\rho A}{k_y}}\omega_2 L)}{\cos(\sqrt{\frac{\rho A}{k_y}}\omega_1 L)}\right] + a_2\cos(\sqrt{\frac{\rho A}{k_y}}\omega_1 S) \end{bmatrix}. \tag{3.30}$$

This corresponds to the example in Krishnaprasad & Marsden [1987].

We now consider computation of additional equilibria. Since we cannot in general solve them analytically we need to resort to the computer. In particular, graphical methods provide an excellent means of investigating more complicated equilibria than we can obtain analytically.

If we restrict ourselves to the case of rotation about one of the principal axes of the augmented inertia then (3.3) reduces to a single scalar equation. In the general case, the solution for (3.8) has the following matrix with $\beta_1 = \sqrt{\frac{\rho A}{k_x}}$, $\beta_2 = \sqrt{\frac{\rho A}{k_y}}$, and $\beta_3 = \sqrt{\frac{\rho A}{k_z}}$ then if we define

$$\phi_{ij} = \begin{cases} \dfrac{\omega_i\omega_j \sin(\beta_i\|\omega\|s) - \omega_i\omega_j\beta_i\|\omega\|s}{\beta_i\|\omega\|^3} & \text{if } i \neq j; \\ \dfrac{(\|\omega\|^2 - \omega_i^2)\sin(\beta_i\|\omega\|s) + \omega_i^2\beta_i\|\omega\|s}{\beta_i\|\omega\|^3} & \text{if } i = j, \end{cases}$$

we have

$$\phi(S,\omega) = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \tag{3.31}$$

With this expression we can explicitly compute $\mathbf{J}_\infty$ by means of numerical integration for any value of $\boldsymbol{\omega}$. More to the point we can compute the righthand side of equation (3.15). the set of equilibrium points then correspond to the zero elevation of this function. These can be searched for by optimization techniques, or displayed graphically.

This technique was used on a Symbolics 3745 Lisp machine using the symbolic computation package Macsyma. In several examples we explored more general cases of equilibria for the rotation vector $\boldsymbol{\omega}$ restricted to lie in the plane $\omega_3 = 0$.

Having computed equilibria for the linear extensible shear beam we next consider linearizing about these equilibria. We will assume that $\tilde{\boldsymbol{p}}$, $\tilde{\boldsymbol{r}}$, and $\tilde{\boldsymbol{m}}$ denote equilibrium values which satisfy (3.3)-(3.5). Letting $\boldsymbol{p}$, $\boldsymbol{m}$, and $\boldsymbol{r}$ denote the variations about the equilibrium, we have, from equation (2.135)

$$\dot{\boldsymbol{p}} = -\mathbf{J}^{-1}\tilde{\boldsymbol{p}} \times \boldsymbol{p} + \tilde{\boldsymbol{p}} \times \mathbf{J}^{-1}\boldsymbol{p} + \int_0^L (\mathrm{K}\frac{\partial^2 \tilde{\boldsymbol{r}}}{\partial S^2} - \tilde{\boldsymbol{r}} \times \frac{\partial^2 \boldsymbol{r}}{\partial S^2})\, dS, \tag{3.32}$$

from equation (2.136)

$$\dot{\boldsymbol{m}} = \mathrm{K}\frac{\partial^2 \boldsymbol{r}}{\partial S^2} - \mathbf{J}^{-1}\tilde{\boldsymbol{p}} \times \boldsymbol{m} + \tilde{\boldsymbol{m}} \times \mathbf{J}^{-1}\boldsymbol{p}, \tag{3.33}$$

and finally from equation (2.137)

$$\dot{\boldsymbol{r}} = \rho_A^{-1}\boldsymbol{m} - \mathbf{J}^{-1}\tilde{\boldsymbol{p}} \times \boldsymbol{r} + \tilde{\boldsymbol{r}} \times \mathbf{J}^{-1}\boldsymbol{p}. \tag{3.34}$$

If we let $S(x)$ denote the skew symmetric matrix of the vector x.

$$\mathrm{S}(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix},$$

then the linearized system is given by the operator $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} -S(\boldsymbol{J}^{-1}\tilde{\boldsymbol{p}}) & \rho_A^{-1}\mathbf{1} & S(\tilde{\boldsymbol{r}})\boldsymbol{J}^{-1} \\ \mathrm{K}\frac{\partial^2}{\partial S^2} & -S(\boldsymbol{J}^{-1}\tilde{\boldsymbol{p}}) & S(\tilde{\boldsymbol{m}})\boldsymbol{J}^{-1} \\ \mathbf{L} & 0 & -S(\boldsymbol{J}^{-1}\tilde{\boldsymbol{p}}) + S(\tilde{\boldsymbol{p}})\boldsymbol{J}^{-1} \end{bmatrix}, \tag{3.35}$$

where $\mathbf{L}\boldsymbol{r} = \int_0^L \mathrm{S}(\mathrm{K}\frac{\partial^2 \tilde{r}}{\partial S^2}\boldsymbol{r} - \mathrm{S}(\tilde{r})\frac{\partial^2 \tilde{r}}{\partial S^2}\boldsymbol{r})\, dS$. In a later section, we will use the resolvent of this operator to compute the transfer function for the system.

3.3.2. The Inextensible, Nonshearable Rod

We now proceed to investigate equilibria associated with the inextensible, nonshearable rod equation in the planar case. The equation associated with this model is (2.48). Clearly, one trivial equilibrium is that of $\alpha(S,t) = \alpha_0$, a constant. Physically this corresponds to a rod which is in an undeformed condition in ambient space.

A more interesting equilibrium can be found by letting $\alpha(S,t) = \omega t + \alpha_0$, corresponding to a constant rotation rate from an arbitrary angle. Clearly $\frac{\partial^2 \alpha}{\partial t^2} = 0$, and $\frac{\partial^2 \alpha}{\partial S^2} = 0$. Furthermore,

$$\begin{aligned} &\cos(\omega t + \alpha_0) \int_0^S \rho_A (\int_0^{\sigma_1} -\sin(\omega t + \alpha_0)\omega^2\, d\sigma_2)\, d\sigma_1 \\ &\quad + \sin(\omega t + \alpha_0) \int_0^S \rho_A (\int_0^{\sigma_1} \cos(\omega t + \alpha_0)\omega^2\, d\sigma_2)\, d\sigma_1 \\ &= (-\cos(\omega t + \alpha_0)\sin(\omega t + \alpha_0) + \sin(\omega t + \alpha_0)\cos(\omega t + \alpha_0)) \int_0^S \rho_A (\int_0^{\sigma_1} \omega^2\, d\sigma_2)\, d\sigma_1, \\ &= 0 \end{aligned} \tag{3.36}$$

from which we conclude that a constant velocity rotation from an arbitrary position is a relative equilibrium.

We next consider a special case for which $\alpha(S,t) = \alpha(S)$. In this case substitution into the equation for the dynamics yields

$$\begin{aligned} 0 &= EI\frac{\partial^2 \alpha}{\partial S^2} - \sin(\alpha)\int_0^S \rho_A \int_0^{\sigma_1} 0\, d\sigma_2\, d\sigma_1 \\ &\qquad + \cos(\alpha)\int_0^S \rho_A \int_0^{\sigma_1} 0\, d\sigma_2\, d\sigma_1, \\ &= EI\frac{\partial^2 \alpha}{\partial S^2}, \end{aligned} \tag{3.37}$$

from which we conclude, by twice integrating, $\alpha(S) = \alpha_0 + S\alpha_1$. To satisfy the boundary condition at the tip, $\frac{\partial \alpha}{\partial S} = 0$, we require $\alpha_1 = 0$. Thus, this reduces to the trivial case.

If we linearize about the equilibrium in the trivial case, $\alpha = \alpha_0$, we find

$$I_\rho \frac{\partial^2 \alpha(S,t)}{\partial t^2} = \int_0^S \rho_A \int_0^{\sigma_1} \frac{\partial^2 \alpha(\sigma_2, t)}{\partial t^2}\, d\sigma_2\, d\sigma_1 + EI\frac{\partial^2 \alpha(S,t)}{\partial S^2}. \tag{3.38}$$

Substituting $\frac{\partial u}{\partial S} = \alpha$ into this expression and differentiating once with respect to S we find

$$I_\rho \frac{\partial^4 u(S,t)}{\partial t^2 \partial S^2} = \rho_A \frac{\partial^2 u(S,t)}{\partial t^2} + EI\frac{\partial^4 u(S,t)}{\partial S^4}, \tag{3.39}$$

which is the classical Euler-Bernoulli beam with rotatory inertia. An important advantage of the formulation in (3.38) over the classical Euler-Bernoulli model is that by integrating we are dimensionally compatible with the rigid body models. Thus connected rigid bodies are naturally incorporated into this formulation. In addition, the integral form leads naturally to the construction of existence proofs for our solutions.

To linearize about the equilibrium corresponding to a constant rotation we first recall

$$\begin{aligned} I_\rho \frac{\partial^2 \alpha}{\partial t^2} &= \cos(\alpha)\int_0^S \rho_A \int_0^{\sigma_1} -\sin(\alpha)\left(\frac{\partial \alpha}{\partial t}\right)^2 + \cos(\alpha)\left(\frac{\partial^2 \alpha}{\partial t^2}\right) d\sigma_2\, d\sigma_1 \\ &\qquad + \sin(\alpha)\int_0^S \rho_A \int_0^{\sigma_1} \cos(\alpha)\left(\frac{\partial \alpha}{\partial t}\right)^2 + \sin(\alpha)\left(\frac{\partial^2 \alpha}{\partial t^2}\right) d\sigma_2\, d\sigma_1 + EI\frac{\partial^2 \alpha}{\partial S^2}. \end{aligned} \tag{3.40}$$

In this equation $\alpha(S,t)$ is an inertially referenced angle between an inertially fixed frame and the tangent to the line of centroids of the rod. In this case our equilibrium corresponds to a reference configuration rotating with constant angular rate ω. The line of centroids in this reference configuration makes an angle ωt at time t with respect to $\mathbf{E}_3$. At a point S along the rod the normal to the cross section makes an angle $\alpha(S,t) = \omega t + \phi(S,t)$ with respect to $\mathbf{E}_3$.

Upon substitution into (3.40) we get

$$\begin{aligned}
I_\rho(\frac{\partial^2 \phi}{\partial t^2}) = \cos(\omega t + \phi) \int_0^S \rho_A \int_0^{\sigma_1} & - \sin(\omega t + \phi)\left(\omega^2 + 2\omega \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial t}^2\right) \\
& + \cos(\omega t + \phi)\left(\frac{\partial^2 \phi}{\partial t^2}\right) d\sigma_2 \, d\sigma_1 \\
+ \sin(\omega t + \phi) \int_0^S \rho_A \int_0^{\sigma_1} & \cos(\omega t + \phi)\left(\omega^2 + 2\omega \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial t}^2\right) \\
& + \sin(\omega t + \phi)\left(\frac{\partial^2 \phi}{\partial t^2}\right) d\sigma_2 \, d\sigma_1 + EI\left(+\frac{\partial^2 \phi}{\partial S^2}\right).
\end{aligned} \tag{3.41}$$

Use of standard trigonometric identities yields

$$\begin{aligned}
I_\rho \frac{\partial^2 \phi}{\partial t^2} = \cos(\phi) \int_0^S \rho_A \int_0^{\sigma_1} & (\cos(\phi)\frac{\partial \phi^2}{\partial t^2} - \sin(\phi)\frac{\partial \phi}{\partial t}^2 \\
& - 2\omega \sin(\phi)\frac{\partial \phi}{\partial t} - \omega^2 \sin(\phi)) \, d\sigma_2 \, d\sigma_1 \\
+ \sin(\phi) \int_0^S \rho_A \int_0^{\sigma_1} & (\sin(\phi)\frac{\partial \phi^2}{\partial t^2} + \cos(\phi)\frac{\partial \phi}{\partial t}^2 \\
& + 2\omega \cos(\phi)\frac{\partial \phi}{\partial t} + \omega^2 \cos(\phi)) \, d\sigma_2 \, d\sigma_1 + EI\frac{\partial^2 \phi}{\partial S^2}.
\end{aligned} \tag{3.42}$$

If we take the linear component about the equilibrium we have

$$\begin{aligned}
I_\rho \frac{\partial^2 \phi(S,t)}{\partial t^2} = \int_0^S \rho_A \int_0^{\sigma_1} & \frac{\partial \phi(\sigma_2,t)^2}{\partial t} - \omega^2 \phi(\sigma_2,t) \, d\sigma_2 \, d\sigma_1 \\
& + \phi(S,t) \int_0^S \rho_A \int_0^{\sigma_1} \omega^2 \, d\sigma_2 \, d\sigma_1 + EI\frac{\partial^2 \phi(S,t)}{\partial S^2},
\end{aligned} \tag{3.43}$$

which is the linearized version of the dynamics. These can be related to the classical Euler-Bernoulli beam equations by the same technique as before. Substituting $\alpha = \frac{\partial u}{\partial S}$, differentiating once with respect to S, and rearranging we obtain

$$\begin{aligned}
\rho_A \frac{\partial^2 u(S,t)}{\partial t^2} + EI\frac{\partial^4 u(S,t)}{\partial S^4} & + \rho_A \omega^2 (S\frac{\partial u(S,t)}{\partial S} - u(S,t)) \\
& = I_\rho \frac{\partial^4 u(S,t)}{\partial t^2 \partial S^2} - \tfrac{1}{2}\rho_A \omega^2 S^2 \frac{\partial^2 u(S,t)}{\partial S^2},
\end{aligned} \tag{3.44}$$

This equation represents the dynamics of the rod with respect to the uniformly rotating reference frame. Such a model is of fundamental importance in application to rotating spacecraft with flexible appendages attached. We note the recent interest in related models as reflected in the papers of Kane,Ryan,Banerjee [1987], and Simo-VuQuoc [1986].

Physically, the additional terms introduce a stiffening effect in the dynamics of the rod. The two terms appearing on the right when the rod is rotating ($\omega \neq 0$) model the components of centrifugal force due respectively to the component in the $\mathbf{a}_3$ direction coupled into the lateral dynamics and that arising from the lateral displacement. Note that no Coriolis force appears since the axial displacement is assumed negligible and the component arising from $\frac{\partial u}{\partial t}$ is of second order. *Furthermore, the term on the right which appears in addition to the force due to the rotatory inertia is different from that of Simo and VuQuoc [1986] who do not prohibit extension.* We will return to this type of model in section 3.5.4.

3.4. Computation of Transfer Functions

Transfer functions relate the frequency response of a system to the response in the time domain by the use of Laplace transforms and associated complex variables methods. In the case of a finite dimensional linear system this technique is straight forward, the frequency response is related to the spectral theory of matrices. Here the domain of the operator is well defined and the spectrum consists of a finite number of isolated eigenvalues. Consequently, we are guaranteed the existence of the appropriate Laplace transforms and their inverses, an important consideration for successful control system design.

In the case of distributed parameter systems the situation is more complicated. The spaces and operators are now infinite dimensional and we need to be concerned with a number of issues which are taken for granted in the finite dimensional setting. In general, the spectrum associated with an infinite dimensional operator has continuous and discrete components. However, the operators which arise from our models will be shown to have compact resolvent. Operators of this class have spectra very similar to operators in a finite dimensional space. In particular these operators have a countable number of eigenvalues at isolated points in the complex plane. Such operators are common in mathematical physics, frequently arising from differential operators associated with boundary value problems.

Physically, the models we consider represent rigid bodies with flexible appendages. We can frequently partition such models into finite and infinite dimensional components, with the rigid bodies described in a finite dimensional space and the flexible components in an infinite dimensional space. Such a model takes the standard, abstract form

$$\frac{dx}{dt} = \mathcal{A}x + Bu, \qquad y = Cx \tag{3.45}$$

where $\mathcal{A}$, B, and C are given linear operators, x is the state of the system, u is a prescribed input, and y is the measured output. In the analysis of such a system, the properties of the operators provide important information about the solution, (see for example [Slemrod, 1987]).

The spaces we work with are a special class of the *Sobolev spaces* [Sobolev, 1938]. They will be denoted as $H^s(\Omega, \mathbb{R}^n)$ and defined as

$$H^s(\Omega, \mathbb{R}^n) = \{x \in C^\infty(\bar{\Omega}, \mathbb{R}^n) \mid \|x\|_{H^s} = \sqrt{\sum_{i=1}^{m} \|D^i x\|_{L^2}} < \infty\}$$

The space $H^s(\Omega, \mathbb{R}^n)$ is a Banach space. In fact, $H^s(\Omega, \mathbb{R}^n)$ is simply the subspace of $L^2(\Omega, \mathbb{R}^n)$ functions whose first s (generalized) derivatives also lie in $L^2(\Omega, \mathbb{R}^n)$. Note that $H^0(\Omega, \mathbb{R}^n) = L^2(\Omega, \mathbb{R}^n)$.

We can endow $H^s(\Omega, \mathbb{R}^n)$ with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{m} \int_\Omega D^i x \cdot D^i y \, d\Omega$$

in order to make it into a Hilbert space. The associated norm is precisely the "energy norm" used in the subsequent examples.

The systems we consider will have the property of skew adjointness of $\mathcal{A}$, that is $\mathcal{A}^* = -\mathcal{A}$. From Stone's theorem [c.f. Yosida, p.253] we know a skew adjoint operator is the generator of a C_0, unitary semigroup $U(t)$ on the associated Hilbert space.

There is a very close relationship between skew adjoint operators and Hamiltonian systems (see Chernoff & Marsden [1970]). Recall that a linear operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D}(\mathcal{A})$ is Hamiltonian if it is ω-skew, $\omega(\mathcal{A}x, y) = -\omega(x, \mathcal{A}y)$, for all $x, y \in$

$\mathcal{D}(\mathcal{A})$,with ω a symmetric bilinear form. Note that the Hamiltonian which is just the energy of the system in a configuration, is given by $H(\boldsymbol{x}) = \frac{1}{2}\omega(\mathcal{A}\boldsymbol{x}, \boldsymbol{x}), \boldsymbol{x} \in \mathcal{D}(\mathcal{A})$

The energy norm can be introduced as the norm associated with the Hilbert space $\mathcal{H}$. This norm is defined by the inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \omega(\mathcal{A}\boldsymbol{x}, \boldsymbol{y})$. Thus,

$$\begin{aligned} \langle \boldsymbol{x}, \mathcal{A}\boldsymbol{y} \rangle &= \omega(\mathcal{A}\boldsymbol{x}, \mathcal{A}\boldsymbol{y}) \\ &= -\omega(\mathcal{A}\boldsymbol{y}, \mathcal{A}\boldsymbol{x}) \\ &= -\langle \mathcal{A}\boldsymbol{x}, \boldsymbol{y} \rangle \end{aligned}$$

from which we conclude $\mathcal{A}^* = -\mathcal{A}$. Thus Hamiltonian systems give rise to skew adjoint operators.

We next turn our attention to the spectral analysis of the operator $\mathcal{A}$. From a theorem in Kato [1976] (see p.187) we know that if $\mathcal{A}$ is a closed operator with compact resolvent then the spectrum of $\mathcal{A}$ is discrete. It will consist of a countable number of isolated eigenvalues with finite multiplicities. The resolvent in this case can then be expressed as an infinite series of the form

$$(Is - \mathcal{A})^{-1}\boldsymbol{x} = \sum_{n=-\infty}^{\infty} \frac{1}{s - \lambda_n} \langle \phi_n, \boldsymbol{x} \rangle \phi_n, \tag{3.46}$$

where λ_n is the n^{th} eigenvalue and ϕ_n is the associated eigenfunction. From the expression for the resolvent we can write the transfer function

$$\begin{aligned} H(s) &= C(Is - \mathcal{A})^{-1}B \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{s - \lambda_n} C\langle \phi_n, B \rangle \phi_n \end{aligned} \tag{3.47}$$

It is often straight forward to show that $\mathcal{A}$ is closed. It is usually more difficult to show compactness of the resolvent. In our examples we can exploit the structure of the underlying Sobolev spaces to settle this.

An important fact related to Sobolev spaces is contained in the *Sobolev embedding theorem* (see Yoshida [1971], Adams, [1975]). In fact, the Sobolev embedding theorem arises as a consequence of the inequalities which also bear his name. While we will not explicitly use these inequalities; here we note that the Poincaré inequality used in the next chapter is a special case. (It is also, and perhaps more naturally, a special case of Gårdings inequality (see Marsden & Hughes [1983], p.324])).

The special case of the Sobolev embedding theorem we will use asserts that $H^\ell([0,L],\mathbb{R})$ is compactly embedded in $C^k([0,L],\mathbb{R})$ when $\ell > \frac{n}{2}+k$ (see, Marsden &Hughes [1983], p.326). From the definition of $H^k([0,L],\mathbb{R})$ it is clear that we also have $C^k([0,L],\mathbb{R}^n) \subset C^\ell([0,L],\mathbb{R}^n)$. From this we can conclude as a corollary to the embedding theorem that $H^\ell([0,L],\mathbb{R})$ is embedded compactly in $H^k([0,L],\mathbb{R})$ when $\ell > \frac{n}{2}+k$. We can use this result to establish that the resolvent of an operator $A: H^\ell \to H^k$ is compact.

Proposition. *Let A be an operator such that $A: H^\ell([0,L]\mathbb{R}^n) \to H^k([0,L],\mathbb{R}^n)$. If $\ell > \frac{n}{2}+k$ then the resolvent of A is compact in the norm on $H^\ell([0,L],\mathbb{R}^n)$.*

Proof: We have $(Is-A)^{-1}: H^k([0,L]\mathbb{R}^n) \to H^\ell([0,L],\mathbb{R}^n)$. Let $\{x_n\} \in H^k([0,L],\mathbb{R}^n)$ be a bounded, convergent sequence. Then we can write

$$\begin{aligned} \|(Is-A)^{-1}(x_n - x_m)\|_{H^k} &\le \|(Is-A)^{-1}\|_{H^k}\|(x_n-x_m)\|_{H^k} \\ &\le K\|(x_n-x_m)\|_{H^k} \qquad 0 < K < \infty \end{aligned}$$

where we have used the compact embedding to assert the existence of K, and the convergence of $\{x_n\}$ in $H^k([0,L],\mathbb{R}^n)$ to establish the result.

In fact, embeddings of this type are known as Hilbert-Schmidt embeddings, these embeddings play an important role in systems arising from differential operators (see Adams, [1975], p.173). More generally the Sobolev spaces and the associated inequalities are essential to the theory of partial differential equations (see for example, Trevés, [1975], section 24).

3.5. Some Examples

In the remainder of this chapter we will compute the transfer functions associated with the two models which we have discussed before. The first model is that of the linear extensible shear beam attached to a rigid body. The second is that of a planar inextensible, nonshearable rod attached to a rigid body and linearized about two different equilibria.

3.5.1. Rigid Body and Linear Extensible Shear Beam

The first example we consider is that of the linear extensible shear beam attached to a rigid body. We have discussed this model in several earlier sections, in particular we introduced a linearized version of this model in section 3.3.1. Recall that linearizing about an equilibrium, $(\tilde{p}, \tilde{r}, \tilde{m}) \in \mathbb{R}^3 \times H^1([0,L], \mathbb{R}^3) \times H^0([0,L], \mathbb{R}^3)$ we obtain the equation for the dynamics for $0 \leq S \leq L$.

$$\begin{bmatrix} \dot{r}(S,t) \\ \dot{m}(S,t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} -S(J^{-1}\tilde{p}) & \rho_A^{-1}\mathbf{1} & S(\tilde{r})J^{-1} \\ \mathrm{K}\frac{\partial^2}{\partial S^2} & -S(J^{-1}\tilde{p}) & S(\tilde{m})J^{-1} \\ \mathbf{L} & 0 & -S(J^{-1}\tilde{p}) + S(\tilde{p})J^{-1} \end{bmatrix} \begin{bmatrix} r(S,t) \\ m(S,t) \\ p(t) \end{bmatrix}. \tag{3.48}$$

where

$$\mathbf{L}r = \int_0^L (\mathrm{K}\frac{\partial^2 \tilde{r}}{\partial S^2} \times r - \tilde{r} \times \mathrm{K}\frac{\partial^2 r}{\partial S^2})\, dS.$$

In this equation $\tilde{p} \in \mathbb{R}^3$ is the rigid body momentum vector at the equilibrium, $\tilde{r}(\cdot) \in H^1([0,L], \mathbb{R}^3)$ is the configuration variable of the flexible appendage at this equilibrium, and finally $\tilde{m}(\cdot) \in H^0([0,L], \mathbb{R}^3)$ is the momentum density of the flexible appendage. Furthermore, J is the inertia matrix of the rigid body, K is a symmetric, positive definite, stiffness matrix, and ρ_A is the (uniform) mass density of the flexible appendage. $S(\cdot)$ is the skew symmetric cross product operator of the vector argument.

The elements $p(t)$, $r(S,t)$, and $m(S,t)$ of the state vector correspond to small excursions about the equilibrium points $\tilde{p}$, $\tilde{r}(S)$, and $\tilde{m}(S)$. Note that the appropriate boundary conditions for $r(S,t)$ are zero if we are to satisfy the boundary values of the original problem, thus $r(0,t) = 0$, and $\partial r(L,t)/\partial S = 0$.

The infinite dimensional component of the operator is of the general form,

$$\left[\frac{\partial}{\partial t} + \mathrm{S}(\omega)\right]^2 r(S,t) - \rho_A \frac{\partial^2 r(S,t)}{\partial S^2} = \mathrm{B}(S)\mathrm{p}(t). \tag{3.49}$$

This is a hyperbolic equation. It is exactly the classical wave equation when $\mathrm{S}(\omega) = 0$. In our case the additional component $\mathrm{S}(\omega)$ is due to the rotation of the entire configuration and the consequent forces which arise in the rotating frame of the flexible appendage. This introduces an effect similar to damping, but fundamentally different. As in the classical case the solution of such a system can be computed by separation of variables. Such an approach gives rise to the modal analysis techniques used extensively in the study of vibrations.

We next add a control torque to the rigid body and assume we can measure tip position. Let $m(t) \in \mathbb{R}^3$ denote the control torque. the measurement of tip position will be denoted $z(t) \in I\!R^3$ If we define

$$\mathcal{A} = \begin{bmatrix} -S(J^{-1}\tilde{p}) & \rho_A^{-1} & S(\tilde{r})J^{-1} \\ -\mathrm{K}\frac{d^2}{dS^2} & -S(J^{-1}\tilde{p}) & S(\tilde{r})J^{-1} \\ \mathrm{L} & 0 & -S(J^{-1}\tilde{p}) + S(\tilde{p})J^{-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [\quad 1|_{S=L} \qquad 0 \qquad 0]$$

where $x_1(t) = p(t)$, $x_2(S,t) = m(S,t)$, and $x_3(t) = p(t)$ then our linearized, distributed parameter system can be put in the form

$$\frac{dx}{dt} = \mathcal{A}x + Bm, \qquad\qquad z = Cx$$

Physically, x_1 corresponds to the displacement at S, x_2 the momentum density at S, and x_3 the rotational velocity of the base mass. In addition, at the base we have the compatibility condition $x_3(t) = J\rho_A^{-1}\partial x_2(S,t)/\partial S$ at $S = 0$. Associated with this we have the following Hilbert space

$$\mathcal{H} = \{(x_1, x_2, x_3) \in H^1([0,L], I\!R^3) \times L^2([0,L], I\!R^3) \times I\!R^3;\; x_1(0) = 0\},$$

with the inner product,

$$\langle x, y \rangle = \tfrac{1}{2}\int_0^L \left\{ \mathrm{K}\frac{dx_1(S)}{dS^2} \cdot \frac{dy_1(S)}{dS} + \rho_A x_2(S) \cdot y_2(S) \right\} dS + \tfrac{1}{2}J^{-1}x_3 \cdot y_3.$$

We note that this norm is the natural norm associated with the total energy of the system, (i.e. $H = \langle x, x \rangle$, see the expression for the Hamiltonian in section (2.6).)

The domain of definition of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = \{(x_1, x_2, x_3) \in H^1([0,L], I\!R^3) \times H^0([0,L], I\!R^3) \times I\!R^3;$$
$$x_1(0) = 0, \frac{dx_1(L)}{dS} = 0, \rho_A\frac{dx_2(0)}{dS} = Jx_3\},$$

where we have explicitly introduced two boundary conditions which are needed for x_1 to satisfy the boundary conditions on $\tilde{r}$. Additionally the compatibility condition is required to match the rigid body motion with that of the base of the appendage.

The dynamics of this system are clearly Hamiltonian having been originally formulated as a Hamiltonian system (see section 3.7). Furthermore, the linearized dynamics are again Hamiltonian (see Marsden-Hughes [1983]). We are therefore assured that $\mathcal{A}$ is a skew-adjoint operator, $\mathcal{A}^* = -\mathcal{A}$ which we are assured by the real version of Stone's theorem will be the generator for a unitary semigroup.

The resolvent can be constructed from the eigenvalues and eigenfunctions associated with $\mathcal{A}$. Hence, we look for solutions to the eigenvalue problem

$$\mathcal{A}\boldsymbol{x} = \lambda \boldsymbol{x}, \tag{3.50}$$

or more explicitly

$$-S(J^{-1}\tilde{p})x_1 + \rho_A^{-1}x_2 + S(\tilde{r})J^{-1}x_3 = \lambda x_1, \tag{3.51}$$

$$\mathrm{K}\frac{d^2x_1}{dS^2} - S(J^{-1}\tilde{p})x_2 + S(\tilde{m})J^{-1}x_3 = \lambda x_2, \tag{3.52}$$

$$\int_0^L (S(J^{-1}\tilde{p} \times \tilde{m})x_1 - S(\tilde{r})\mathrm{K}\frac{d^2x_1}{dS^2}$$
$$-S(J^{-1}\tilde{p})x_3 + S(\tilde{p})J^{-1}x_3 = \lambda x_3. \tag{3.53}$$

Eliminating x_2 from the first two equations we obtain

$$\mathrm{K}\frac{d^2x_1}{dS} - \rho_A(S(J^{-1}\tilde{p}) + \lambda)^2 x_1 = -\rho_A(S(J^{-1}\tilde{p}) + \lambda)S(\tilde{r})J^{-1}x_3 - S(\tilde{m})J^{-1}x_3.$$

This equation describes the dynamics associated with the flexible appendage.

Integrating the third equation by parts we have

$$(S(J^{-1}\tilde{p}) - S(\tilde{p})J^{-1} + \mathrm{I}\lambda)x_3 =$$
$$S(\mathrm{K}\frac{\partial \tilde{r}}{\partial S})x_1|_{S=L} + S(\tilde{r})\mathrm{K}\frac{\partial x_1}{\partial S}|_{S=0}$$
$$+ \int_0^L S(\frac{\partial \tilde{r}}{\partial S})\mathrm{K}\frac{\partial x_1}{\partial S} - S(\mathrm{K}\frac{\partial \tilde{r}}{\partial S})\frac{\partial x_1}{\partial S}\, dS.$$

which describes the dynamics associated with the rigid body connected to the base of the appendage.

If we restrict ourselves to the equilibrium computed in the earlier example for which K is diagonal and $k_2 = k_3$, with $r_1 = 0$, and $r_3 = 0$ then we have for all S.

$$S(\frac{\partial \tilde{r}}{\partial S})\mathrm{K}\frac{\partial x_1}{\partial S} = S(\mathrm{K}\frac{\partial \tilde{r}}{\partial S})\frac{\partial x_1}{\partial S},$$

and the integral term vanishes. In this case the dynamics of the rigid body are coupled with the rod through the boundary conditions.

3.5.2. Nonshearable, Inextensible Rod with No Rotatory Inertia

We now turn our attention to the state space formulation and the computation of transfer functions for a nonshearable, inextensible rod connected to a rigid body. In the initial case we will treat the rotatory inertia to be negligible and we assume our linearization is done about a nonrotating configuration. The control torque will be applied to the rigid body at the base, our measurement is assumed to be either the rotational velocity of the base or the acceleration at the tip of the appendage.

An alternative method one could follow is based on a technique of Mindlin [1950] for solving a beam equation with time varying functions at the boundary. In this case one considers the dynamics associated with the rigid body at the base to be the time dependent functions. In our case these correspond to the control torque as well as the reaction moment of the rigid body which show up in the boundary condition at the base. By a suitable transformation one can transform the system to a nonhomogeneous equation with stationary boundary conditions. The solution is then found by standard techniques. In fact this generalizes the method of separation of variables enabling us to explicitly solve the equations for the model of this section. In contrast, the technique we use is based on spectral analysis of the generator of a one parameter, unitary semigroup. This method makes clear the relationship between our model and the spectrum of the associated generator.

The rod equation, in Euler-Bernoulli form and neglecting rotatory inertia is described in Love [1944] . It is of course

$$EI\frac{\partial^4 u(S,t)}{\partial S^4} + \rho_A \frac{\partial^2 u(S,t)}{\partial t^2} = 0. \tag{3.54}$$

Associated with this are the four boundary conditions

$$\begin{aligned} u(0,t) &= 0, & \frac{\partial^2 u(L,t)}{\partial S^2} &= 0, \\ I_B \frac{\partial^3 u(0,t)}{\partial S \partial t^2} - EI\frac{\partial^2 u(0,t)}{\partial S^2} &= m(t), & \frac{\partial^3 u(L,t)}{\partial S^3} &= 0. \end{aligned}$$

The boundary conditions at the base correspond to the hinged condition and the dynamics of the rigid body. The boundary conditions at the tip reflect the absence of a force or moment. The tip position is given by $u(L,t)$, the tip acceleration by $\partial^2 u(L,t)/\partial t^2$, and the base rotation rate $\partial^2 u(0,t)/\partial S \partial t$.

The system can be recast in an infinite dimensional state space representation as follows; first define x_1, x_2, and x_3 as

$$x_1(S,t) = u(S,t), \qquad x_2(S,t) = \frac{\partial u(S,t)}{\partial S}, \qquad x_3(S,t) = \left.\frac{\partial^2 u(S,t)}{\partial S \partial t}\right|_{S=0}.$$

Clearly x_1 corresponds to the lateral displacement, x_2 the lateral velocity, and x_3 the angular velocity of the base mass. In addition, at the base we have the compatibility condition $x_3(t) = \partial x_2(S,t)/\partial t$ at $S = 0$. We define;

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{EI}{\rho_A}\frac{d^4}{dS^4} & 0 & 0 \\ \left.\frac{EI}{I_B}\frac{d^2}{dS^2}\right|_{S=0} & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_B} \end{bmatrix},$$

$$C = \begin{bmatrix} \left.-\frac{EI}{\rho_A}\frac{d^4}{dS^4}\right|_{S=L} & 0 & 0 \end{bmatrix} \qquad \text{(tip acceleration)},$$

or

$$C = [\ 0 \quad 0 \quad 1] \qquad \text{(base rotation rate)}.$$

Associated with this set up we have the following Hilbert space,

$$\mathcal{H} = \{(x_1, x_2, x_3) \in H^2(0,L) \times L^2(0,L) \times \mathbb{R};\ x_1(0) = 0\},$$

with the inner product,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \tfrac{1}{2} \int_0^L \left\{ EI \frac{d^2 x_1(S)}{dS^2} \frac{d^2 y_1(S)}{dS^2} + \rho_A x_2(S) y_2(S) \right\} dS + \tfrac{1}{2} I_B x_3 y_3.$$

We note that the norm is the natural norm associated with the total energy of the system, (i.e. $H = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$, the Hamiltonian of the system).

The domain of definition of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = \{(x_1, x_2, x_3) \in H^4(0,L) \times H^2(0,L) \times \mathbb{R};$$
$$x_1(0) = 0, \frac{d^2 x_1(L)}{dS^2} = 0, \frac{d^3 x_1(L)}{dS^3} = 0, \frac{dx_2(0)}{dS} = x_3\}.$$

Thus we can write the system in state space representation as

$$\frac{d\boldsymbol{x}}{dt} = \mathcal{A}\boldsymbol{x} + Bm, \qquad z = C\boldsymbol{x} \tag{3.55}$$

where $\mathcal{A}$,B, and C are as defined as above and $\boldsymbol{x} = (x_1,\, x_2,\, x_3)^T$, $m(t)$ is the torque applied to the base, and z is the measurement of tip acceleration.

It is a straight forward computation to show that the operator $\mathcal{A}$ is skew adjoint, $\mathcal{A} = -\mathcal{A}^*$. Thus

$$\begin{aligned}\langle \mathcal{A}\boldsymbol{x}, \boldsymbol{y}\rangle &= \tfrac{1}{2}\int_0^L EI\frac{d^2x_2}{dS^2}\frac{d^2y_1}{dS^2} + \rho_A\left(-\frac{EI}{\rho_A}\frac{d^4x_1}{dS^4}\right)y_2\,dS + \tfrac{1}{2}I_B\left(\frac{EI}{I_B}\frac{d^2x_1}{dS^2}\bigg|_{S=0}\right)y_3 \\ &= \tfrac{1}{2}EI\Bigg\{\frac{dx_2}{dS}\frac{d^2y_1}{dS^2}\bigg|_0^L - x_2\frac{d^3y_1}{dS^3}\bigg|_0^L + \int_0^L x_2\frac{d^4y_1}{dS^4}\,dS \\ &\quad - \frac{d^3x_1}{dS}y_2\bigg|_0^L + \frac{dx_1}{dS}\frac{dy_2}{dS}\bigg|_0^L - \int_0^L \frac{d^2x_1}{dS^2}\frac{d^2y_2}{dS^2}\,dS + \frac{d^2x_1}{dS^2}\bigg|_{S=0}y_3\Bigg\}.\end{aligned}$$

If we evaluate, rearrange, and use the compatibility condition we find;

$$\begin{aligned}\langle \mathcal{A}\boldsymbol{x}, \boldsymbol{y}\rangle = \tfrac{1}{2}EI\Bigg\{&\int_0^L \left\{\frac{d^2x_1}{dS^2}\left(-\frac{d^2y_2}{dS^2}\right) + x_2\left(\frac{d^4y_1}{dS^4}\right)\right\}dS + x_3\left(-\frac{d^2y_1(0)}{dS^2}\right) \\ &+ \frac{dx_2(L)}{dS}\frac{d^2y_1(L)}{dS^2} - x_2(L)\frac{d^3y_1(L)}{dS^3} + x_2(0)\frac{d^3y_1(0)}{dS^3} \\ &- \frac{d^3x_1(L)}{dS}y_2(L) + \frac{d^3x_1(0)}{dS}y_2(0) + \frac{d^2x_1(L)}{dS^2}\frac{dy_2(L)}{dS} + \frac{d^2x_1(0)}{dS^2}\left(-\frac{dy_2(0)}{dS} + y_3\right)\Bigg\}.\end{aligned}$$

To satisfy $\langle \mathcal{A}\boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, \mathcal{A}^*\boldsymbol{y}\rangle$ we require that the terms on the second and third lines all be zero. We can go through them term by term thus; Since $dx_2(L)/dS \neq 0$ we require that $d^2y_1(L)/dS^2 = 0$. The free tip means $x_2(L) \neq 0$, thus we require $d^3y_1(L)/dS^3 = 0$. We have $x_2(0)d^3y_1(0)/dS^3 = 0$ since $x_2(0) = 0$ Similarly, since $d^3x_1(L)/dS^3 = 0$ we have $d^3x_1(L)/dSy_2(L) = 0$. Next $d^3x_1(0)/dS^3 \neq 0$ requires $y_2(0)$. Since $d^2x_1(L)/dS^2 = 0$ we conclude that $d^2x_1(L)/dS^2 dy_2(L)/dS = 0$. Finally, $d^2x_1(0)/dS^2 \neq 0$ requires $dy_2(0)/dS = y_3$.

Note that we have that $y_2(0) = 0$. This implies that $y_1(0)$ is equal to a constant. However, for the domain to be a vector subspace we need this constant to be zero. Thus we replace $y_2(0) = 0$ by the more restrictive requirement that $y_1(0) = 0$ and find that the adjoint operator, defined on the same Hilbert space as before is

$$\mathcal{A}^* = \begin{bmatrix} 0 & -1 & 0 \\ \frac{EI}{\rho_A}\frac{d^4}{dS^4} & 0 & 0 \\ -\frac{EI}{I_B}\frac{d^2}{dS^2}\Big|_{S=0} & 0 & 0 \end{bmatrix},$$

with the associated domain

$$\mathcal{D}(\mathcal{A}^*) = \{(y_1, y_2, y_3) \in H^4(0,L) \times H^2(0,L) \times I\!R; \\ y_1(0) = 0, \frac{d^2 y_1(L)}{dS^2} = 0, \frac{d^3 y_1(L)}{dS^3} = 0, \frac{dy_2(0)}{dS} = y_3\}.$$

Thus we have established that $\mathcal{A}$ is skew adjoint. From (a real version of) Stone's theorem we then conclude that $\mathcal{A}$ is the generator of a one parameter group of unitary operators.

We next consider the spectral properties of the operator $\mathcal{A}$. We first observe that this operator is closed. Next we would like to show that it has compact resolvent. To show this we note that $\mathcal{D}(\mathcal{A})$ is embedded in $\mathcal{H}$ since $H^4(0,L) \times H^2(0,L)$ is embedded in $H^2(0,L) \times H^0(0,L)$. By the Sobolev embedding theorem (see Yosida [1971], p.174, or Adams [1975]), we know that this embedding is compact. Hence the resolvent $(sI - \mathcal{A})^{-1}$ is compact for any real s.

To find the transfer function associated with the system we will use spectral representation theory to compute the resolvent. We compute the eigenvalues and eigenfunctions of $\mathcal{A}$. The eigenvalues and eigenfunctions satisfy

$$(I\lambda - \mathcal{A})x = 0$$

from which x_1, x_2, and x_3 satisfy the following three equations for $0 < S < L$

$$x_2(S) = \lambda x_1(S),$$
$$-\frac{EI}{\rho_A}\frac{d^4 x_1(S)}{dS^4} = \lambda x_2(S),$$
$$\frac{EI}{I_B}\frac{d^2 x_1(0)}{dS^2} = \lambda x_3.$$

with the boundary conditions $x_1(0) = 0$, $d^2x_1(L)/dS^2 = 0$, $d^3x_1(L)/dS^3 = 0$, and the compatibility condition $dx_2(0)/dS = x_3$. Substituting the first equation into the second we get

$$-\frac{EI}{\rho_A}\frac{d^4 x_1(S)}{dS^4} = \lambda^2 x_1(S). \tag{3.56}$$

We already have three boundary conditions for this equation, we get a fourth by using the compatibility condition in the third equation

$$\frac{EI}{I_B}\frac{d^2 x_1(0)}{dS^2} = \lambda^2 \frac{dx_1(0)}{dS}.$$

The general solution for (3.56) is of the form

$$x_1(S) = \xi_1 \cos(\beta S) + \xi_2 \sin(\beta S) + \xi_3 \cosh(\beta S) + \xi_4 \sinh(\beta S). \tag{3.57}$$

Substituting this into (3.56) we conclude

$$\lambda^2 = -\frac{EI}{\rho_A}\beta^4. \tag{3.58}$$

Using the first boundary condition $x_1(0) = 0$ in (3.57), we have $\xi_1 = -\xi_3$ which enables us to eliminate ξ_3. Thus, we have

$$x_1(S) = \xi_1(\cos(\beta S) - \cosh(\beta S)) + \xi_2 \sin(\beta S) + \xi_4 \sinh(\beta S),$$

which we will differentiate three times

$$\frac{dx_1(S)}{dS} = -\xi_1\beta(\sin(\beta S) + \sinh(\beta S)) + \xi_2\beta\cos(\beta S) + \xi_4\beta\cosh(\beta S), \tag{3.59}$$

$$\frac{d^2x_1(S)}{dS^2} = -\xi_1\beta^2(\cos(\beta S) + \cosh(\beta S)) - \xi_2\beta^2\sin(\beta S) + \xi_4\beta^2\sinh(\beta S), \tag{3.60}$$

$$\frac{d^3x_1(S)}{dS^3} = \xi_1\beta^3(\sin(\beta S) - \sinh(\beta S)) - \xi_2\beta^3\cos(\beta S) + \xi_4\beta^3\cosh(\beta S). \tag{3.61}$$

The remaining boundary conditions can then be used with these expressions to give

$$-2\frac{EI}{I_B}\xi_1\beta = \lambda^2(\xi_2 + \xi_4), \tag{3.62}$$

$$0 = -\xi_1(\cos(\beta L) + \cosh(\beta L)) - \xi_2 \sin(\beta L) + \xi_4 \sinh(\beta L), \tag{3.63}$$

$$0 = \xi_1(\sin(\beta L) - \sinh(\beta L)) - \xi_2 \cos(\beta L) + \xi_4 \cosh(\beta L). \tag{3.64}$$

Using the expression for λ in terms of β and solving the first of these equations for ξ_1 we have

$$\xi_1 = \frac{I_B\beta^3}{2\rho_A}(\xi_2 + \xi_4).$$

Substitution into the remaining two equations gives two equations in ξ_2, and ξ_4

$$\begin{aligned} 0 &= -\frac{I_B\beta^3}{2\rho_A}(\xi_2 + \xi_4)(\cos(\beta L) + \cosh(\beta L)) - \xi_2 \sin(\beta L) + \xi_4 \sinh(\beta L), \\ &= -(\frac{I_B\beta^3}{2\rho_A}(\cos(\beta L) + \cosh(\beta L)) + \sin(\beta L))\xi_2 \\ &\qquad - (\frac{I_B\beta^3}{2\rho_A}(\cos(\beta L) + \cosh(\beta L)) - \sinh(\beta L))\xi_4, \end{aligned} \tag{3.65}$$

$$\begin{aligned} 0 &= \frac{I_B\beta^3}{2\rho_A}(\xi_2 + \xi_4)(\sin(\beta L) - \sinh(\beta L)) - \xi_2 \cos(\beta L) + \xi_4 \cosh(\beta L), \\ &= (\frac{I_B\beta^3}{2\rho_A}(\sin(\beta L) - \sinh(\beta L)) - \cos(\beta L))\xi_2 \\ &\qquad + (\frac{I_B\beta^3}{2\rho_A}(\sin(\beta L) - \sinh(\beta L)) + \cosh(\beta L))\xi_4. \end{aligned} \tag{3.66}$$

For these equations to have a nontrivial solution for ξ_2, and ξ_4 we require that the determinant be zero. Thus β satisfies

$$
\begin{aligned}
0 = -(&\frac{I_B\beta^3}{2\rho_A}(\cos(\beta L) + \cosh(\beta L)) + \sin(\beta L)) \\
&\cdot (\frac{I_B\beta^3}{2\rho_A}(\sin(\beta L) - \sinh(\beta L)) + \cosh(\beta L)) \\
- (&\frac{I_B\beta^3}{2\rho_A}(\cos(\beta L) + \cosh(\beta L)) - \sinh(\beta L)) \\
&\cdot (\frac{I_B\beta^3}{2\rho_A}(\sin(\beta L) - \sinh(\beta L)) - \cos(\beta L)).
\end{aligned}
$$

Expanding this and canceling terms we obtain

$$
\begin{aligned}
0 &= -\frac{I_B\beta^3}{2\rho_A}(\cos(\beta L) + \cosh(\beta L))(\cos(\beta L) + \cosh(\beta L)) \\
&\quad - \frac{I_B\beta^3}{2\rho_A}(\sin(\beta L) - \sinh(\beta L))(\sin(\beta L) - \sinh(\beta L)) \\
&\quad\quad - \sin(\beta L)\cosh(\beta L) + \sinh(\beta L)\cos(\beta L), \\
&= -\frac{I_B\beta^3}{\rho_A}(1 + \cos(\beta L)\cosh(\beta L)) \\
&\quad\quad - \sin(\beta L)\cosh(\beta L) + \sinh(\beta L)\cos(\beta L).
\end{aligned} \tag{3.67}
$$

In general there are a countable number of β which satisfy this equation, subsequently we will denote these by β_n, $n = 1, 2, \ldots$. The countable number of β_n give rise to the set of discrete eigenvalues, $\lambda_n = \pm i\sqrt{\frac{EI}{\rho_A}}\beta_n^2$.

We note that for $I_B = 0$ (3.67) reduces to

$$\sin(\beta_n L)\cosh(\beta_n L) = \sinh(\beta_n L)\cos(\beta_n L),$$

which is the equation for a hinged-free rod. For $I_B \to \infty$ it reduces to

$$-1 = \cosh(\beta_n L)\cos(\beta_n L),$$

the equation for a clamped-free rod.

Solving for ξ_2, and ξ_4 in a particular solution have

$$
\begin{aligned}
\xi_2 &= \frac{I_B\beta_n^3}{2\rho_A}(\sin(\beta_n L) - \sinh(\beta_n L) + \cosh(\beta_n L), \\
\xi_4 &= -\frac{I_B\beta_n^3}{2\rho_A}(\sin(\beta_n L) - \sinh(\beta_n L)) - \cos(\beta_n L).
\end{aligned}
$$

From which we can compute

$$\xi_1 = \frac{I_B\beta_n^3}{2\rho_A}(\cos(\beta_n L) + \cosh(\beta_n L)).$$

Substitution of these into the expression for the eigenfunctions gives

$$\begin{aligned}\phi_n(S) = \frac{I_B\beta_n^3}{2\rho_A}&(\cos(\beta_n L) + \cosh(\beta_n L))(\cos(\beta_n S) - \cosh(\beta_n S)) \\ &+ (\frac{I_B\beta_n^3}{2\rho_A}(\sin(\beta_n L) - \sinh(\beta_n L)) + \cosh(\beta_n L))\sin(\beta_n S) \\ &\quad - (\frac{I_B\beta_n^3}{2\rho_A}(\sin(\beta_n L) - \sinh(\beta_n L)) - \cos(\beta_n L))\sinh(\beta_n S). \quad (3.68)\end{aligned}$$

where we have denoted the lefthand side of (3.56) by $\phi_n(S)$.

From the above we conclude that the eigenvalues are given by

$$\lambda_n = \pm i\sqrt{\frac{EI}{\rho_A}}\beta_n^2, \qquad n = 1, 2, \ldots$$

and the associated eigenvectors by

$$\boldsymbol{x}^n = \begin{bmatrix} \phi_n(S) \\ \pm i\sqrt{\frac{EI}{\rho_A}}\beta_n^2\phi_n(S) \\ \pm i\sqrt{\frac{EI}{\rho_A}}\beta_n^2\frac{d\phi_n(0)}{dS} \end{bmatrix}. \quad (3.69)$$

Note that the spectrum is discrete as we expected.

We need to show that the set of eigenvectors form a complete, orthonormal set in $\mathcal{H}$. In this we follow standard techniques. For $\lambda = \lambda_n$ equation (3.56) is satisfied by ϕ_n, in which case if we multiply both sides by ϕ_m and integrate from 0 to L we get

$$\begin{aligned}-\lambda_n^2\int_0^L \phi_n(S)\phi_m(S)\,dS &= \int_0^L \frac{EI}{\rho_A}\frac{d^4\phi_n(S)}{dS^4}\phi_m(S)\,dS \\ &= \frac{EI}{\rho_A}\frac{d^2\phi_n(0)}{dS^2}\frac{d\phi_m(0)}{dS} - \int_0^L \frac{EI}{\rho_A}\frac{d^2\phi_n}{dS^2}\frac{d^2\phi_m}{dS^2}\,dS,\end{aligned}$$

where we have integrated by parts and exploited the boundary conditions. If we now interchange n and m and subtract

$$\begin{aligned}(\lambda_n^2 - \lambda_m^2)\int_0^L \phi_n(S)\phi_m(S)\,dS &= \frac{EI}{\rho_A}\frac{d^2\phi_n(0)}{dS^2}\frac{d\phi_m(0)}{dS} - \frac{EI}{\rho_A}\frac{d^2\phi_m(0)}{dS^2}\frac{d\phi_n(0)}{dS}, \\ &= (\lambda_n^2 - \lambda_m^2)\frac{I_B}{\rho_A}(\frac{d\phi_n(0)}{dS}\frac{d\phi_m(0)}{dS}.\end{aligned}$$

where we have used the fourth boundary condition. Thus, we conclude

$$\int_0^L \phi_n \phi_m \, dS - \frac{I_B}{\rho_A} \frac{d\phi_n(0)}{dS} \frac{d\phi_m(0)}{dS}; = 0 \qquad\qquad n \neq m,$$

and also

$$\int_0^L \frac{d^2\phi_n}{dS^2} \frac{d^2\phi_m}{dS^2} \, dS = 0; \qquad\qquad n \neq m.$$

From the above and the definition of the norm we conclude

$$\langle \boldsymbol{x}^n, \boldsymbol{x}^m \rangle = \begin{cases} c_n^2, & \text{if } n = m; \\ 0, & \text{if } n \neq m. \end{cases} \tag{3.70}$$

where c_n is a constant which can be used to normalize the eigenvectors.

We now have everything we need to compute the transfer functions associated with this model. In what follows we will compute two transfer functions, the first is an example of a colocated actuator and sensor, the second an example of a noncolocated actuator and sensor.

From spectral theory, we know that the semigroup associated with the generator $\mathcal{A}$ can be explicitly computed as

$$e^{\mathcal{A}t} \boldsymbol{x} = \sum_{n=-\infty}^{\infty} e^{\lambda_n t} \langle \boldsymbol{x}^n, \boldsymbol{x} \rangle \boldsymbol{x}^n,$$

where $\boldsymbol{x}^n$ is the eigenfunction associated with the n-th eigenvalue and $\boldsymbol{x}_0$ is an initial condition. An expression for the transfer function follows easily from the above

$$C(Is - \mathcal{A})^{-1} B \hat{m}(s) = \sum_{n=-\infty}^{\infty} \frac{1}{s - \lambda_n} (B^* \boldsymbol{x}^n)(C \boldsymbol{x}^n) \hat{m}(s). \tag{3.71}$$

We can use this expression to compute explicitly the transfer function for this example using the inner product on $\mathcal{H}$ and the eigenfunctions computed above.

Explicit computation of $e^{\mathcal{A}t}\boldsymbol{x}_0$ is as follows, we assume that $\boldsymbol{x}_0 = (x_0^{(1)}\, x_0^{(2)}\, x_0^{(3)})^T \in \mathcal{D}(\mathcal{A})$ is the initial condition, then from our definition of the norm

$$\begin{aligned} \langle \boldsymbol{x}^n, \boldsymbol{x}_0 \rangle &= \tfrac{1}{2} \int_0^L EI \frac{d^2\phi_n}{dS^2} \frac{d^2 x_0^{(1)}}{dS^2} + i\rho_A \Big(\sqrt{\frac{EI}{\rho_A}} \beta_n^2 \phi_n\Big) x_0^{(2)} \, dS \\ &\qquad\qquad + \tfrac{1}{2} i I_B \Big(\sqrt{\frac{EI}{\rho_A}} \beta_n^3 (\cos(\beta_n L) + \cosh(\beta_n L))\Big) x_0^{(3)}, \\ &= a_n + i b_n. \end{aligned}$$

where we have defined

$$a_n = \tfrac{1}{2}\int_0^L EI \frac{d^2\phi_n}{dS^2}\frac{d^2 x_0^{(1)}}{dS^2}\, dS, \tag{3.72}$$

$$b_n = \tfrac{1}{2}\int_0^L \rho_A(\sqrt{\frac{EI}{\rho_A}}\beta_n^2\phi_n)x_0^{(2)}\, dS. + \tfrac{1}{2}I_B(\sqrt{\frac{EI}{\rho_A}}\beta_n^3(\cos(\beta_n L) + \cosh(\beta_n L)))x_0^{(3)}. \tag{3.73}$$

We then have

$$e^{\lambda_n t}\langle \boldsymbol{x}^n, \boldsymbol{x}_0\rangle \boldsymbol{x}^n = (a_n + ib_n)(\cos(\lambda_n t) + i\sin(\lambda_n t))\boldsymbol{x}^n.$$

Note that $x_{-n} = \bar{x_n}$, and $\lambda_{-n} = \bar{\lambda_n}$, and furthermore $\boldsymbol{x}_0$ is real. Thus, we can write

$$\begin{aligned} \mathrm{e}^{\mathcal{A}t}\boldsymbol{x}_0 &= \sum_{n=1}^{\infty} \mathrm{Re}(\mathrm{e}^{\lambda_n t}\langle \boldsymbol{x}^n, \boldsymbol{x}_0\rangle \boldsymbol{x}^n), \\ &= \sum_{n=1}^{\infty} \begin{bmatrix} (a_n\cos(\lambda_n t) - b_n\sin(\lambda_n t))\phi_n(S) \\ -(b_n\cos(\lambda_n t) + a_n\sin(\lambda_n t))\sqrt{\frac{EI}{\rho_A}}\beta_n^2\phi_n(S) \\ -(b_n\cos(\lambda_n t) + a_n\sin(\lambda_n t))\sqrt{\frac{EI}{\rho_A}}\beta_n^3(\cos(\beta_n L) + \cosh(\beta_n L)) \end{bmatrix} \end{aligned} \tag{3.74}$$

We say that we have a colocated actuator and sensor if external base torque is the control and a measurement of rigid body base rotation rate is available. In this case the output will be $y(t) = x_3(t)$. The bounded operator C is then a matrix

$$C = [0 \quad 0 \quad 1].$$

Note that $B^* = I_B^{-1}C$, and in this sense the C matrix is adjoint to the B matrix, a general property associated with colocated actuators and sensors.

Using the expression for the transfer function (3.71) we can write

$$\begin{aligned} C(Is - \mathcal{A})^{-1}B &= \sum_{n=-\infty}^{\infty} \frac{1}{s-\lambda_n}\frac{1}{2I_B}(i\sqrt{\frac{EI}{\rho_A}}\frac{\beta_n^2}{c_n}\frac{d\phi_n(0)}{dS})(i\sqrt{\frac{EI}{\rho_A}}\frac{\beta_n^2}{c_n}\frac{d\phi_n(0)}{dS}), \\ &= \sum_{n=1}^{\infty} \frac{s}{s^2-\lambda_n^2}\frac{1}{I_B}(-\frac{EI}{\rho_A}\frac{\beta_n^4}{c_n^2}(\frac{d\phi_n(0)}{dS})^2), i \\ &= \sum_{n=1}^{\infty} \frac{s\lambda_n^2}{s^2 + \frac{EI}{\rho_A}\beta_n^4}\frac{\beta_n}{I_B c_n^2}(\cos(\beta_n L) + \cosh(\beta_n L))^2. \end{aligned}$$

which describes the effect of an input torque on the angular velocity of the rigid body at the base.

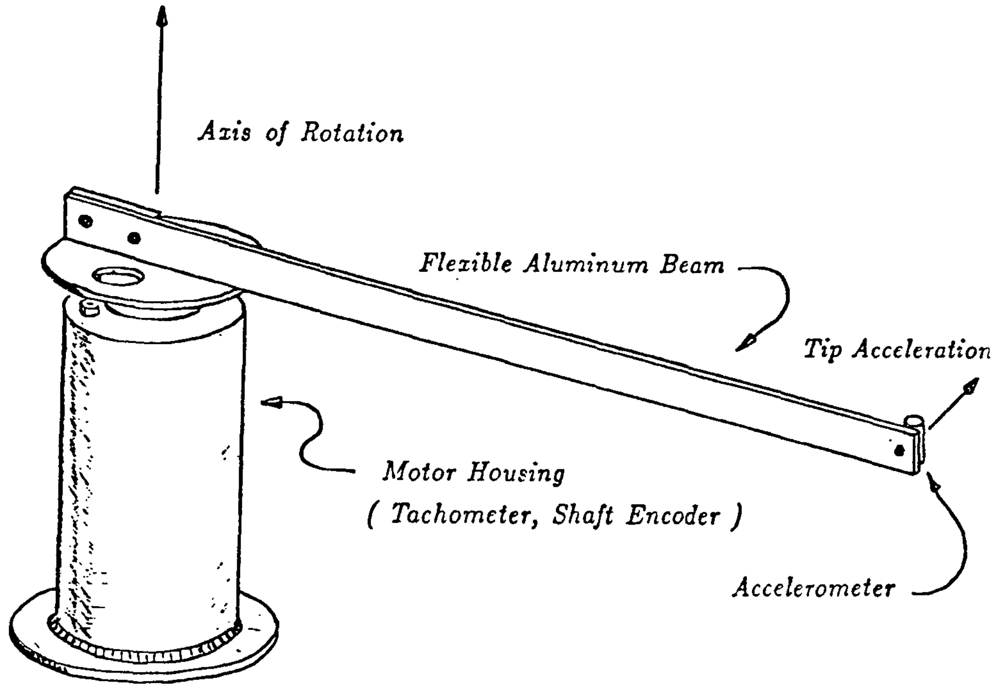

Figure 3.1. Planar Rigid Body and Flexible Appendage Experiment.

For the model in the case when the sensor is an accelerometer which is not colocated with the actuator we have a slightly more complicated model. In this case, for the accelerometer output

$$
\begin{aligned}
C(Is-A)^{-1}B &= \sum_{n=-\infty}^{\infty} \frac{1}{s-\lambda_n}\frac{1}{2I_B}\left(i\sqrt{\frac{EI}{\rho_A}}\frac{\beta_n^2}{c_n}\frac{d\phi_n(0)}{dS}\right)\lambda_n^2\phi_n(L) \\
&= \sum_{n=1}^{\infty} \frac{\lambda_n^4}{s^2+\frac{EI}{\rho_A}\beta_n^4}\frac{\beta_n}{I_B c_n^2}(\cos(\beta_n L)+\cosh(\beta_n L))\phi_n(L)
\end{aligned}
$$

where $\phi_n(L)$ is (3.68) evaluated at the $S = L$. This is the transfer function describing the response of the acceleration at the tip of the appendage to torques applied to the rigid body at the base.

Example of an Aluminum Beam and Hub

We can illustrate the model for this section by considering a laboratory experiment (see Frank [1986]). Consider an aluminum disk which has one degree of freedom so it can rotate about it's central axis. To this disk we attach (clamp) a long, flexible aluminum beam. This configuration is illustrated in figure 3.1.

The inertia of the disk, which we will subsequently call the *hub* is easily computed from its dimesnsions. This particular hub is of radius $r = 11.4\,cm$, with a thickness of $t = 0.87\,cm$. For aluminum, the weight is $w = 26.6\,kN/m^3$. The mass density can be computed from the weight, thus $\rho_V = 26.6 \times 10^3/9.806 = 2.71262 \times 10^3\,kg/m^3$. The total mass of the hub is therefore $M = \rho_V \pi r^2 t$, or $0.8971\,kg$. In this case, the inertia about the center is $I = M\frac{r^2}{2}$, which in our case can be computed to be $5.427 \times 10^{-3}\,\frac{kg}{m^3}$.

We next consider the parameters associated with the aluminum beam. For aluminum, the modulus of elasticity is $E = 71.0\,GPa$. The dimensions of our beam in cm are $0.30625 \times 4.826 \times 1000.0$ where we have assumed a length of 1 meter. The bending takes place in the $x - z$ plane.

The geometric inertia about the y-axis is computed

$$I = \int_A x^2\,dA = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\,dxdy = b\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\,dx = \frac{ba^3}{12}$$

Thus, for the dimensions of our beam

$$I = \frac{(4.826 \times 10^{-2})(3.0625 \times 10^{-3})^3}{12} = 1.155 \times 10^{-10}\,m^4$$

And it follows that, $EI = (71.0 \times 10^9)(1.155 \times 10^{-10}) = 8.201\,Nm^3$

The linear density of the aluminum is the mass density times the cross sectional area

$$\rho_A = \rho_V A = (2.713 \times 10^3)(3.0625 \times 10^{-3})(4.826 \times 10^{-2}) = 0.401\,kg/m$$

We have now computed the physical parameters we need. Note that in SI the units of the ratio $\frac{EI}{\rho}$ will be $\frac{Nm^3}{kg}$.

In table 3.1 we have tabulated the first few solutions to (A.7) for several values of hub inertia. Note that for large hub inertia the values compare favorably with that of a fixed free beam (although we still have the eigenvalue at zero associated with the hub). For negligible hub inertia, the values compare favorable with those of a hinged free beam. The values associated with the hub inertia in our case compare favorably with the empirically determined values reported by Frank.

Hub inertia, $I_B = 1.0 \times 10^{-6}$

$n = 1$	$\beta_n L = 0.0$	$\lambda_{\pm n} = 0.0$	$f = 0.0$ Hz
2	3.926	$\pm i$ 69.724	11.097
3	7.068	$\pm i$ 225.93	35.958
4	10.209	$\pm i$ 471.32	75.013
5	13.348	$\pm i$ 805.83	128.25

Hub inertia, $I_B = 5.4 \times 10^{-3}$

$n = 1$	$\beta_n L = 0.0$	$\lambda_{\pm n} = 0.0$	$f = 0.0$ Hz
2	3.550	$\pm i$ 57.008	9.073
3	5.419	$\pm i$ 132.79	21.134
4	8.021	$\pm i$ 290.97	46.309
5	11.054	$\pm i$ 552.55	87.941

Hub inertia, $I_B = 1.0 \times 10^{6}$

$n = 1$	$\beta_n L = 0.0$	$\lambda_{\pm n} = 0.0$	$f = 0.0$ Hz
2	1.875	$\pm i$ 15.901	2.531
3	4.694	$\pm i$ 99.647	15.859
4	7.854	$\pm i$ 279.01	44.407
5	10.996	$\pm i$ 546.76	87.019

Table 3.1. Rigid Body and Rod Modes

3.5.3. Nonshearable, Inextensible Rod with Rotatory Inertia

In this section we will consider transfer functions for the nonshearable, inextensible rod. Our model is that of a rod with a rigid body attached to the base. As in the previous section, we again assume the mass of the rod is small compared to that of the base mass. In this section we allow the rod to have nonzero cross section inertia and the associated rotatory inertia effects.

For this model we will assume that deformation is restricted to lie in a plane, control torques will be applied to the rigid body, measurements will be tangential tip acceleration. For this model, if we linearize first about the trivial equilibrium, we get;

$$I_\rho \frac{\partial^2 \alpha(S,t)}{\partial t^2} - I_\rho \frac{\partial^2 \alpha(0,t)}{\partial t^2} = \int_0^S \rho_A \int_0^{\sigma_1} \frac{\partial^2 \alpha(\sigma_2,t)}{\partial t^2} d\sigma_2 d\sigma_1$$

$$+ EI\frac{\partial^2\alpha(S,t)}{\partial S^2} - EI\frac{\partial^2\alpha(0,t)}{\partial S^2}, \quad (3.75)$$

with the associated boundary conditions at the base

$$I_B\frac{\partial^2\alpha(0,t)}{\partial t^2} - EI\frac{\partial\alpha(0,t)}{\partial S} = m(t), \quad (3.76)$$

and at the tip

$$I_\rho\frac{\partial^2\alpha(L,t)}{\partial t^2} - EI\frac{\partial^2\alpha(L,t)}{\partial S^2} = 0, \quad (3.77)$$

$$EI\frac{\partial\alpha(L,t)}{\partial S} = 0. \quad (3.78)$$

Note that the dynamics in the boundary condition at the base couple the rigid body to the rod. This equation couples the dynamics of the rigid body to the rod by balancing the moment at the base. An important feature of this model is that the assumption of no shear requires the cross sections to be perpendicular to the line of centroids of the rod. Consequently, the tangent vector to the line of centroids will be normal to the cross sections for our model.

The measurements at the tip of the rod will be tangential to the tip displacement (about the equilibrium). Thus

$$y(L,t) = \int_0^L \frac{\partial^2\alpha(\sigma,t)}{\partial t^2}\,d\sigma \quad (3.79)$$

represents the linearized acceleration measurement.

This system is equivalent to an Euler-Bernoulli type system with rotatory inertia which can be obtained by substituting $\alpha(S,t) = \partial u(S,t)/\partial S$ into (3.78) and differentiating once with respect to S. For such a system we pick up an additional boundary condition, $u(0,t) = 0$ and obtain a fourth order partial differential equation in S, and t,

$$EI\frac{\partial^4 u(S,t)}{\partial S^4} + \rho_A\frac{\partial^2 u(S,t)}{\partial t^2} = I_\rho\frac{\partial^4 u(S,t)}{\partial S^2\partial t^2}, \quad (3.80)$$

with the associated boundary conditions

$$u(0,t) = 0, \qquad \frac{\partial^2 u(L,t)}{\partial S^2} = 0,$$

$$I_B\frac{\partial^3 u(0,t)}{\partial S\partial t^2} - EI\frac{\partial^2 u(0,t)}{\partial S^2} = m(t), \qquad EI\frac{\partial^3 u(L,t)}{\partial S^3} = I_\rho\frac{\partial^3 u(L,t)}{\partial S\partial t^2}.$$

This system, along with the measurement equation can be recast in state space representation. We first define $x_1(S,t) = u(S,t)$, $x_2(S,t) = \frac{\partial u(S,t)}{\partial t}$, and $x_3(t) = \frac{\partial^2 u(S,t)}{\partial t \partial S}|_{S=0}$. We let these take values on the Hilbert space

$$\mathcal{H} = \{(x_1, x_2, x_3) \in H^2([0,L], I\!R) \times L^2([0,L], I\!R) \times I\!R; \quad x_1(0) = 0\},$$

endowed with the inner product,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \tfrac{1}{2} \int_0^L \left\{ EI \frac{d^2 x_1(S)}{dS^2} \frac{d^2 y_1(S)}{dS^2} + I_\rho \frac{dx_2(S)}{dS} \frac{dy_2(S)}{dS} + \rho_A x_2(S) y_2(S) \right\} dS + \tfrac{1}{2} I_B x_3 y_3$$

As before this inner product corresponds to the energy norm. The Hamiltonian for this system can be written as $H(x_1, x_2, x_3) = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$ where the terms of the integrand correspond respectively to the potential, rotational, and linear kinetic energies of a point on the rod. The last term outside the integral corresponds to the rotational kinetic energy of the rigid body.

Note that in general one needs an additional state to describe the dynamics of the cross section. However, in our case the condition of no shear reduces the cross section dynamics to an algebraic constraint. Consequently, the normal for each cross section is always parallel to the tangent to the line of centroids at each point on the line of centroids.

Equation (3.80) can be written in terms of the three equations,

$$\frac{\partial x_1}{\partial t} = x_2, \tag{3.81}$$

$$\left(1 - \frac{I_\rho}{\rho_A} \frac{\partial^2}{\partial S^2}\right) \frac{\partial x_2}{\partial t} = -\frac{EI}{\rho_A} \frac{\partial^4 x_1}{\partial S^4}, \tag{3.82}$$

$$\frac{dx_3}{dt} = \frac{EI}{I_B} \frac{\partial^2 x_1}{\partial S^2}\bigg|_{S=0} + \frac{1}{I_B} m(t). \tag{3.83}$$

This system can be written as the operator equation

$$\boldsymbol{\Gamma} \frac{d\boldsymbol{x}}{dt} = \mathbf{A}\boldsymbol{x} + \mathbf{B}m, \qquad \boldsymbol{z} = \mathbf{C}\boldsymbol{x}, \tag{3.84}$$

where the operators $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are as defined in (3.55). In addition we now have the operator, $\boldsymbol{\Gamma}: \mathcal{D}(\boldsymbol{\Gamma}) \rightarrow \mathcal{H}$ where

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{I_\rho}{\rho_A} \frac{d^2}{dS^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.85}$$

with the natural domain of definition

$$\mathcal{D}(\Gamma) = \{x \in L^2([0,L],I\!R) \times H^2([0,L],I\!R) \times I\!R; x_2(0) = 0, \frac{dx_2(0)}{dS} = x_3\}.$$

Note that $\mathcal{D}(\mathbf{A}) \subset \mathcal{D}(\Gamma)$ and Γ is a second order differential operator. In the domain of definition, two initial boundary conditions are specified for x_2. Such an initial value problem is always solvable in a unique way from which fact we infer that Γ^{-1} exists. Thus we can recast (3.84) in the form of (3.55) by defining $\tilde{\mathbf{A}} = \Gamma^{-1}\mathbf{A}$, and $\tilde{\mathbf{B}} = \Gamma^{-1}\mathbf{B}$.

We can take the inverse of the operator $(1 - \frac{I_\rho}{\rho_A}\frac{\partial^2}{\partial S^2})$ in order to move it to the right hand side of the second equation. Thus, for functions $f(S,t)$, and $g(S,t)$ we have

$$\begin{aligned}(1 - \frac{I_\rho}{\rho_A}\frac{\partial^2}{\partial S^2})^{-1} g(S,t) = &- \cosh(\sqrt{\frac{\rho_A}{I_\rho}}S) f(0,t) - \sqrt{\frac{I_\rho}{\rho_A}} \sinh(\sqrt{\frac{\rho_A}{I_\rho}}S)\frac{\partial f(0,t)}{\partial S} \\ &+ \int_0^S \sinh(\sqrt{\frac{\rho_A}{I_\rho}}(S-\sigma)) g(\sigma)\, d\sigma.\end{aligned}$$

The inverse operator is a convolution, an elementary result. Furthermore we can identify the kernel of the integrand $\sinh(\sqrt{\frac{\rho_A}{I_\rho}}(S-\sigma))$ as the *Green's function* associated with the differential operator Γ_{22}.

The boundary conditions associated with this system at the base remain the same as in the previous section. we recall these modeled the physical condition corresponding to no displacement, $x_1(0,t) = 0$, and the compatibility condition, $x_3 = \frac{\partial x_2(0,t)}{\partial S}$, coupling the rigid body to the base of the rod.

At the tip of the rod we still have the condition $\frac{\partial^2 x_1(L,t)}{\partial S^2} = 0$ because of the absence of any external force. For the equation modeling the moment at the tip we have

$$\begin{aligned}EI\frac{\partial^3 x_1(L,t)}{\partial S^3} &= I_\rho \frac{\partial}{\partial S}(\frac{\partial x_2(L,t)}{\partial t}) \\ &= I_\rho \frac{\partial}{\partial S}((1 - \frac{I_\rho}{\rho_A}\frac{\partial^2}{\partial S^2})^{-1}(-\frac{EI}{\rho_A}\frac{\partial^4 x_1(L,t)}{\partial S^4})),\end{aligned}$$

which is the remaining boundary condition.

We can recast this in the form of a state space representation by defining

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 \\ -(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{EI}{\rho_A}\frac{d^4}{dS^4} & 0 & 0 \\ \frac{EI}{I_B}\frac{d^2}{dS^2}\Big|_{S=0} & 0 & 0 \end{bmatrix}, \qquad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ ,0 \\ \frac{1}{I_B} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{EI}{\rho_A}\frac{d^4}{dS^4}\Big|_{S=L} & 0 & 0 \end{bmatrix} \qquad \text{(tip acceleration)},$$

with the associated domain of definition

$$\mathcal{D}(\mathbf{A}) = \{(x_1, x_2, x_3) \in H^4([0,L], I\!R) \times H^2([0,L], I\!R) \times I\!R;$$
$$x_1(0) = 0, \frac{dx_2(0)}{dS} = x_3, \frac{d^2x_1(L)}{d2^3} = 0,$$
$$\frac{\partial^3 x_1(L,t)}{\partial S^3} = -\frac{I_\rho}{\rho_A}\frac{d}{dS}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4}{dS^4})\}.$$

With the above definitions we have shown that the system can be put in the form of (3.55). The next question is whether or not $\tilde{\mathbf{A}}$ is the generator of a semigroup. As in the previous section our first step is to demonstrate that $\tilde{\mathbf{A}}$ is skew adjoint. As in the previous section we begin with $\langle \boldsymbol{x}, \tilde{\mathbf{A}}\boldsymbol{y}\rangle$, perform suitable manipulations to put this in the form $\langle \tilde{\mathbf{A}}^* \boldsymbol{x}, \boldsymbol{y}\rangle$, and from this identify the adjoint operator $\tilde{\mathbf{A}}^*$. Thus, using (3.81)-(3.83) in the expression for the inner product we have

$$\begin{aligned}
\langle \tilde{\mathbf{A}}\boldsymbol{x}, \boldsymbol{y}\rangle &= \tfrac{1}{2}\int_0^L EI\frac{d^2x_2}{dS^2}\frac{d^2y_1}{dS^2} + I_\rho\frac{d}{dS}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(-\frac{EI}{\rho_A}\frac{d^4x_1}{dS^4}))\frac{dy_2}{dS} \\
&\qquad + \rho_A(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(-\frac{EI}{\rho_A}\frac{d^4x_1}{dS^4})y_2\,dS + \tfrac{1}{2}I_B(\frac{EI}{I_B}\frac{d^2x_1}{dS^2}\bigg|_{S=0})y_3, \\
&= \tfrac{1}{2}EI\bigg\{\frac{dx_2}{dS}\frac{d^2y_1}{dS^2}\bigg|_0^L - x_2\frac{d^3y_1}{dS^3}\bigg|_0^L + \int_0^L x_2\frac{d^4y_1}{dS^4}\,dS \\
&\qquad + \frac{I_\rho}{\rho_A}\frac{dy_2}{dS}\frac{d}{dS}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(\frac{d^4x_1}{dS^4}))\bigg|_0^L - \frac{d^2x_1}{dS^2}\frac{dy_2}{dS}\bigg|_0^L \\
&\qquad + \int_0^L(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(\frac{d^4x_1}{dS^4}))(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(y_2))\,dS, \\
&= \tfrac{1}{2}EI\bigg\{\frac{dx_2}{dS}\frac{d^2y_1}{dS^2}\bigg|_0^L - x_2\frac{d^3y_1}{dS^3}\bigg|_0^L + \int_0^L x_2\frac{d^4y_1}{dS^4}\,dS \\
&\qquad - \frac{d^3x_1}{dS}y_2\bigg|_0^L + \frac{dx_1}{dS}\frac{dy_2}{dS}\bigg|_0^L - \int_0^L\frac{d^2x_1}{dS^2}\frac{d^2y_2}{dS^2}\,dS \\
&\qquad - \frac{I_\rho}{\rho_A}y_2\frac{d}{dS}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(\frac{d^4x_1}{dS^4}))\bigg|_0^L + \frac{d^2x_1}{dS^2}\bigg|_{S=0}y_3\bigg\},
\end{aligned}$$

where we have made extensive use of integration by parts, as well as the fact that

$$\begin{aligned}
\int_0^L(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})y_2(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4x_1}{dS^4} &= -\frac{I_\rho}{\rho_A}\frac{dy_2}{dS}(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4x_1}{dS^4}\bigg|_0^L \\
&\qquad - \frac{I_\rho}{\rho_A}y_2\frac{d}{dS}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4x_1}{dS^4})\bigg|_0^L + \int_0^L y_2\frac{d^4x_1}{dS^4}\,dS.
\end{aligned}$$

Furthermore, note that

$$\int_0^L\frac{d^4y_1}{dS^4}x_2\,dS = \int_0^L(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4y_1}{dS^4}x_2\,dS,$$

$$
\begin{aligned}
&= \int_0^L (1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4 y_1}{dS^4} x_2\, dS \\
&\qquad\qquad - \int_0^L \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4 y_1}{dS^4}) x_2\, dS, \\
&= \int_0^L (1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4 y_1}{dS^4} x_2\, dS - \frac{I_\rho}{\rho_A}\frac{d}{dS}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4 y_1}{dS^4} x_2\Big|_0^L \\
&\qquad\qquad + \int_0^L \frac{I_\rho}{\rho_A}\frac{d}{dS}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4 y_1}{dS^4}) x_2\, dS.
\end{aligned}
$$

Using this in the expression for $\langle \boldsymbol{x}, \tilde{\mathbf{A}}\boldsymbol{y}\rangle$, and rearranging slightly, we find

$$
\begin{aligned}
\langle \tilde{\mathbf{A}}\boldsymbol{x}, \boldsymbol{y}\rangle = \tfrac{1}{2}\int_0^L & EI\frac{d^2 x_1}{dS^2}\frac{d^2(-y_2)}{dS^2} + I_\rho \frac{d}{dS}\frac{dx_2}{dS}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(\frac{EI}{\rho_A}\frac{d^4 y_1}{dS^4})) \\
&+ \rho_A x_2 (1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(\frac{EI}{\rho_A}\frac{d^4 y_1}{dS^4})\, dS + \tfrac{1}{2} I_B x_3 (-\frac{EI}{I_B}\frac{d^2 y_1}{dS^2}\Big|_{S=0}) \\
+ \tfrac{1}{2}EI\Bigg\{ & \frac{dx_2}{dS}\frac{d^2 y_1}{dS^2}\Big|_{S=L} + \frac{d^2 x_1}{dS^2}\frac{d^2 y_2}{dS^2}\Big|_0^L + \frac{d^2 x_1}{dS^2}\Big|_{S=0} y_3 \\
&- (\frac{d^3 x_1}{dS} + (1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(\frac{EI}{\rho_A}\frac{d^4 x_1}{dS^4})) y_2\Big|_0^L \\
&\qquad - (\frac{d^3 y_1}{dS} + (1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(\frac{d^4 y_1}{dS^4})) x_2\Big|_0^L \Bigg\}.
\end{aligned}
$$

To satisfy $\langle \tilde{\mathbf{A}}\boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, \tilde{\mathbf{A}}^*\boldsymbol{y}\rangle$ we need to verify that the terms in braces in the above expression are zero. This can be done by using the boundary conditions and defining a suitable domain. For the first term in braces we note that at $S = L$, we require $d^2 y_1/dS^2 = 0$ to assure the term is zero. For the second term observe that $d^2 x_1/dS^2 = 0$ at $S = L$. At $S = 0$, we can combine it with the third term and require $y_3 = dy_2(0)/dS$. The fourth term can be made zero at $S = 0$ if we require $y_2(0) = 0$, at $S = L$ this term is zero because of the boundary condition modeling the moment at the tip. Conversely, the fifth term is zero at $S = 0$ since $x_2 = 0$, for $S = L$ this term becomes zero if we require $(\frac{d^3 y_1}{dS} = -(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(\frac{EI}{\rho_A}\frac{d^4 y_1}{dS^4}))$.

As before we note that we have that $y_2(0) = 0$. This implies that $y_1(0)$ is equal to a constant. However, for the domain to be a vector subspace we need this constant to be zero. Thus we replace $y_2(0) = 0$ by the more restrictive requirement that $y_1(0) = 0$

and find that the adjoint operator, defined on the same Hilbert space as before is

$$\tilde{\mathbf{A}}^* = \begin{bmatrix} 0 & -1 & 0 \\ (1-\frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{EI}{\rho_A}\frac{d^4}{dS^4} & 0 & 0 \\ -\frac{EI}{I_B}\frac{d^2}{dS^2}\Big|_{S=0} & 0 & 0 \end{bmatrix},$$

with the associated domain of definition

$$\begin{aligned} \mathcal{D}(\tilde{\mathbf{A}}^*) = \{(y_1, y_2, y_3) \in & H^4([0,L], I\!R) \times H^2([0,L], I\!R) \times I\!R; \\ & y_1(0) = 0, \frac{dy_2(0)}{dS} = y_3, \frac{d^2y_1(L)}{d2^3} = 0, \\ & EI\frac{\partial^3 y_1(L,t)}{\partial S^3} = -I_\rho\frac{d}{dS}((1-\frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{EI}{\rho_A}\frac{d^4y_1}{dS^4})\}. \end{aligned}$$

Thus we have established that $\tilde{\mathbf{A}}$ is skew adjoint. From Stone's theorem we then conclude that $\tilde{\mathbf{A}}$ is the generator of a unitary group.

We next turn our attention to the computation of the eigenvalues and eigenfunctions associated with this problem. These will subsequently be used when we employ the spectral theorem to compute the transfer function. For this case the eigenvalues and eigenfunctions satisfy

$$\lambda x_1 = x_2, \tag{3.86}$$

$$\lambda x_2 = -(1-\frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{EI}{\rho_A}\frac{d^4x_1}{dS^4}, \tag{3.87}$$

$$\lambda x_3 = \frac{EI}{I_B}\frac{d^2x_1}{dS^2}\Big|_{S=0}, \tag{3.88}$$

with the four associated boundary conditions;

$$x_1(0) = 0, \qquad \frac{d^3x_1(L)}{dS^3} = 0,$$

$$\frac{EI}{I_B}\frac{d^2x_1(0)}{dS^2} = \lambda^2\frac{dx_1(0)}{dS}, \qquad \frac{EI}{I_\rho}\frac{d^3x_1(L)}{dS^3} = \lambda^2\frac{dx_1(L)}{dS}.$$

The first three equations are the same as in the previous section, the fourth is obtained from (3.85), (3.86) and the boundary condition associated with the moment at the tip of the rod. Note that this condition is essentially the same as the boundary condition for the moment at the base of the rod.

From the first two equations associated with the eigenvalue problem we can obtain an expression for x_1

$$\lambda^2(x_1 - \frac{I_\rho}{\rho_A}\frac{d^2x_1}{dS^2}) = -\frac{EI}{\rho_A}\frac{d^4x_1}{dS^4} \tag{3.89}$$

This equation can be solved by assuming a solution of the form

$$x_1(S) = \xi_1\cos(\gamma_1 S) + \xi_2\sin(\gamma_1 S) + \xi_3\cosh(\gamma_2 S) + \xi_4\sinh(\gamma_2 S). \tag{3.90}$$

From the conditions at the base we can eliminate two of the ξ_i, thus

$$\begin{aligned} 0 &= x_1(S)\Big|_{S=0}, \\ &= \xi_1 + \xi_3. \end{aligned} \tag{3.91}$$

from which we conclude $\xi_3 = -\xi_1$. From the second condition at the base

$$\begin{aligned} 0 &= \lambda^2\frac{dx_1}{dS}\Big|_{S=0} - \frac{EI}{I_B}\frac{d^2x_1}{dS^2}\Big|_{S=0}, \\ &= \lambda^2(\xi_2\gamma_1 + \xi_4\gamma_2) + \frac{EI}{I_B}(\xi_1\gamma_1^2 + \xi_1\gamma_2^2), \end{aligned} \tag{3.92}$$

which gives an expression for ξ_1,

$$\xi_1 = -\frac{\lambda^2 I_B}{EI}(\frac{\xi_2\gamma_1 + \xi_4\gamma_2}{\gamma_1^2 + \gamma_2^2}). \tag{3.93}$$

Substituting this into the expression for x_1 and rearranging we find

$$\begin{aligned} x_1(S) = (&\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\cosh(\gamma_2 S) - \cos(\gamma_1 S)) + \sin(\gamma_1 S))\xi_2 \\ &+ (\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\cosh(\gamma_2 S) - \cos(\gamma_1 S)) + \sinh(\gamma_2 S))\xi_4. \end{aligned} \tag{3.94}$$

Now we can use the boundary conditions at the base to find ξ_2, ξ_4 in the above. We have,

$$\begin{aligned} 0 &= \frac{d^2x_1}{dS^2}\Big|_{S=L}, \\ &= (\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2^2\cosh(\gamma_2 L) + \gamma_1^2\cos(\gamma_1 L)) - \gamma_1^2\sin(\gamma_1 L))\xi_2 \\ &\quad + (\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2^2\cosh(\gamma_2 L) + \gamma_1^2\cos(\gamma_1 L)) + \gamma_2^2\sinh(\gamma_2 L))\xi_4, \end{aligned} \tag{3.95}$$

$$
\begin{aligned}
0 &= \left.\frac{d^3x_1}{dS^3}\right|_{S=L} - \left.\frac{\lambda^2 I_\rho}{EI}\frac{dx_1}{dS}\right|_{S=L}, \\
&= (\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2^3\sinh(\gamma_2 L) - \gamma_1^3\sin(\gamma_1 L)) - \gamma_1^3\cos(\gamma_1 L))\xi_2 \\
&\quad + (\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2^3\sinh(\gamma_2 L) - \gamma_1^3\sin(\gamma_1 L)) + \gamma_2^3\cosh(\gamma_2 L))\xi_4 \\
&\quad - \frac{\lambda^2 I_\rho}{EI}\Big\{(\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2\sinh(\gamma_2 L) + \gamma_1\sin(\gamma_1 L)) + \gamma_1\cos(\gamma_1 L))\xi_2 \\
&\quad + (\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2\sinh(\gamma_2 L) + \gamma_1\sin(\gamma_1 L)) + \gamma_2\cosh(\gamma_1 L))\xi_4\Big\}.
\end{aligned}
\tag{3.96}
$$

These equations have a nontrivial solution for ξ_2, and ξ_4 when the determinant of the coefficient matrix is zero. Thus, we require

$$
\begin{aligned}
0 &= (\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2^2\cosh(\gamma_2 L) + \gamma_1^2\cos(\gamma_1 L)) - \gamma_1^2\sin(\gamma_1 L)) \\
&\quad \Big\{(\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2^3\sinh(\gamma_2 L) - \gamma_1^3\sin(\gamma_1 L)) + \gamma_2^3\cosh(\gamma_2 L)) \\
&\quad\quad - \frac{\lambda^2 I_\rho}{EI}(\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2\sinh(\gamma_2 L) + \gamma_1\sin(\gamma_1 L)) + \gamma_2\cosh(\gamma_1 L))\Big\} \\
&\quad - (\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2^2\cosh(\gamma_2 L) + \gamma_1^2\cos(\gamma_1 L)) + \gamma_2^2\sinh(\gamma_2 L)) \\
&\quad \Big\{(\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2^2\cosh(\gamma_2 L) + \gamma_1^2\cos(\gamma_1 L)) + \gamma_2^2\sinh(\gamma_2 L)) \\
&\quad\quad - \frac{\lambda^2 I_\rho}{EI}(\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2\sinh(\gamma_2 L) + \gamma_1\sin(\gamma_1 L)) + \gamma_1\cos(\gamma_1 L))\Big\}, \\
&= \frac{\lambda^2 I_B\gamma_1\gamma_2}{EI(\gamma_1^2+\gamma_2^2)}\Big\{\gamma_1^4 + \gamma_2^4 + 2\gamma_1^2\gamma_2^2\cosh(\gamma_2 L)\cos(\gamma_1 L)\gamma_1\gamma_2(\gamma_1^2-\gamma_2^2)\sinh(\gamma_2 L)\sin(\gamma_1 L)\Big\} \\
&\quad - \gamma_1^2\gamma_2^2(\gamma_2\cosh(\gamma_2 L)\sin(\gamma_1 L) + \gamma_1\cos(\gamma_1 L)\sinh(\gamma_2 L)) \\
&\quad - \frac{\lambda^2 I_\rho}{EI}\frac{\lambda^2 I_B\gamma_1\gamma_2}{EI(\gamma_1^2+\gamma_2^2)}\Big\{\gamma_2^2 - \gamma_1^2 + (\gamma_1^2-\gamma_2^2)\cosh(\gamma_2 L)\cos(\gamma_1 L) \\
&\quad\quad - \gamma_1\gamma_2\sinh(\gamma_2 L)\sin(\gamma_1 L)\Big\} \\
&\quad - \gamma_1\gamma_2(\gamma_1\cosh(\gamma_2 L)\sin(\gamma_1 L) - \gamma_2\cos(\gamma_1 L)\sinh(\gamma_2 L)).
\end{aligned}
\tag{3.97}
$$

Since this is an equation in three unknowns λ, γ_1, and γ_2 we need two more equations. These equations are obtained by substituting the expression for the eigenfunctions into the original expression for x_1 and equating coefficients. Doing this we obtain

$$0 = \lambda^2\rho_A + \lambda^2 I_\rho\gamma_1^2 + EI\gamma_1^4, \tag{3.98}$$

$$0 = \lambda^2\rho_A - \lambda^2 I_\rho\gamma_1^2 + EI\gamma_1^4, \tag{3.99}$$

from which

$$\lambda^2 \frac{I_\rho}{EI} = \gamma_2^2 - \gamma_1^2, \tag{3.100}$$

$$-\lambda^2 \frac{\rho_A}{EI} = \gamma_2^2\gamma_1^2, \tag{3.101}$$

thus γ_1 and γ_2 are related to each other as

$$\rho_A(\gamma_1^2 - \gamma_2^2) = I_\rho \gamma_1^2 \gamma_2^2. \tag{3.102}$$

The eigenvalues can be computed from (3.96), (3.99), and (3.100). Note that for I_ρ, we have $\gamma_1^2 = \gamma_1^2$, in which case (3.96) reduces to the expression of the previous section.

From expression (3.100) we have that

$$\lambda = \pm i\gamma_1\gamma_2\sqrt{\frac{EI}{\rho_A}}, \tag{3.103}$$

which is the expression for the eigenvalues, in this case all of which lie on the imaginary axis. This expression is analogous to (3.67). As before we denote the n^{th} eigenvalue by λ_n and similarly denote $\gamma_{1,n}$ and $\gamma_{2,n}$.

From equations (3.95) and (3.96), along with (3.97), we can solve for ξ_2, and ξ_4,

$$\begin{aligned}\xi_2 = &(\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2^3\sinh(\gamma_2 L) - \gamma_1^3\sin(\gamma_1 L)) + \gamma_2^3\cosh(\gamma_2 L))\\ &\quad - \frac{\lambda^2 I_\rho}{EI}(\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2\sinh(\gamma_2 L) + \gamma_1\sin(\gamma_1 L)) + \gamma_2\cosh(\gamma_1 L)),\\ = &-(\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2^2\cosh(\gamma_2 L) + \gamma_1^2\cos(\gamma_1 L)) + \gamma_2^2\sinh(\gamma_2 L)),\end{aligned} \tag{3.104}$$

$$\begin{aligned}\xi_4 = &-(\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2^3\sinh(\gamma_2 L) - \gamma_1^3\sin(\gamma_1 L)) - \gamma_1^3\cos(\gamma_1 L))\\ &\quad + \frac{\lambda^2 I_\rho}{EI}(\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2\sinh(\gamma_2 L) + \gamma_1\sin(\gamma_1 L)) + \gamma_1\cos(\gamma_1 L)),\\ = &(\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2^2\cosh(\gamma_2 L) + \gamma_1^2\cos(\gamma_1 L)) - \gamma_1^2\sin(\gamma_1 L)).\end{aligned} \tag{3.105}$$

Using these in the expression for x_1, the first component of the eigenfunction we find

$$\begin{aligned}x_1(S) = &(\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\cosh(\gamma_2 S) - \cos(\gamma_1 S)) + \sin(\gamma_1 S))\\ &\quad(-\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\gamma_2^2\cosh(\gamma_2 L) + \gamma_1^2\cos(\gamma_1 L)) - \gamma_2^2\sinh(\gamma_2 L))\\ &+ (\frac{\lambda^2 I_B}{EI}\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}(\cosh(\gamma_2 S) - \cos(\gamma_1 S)) + \sinh(\gamma_2 S))\\ &\quad(\frac{\lambda^2 I_B}{EI}\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}(\gamma_2^2\cosh(\gamma_2 L) + \gamma_1^2\cos(\gamma_1 L)) - \gamma_1^2\sin(\gamma_1 L)).\end{aligned} \tag{3.106}$$

From the equations corresponding to the eigenvalue problem we can now obtain an explicit expression for the elements of the eigenfunction;

$$x_2^n(S) = \pm i\gamma_{1,n}\gamma_{2,n}\sqrt{\frac{EI}{\rho_A}}x_1^n(S),$$

$$x_3^n(S) = \pm i(\gamma_{1,n}\gamma_{2,n})^2\sqrt{\frac{EI}{\rho_A}}(\gamma_{2,n}\sinh(\gamma_{2,n}L) - \gamma_{1,n}\sin(\gamma_{1,n}L)).$$

We can now proceed to compute the transfer function for the case of the colocated actuator and sensor as in the previous section.

Recall that

$$\mathbf{C}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{B} = \sum_{n=-\infty}^{\infty} \frac{(\mathbf{B}^*\boldsymbol{x}^n)(\mathbf{C}\boldsymbol{x}^n)}{s - \lambda_n}. \tag{3.107}$$

We compute

$$\mathbf{B}^*\boldsymbol{x}^n = \frac{1}{I_B}\frac{EI}{I_B\lambda_n}\frac{d^2x_1(0)}{dS^2},$$

$$\mathbf{C}\boldsymbol{x}^n = \frac{EI}{I_B\lambda_n}\frac{d^2x_1(0)}{dS^2}.$$

Using the conjugacy of the eigenfunctions $\mathrm{Re}\boldsymbol{x}^n = \mathrm{Re}\boldsymbol{x}^{-n}$ and $\mathrm{Im}\boldsymbol{x}^n = -\mathrm{Im}\boldsymbol{x}^{-n}$ we can write

$$\begin{aligned}\mathbf{C}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{B} &= \sum_{n=-\infty}^{\infty} \frac{(\mathbf{B}^*\boldsymbol{x}^{-n})(\mathbf{C}\boldsymbol{x}^{-n})}{2(s - \lambda_{-n})} + \frac{(\mathbf{B}^*\boldsymbol{x}^n)(\mathbf{C}\boldsymbol{x}^n)}{2(s - \lambda_n)}, \\ &= \sum_{n=1}^{\infty} \frac{(\mathbf{B}^*\boldsymbol{x}^n)(\mathbf{C}\boldsymbol{x}^n)s}{s^2 - \lambda_n^2}.\end{aligned}$$

3.5.4. Nonshearable, Inextensible Rod in Rotating Configuration

If we assume that the rigid body with the nonshearable, inextensible rod is linearized about a reference moving with a constant rate of rotation. From previous computations we know that the equation in this case satisfies

$$\begin{aligned}I_\rho\frac{\partial^2\phi}{\partial t^2} = \int_0^S \rho_A \int_0^{\sigma_1} (\frac{\partial^2\phi}{\partial t^2} - \omega^2\phi)\, d\sigma_2\, d\sigma_1 \\ + \phi\int_0^S \rho_A \int_0^{\sigma_1} \omega^2\phi\, d\sigma_2\, d\sigma_1 + EI\frac{\partial^2\phi}{\partial S^2},\end{aligned} \tag{3.108}$$

with the associated boundary conditions at the base

$$I_B \frac{\partial^2 \phi(0,t)}{\partial t^2} - EI \frac{\partial \phi(0,t)}{\partial S} = m(t), \tag{3.109}$$

and at the tip

$$I_\rho \frac{\partial^2 \phi(L,t)}{\partial t^2} - EI \frac{\partial^2 \phi(L,t)}{\partial S^2} = 0, \tag{3.110}$$

$$EI \frac{\partial \phi(L,t)}{\partial S} = 0. \tag{3.111}$$

By substituting $\phi = \frac{\partial u}{\partial S}$, differentiating once with respect to S, and rearranging this can be put in the familiar form

$$\rho_A \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial S^4} + \rho_A \omega^2 (S \frac{\partial u}{\partial S} - u) = I_\rho \frac{\partial^4}{\partial S^2 \partial t^2} - \rho_A \omega^2 \tfrac{1}{2} S^2 \frac{\partial^2 u}{\partial S^2},$$

with the associated boundary conditions

$$u(0,t) = 0, \qquad \frac{\partial^2 u(L,t)}{\partial S^2} = 0,$$

$$I_B \frac{\partial^3 u(0,t)}{\partial S \partial t^2} - EI \frac{\partial^2 u(0,t)}{\partial S^2} = m(t), \qquad EI \frac{\partial^3 u(L,t)}{\partial S^3} = I_\rho \frac{\partial^3 u(L,t)}{\partial S \partial t^2}.$$

The measurements at the tip of the rod will be tangential to the tip displacement (about the equilibrium). Thus

$$y(L,t) = \frac{\partial^2 u(L,t)}{\partial t^2}, \tag{3.112}$$

represents the linearized measurement.

As before, we can recast this in the form of a state equation. If we define the operator $f(u)$ by

$$f(u) = (S \frac{\partial u}{\partial S} - u + \tfrac{1}{2} S^2 \frac{\partial^2 u}{\partial S^2}), \tag{3.113}$$

then we can define x_1, x_2, and x_3 as in the previous section. We first define $x_1(S,t) = u(S,t)$, $x_2(S,t) = \frac{\partial u(S,t)}{\partial t}$, and $x_3(t) = \frac{\partial^2 u(S,t)}{\partial t \partial S}|_{S=0}$. we let these take values on the Hilbert space

$$\mathcal{H} = \{(x_1, x_2, x_3) \in H^2([0,L], I\!R) \times L^2([0,L], I\!R) \times I\!R;\ x_1(0) = 0\},$$

endowed with the inner product,

$$\langle x, y \rangle = \tfrac{1}{2} \int_0^L \left\{ EI \frac{d^2 x_1(S)}{dS^2} \frac{d^2 y_1(S)}{dS^2} + I_\rho \frac{dx_2(S)}{dS} \frac{dy_2(S)}{dS} + \rho_A (\omega S + x_2(S))(\omega S + y_2(S)) \right\} dS + \tfrac{1}{2} I_B x_3 y_3.$$

As before this inner product corresponds to the energy norm. The Hamiltonian for this system can be written as $H(x_1, x_2, x_3) = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$ where the terms of the integrand correspond respectively to the potential, rotational, and linear kinetic energies of a point on the rod. Note that we have included the energy arising from the rotation of the rod.

This equation can be written in terms of the three equations,

$$\frac{\partial x_1}{\partial t} = x_2, \tag{3.114}$$

$$(1 - \frac{I_\rho}{\rho_A}\frac{\partial^2}{\partial S^2})\frac{\partial x_2}{\partial t} = -(\frac{EI}{\rho_A}\frac{\partial^4 x_1}{\partial S^4} + f(x_1)), \tag{3.115}$$

$$\frac{dx_3}{dt} = -\frac{EI}{I_B}\frac{\partial^2 x_1}{\partial S^2}\bigg|_{S=0} + \frac{1}{I_B}m(t). \tag{3.116}$$

This system can be written as the operator equation

$$\boldsymbol{\Gamma}\frac{d\boldsymbol{x}}{dt} = (\mathbf{A} + \omega^2\mathbf{F})\boldsymbol{x} + \mathbf{B}m, \qquad \boldsymbol{z} = \mathbf{C}\boldsymbol{x}, \tag{3.117}$$

where the operators $\boldsymbol{\Gamma}$, $\mathbf{F}$, $\mathbf{B}$, and $\mathbf{C}$ are as defined in the previous section. In addition we have the perturbation operator

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.118}$$

where f is as defined in (3.113).

The boundary conditions associated with this system at the base remain the same as in the previous section. we recall these modeled the physical condition corresponding to no displacement, $x_1(0,t) = 0$, and the compatibility condition, $x_3 = \frac{\partial x_2(0,t)}{\partial S}$, coupling the rigid body to the base of the rod.

At the tip of the rod we now have

$$EI\frac{\partial^2 x_1(L,t)}{\partial S^2} = \rho_A\omega^2(L\frac{\partial x_1}{\partial S} - x_1),$$

balancing the tip force. For the equation modeling the moment at the tip we again have

$$EI\frac{\partial^3 x_1(L,t)}{\partial S^3} = I_\rho\frac{\partial}{\partial S}((1 - \frac{I_\rho}{\rho_A}\frac{\partial^2}{\partial S^2})^{-1}(-\frac{EI}{\rho_A}\frac{\partial^4 x_1(L,t)}{\partial S^4})).$$

We can recast this in the form of a state space representation by defining

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 \\ -(1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}(\frac{EI}{\rho_A}\frac{d^4}{dS^4} + f) & 0 & 0 \\ \frac{EI}{I_B}\frac{d^2}{dS^2}\Big|_{S=0} & 0 & 0 \end{bmatrix}, \qquad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_B} \end{bmatrix},$$

$$\mathbf{C} = \left[-(1 - \tfrac{I_\rho}{\rho_A}\tfrac{d^2}{dS^2})^{-1}\tfrac{EI}{\rho_A}\tfrac{d^4}{dS^4}\Big|_{S=L} \quad 0 \quad 0 \right] \qquad \text{(tip acceleration)},$$

with the associated domain of definition

$$\mathcal{D}(\mathbf{A}) = \{(x_1, x_2, x_3) \in H^4([0,L], I\!R) \times H^2([0,L], I\!R) \times I\!R;$$
$$x_1(0) = 0, \frac{dx_2(0)}{dS} = x_3, \frac{d^2x_1(L)}{d2^3} = \rho_A\omega^2(L\frac{\partial x_1}{\partial S} - x_1),$$
$$\frac{\partial^3 x_1(L,t)}{\partial S^3} = -\frac{I_\rho}{\rho_A}\frac{d}{dS}((1 - \frac{I_\rho}{\rho_A}\frac{d^2}{dS^2})^{-1}\frac{d^4}{dS^4})\}.$$

With the above definitions we have shown that the system can be put in the form of (3.55).

3.5.5. Nonshearable, Inextensible Rod with Rate Damping

We now consider a configuration consisting of a rigid body to which a long, flexible rod is attached. As before we assume the mass of the rod is negligible when compared to the mass of the rigid body. If we linearize about a nonrotating equilibrium and incorporate Kelvin-Voigt type damping in the constitutive equations (see section 2.6) then the equations of motion for the planer problem become

$$EI\frac{\partial^4 u(S,t)}{\partial S^4} + \rho_A\frac{\partial^2 u(S,t)}{\partial t^2} + c\frac{\partial^5 u(S,t)}{\partial S^4 \partial t} = 0, \tag{3.119}$$

with the four boundary conditions

$$u(0,t) = 0, \qquad \frac{\partial^2 u(L,t)}{\partial S^2} = 0,$$
$$I_B\frac{\partial^3 u(0,t)}{\partial S \partial t^2} - EI\frac{\partial^2 u(0,t)}{\partial S^2} = m(t), \qquad \frac{\partial^3 u(L,t)}{\partial S^3} = 0.$$

In the equation for the rod dynamics the additional term corresponds to a stress which is proportional to the rate of change of strain.

In addition we will assume that we have measurements available for the tangential acceleration of the tip of the rod. These take the form

$$y(t) = -\frac{EI}{\rho_A}\frac{\partial^4 u(L,t)}{\partial S^4} - \frac{c}{\rho_A}\frac{\partial^5 u(L,t)}{\partial S^4 \partial t}.$$

As before we can recast this into the form of an infinite state space representation by defining the states as in the case of the nonshearable, inextensible rod with no rotatory inertia

$$x_1(S,t) = u(S,t), \qquad x_2(S,t) = \frac{\partial u(S,t)}{\partial S}, \qquad x_3(S,t) = \frac{\partial^2 u(S,t)}{\partial S \partial t}\Big|_{S=0}.$$

With these definitions the dynamics are now written

$$\frac{dx_1}{dt} = x_2,$$
$$\frac{dx_2}{dt} = -\frac{EI}{\rho_A}\frac{d^4x_1}{dS^4} - \frac{c}{\rho_A}\frac{d^4x_2}{dS^4},$$
$$\frac{dx_3}{dt} = \frac{EI}{I_B}\frac{d^2x_1}{dS^2}\Big|_{S=0},$$

with the obvious definitions of the operators **A**, **B**.

Similarly, we define the Hilbert space $\mathcal{H}$, on which $\boldsymbol{x}$ takes values as

$$\mathcal{H} = \{\boldsymbol{x} \in H^2(0,L) \times L^2(0,L) \times I\!R;\ x_1(0) = 0\},$$

endowed with the inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \tfrac{1}{2}\int_0^L \left\{ EI\frac{d^2x_1(S)}{dS^2}\frac{d^2y_1(S)}{dS^2} + \rho_A x_2(S) y_2(S) \right\} dS + \tfrac{1}{2} I_B x_3 y_3.$$

For this particular system the associated domain of definition is

$$\mathcal{D}(\mathcal{A}) = \{(x_1, x_2, x_3) \in H^4(0,L) \times H^2(0,L) \times I\!R;$$
$$x_1(0) = 0, \frac{d^2x_1(L)}{dS^2} = 0, \frac{d^3x_1(L)}{dS^3} = 0, \frac{dx_2(0)}{dS} = x_3\}.$$

For a discussion of the conditions under which this system generates a contraction semigroup see Marsden and Hughes [1983], page 357. The relevant theorem is Proposition 3.12 in the cited reference. Note that a straight forward computation shows that for $c > 0$ we have $\frac{d}{dt}\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, corresponding to energy dissipation.

For this system the associated eigenvalue problem will be

$$x_2 = \lambda x_1, \tag{3.120}$$
$$-\frac{EI}{\rho_A}\frac{d^4x_1}{dS^4} - \frac{c}{\rho_A}\frac{d^4x_2}{dS^4} = \lambda x_2, \tag{3.121}$$
$$\frac{EI}{I_B}\frac{d^2x_1}{dS^2}\Big|_{S=0} = \lambda x_3. \tag{3.122}$$

Substituting the first equation into the second we obtain an equation in x_1

$$-\left(\frac{EI + \lambda c}{\rho_A}\right)\frac{d^4x_1}{dS^4} = \lambda^2 x_1. \tag{3.123}$$

Solution of this equation proceeds as before, we first assume a solution of the form

$$x_1(S) = \xi_1 \cos(\beta S) + \xi_2 \sin(\beta S) + \xi_3 \cosh(\beta S) + \xi_4 \sinh(\beta S). \tag{3.124}$$

Substituting this into (3.123) we obtain an expression relating λ and β. This can be put in the form of a quadratic equation

$$EI\beta^4 + \lambda c\beta^4 + \lambda^2 \rho_A,$$

solution of which for λ gives an expression for the eigenvalues in terms of β

$$\lambda = -\frac{c\beta^4}{2\rho_A} \pm \frac{c\beta^4}{2\rho_A}\sqrt{1 - \frac{4\rho_A EI}{\beta^4 c^2}}.$$

Note that λ is real for $\beta^4 > \frac{4\rho_A EI}{c^2}$, and as expected, all the roots are in the left half plane.

Next we exploit the boundary conditions to evaluate the ξ_i. From the the base condition $x_1(0) = 0$ we have $\xi_3 = -\xi_1$. The second condition is found by substituting (3.120) into the compatibility condition and using this in (3.122)

$$EI\frac{d^2 x_1}{dS^2}\bigg|_{S=0} = I_B \lambda^2 \frac{dx_1}{dS}\bigg|_{S=0}.$$

If we now use the expression for x_1 in this we find

$$\xi_1 = -\frac{I_B \lambda^2}{2\beta EI}(\xi_2 + \xi_4).$$

Eliminating ξ_1 and ξ_2 we can use the boundary conditions at the tip to obtain the two equation in ξ_2 and ξ_4

$$\begin{aligned} 0 = {} & \frac{I_B \lambda^2}{2\beta EI}(\xi_2 + \xi_4)(\cos(\beta L) + \cosh(\beta L)) \\ & - \xi_2 \sin(\beta L) + \xi_4 \sinh(\beta L), \end{aligned} \tag{3.125}$$

$$\begin{aligned} 0 = {} & -\frac{I_B \lambda^2}{2\beta EI}(\xi_2 + \xi_4)(\sin(\beta L) - \sinh(\beta L)) \\ & - \xi_2 \cos(\beta L) + \xi_4 \cosh(\beta L). \end{aligned} \tag{3.126}$$

These equations have a nontrivial solution for ξ_2, and ξ_4 when

$$\begin{aligned} 0 = {} & \frac{I_B \lambda^2}{\beta EI}(1 + \cos(\beta L)\cosh(\beta L)) \\ & - \sin(\beta L)\cosh(\beta L) + \sinh(\beta L)\cos(\beta L). \end{aligned} \tag{3.127}$$

In general there are a countable number of β which satisfy this equation, subsequently we will denote these by β_n, $n = 1, 2, \ldots$.

Solving for ξ_2, and ξ_4 in a particular solution we get

$$\xi_2 = -\frac{I_B \lambda_n^2}{2\beta_n EI}(\sin(\beta_n L) - \sinh(\beta_n L)) + \cosh(\beta_n L),$$
$$\xi_4 = \frac{I_B \lambda_n^2}{2\beta_n EI}(\sin(\beta_n L) - \sinh(\beta_n L)) - \cos(\beta_n L),$$

and from the expression for ξ_1

$$\xi_1 = -\frac{I_B \lambda_n^2}{2\beta_n EI}(\cos(\beta_n L) + \cosh(\beta_n L)).$$

If we denote the solutions for x_1 as ϕ_n, $n = 1, 2, \ldots$ then

$$\begin{aligned}\phi_n(S) = &-\frac{I_B \lambda_n^2}{2\beta_n EI}(\cos(\beta_n L) + \cosh(\beta_n L))(\cos(\beta_n S) - \cosh(\beta_n S)) \\ &-(\frac{I_B \lambda_n^2}{2\beta_n EI}(\sin(\beta_n L) - \sinh(\beta_n L)) + \cosh(\beta_n L))\sin(\beta_n S) \\ &+(\frac{I_B \lambda_n^2}{2\beta_n EI}(\sin(\beta_n L) - \sinh(\beta_n L)) - \cos(\beta_n L))\sinh(\beta_n S).\end{aligned} \quad (3.128)$$

From the above we conclude that the eigenfunctions of our system are given by

$$\boldsymbol{x}_n = \begin{bmatrix} \phi_n(S) \\ \lambda_n \phi_n(S) \\ \lambda_n \frac{d\phi_n(0)}{dS} \end{bmatrix}. \quad (3.129)$$

Subsequently we will assume that the eigenvectors have been normalized.

Note that for any n, $\mathrm{Re}(\lambda_n) < 0$, while for $4\rho_A EI > \beta_n^4 c^2$, the roots are complex conjugate. Furthermore, for $4\rho_A EI < \beta_n^4 c^2$ the roots are on the real axis. For these roots we will be interested in $\sup_n \mathrm{Re}(\lambda_n)$, the closest they get to the right half plane. Therefore we consider the subsequence of roots which approach the origin along the real axis

$$\lambda_{n_i} = -\frac{c\beta_{n_i}^4}{2\rho_A}(1 - \sqrt{1 - \frac{4\rho_A EI}{\beta_{n_i}^4 c^2}}).$$

Now, with a little manipulation

$$\begin{aligned}\lambda_{n_i} &= -\frac{c\beta_{n_i}^4}{2\rho_A}\left(1 - \sqrt{1 - \frac{4\rho_A EI}{\beta_{n_i}^4 c^2}}\right)\left(1 + \sqrt{1 - \frac{4\rho_A EI}{\beta_{n_i}^4 c^2}}\right)\left(1 + \sqrt{1 - \frac{4\rho_A EI}{\beta_{n_i}^4 c^2}}\right)^{-1}, \\ &= -\frac{2EI}{c}\left(1 + \sqrt{1 - \frac{4\rho_A EI}{\beta_{n_i}^4 c^2}}\right)^{-1}.\end{aligned}$$

Since the term in parentheses has an upper bound of 2 as $\beta_{n_i}^4 \to \infty$ we can conclude that

$$\sup_n \lambda_n = -\frac{EI}{c}. \tag{3.130}$$

The transfer function in this case is computed as

$$C(Is-A)^{-1}B = \sum_{n=-\infty}^{\infty} \frac{1}{s-\lambda_n} \mathbf{B}^* \tilde{\boldsymbol{x}}_n \cdot \mathbf{C}\boldsymbol{x}_n,$$

where $\boldsymbol{x}_i$ are the eigenvectors of A and $\tilde{\boldsymbol{x}}_i$ are the eigenvectors of the adjoint system.

Computation of the transfer function relating the torque applied to the rigid body to a measurement of tip acceleration is now straight forward. We first compute

$$\begin{aligned} B^*\tilde{\boldsymbol{x}}^n &= \frac{1}{I_B}\tilde{x}_{3,n}, \\ &= \frac{1}{I_B}\left(-\frac{EI}{\lambda_n I_B}\frac{d^2 x_{1,n}}{dS^2}\bigg|_{S=0}\right), \\ &= \frac{1}{I_B}\left(-\frac{EI}{\lambda_n I_B}\frac{d^2 x_{1,n}}{dS^2}\bigg|_{S=0}\right); \end{aligned} \tag{3.131}$$

$$\begin{aligned} C\boldsymbol{x}^n &= -\left(\frac{EI + c\lambda_n}{\rho_A}\right)\frac{d^4 x_{1,n}}{dS^4}\bigg|_{S=L}, \\ &= \lambda_n^2 x_{1,n}\bigg|_{S=L}. \end{aligned} \tag{3.132}$$

Using the expression for the transfer function (3.71) we can write

$$\begin{aligned} &C(Is-A)^{-1}B \\ &\quad = \sum_{n=-\infty}^{\infty} \frac{B^*\tilde{\boldsymbol{x}}^n \cdot C\boldsymbol{x}^n}{2(s-\lambda_n)}, \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{2}\frac{(s(C\boldsymbol{x}_n B^*\tilde{\boldsymbol{x}}^n + C\boldsymbol{x}_{-n}B^*\tilde{\boldsymbol{x}}^{-n}) + C\boldsymbol{x}_n B^*\tilde{\boldsymbol{x}}^n\lambda_n + C\boldsymbol{x}_{-n}B^*\tilde{\boldsymbol{x}}^{-n}\lambda_{-n})}{s^2 - (\lambda_n + \lambda_{-n})s + \lambda_n\lambda_{-n})}, \\ &\quad = -\frac{1}{I_B}\frac{dx_1}{dS}\bigg|_{S=0} x_1\bigg|_{S=L} \frac{(s(\lambda_n^3 + \lambda_{-n}^3) + \lambda_n\lambda_{-n}(\lambda_n^2 + \lambda_{-n}^2))}{s^2 - (\lambda_n + \lambda_{-n})s + \lambda_n\lambda_{-n})}. \end{aligned} \tag{3.133}$$

which describes the effect of an input torque on the rotational tangential acceleration of the tip of the appendage.

In this expression we have;

$$\begin{aligned} \frac{dx_1}{dS}\bigg|_{S=0} &= \beta_n(\cosh(\beta_n L) + \cos(\beta_n L)), \\ x_1\bigg|_{S=L} &= -\frac{I_B\lambda_n^2}{2\beta_n EI}(\sin(\beta_n L) - \sinh(\beta_n L))^2 \\ &\qquad + \cosh(\beta_n L)\sin(\beta_n L) - \cos(\beta_n L)\sinh(\beta_n L), \end{aligned}$$

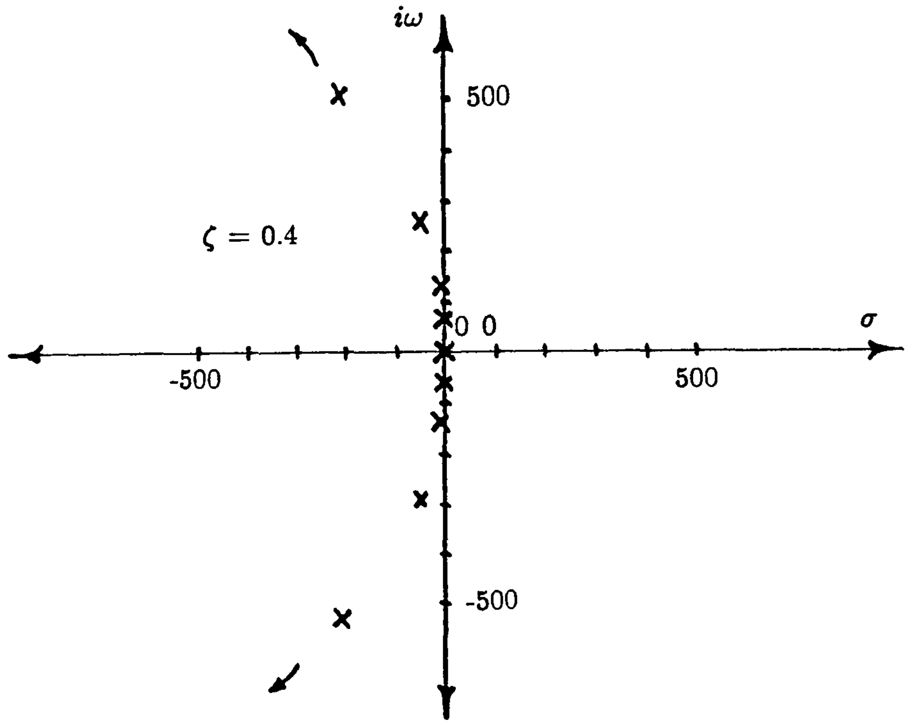

Figure 3.2. Eigenvalues of a Damped Beam.

with

$$\lambda_n = \frac{c\beta_n^4}{2\rho_A} + \frac{c\beta_n^4}{2\rho_A}\sqrt{1 - \frac{4\rho_A EI}{\beta_n^4 c^2}}, \qquad \text{and} \qquad \lambda_{-n} = \frac{c\beta_n^4}{2\rho_A} - \frac{c\beta_n^4}{2\rho_A}\sqrt{1 - \frac{4\rho_A EI}{\beta_n^4 c^2}}.$$

Example of an Aluminum Beam and Hub

Again we can get a better feeling for the physical meaning of some of the results of this section by returning to the model of the aluminum hub with attached beam. we can make a reasonable guess at the value of the damping constant c as follows; for a particular mode the eigenvalues are the roots of

$$\begin{aligned} 0 &= s^2 + \frac{c\beta_n^4}{\rho_A}s + \frac{\beta_n^4 EI}{\rho_A}, \\ &= s^2 + 2\zeta_n\omega_n s + \omega_n^2, \end{aligned} \tag{3.134}$$

where we have used the standard form for a second order system. In a control context one refers to ω as the *natural frequency* and ζ_n as the *damping ratio.* From these expressions we have;

$$\zeta_n = \frac{\beta_n^2 c}{2\sqrt{EI\rho_A}}, \qquad \text{and} \qquad \omega_n = \beta_n^2\sqrt{\frac{EI}{\rho_A}}. \tag{3.135}$$

Hub inertia, $I_B = 5.4 \times 10^{-3}\ kgm^2$, $\zeta_1 = 0.0$

$n = 1$	$\beta_n L = 0.0$	$\lambda_{\pm n} = 0.0$	$f = 0.0$	Hz
2	3.550	$\pm i\ 57.008$	9.073	
3	5.419	$\pm i\ 132.79$	21.134	
4	8.021	$\pm i\ 290.97$	46.309	
5	11.054	$\pm i\ 552.55$	87.941	

Hub inertia, $I_B = 5.4 \times 10^{-3}\ kgm^2$, $\zeta_1 = 0.02$

$n = 1$	$\lambda_{\pm n} = 0.0$	$f = 0.0$	Hz
2	$-0.709 \pm i\ 57.016$	9.075	
3	$-3.361 \pm i\ 132.76$	21.137	
4	$-27.070 \pm i\ 289.66$	46.302	
5	$-104.80 \pm i\ 542.48$	87.935	

Hub inertia, $I_B = 5.4 \times 10^{-3}\ kgm^2$, $\zeta_1 = 0.04$

$n = 1$	$\lambda_{\pm n} = 0.0$	$f = 0.0$	Hz
2	$-1.419 \pm i\ 57.038$	9.081	
3	$-6.709 \pm i\ 132.69$	21.145	
4	$-54.141 \pm i\ 285.69$	46.279	
5	$-209.61 \pm i\ 511.08$	87.916	

Table 3.2. Rigid Body and Damped Rod Modes.

We can use the expression for ζ_n to find c for a given damping ratio associated with a particular mode. If we use a constant c for *all* n then the eigenvalues associated with a particular n all lay on the real axis for $4\rho_A EI > \beta_n^4 c^2$ will all lay on the real axis. Assuming a lightly damped beam we can compute c on the basis of β_1 for and undamped rod and a given ζ_1. We have tabulated the results in table 3.2 for the rigid body and rod we have previously discussed. These are solutions of equation (3.67) with the physical parameters associated with the aluminum hub and one meter aluminum beam of the earlier section.

3.6. Stabilization and Control

For the models which we have developed in the earlier part of this chapter a natural question to ask is does there exist a control which will assure is of asymptotic stability?

In other words can we always drive the state to zero?. In the general sense this is a question which deals with the stabilizability of the infinite dimensional systems we have obtained.

The answer to this question is yes. In Slemrod [1987] it is shown that a class of saturating feedback controllers can always be found which will assure asymptotic stability at the origin In this report the stabilization of a distributed parameter system is investigated. In particular, two theorems are presented which establish asymptotic stability of $\{0\}$ for a class of saturating feedback controllers.

The systems considered by Slemrod have the abstract form;

$$\frac{dx}{dt} = \mathbf{A}x + \mathbf{B}u, \qquad x = x_0, \tag{3.136}$$

with $\mathbf{A}$ an infinite dimensional operator defined on a Hilbert space $\mathcal{H}$. For these systems a feedback control of the form

$$u = G(x),$$

is used, where,

$$G(x) = \begin{cases} \dfrac{-r\mathrm{BB}^*x}{\|\mathrm{B}^*x\|_E}, & \text{if } \|\mathrm{B}^*x\|_E \geq r; \\ -\mathrm{BB}^*x, & \text{if } \|\mathrm{B}^*x\|_E < r. \end{cases} \tag{3.137}$$

In this case $G(x) \in \mathcal{H}$ takes values inside or on the boundary of a ball of radius $\|\mathbf{B}\|r$, with r a given value. Here $\|\cdot\|_E$ is the norm on the space of inputs and outputs.

Slemrod proves the following theorem;

Theorem. *(Slemrod, 1987) Assume that for each $x_0 \in \mathcal{H}$ there is a unique weak solution*

$x(t; x_0) = T(t)x_0$, of (3.136) and (3.137) defined for all $t \geq 0$ with $\{0\}$ a stable equilibrium. If in addition $\mathbf{B}$ is compact and the only solution of the equation

$$\mathbf{B}^* e^{\mathbf{A}t}\psi = 0 \qquad \text{for all } t \geq 0$$

is $\psi = 0$, then $x(t; x) \to 0$ as $t \to \infty$ for all $x_0 \in \mathcal{H}$.

Proof: The proof is based on a result of Ball and Slemrod [1979] which gives us conditions for the weak solution to a semilinear equation. In the proof it is shown that

(i) $G(x)$ is globally Lipschitz, in other words there exists a constant K such that $\|G(x_1) - G(x_2)\| \leq K\|x_1 - x_2\|$ for all $x_1, x_2 \in \mathcal{H}$,

(ii) B compact implies $G(\psi_n) \to G(\psi)$ when $\psi_n \to \psi$,

(iii) $G(\psi)$ is dissipative, $\langle G(\psi), \psi \rangle \leq 0$.

These conditions are precisely the ones needed for the theorem of Ball and Slemrod to be applicable. From the theorem, we conclude that for each $\boldsymbol{x}_0 \in \mathcal{H}$, $\omega_W(\boldsymbol{x}_0)$ is a nonempty, invariant set in $\mathcal{H}$. Also, for each $\psi \in \omega_W(\boldsymbol{x}_0)$

$$\langle T(t)\psi, G(T(t)\psi) \rangle = 0 \qquad \text{for all } t \geq 0$$

thus from the definition of G we have that $B^* T(t)\psi = 0$ from which we then conclude $G(\boldsymbol{x}) = 0$ for all $t \geq 0$. But from the variation of constants formula, with $G(\boldsymbol{x}(S)) = 0$ we have $B^* T(t)\psi = B^* e^{At}\psi$. This implies that $\psi = 0$ only so we must have $\omega_W(\boldsymbol{x}_0) = \{0\}$. One of course trivially has $\|T(t)\boldsymbol{x}_0\| \leq \|\boldsymbol{x}_0\|$.

In the case of noncolocated actuators and sensors this theorem may fail. The problem here is that the output operator C may be unbounded. However, as it turns out in our case $x_1 \in H^2([0, L], I\!R)$ assures us that if we measure tip displacement the operator C is a bounded operator. By definition,

$$\begin{aligned} \|C\| &= \sup_{\|x\|=1} \|C\boldsymbol{x}\|, \\ &= \sup_{\|x\|=1} \|x_1(S)|_{S=L}\|, \end{aligned}$$

since $\partial x_1(S)/\partial S \in L_2([0, L], I\!R)$, we have for $x_1(0) = 0$,

$$\begin{aligned} \|x_1(S)\| &= \sqrt{\int_0^L \frac{\partial x_1(\sigma)}{\partial S} d\sigma}, \\ &= L^{\frac{1}{2}} \sqrt{\int_0^L \frac{\partial x_1(\sigma)}{\partial S}^2 d\sigma}, \\ &= L^{\frac{1}{2}} K^{\frac{1}{2}} < \infty, \end{aligned}$$

where K is a constant and we have used the Cauchy-Schwarz inequality. Thus we can conclude that C is a *bounded* operator in the norm defined on $H^2([0, L], I\!R)$. Consequently the results of the theorem of Slemrod apply when our output is a measurement of tip displacement.

If however we want to use the measurement of acceleration the proof of the theorem breaks down. In this case the tip acceleration can be expressed

$$\left.\frac{\partial x_2}{\partial t}\right|_{S=L} = -\left.\frac{EI}{A_\rho}\frac{\partial^4 x_1}{\partial S^4}\right|_{S=0},$$

which is clearly unbounded.

The second theorem of Slemrod's gives conditions for the *strong convergence* of (3.136) to zero. It is based on a result of Dafermos and Slemrod [1973] related to nonlinear contraction semigroups.

Theorem. *For each $x_0 \in \mathcal{H}$ there exists a unique solution of (3.55) for all $t \geq 0$ with $\{0\}$ a stable equilibrium of (3.55). If in addition, $E = I\!R$, $(\lambda\mathbf{1} - \mathbf{A})^{-1}$ is compact for some $\lambda > 0$, and the only solution of*

$$\mathbf{B}^* e^{\mathbf{A}t}\psi = 0 \qquad\qquad t \geq 0$$

is $\psi = 0$, then $x(t; x_0) \to 0$ as $t \to \infty$ for all $x_0 \in \mathcal{H}$.

Proof: See Slemrod [1987].

We can use this theorem to find the stabilizing controls for the systems we have considered. We will illustrate the application for the model in section (3.5.2), the other models are treated in an analogous fashion.

For the systems we consider we have shown that they generate contraction semigroups on the appropriate Hilbert space. We have also shown that $\lambda\mathbf{1} - \mathbf{A}$ has compact resolvent by the use of the Sobolev embedding theorem. Thus, to satisfy the conditions of the theorem it remains to show that for $\psi \in \mathcal{H}$, and all $t \geq 0$ that $\mathbf{B}^* e^{\mathbf{A}t}\psi = 0$ implies $\psi = 0$. As we have remarked earlier this is an observability condition, a fact that will be important below. For our model we will proceed in essentially the same way as in Slemrod [1987].

We first note that if $\mathbf{B}^* e^{\mathbf{A}t}\psi = 0$, then $\langle \mathbf{B}u, e^{\mathbf{A}t}\psi\rangle = 0$ for all $u \in I\!R$, or if we recall the definition of $\mathbf{B}$ this implies $u\mathbf{I}_B^{-1}\psi_3(t) = 0$, for $t \geq 0$, or $\psi_3(t) = 0$ for all $t \geq 0$. Thus we would like to show that $\psi_3(t) = 0$ with $u \in I\!R$ for all $t \geq 0$ implies $\psi = 0$. This we can do by explicit computation of $e^{\mathbf{A}t}\psi$ for $\psi \in \mathcal{D}(\mathbf{A})$.

We recall that;

$$e^{\mathcal{A}t}x_0 = \sum_{n=-\infty}^{\infty} e^{\lambda_n t}\langle x^n, x\rangle x^n,$$

where λ_n is the n^{th} eigenvalue of $\mathbf{A}$ and x^n is the associated eigenvector (more generally we need the eigenvectors of both $\mathbf{A}$ and $\mathbf{A}^*$.) For the this model we have previously computed the eigenvectors to be,

$$\boldsymbol{x}^n = \begin{bmatrix} \phi_n(S) \\ \pm i\sqrt{\frac{EI}{A_\rho}}\beta_n^2\phi_n(S) \\ \pm i\sqrt{\frac{EI}{A_\rho}}\beta_n^2\frac{d\phi_n(0)}{dS} \end{bmatrix}. \tag{3.138}$$

As before we have

$$\langle \boldsymbol{x}^n, \boldsymbol{x}^0 \rangle = a_n + ib_n,$$

with a_n and b_n as defined in section (3.5.3).

Proceeding as in (3.71), we compute the third component of $e^{\mathcal{A}t}\psi$,

$$\psi_3(t) = \sum_{n=1}^{\infty} -(b_n \cos(\lambda_n t) + a_n \sin(\lambda_n t))\sqrt{\frac{EI}{A_\rho}}\beta_n^2 \left.\frac{d\phi_n}{dS}\right|_{S=0}.$$

From the uniqueness theorem of almost periodic functions, $\psi_3(t) = 0$ if the coefficients on the right hand side are zero, i.e.

$$b_n\sqrt{\frac{EI}{A_\rho}}\beta_n^2 \left.\frac{d\phi_n}{dS}\right|_{S=0} = 0,$$
$$a_n\sqrt{\frac{EI}{A_\rho}}\beta_n^2 \left.\frac{d\phi_n}{dS}\right|_{S=0} = 0.$$

We have $\sqrt{\frac{EI}{A_\rho}}\beta_n^2 \neq 0$ for $n = 1, 2, \ldots$. The conditions under which $\left.\frac{d\phi_n}{dS}\right|_{S=0} = 0$ are found as follows; recall that for the nonshearable, inextensible beam with no rotatory inertia the eigenfunctions $\phi_n(S)$ was given by (3.57), i.e.,

$$\phi_n(S) = \xi_1(\cos(\beta S) - \cosh(\beta S)) + \xi_2 \sin(\beta S) + \xi_4 \sinh(\beta S). \tag{3.139}$$

With our additional constraint we find that the second boundary condition at the base now requires

$$\left.\frac{d^2\phi_n}{dS^2}\right|_{S=0} = 0. \tag{3.140}$$

Differentiating the expression for ϕ_n once and evaluating it at $S = 0$, to satisfy (3.140) we require $\xi_1 = 0$. The eigenfunctions which satisfy our three boundary conditions at the base are thus of the form;

$$\phi_n(S) = \xi_2(\sin(\beta S) - \sinh(\beta S)). \tag{3.141}$$

If we now use the conditions at $S = L$, we find that we must have

$$0 = \sin(\beta L) + \sinh(\beta L),$$
$$\text{and} \qquad 0 = \cos(\beta L) + \cosh(\beta L).$$

The second condition can never be satisfied, consequently we conclude $\left.\frac{d\phi_n}{dS}\right|_{S=0} \neq 0$. Thus we must have $a_n = 0$, and $b_n = 0$, in which case we conclude $\psi = 0$. Thus, with our controls taking values in $\mathbb{R}$ the conditions of the theorem are satisfied and we are assured that there exists a saturating controller which will drive any initial state to zero.

In our case we can use the theorem to explicitly construct a control law. We first observe

$$\begin{aligned}\langle x, Bx\rangle &= I_B x_3(\frac{y_3}{I_B}),\\ &= I_B(\frac{x_3}{I_B})y_3,\\ &= \langle B^* x, y\rangle,\end{aligned}$$

from which we conclude B^* is simply the transpose of B. The function $G(x)$ is then straight forward to construct.

$$G(x) = \begin{cases} -\frac{r}{I_B}\operatorname{sign}(x_3(t)), & \text{if } \frac{x_3(t)}{I_B} \geq r;\\ -\frac{r}{I_B^2}x_3(t), & \text{if } \frac{x_3(t)}{I_B} < r.\end{cases} \tag{3.142}$$

Of course the actual design of such a control must take into account more than just stabilizability. In this regard we note that within the radius r of the origin the feedback control is linear. Within this region we can contemplate the application of a linear feedback control methodology to do the design. (However, keep in mind the system is infinite dimensional). One such methodology which we may wish to apply is the so called L_∞ technique of Curtain and Glover [1986] which is discussed in chapter 5.

Comparison of Some Models

Slemrod considers a special case of a model due to Bailey and Hubbard [1985] which describes the dynamics of a cantilever beam with a tip mass and inertia. This beam was assumed to have a control torque at the tip. The dynamics are described by

$$EI\frac{\partial^4 u(S,t)}{\partial S^4} + A_\rho \frac{\partial^2 u(S,t)}{\partial t} = 0, \qquad \text{for} \quad 0 < S < L, \tag{3.143}$$

with the boundary conditions at the base,

$$u(S,t)\Big|_{S=0} = 0, \qquad \frac{\partial u(S,t)}{\partial S}\Big|_{S=0} = 0, \tag{3.144}$$

and at the tip

$$\begin{aligned} EI\frac{\partial^2 u(S,t)}{\partial S^2}\Big|_{S=L} &= -I_{tip}\frac{\partial^3 u(S,t)}{\partial S \partial t^2}\Big|_{S=L} + f(t), \\ EI\frac{\partial^3 u(S,t)}{\partial S^3}\Big|_{S=L} &= M_{tip}\frac{\partial^2 u(S,t)}{\partial t^2}\Big|_{S=L}. \end{aligned} \tag{3.145}$$

Slemrod defines the 4 state variables;

$$\begin{aligned} w(S,t) &= u(S,t), \qquad v(S,t) = \frac{\partial u(S,t)}{\partial t}, \\ a(t) &= \frac{\partial u(S,t)}{\partial t}\Big|_{S=L}, \quad b(t) = \frac{\partial^2 u(S,t)}{\partial S \partial t}\Big|_{S=L}. \end{aligned}$$

where w corresponds to the configuration variable, v is the velocity density, a is the velocity of displacement at the tip, and b is the velocity of rotation at the tip.

The model which we have used in this dissertation is that of a beam which is hinged at the base and free at the tip. In addition we have assumed that there is a rigid body attached to the base which has mass and inertia. Since there is no longer a mass at the tip in our model our boundary conditions at the tip reflect the absence of any external torques or forces (in other words the right hand side of Slemrod's boundary conditions for the tip are zero in our case.) At the base we have the boundary condition $u(S,t)|_{S=0}$, the same as Slemrod. However the remaining boundary condition reflects the dynamics of our base mass and is given by

$$I_{base}\frac{\partial^3 u(S,t)}{\partial S \partial t^2}\Big|_{S=0} - EI\frac{\partial^2 u(S,t)}{\partial S^2}\Big|_{S=0} = f(t).$$

To summarize, the difference between the model used by Slemrod, and that discussed in this chapter is the location of the rigid body at different ends of a hinged–free beam.

CHAPTER FOUR

STABILITY

The notion of stability is fundamental to the study of dynamical systems. Knowledge of the stability of an equilibrium provides important insight into the local structure of the phase space. From a physical standpoint an unstable system can have undesirable and even catastrophic consequences. Consequently, an understanding of the stability of the equilibria is essential to the control system design.

There are many definitions of stability, the one we will be mostly concerned with here is that of Liapunov stability (Liapunov, [1947]). The basic idea of Liapunov stability is that, for any given point near an equilibrium, an orbit passing through that point remains near the equilibrium. Of course this presupposes we have enough structure associated with the manifold to quantify this notion.

In addition to the stability of an equilibrium we will be interested in extending these concepts to reduced phase spaces. In a loose sense we can think of this as stability of the steady state. In the analysis of fluids and plasmas one deals with the reduced spaces corresponding to fluid flows, ignoring the positions of individual particles (Holm, et.al. [1983]). In particular, Arnold [1965,1966] was able to formulate a methodology for investigating the Liapunov stability of planer ideal incompressible fluid motion. In recent years this technique has come to be known as the energy-Casimir method and has been used to study MHD, multifluid plasmas, and the Maxwell-Vlasov equation (Holm, Marsden, Ratiu, Weinstein [1984].) Classical stability results for the rigid body and heavy top were reproduced by Holm using this technique.

A natural application of this technique for infinite dimensional mechanical systems is that of the rigid body with flexible appendage as was investigated by Krishnaprasad and Marsden [1987]. Interestingly enough, in more complicated models of this type, using geometrically exact rod theory, the Casimirs which are crucial to this technique are unknown or do not exist. As a consequence, a more general method called the energy-momentum method is currently employed in investigations these models (Marsden,Simo,Posbergh,Krishnaprasad [1988].)

In this chapter we will first present the classical notions of stability, recast in the geometric setting. We will also show how these notions are extended to the reduced systems. In the second section we will describe the energy-Casimir method, and a more recently explored method called the energy-momentum method. In the last section we will apply this technique to the case of the linear extensible shear beam.

4.1. Fundamental Stability Concepts

To discuss stability we must first define precisely what we mean. Several interelated concepts of stability will be of interest with regard to assessing the stability of Hamiltonian systems. The most important concept is that of *nonlinear* or *Liapunov stability* of an equilibrium point. The idea of stability in this case is that an equilibrium point is stable if an orbit passing through a point nearby remains in the vicinity for all subsequent time. We will also be concerned with two other concepts of stability.

Definition (4.1): *An equilibrium point is called linearly stable relative to a norm* $\|\delta \mathbf{x}\|$ *provided that for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\|\delta \mathbf{x}\| < \delta$ *at* $t = t_0$ *implies* $\|\delta \mathbf{x}\| < \epsilon$ *for* $t > t_0$ *where* $\delta \mathbf{x}$ *evolves according to* $d(\delta \mathbf{x})/dt = DX(\mathbf{x}_e) \cdot \delta \mathbf{x}$

In this case we are applying the stability criteria to the linearized system. We note that *spectral stability*, which requires that the spectrum of the linearized operator $DX(\mathbf{x}_e)$ have no strictly positive real part, is implied by linear stability. However, the converse is not generally true. In finite dimensions, a sufficient condition for linearized stability is that $DX(\mathbf{x}_e)$ have distinct eigenvalues on the imaginary axis. In infinite dimensions, a sufficient condition for linearized stability is that $DX(\mathbf{x}_e)$ have a complete set of eigenfunctions with purely imaginary eigenvalues of multiplicity one.

Definition (4.2): *An equilibrium solution* $\mathbf{x}_e$ *is formally stable if a conserved quantity is found whose first variation vanishes at the solution and whose second variation at the equilibrium is positive or negative definite.*

Formal stability implies linearized stability, however the converse is not true. Formal stability of fluids and plasmas has been investigated by many researchers (see Holm, et. al.[1983] and the references therein).

Nonlinear or Liapunov stability quantifies the notion that if $\mathbf{x}_0$ is an equilibrium point then the flow through a nearby point remains for all time in the vicinity of the equilibrium point. More precisely, we have the following (Abraham & Marsden, [1978], p.737).

Definition (4.3): *Let $\mathbf{x}_0$ be an equilibrium point of the vector field X. Let $\Phi(\cdot)$ denote the flow of X. Then;*

(i) $\mathbf{x}_0$ *is stable in the sense of Liapunov if for any neighborhood U of $\mathbf{x}_0$, there is a neighborhood V of $\mathbf{x}_0$, such that if $\mathbf{x} \in V$, then $\Phi_\lambda(\mathbf{x}) \in U$ for all $\lambda \geq 0$.*

(ii) $\mathbf{x}_0$ *is asymptotically stable if there is a neighborhood V of $\mathbf{x}_0$ such that if $\mathbf{x} \in V$, then $\Phi_t(V) \subset \Phi_s(V)$ if $t > s$ and*

$$\lim_{t \to +\infty} \Phi_t(V) = \{\mathbf{x}_0\}, \tag{4.1}$$

(i.e. for any U, $\mathbf{x}_0 \in U$, there is a T s.t. $\Phi_t(V) \subset U$ if $t \geq T$).

Neither linearized nor formal stability is sufficient for nonlinear stability. (see Holm, et. al.[1983] for counter examples). However, in finite dimensions, formal stability implies stability. In infinite dimensions this is not generally true and there exist physically meaningful counter examples (as in Ball & Marsden [1976]. Generally one neeeds additional convexity estimates to extend formal stability results to establish nonlinear stability.

In addition, the relative stability of a dynamical system is frequently of interest. In this case we consider a reduced manifold M_μ and the associated Hamiltonian H_μ as defined in chapter 2.

Definition (4.4): *Let (M,ω) be a symplectic manifold and G a Lie group acting symplectically on M and leaving a Hamiltonian H invariant. Under the conditions of (4.3.1) and (4.3.5), a relative equilibrium $\mathbf{x} \in M$ is relatively stable if $\pi_\mu(\mathbf{x})$ is stable for the induced dynamical system X_{H_μ} on M_μ where $\pi_\mu(\mathbf{x})$ appears as an equilibrium point of X_{H_μ}.*

4.2. The Energy-Casimir method

The energy-Casimir method is a technique based on an original idea of V. I. Arnold and used in his investigation of the stability of a planer, ideal incompressible fluid (Arnold, [1965], Arnold, [1969]). These nonlinear stability results extended the classical linear theory of Rayleigh. In this analysis, Arnold adds a conserved quantity C called a Casimir (see section 2.8) to the energy of the system, the Hamiltonian. The Casimir term is kinematic in the sense that it will be conserved for any Hamiltonian system. The Casimir is chosen so that $H + C$ is stationary at a critical point. Finally, convexity estimates for $H + C$ were used to establish rigorous nonlinear stability.

More precisely we have the following methodology:

(i) Choose a manifold M, and associated bracket $\{\cdot, \cdot\}$ such that the equations of motion can be written in Hamiltonian form

$$\dot{F} = \{F, H\} \tag{4.2}$$

where H is the Hamiltonian of the system and F is an arbitrary function defined on M. We recall H is a conserved quantity, $dH(\Phi_t \circ \boldsymbol{x}_0)/dt$ where F_t is the flow through some point $\boldsymbol{x}_0$.

(ii) Find a family of constants for the motion of (4.2), that is C such that $dC(F_t \circ \boldsymbol{x}_0)/dt = 0$ Note that for any F defined on M we have $\{F, C\} = 0$. These functionals may be associated with symmetries of the Hamiltonian. In general, the larger the family of C the better, however we may not be able to find any C.

(iii) Next take the first variation $\delta(H + C)$ of $H + C$ and relate this to a critical point of (4.2) by requiring that $H + C$ have a critical point at $\boldsymbol{x}_e$

(iv) Take the second variation, $\delta^2(H + C)$, to examine formal stability of the system. In certain cases, such as finite dimensional systems the definiteness of the second variation is sufficient to establish formal stability.

In general, as we have remarked earlier, formal stability will be insufficient to establish nonlinear stability. In that case we need to proceed with the following two steps.

(v) Construct convexity estimates. Find quadratic forms Q_1 and Q_2 on M such that

$$Q_1(\Delta \boldsymbol{x}) \leq H(\boldsymbol{x}_e + \Delta \boldsymbol{x}) - H(\boldsymbol{x}_e) - DH(\boldsymbol{x}_e) \cdot \Delta \boldsymbol{x}, \tag{4.3}$$

$$Q_2(\Delta \boldsymbol{x}) \leq C(\boldsymbol{x}_e + \Delta \boldsymbol{x}) - C(\boldsymbol{x}_e) - DC(\boldsymbol{x}_e) \cdot \Delta \boldsymbol{x}, \tag{4.4}$$

for all finite variations, $\Delta \boldsymbol{x}$ in M we require

$$Q_1(\Delta \boldsymbol{x}) + Q_2(\Delta \boldsymbol{x}) > 0, \qquad \text{for all } \Delta \boldsymbol{x} \text{ in } M, \Delta \boldsymbol{x} \neq 0. \tag{4.5}$$

(vi) With this definition we have, for any solution $\boldsymbol{x}(t)$ of (4.2) the a-priori estimate on $\Delta \boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_e(t)$,

$$Q_1(\Delta \boldsymbol{x}(t)) + Q_2(\Delta \boldsymbol{x}(t)) \leq (H + C)(\boldsymbol{x}(0)) - (H + C)(\boldsymbol{x}_e(t)). \tag{4.6}$$

(vii) Nonlinear stability is established as follows; We set

$$||\mathbf{v}||^2 = Q_1(\mathbf{v}) + Q_2(\mathbf{v}) > 0, \qquad (\text{for } \mathbf{v} \neq 0), \tag{4.7}$$

which defines a norm on M. If $H + C$ is continuous on this norm at $\mathbf{x}_e$, and solutions to (4.2) exist for all time $\boldsymbol{x}_e$ is stable. If on the other hand solutions to (4.2) are not known to exist for all time, we still have conditional stability.

We note that a sufficient condition for continuity of $H + C$ is the existence of positive constants C_1, and C_2 such that

$$H(\boldsymbol{x}_e + \Delta \boldsymbol{x}) - H(\boldsymbol{x}_e) - DH(\boldsymbol{x}_e) \cdot \Delta \boldsymbol{x} \leq C_1 ||\Delta \boldsymbol{x}||^2,$$
$$C(\boldsymbol{x}_e + \Delta \boldsymbol{x}) - C(\boldsymbol{x}_e) - DC(\boldsymbol{x}_e) \cdot \Delta \boldsymbol{x} \leq C_2 ||\Delta \boldsymbol{x}||^2.$$

From which we construct the stability estimate: for all $\Delta \boldsymbol{x}(0)$ in M.

$$||\Delta \boldsymbol{x}(t)||^2 = Q_1(\Delta \boldsymbol{x}(t)) + Q_2(\Delta \boldsymbol{x}(t)) \leq (C_1 + C_2)||\Delta \boldsymbol{x}(0)||^2. \tag{4.8}$$

The assertion of the stability estimate is proved in Holm, et. al [1983].

The standard example for the energy-Casimir method which we use to illustrate the procedure is the stability of rotation of the rigid body. Arnold showed that the classical results of rigid body motion were reproduced by this method. This example is discussed at length in Holm, et.al. [1983].

4.2.1. Energy-Casimir analysis of a rigid body

In this section we consider the example of a free rigid body. In particular we wish to asses nonlinear stability about the equilibria associated with a freely rotating rigid body. This example is standard and will serve to illustrate the energy-Casimir method and it relationship with the energy-momentum method. For the rigid body we have the configuration space $\mathcal{C} = SO(3)$, with the Hamiltonian given by

$$H = \tfrac{1}{2}\mathbf{\Pi} \cdot \mathbf{J}^{-1}\mathbf{\Pi} \tag{4.9}$$

where $\mathbf{\Pi} \in so(3)^*$ is the convected momentum vector and $\mathbf{J}^{-1}$ is the time invariant inertia matrix.

Observing that H is invariant under spatial isometries we we can reduce it by this symmetry. Physically this means that we ignore rotation by considering the reduced phase space $\mathcal{P} = T^*SO(3)/SO(3)$. The Lie-Poisson bracket on this space is

$$\{F, G\} = -\mathbf{\Pi} \cdot (\nabla_\pi F \times \nabla_\pi G) \tag{4.10}$$

(see Simo, Marsden, Krishnaprasad [1988] for details).

We note that

$$C_\phi = \tfrac{1}{2}\phi(||\mathbf{\Pi}||^2), \tag{4.11}$$

is a family of Casimirs for this system since, for any F

$$\begin{aligned}\{F, G\} &= -\mathbf{\Pi} \cdot (\phi'\mathbf{\Pi} \times \nabla_\pi F),\\ &= -\mathbf{\Pi} \times \phi'\mathbf{\Pi} \cdot \nabla_\pi F,\\ &= 0.\end{aligned} \tag{4.12}$$

If we add this Casimir to the Hamiltonian and take the first variation,

$$\delta(H + C) = \mathbf{J}^{-1}\mathbf{\Pi} \cdot \delta\mathbf{\Pi} + \phi'(||\mathbf{\Pi}||^2)\mathbf{\Pi} \cdot \delta\mathbf{\Pi}. \tag{4.13}$$

The conditions for $\delta(H + C) = 0$ can be written

$$(\mathbf{J}^{-1} + \mathbf{1}\phi'(||\mathbf{\Pi}||^2))\mathbf{\Pi} = 0 \tag{4.14}$$

which is in the form of an eigenvalue problem. Clearly, there are three values of ϕ' $(\lambda_1, \lambda_2, \lambda_3)$ and three associated values of $\mathbf{\Pi}$. These correspond to the inverses of the principle values of inertia of $\mathbf{J}$ and the associated directions. The three values of $\mathbf{\Pi}$ correspond to the reduced equilibria of the rigid body.

Taking the second variation

$$\delta^2(H+C) = \mathbf{J}^{-1}\delta\mathbf{\Pi}\cdot\delta\mathbf{\Pi} + 2\phi'(\|\mathbf{\Pi}\|^2)(\mathbf{\Pi}\cdot\delta\mathbf{\Pi})^2 + \phi'(\|\mathbf{\Pi}\|^2)\delta\mathbf{\Pi}\cdot\delta\mathbf{\Pi}. \tag{4.15}$$

Since this is finite dimensional we need only establish formal stability to ascertain the stability of rotations about particular equilibrium. Thus, we seek ϕ'' which make $\delta^2(H+C)$ positive or negative definite while at the same time satisfy $\delta(H+C)=0$. Since $\mathbf{J}^{-1}$ is symmetric we can transform it into a diagonal matrix. In this case $\mathbf{J}^{-1}\mathbf{P} = \mathbf{PS}$ where $\mathbf{P} = [\mathbf{\Pi}^{(1)}, \mathbf{\Pi}^{(2)}, \mathbf{\Pi}^{(3)}]$, and $\mathrm{S} = \mathrm{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})$. Thus for $\phi' = \lambda_1^{-1}$, $\mathbf{\Pi} = \mathbf{\Pi}^{(1)}$ we have for positive definiteness the requirement

$$0 < (\lambda_2^{-1} - \lambda_1^{-1})\delta\mathbf{\Pi}_2\cdot\delta\mathbf{\Pi}_2 + (\lambda_3^{-1} - \lambda_1^{-1})\delta\mathbf{\Pi}_3\cdot\delta\mathbf{\Pi}_3 + \phi''(\tfrac{1}{2}\|\mathbf{\Pi}^{(1)}\|^2)\delta\mathbf{\Pi}_1\cdot\delta\mathbf{\Pi}_1 \tag{4.16}$$

which is satisfied for

$$\lambda_2 < \lambda_1 \qquad \lambda_3 < \lambda_1 \qquad \phi(x) = (-2/\lambda_1)x + (x-\tfrac{1}{2})^2$$

implying that rotation about the longest axis is stable.

Similarly, for rotation about the shortest axis we require

$$\lambda_3 < \lambda_1 \qquad \lambda_3 < \lambda_2$$

and choose $\phi(x) = (-2/\lambda_3)x - (x-\frac{1}{2})^2$ to assure that $\delta^2(H+C) < 0$

Note that in the above we were able to find a Casimir and exploit the freedom it gave us to establish positive or negative definiteness in establishing formal stability. Suppose however we were unable to find a large enough family of Casimirs or any Casimirs at all. Such is the case for the convected representation of geometrically nonlinear rods. Similar results have been reported by other researchers in the area of compressible Euler plasma described in spatial coordinates [Marsden]. In this case we employ a more general method known as the energy-momentum method. This method is discussed in the second half of this chapter.

4.3. Energy-Casimir Examples

In this section we apply the energy-Casimir method to the example of the linear extensible shear beam attached to the rigid body. This example was discussed earlier sections 2.8 and 3.2. Recall that we consider a rigid body to which a long, flexible appendage is attached. A coordinate reference frame is fixed in the rigid body with the origin at the center of mass of the rigid body. The flexible attachment is assumed to lie along the second coordinate axis when the configuration is at rest. We note that this section is an extension of the analysis done in Krishnaprasad and Marsden [1986]. In what follows we work in the convected representation.

We are interested in the stability of the system about equilibria points. Recall that these equilibria will satisfy,

$$0 = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{a} \times \mathbf{K}\frac{\partial \boldsymbol{r}}{\partial s}\bigg|_{s=0} - \boldsymbol{r}(L) \times \mathbf{K}\,\hat{\mathbf{e}}_2 + \int_0^L \frac{\partial \boldsymbol{r}}{\partial s} \times \mathbf{K}\frac{\partial \boldsymbol{r}}{\partial s}\,ds, \tag{4.17}$$

$$0 = \rho_A^{-1}\boldsymbol{m} + \boldsymbol{r} \times \boldsymbol{\omega}, \tag{4.18}$$

$$0 = \mathbf{K}\frac{\partial^2 \boldsymbol{r}}{\partial s^2} + \boldsymbol{m} \times \boldsymbol{\omega}. \tag{4.19}$$

Two boundary values are associated with these equations,

$$\frac{\partial \boldsymbol{r}}{\partial s}\bigg|_{s=L} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2, \qquad \text{and} \qquad \boldsymbol{r}\bigg|_{s=0} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} = \mathbf{a}. \tag{4.20}$$

We now proceed to carry out the first steps of the energy-Casimir method to verify formal stability.

4.3.1. Computation of the First and Second Variations

In this section we compute the first and second variations of the Hamiltonian plus the Casimir function, $\mathbf{H} + \mathbf{C}_\phi$. From the previous definitions of these we know

$$\mathbf{H} = \tfrac{1}{2}\mathbf{J}^{-1}\boldsymbol{p}\cdot\boldsymbol{p} + \tfrac{1}{2}\int_0^L \rho_A^{-1}\|\boldsymbol{m}(s)\|^2\,ds + \tfrac{1}{2}\int_0^L \mathbf{K}\frac{\partial \boldsymbol{r}}{\partial s}\cdot\frac{\partial \boldsymbol{r}}{\partial s}\,ds, \tag{4.21}$$

and the Casimir function may be taken to be

$$\mathbf{C}_\phi = \tfrac{1}{2}\phi(\|\boldsymbol{p} + \int_0^L \boldsymbol{r} \times \boldsymbol{m}\;ds\|^2). \tag{2.2}$$

We will denote the first and second variations by $\delta(\mathbf{H} + \mathbf{C}_\phi)$, and $\delta^2(\mathbf{H} + \mathbf{C}_\phi)$. Note that because of the distributed nature of the system we are dealing with we will need to compute variational derivatives instead of ordinary gradients.

4.3.1.1. Computation of the First Variation

For the first variation we get

$$
\begin{aligned}
\delta(\mathrm{H}+\mathrm{C}_\phi) &= \mathbf{J}^{-1}\boldsymbol{p}\cdot\delta\boldsymbol{p} + \int_0^L \rho_A^{-1}\boldsymbol{m}\cdot\delta\boldsymbol{m}\,ds - \int_0^L \mathrm{K}\frac{\partial^2\boldsymbol{r}}{\partial s^2}\cdot\delta\boldsymbol{r}\,ds \\
&\quad + \phi'(\|\boldsymbol{\alpha}\|^2)\,\boldsymbol{\alpha}\cdot(\delta\boldsymbol{p} + \int_0^L \boldsymbol{r}\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}\,ds), \qquad (4.22)\\
&= (\mathbf{J}^{-1}\boldsymbol{p} + \phi'(\|\boldsymbol{\alpha}\|^2))\cdot\delta\boldsymbol{p} \\
&\quad \int_0^L (\rho_A\boldsymbol{m} + \phi'(\|\boldsymbol{\alpha}\|^2)\boldsymbol{\alpha}\times\boldsymbol{r})\cdot\delta\boldsymbol{m}\,ds \\
&\qquad \int_0^L (-\mathrm{K}\frac{\partial^2\boldsymbol{r}}{\partial s^2} - \phi'(\|\boldsymbol{\alpha}\|^2)\boldsymbol{\alpha}\times\boldsymbol{m})\cdot\delta\boldsymbol{r}\,ds. \qquad (4.23)
\end{aligned}
$$

where we have integrated by parts and used the vector identity $\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c} = \boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}$. At an equilibrium we require $\delta(H + C_\phi) = 0$, which is satisfied for

$$\phi'(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e = -\boldsymbol{\omega}^e, \qquad (4.24)$$

$$\phi'(\|\boldsymbol{\alpha}^e\|^2)\,\boldsymbol{\alpha}^e\times\boldsymbol{r}^e = -\rho_A^{-1}\boldsymbol{m}^e, \qquad (4.25)$$

$$\phi'(\|\boldsymbol{\alpha}^e\|^2)\,\boldsymbol{\alpha}^e\times\boldsymbol{m}^e = -\mathrm{K}\frac{\partial^2\boldsymbol{r}^e}{\partial s^2}, \qquad (4.26)$$

where $\boldsymbol{\omega}^e = \mathbf{J}^{-1}\boldsymbol{p}^e$, and

$$\boldsymbol{\alpha}^e = \boldsymbol{p}^e + \int_0^L \boldsymbol{r}^e\times\boldsymbol{m}^e\,ds. \qquad (4.27)$$

We use the superscript e to denote evaluation at an equilibrium. If we dot (4.24) with $\boldsymbol{\alpha}^e$ we have

$$\phi'(\|\boldsymbol{\alpha}^e\|^2) = -\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}. \qquad (4.28)$$

Physically, the first condition implies that rotation takes place about a principle axis of inertia of the extended inertia matrix. If we substitute (4.28) into (4.25) and (4.26) we get

$$
\begin{aligned}
0 &= \rho_A^{-1}\boldsymbol{m}^e - \mathbf{J}^{-1}\boldsymbol{p}^e\times\boldsymbol{r}^e,\\
0 &= \mathrm{K}\frac{\partial^2\boldsymbol{r}^e}{\partial s^2} - \mathbf{J}^{-1}\boldsymbol{p}^e\times\boldsymbol{m}^e,
\end{aligned}
$$

which are exactly the equilibrium equations in Krishnaprasad and Marsden [1987]. Using the second of these equations in the definition of $\boldsymbol{\alpha}$ we get

$$\boldsymbol{\alpha}^e = \boldsymbol{p}^e + \int_0^L \boldsymbol{r}^e\times\rho_A(\mathbf{J}^{-1}\boldsymbol{p}^e\times\boldsymbol{r}^e)\,ds,$$

$$= \boldsymbol{p}^e - \int_0^L \rho_A \boldsymbol{r}^e \times (\boldsymbol{r}^e \times \mathbf{J}^{-1}\boldsymbol{p}^e)\, ds,$$
$$= \mathbf{J}\boldsymbol{\omega}^e - \int_0^L \rho_A(\|\boldsymbol{r}^e\|^2 \mathbf{1} - \boldsymbol{r}^e \otimes \boldsymbol{r}^e)\boldsymbol{\omega}^e\, ds,$$
$$= \mathbf{J}_\infty \boldsymbol{\omega}^e,$$

where we define the extended inertia dyadic as

$$\mathbf{J}_\infty = \mathbf{J} - \int_0^L \rho_A(\|\boldsymbol{r}^e\|^2 \mathbf{1} - \boldsymbol{r}^e \otimes \boldsymbol{r}^e)\, ds.$$

4.3.1.2. Computation of the Second Variation

For the second variation, the starting point is the expression for the first variation. The terms arising from the original Hamiltonian are straight forward to compute, they are

$$\delta(\mathbf{J}^{-1}\boldsymbol{p} \cdot \delta\boldsymbol{p}) = \mathbf{J}^{-1}\delta\boldsymbol{p} \cdot \delta\boldsymbol{p}, \tag{4.29}$$
$$\delta\Big(\int_0^L \rho_A^{-1}\boldsymbol{m} \cdot \delta\boldsymbol{m}\, ds\Big) = \int_0^L \rho_A^{-1}\delta\boldsymbol{m} \cdot \delta\boldsymbol{m}\, ds, \tag{4.30}$$
$$\delta\Big(\int_0^L \mathbf{K}\frac{\partial^2 \boldsymbol{r}}{\partial s^2} \cdot \delta\boldsymbol{r}\, ds\Big) = \int_0^L \mathbf{K}\frac{\partial^2 \delta\boldsymbol{r}}{\partial s^2} \cdot \delta\boldsymbol{r}\, ds. \tag{4.31}$$

Note that we can use the boundary conditions on $\delta\boldsymbol{r}$ to get

$$\int_0^L \mathbf{K}\frac{\partial^2 \delta\boldsymbol{r}}{\partial s^2} \cdot \delta\boldsymbol{r}\, ds = -\int_0^L \mathbf{K}\frac{\partial \delta\boldsymbol{r}}{\partial s} \cdot \frac{\partial \delta\boldsymbol{r}}{\partial s}\, ds. \tag{4.32}$$

Next we consider the component which arises from the Casimir function which we added to the Hamiltonian. From the first factor of this term we compute,

$$\delta\phi'(\boldsymbol{\alpha}) = 2\,\phi''(\|\boldsymbol{\alpha}\|^2)\,\boldsymbol{\alpha} \cdot \Big(\delta\boldsymbol{p} + \int_0^L \boldsymbol{r} \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}\, ds\Big). \tag{4.33}$$

From the second factor of the Casimir term we compute

$$\begin{aligned}\delta\Big(\boldsymbol{\alpha} \cdot \Big(\delta\boldsymbol{p} + \int_0^L \boldsymbol{r} \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}\, ds\Big)\Big) = &\\ \Big\|\delta\boldsymbol{p} + \int_0^L \boldsymbol{r} \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}\, ds\Big\|^2 &\\ + 2\Big(\boldsymbol{p} + \int_0^L \boldsymbol{r} \times \boldsymbol{m}\, ds\Big) \cdot \Big(\int_0^L \delta\boldsymbol{r} \times \delta\boldsymbol{m}\, ds\Big).&\end{aligned} \tag{4.34}$$

We use the above to get the expression for the second variation

$$
\begin{aligned}
\delta^2(\mathbf{H}+\mathbf{C}_\phi) = {} & \mathbf{J}^{-1}\delta \boldsymbol{p}\cdot\delta\boldsymbol{p} + \int_0^L \rho_A^{-1}\delta\boldsymbol{m}\cdot\delta\boldsymbol{m}\,ds + \int_0^L \mathbf{K}\frac{\partial\delta\boldsymbol{r}}{\partial s}\cdot\frac{\partial\delta\boldsymbol{r}}{\partial s}\,ds \\
& + 2\,\phi''(\|\boldsymbol{\alpha}\|^2)\,(\boldsymbol{\alpha}\cdot(\delta\boldsymbol{p} + \int_0^L \boldsymbol{r}\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}\,ds))^2 \\
& + \phi'(\|\boldsymbol{\alpha}\|^2)\{\|\delta\boldsymbol{p} + \int_0^L \boldsymbol{r}\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}\,ds\|^2 \\
& + 2(\boldsymbol{p} + \int_0^L \boldsymbol{r}\times\boldsymbol{m}\,ds)\cdot(\int_0^L \delta\boldsymbol{r}\times\delta\boldsymbol{m}\,ds)\}.
\end{aligned}
\tag{4.35}
$$

4.3.2. Computation of a Stability Criterion

If we evaluate the second variation at an equilibrium we can derive conditions which assure the stability of the equilibrium. In the following sequence of steps we demonstrate how this is done.

Step 1 : Evaluate the Second Variation at an Equilibrium

Recall the second variation. If we use the above to substitute for $\phi'(\|\boldsymbol{\alpha}^e\|^2)$ in this expression and rearrange slightly we find that

$$
\begin{aligned}
\delta^2(\mathbf{H}+\mathbf{C}_\phi)_{(p^e,r^e,m^e)} = {} & \mathbf{J}^{-1}\delta \boldsymbol{p}\cdot\delta\boldsymbol{p} + \int_0^L \rho_A^{-1}\delta\boldsymbol{m}\cdot\delta\boldsymbol{m}\,ds + \int_0^L \mathbf{K}\frac{\partial\delta\boldsymbol{r}}{\partial s}\cdot\frac{\partial\delta\boldsymbol{r}}{\partial s}\,ds \\
& - \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\,\|\delta\boldsymbol{p} + \int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds\|^2 \\
& - 2\,\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\,\boldsymbol{\alpha}^e\cdot(\int_0^L \delta\boldsymbol{r}\times\delta\boldsymbol{m}\,ds) \\
& + 2\,\phi''(\|\boldsymbol{\alpha}^e\|^2)\,(\boldsymbol{\alpha}^e\cdot(\delta\boldsymbol{p} + \int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds))^2,
\end{aligned}
\tag{4.36}
$$

which corresponds to expression (5.5) in Krishnaprasad and Marsden [1986]. In that paper, ϕ is required to satisfy the condition:

$$
\phi''(\|\boldsymbol{\alpha}^e\|^2) = \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{2\,\|\boldsymbol{\alpha}^e\|^4},
\tag{4.37}
$$

which is consistent with (4.28). In the following development we impose no conditions on $\phi''(\|\boldsymbol{\alpha}^e\|^2)$ at this time.

Step 2: Expand Terms Containing δp

We first note that the fourth and sixth terms in (4.36) can be expanded. For the fourth term we have

$$
\begin{aligned}
-\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\,\|\delta\boldsymbol{p}+\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds\|^2 \\
&= -\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\delta\boldsymbol{p}\cdot\delta\boldsymbol{p} \\
&\quad - 2\,\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\delta\boldsymbol{p}\cdot(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds) \\
&\quad - \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\|\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds\|^2,
\end{aligned}
\tag{4.38}
$$

while for the sixth term,

$$
\begin{aligned}
2\,\phi''(\|\boldsymbol{\alpha}^e\|^2)\,(\boldsymbol{\alpha}^e\cdot(\delta\boldsymbol{p}+\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds))^2 \\
&= 2\phi''(\|\boldsymbol{\alpha}^e\|^2)(\boldsymbol{\alpha}^e\cdot\delta\boldsymbol{p})^2 \\
&\quad + 4\phi''(\|\boldsymbol{\alpha}^e\|^2)(\boldsymbol{\alpha}^e\cdot\delta\boldsymbol{p})\,(\boldsymbol{\alpha}^e\cdot(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds)) \\
&\quad + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)(\boldsymbol{\alpha}^e\cdot(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds))^2.
\end{aligned}
\tag{4.39}
$$

Step 3: Collect Terms Containing δp

Now, collect together terms in which the quantity $\delta\boldsymbol{p}$ appears. Our expression for the second variation at an equilibrium can then be written

$$
\begin{aligned}
\delta^2(\mathbf{H}+\mathbf{C}_\phi) = \Big[&\mathbf{J}^{-1}\delta\boldsymbol{p}\cdot\delta\boldsymbol{p}-\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}(\delta\boldsymbol{p}\cdot\delta\boldsymbol{p}+2\delta\boldsymbol{p}\cdot(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds)) \\
&+2\phi''(\|\boldsymbol{\alpha}^e\|^2)\{(\boldsymbol{\alpha}^e\cdot\delta\boldsymbol{p})^2+2(\boldsymbol{\alpha}^e\cdot\delta\boldsymbol{p})(\boldsymbol{\alpha}^e\cdot(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds))\}\Big] \\
&-\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\|\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds\|^2 \\
&+2\phi''(\|\boldsymbol{\alpha}^e\|^2)(\boldsymbol{\alpha}^e\cdot(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds+\int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds))^2 \\
&-2\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e\cdot(\int_0^L \delta\boldsymbol{r}\times\delta\boldsymbol{m}\,ds)+\int_0^L \rho_A^{-1}\delta\boldsymbol{m}\cdot\delta\boldsymbol{m}\,ds+\int_0^L \mathbf{K}\frac{\partial\delta\boldsymbol{r}}{\partial s}\cdot\frac{\partial\delta\boldsymbol{r}}{\partial s}\,ds.
\end{aligned}
\tag{4.40}
$$

Step 4: Complete the Square

The term in square brackets which contains the $\delta\boldsymbol{p}$ terms can be rewritten

$$\Big[\ \cdot\ \Big] = (\mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{1} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e)\delta\boldsymbol{p}\cdot\delta\boldsymbol{p}$$
$$+\,2(-\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{1} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e)\delta\boldsymbol{p}\cdot(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds). \tag{4.41}$$

In this expression we use $\otimes$ to denote the tensor product and $\mathbf{1}$ the identity. Note that $\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e$ is a tensor of rank 2. We can complete the square for this expression provided the quantity

$$\mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{1} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e, \tag{4.42}$$

has an inverse.

We next assume this inverse exists and define the two symmetric matrices M and N by,

$$\mathrm{M}^T\mathrm{M} = \mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{1} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e,$$
$$\triangleq \mathbf{J}_e^{-1}, \tag{4.43}$$

$$\mathrm{N}^T\mathrm{M} = -\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{1} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e,$$
$$\triangleq \mathbf{Q}_e. \tag{4.44}$$

Completing the square for the term in brackets we now get

$$\Big[\cdot\Big] = \|\,\mathrm{M}\delta\boldsymbol{p} + \mathrm{N}(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds)\,\|^2$$
$$-\,\mathrm{N}^T\mathrm{N}(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds)\cdot(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds). \tag{4.45}$$

The term in braces is bounded below by a perfect square when $\mathrm{N}^T\mathrm{N}\geq \mathbf{0}$. For this to be the case we need to assume that the inverted matrix, $\mathbf{J}_e^{-1}$ is positive definite, in general it need not be. Note that this assumption will impose conditions on $\phi''(\|\boldsymbol{\alpha}^e\|^2)$. The requirements on the parameters in this matrix to assure it is strictly positive definite will be expressed in the form of inequalities. These inequalities will be the first conditions that we need to assure stability.

Step 5: The Reformulated Second Variation

The second variation at an equilibrium is thus of the form

$$
\begin{aligned}
\delta^2(\mathbf{H}+\mathbf{C}_\phi) = \quad & (square) \\
& - \mathbf{N}^T\mathbf{N}(\int_0^L \boldsymbol{r}^e \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}^e\, ds) \\
& \qquad\qquad\qquad \cdot (\int_0^L \boldsymbol{r}^e \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}^e\, ds) \\
& - \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\|(\int_0^L \boldsymbol{r}^e \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}^e\, ds)\|^2 \\
& + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e(\int_0^L \boldsymbol{r}^e \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}^e\, ds) \\
& \qquad\qquad\qquad \cdot (\int_0^L \boldsymbol{r}^e \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}^e\, ds) \\
& - 2\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e \cdot (\int_0^L \delta\boldsymbol{r} \times \delta\boldsymbol{m}\, ds) + \int_0^L \rho_A^{-1}\delta\boldsymbol{m} \cdot \delta\boldsymbol{m}\, ds + \int_0^L \mathbf{K}\frac{\partial \delta\boldsymbol{r}}{\partial s} \cdot \frac{\partial \delta\boldsymbol{r}}{\partial s}\, ds.
\end{aligned}
\tag{4.46}
$$

Where we note that

$$
\begin{aligned}
\mathbf{N}^T\mathbf{N} = & (-\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{1} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e) \\
& (\mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{1} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e)^{-1}(-\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{1} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e), \\
= & \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e.
\end{aligned}
\tag{4.47}
$$

Step 6: Collect Integrals of Cross Products

Collecting terms containing the integrals of cross products the second variation can be written

$$
\begin{aligned}
\delta^2(\mathbf{H}+\mathbf{C}_\phi) = \quad & (square) \\
& - (\mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e - \mathbf{Q}_e)(\int_0^L \boldsymbol{r}^e \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}^e\, ds) \\
& \qquad\qquad\qquad \cdot (\int_0^L \boldsymbol{r}^e \times \delta\boldsymbol{m}\, ds + \int_0^L \delta\boldsymbol{r} \times \boldsymbol{m}^e\, ds) \\
& - 2\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e \cdot (\int_0^L \delta\boldsymbol{r} \times \delta\boldsymbol{m}\, ds) + \int_0^L \rho_A^{-1}\delta\boldsymbol{m} \cdot \delta\boldsymbol{m}\, ds + \int_0^L \mathbf{K}\frac{\partial \delta\boldsymbol{r}}{\partial s} \cdot \frac{\partial \delta\boldsymbol{r}}{\partial s}\, ds.
\end{aligned}
\tag{4.48}
$$

Step 7: A Vector Identity

Observe that a simple vector identity enables us to write

$$2\,\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e\cdot\Big(\int_0^L \delta\boldsymbol{r}\times\delta\boldsymbol{m}\,ds\Big) = 2\int_0^L\Big(\frac{\boldsymbol{\alpha}^e\cdot\boldsymbol{\omega}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e\times\delta\boldsymbol{r}\Big)\cdot\delta\boldsymbol{m}\,ds$$
$$= 2\int_0^L \delta\boldsymbol{m}^T\mathrm{S}\Big(\frac{(\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e)\boldsymbol{\omega}^e}{\|\boldsymbol{\alpha}^e\|^2}\Big)\delta\boldsymbol{r}\,ds, \qquad (4.49)$$

where we have used the skew-symmetric matrix $S(\mathbf{x})$ associated with the cross-product

$$\mathrm{S}(\mathbf{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \qquad (4.50)$$

Step 8: A Quadratic Form

Now define the symmetric matrix

$$\mathbf{R} \triangleq \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e - \mathbf{Q}_e. \qquad (4.51)$$

We will see below, that an eigenvalue estimate relies on having $\mathbf{R}$ nonnegative definite. We thus require that conditions on the parameters of the problem and $\phi''(\|\boldsymbol{\alpha}^e\|^2)$ hold such that $\mathbf{J}_e^{-1}$ defined in (4.43) is positive definite *and* $\mathbf{R}$ defined in (4.51) is nonnegative definite. The latter will assure that $\mathbf{R}$ has a square root $\mathbf{R}^{1/2}$. We will examine these assumptions again in remark 2 below.

Expanding the second term in (4.48), we can re-express it as a quadratic form,

$$\mathbf{R}\Big(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds\Big)\cdot\Big(\int_0^L \boldsymbol{r}^e\times\delta\boldsymbol{m}\,ds + \int_0^L \delta\boldsymbol{r}\times\boldsymbol{m}^e\,ds\Big)$$
$$= \int_0^L\int_0^L \mathbf{R}(\mathrm{S}(\boldsymbol{r}^e(s))\delta\boldsymbol{m}(s) - \mathrm{S}(\boldsymbol{m}^e(s))\delta\boldsymbol{r}(s))$$
$$\cdot(\mathrm{S}(\boldsymbol{r}^e(\sigma))\delta\boldsymbol{m}(\sigma) - \mathrm{S}(\boldsymbol{m}^e(\sigma))\delta\boldsymbol{r}(\sigma))\,ds\,d\sigma,$$
$$= \int_0^L\int_0^L [\,\delta\boldsymbol{m}^T(s)\quad \delta\boldsymbol{r}^T(s)\,]\begin{bmatrix}\mathrm{S}^T(\boldsymbol{r}^e(s))\\ -\mathrm{S}^T(\boldsymbol{m}^e(s))\end{bmatrix}$$
$$\mathbf{R}\,[\,\mathrm{S}(\boldsymbol{r}^e(\sigma))\quad -\mathrm{S}(\boldsymbol{m}^e(\sigma))\,]\begin{bmatrix}\delta\boldsymbol{m}(\sigma)\\ \delta\boldsymbol{r}(\sigma)\end{bmatrix}\,ds\,d\sigma,$$
$$= \int_0^L\int_0^L [\,\delta\boldsymbol{m}^T(s)\quad \delta\boldsymbol{r}^T(s)\,]\,\mathbf{A}^T(s)\mathbf{A}(\sigma)\begin{bmatrix}\delta\boldsymbol{m}(\sigma)\\ \delta\boldsymbol{r}(\sigma)\end{bmatrix}\,ds\,d\sigma. \qquad (4.52)$$

We now can find a lower bound on the above. The bound we want is obtained from an eigenvalue inequality which we introduce by way of the following lemma

Step 9: An Eigenvalue Inequality

Lemma 4.5. *Let* $\mathbf{A}(s) \in L_2^{n\times n}(0,L)$, *and* $\mathbf{x}(s) \in L_2^n(0,L)$ *then*

$$\int_0^L \int_0^L \mathbf{x}^T(s)\mathbf{A}^T(s)\mathbf{A}(\sigma)\mathbf{x}(\sigma)\,d\sigma\,ds \leq \int_0^L \mathbf{x}^T(s)\{\int_0^L \mathbf{1}\lambda^2(\sigma)\,d\sigma\}\mathbf{x}(s)\,ds, \qquad (4.53)$$

where $\lambda^2(s)$ *is the maximum eigenvalue of* $\mathbf{A}^T(s)\mathbf{A}(s)$.

Proof: Let $||\cdot||$ denote the standard norm in Euclidean space and also the induced matrix norm associated with it. Then

$$\begin{aligned}\int_0^L \int_0^L \mathbf{x}^T(s)\mathbf{A}^T(s)\mathbf{A}(\sigma)\mathbf{x}(\sigma)\,d\sigma\,ds &\leq \int_0^L \int_0^L |\mathbf{x}^T(s)\mathbf{A}^T(s)\mathbf{A}(\sigma)\mathbf{x}(\sigma)|\,d\sigma\,ds,\\ &\leq \int_0^L \int_0^L ||\mathbf{A}(s)\mathbf{x}(s)||\,||\mathbf{A}(\sigma)\mathbf{x}(\sigma)||\,d\sigma\,ds,\\ &\leq \int_0^L ||\mathbf{A}(s)||\,||\mathbf{x}(s)||\,ds \int_0^L ||\mathbf{A}(\sigma)||\,||\mathbf{x}(\sigma)||\,d\sigma,\end{aligned}$$

where we have used $||\mathbf{A}(s)\mathbf{x}(s)|| \leq ||\mathbf{A}(s)||\,||\mathbf{x}(s)||$. We can now use the Schwarz inequality

$$(\int_0^L ||\mathbf{A}(s)||\,||\mathbf{x}(s)||\,ds)^2 \leq \int_0^L ||\mathbf{A}(s)||^2\,ds \int_0^L ||\mathbf{x}(s)||^2\,ds.$$

Finally noting that the value of $||\mathbf{A}(s)||$ is simply the square root of the maximum eigenvalue of $\mathbf{A}^T(s)\mathbf{A}(s)$ establishes the result. ∎

If we let $\lambda^2(s)$ be the maximum eigenvalue of

$$\mathbf{A}^T(s)\mathbf{A}(s) = \begin{bmatrix} \mathbf{S}^T(\boldsymbol{r}^e(s))\mathbf{R}\mathbf{S}(\boldsymbol{r}^e(s)) & -\mathbf{S}^T(\boldsymbol{r}^e(s))\mathbf{R}\mathbf{S}(\boldsymbol{m}^e(s)) \\ -\mathbf{S}^T(\boldsymbol{m}^e(s))\mathbf{R}\mathbf{S}(\boldsymbol{r}^e(s)) & \mathbf{S}^T(\boldsymbol{m}^e(s))\mathbf{R}\mathbf{S}(\boldsymbol{m}^e(s)) \end{bmatrix}, \qquad (4.54)$$

and let $\tilde{\lambda}^2 = \int_0^L \lambda^2(s)\,ds$ then we have by way of lemma 3.1 a lower bound on the second variation

$$\begin{aligned} D^2(\mathbf{H}+\mathbf{C}_\phi)_{(p^e,r^e,m^e)} \geq\ & (\textit{square}) \\ & - \tilde{\lambda}^2 \int_0^L \delta\boldsymbol{m}^T\delta\boldsymbol{m}\,ds - \tilde{\lambda}^2 \int_0^L \delta\boldsymbol{r}^T\delta\boldsymbol{r}\,ds \\ & - 2\int_0^L \delta\boldsymbol{m}^T\mathbf{S}(\frac{(\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e)\omega^e}{||\boldsymbol{\alpha}^e||^2})\delta\boldsymbol{r}\,ds \\ & + \int_0^L \rho_A^{-1}\delta\boldsymbol{m}^T\delta\boldsymbol{m}\,ds + \int_0^L \mathbf{K}\frac{\partial\delta\boldsymbol{r}}{\partial s}\cdot\frac{\partial\delta\boldsymbol{r}}{\partial s}\,ds. \end{aligned} \qquad (4.55)$$

Step 10: A Poincaré Type Inequality

If we assume that $\mathbf{K}$ is diagonal and use a Poincaré-type inequality

$$\int_0^L \mathbf{K}\frac{\partial \delta r}{\partial s} \cdot \frac{\partial \delta r}{\partial s}\, ds \geq c \int_0^L \mathbf{K}\delta r \cdot \delta r\, ds, \tag{4.56}$$

with $c = (\frac{\pi}{2L})^2$, then the second variation can be bounded below as

$$\begin{aligned} D^2(\mathbf{H} + \mathbf{C}_\phi)_{(p^e,r^e,m^e)} \geq (\quad & square \quad) \\ & - \tilde{\lambda}^2 \int_0^L \delta m^T \delta m\, ds - \tilde{\lambda}^2 \int_0^L \delta r^T \delta r\, ds \\ & - 2 \int_0^L \delta m^T \mathbf{S}(\frac{(\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e)\omega^e}{\|\boldsymbol{\alpha}^e\|^2})\delta r\, ds \\ & + \int_0^L \rho_A^{-1} \delta m^T \delta m\, ds + c \int_0^L \delta r^T \mathbf{K} \delta r\, ds. \end{aligned} \tag{4.57}$$

Step 11: Rewrite The Lower Bound

We can reformulate the lower bound in a clearer form as follows

$$\begin{aligned} D^2(\mathbf{H} + \mathbf{C}_\phi)_{(p^e,r^e,m^e)} \geq \; & (square) \\ + \int_0^L & \begin{bmatrix} \rho_A^{-1}\mathbf{1} - \mathbf{1}\tilde{\lambda}^2 & -\mathbf{S}(\frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\omega^e) \\ -\mathbf{S}^T(\frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\omega^e) & c\mathbf{K} - \mathbf{1}\tilde{\lambda}^2 \end{bmatrix} \begin{bmatrix} \delta m \\ \delta r \end{bmatrix} \cdot \begin{bmatrix} \delta m \\ \delta r \end{bmatrix} ds \end{aligned} \tag{4.58}$$

If we define the matrix

$$\mathbf{D}(p^e, r^e, m^e) = \begin{bmatrix} \rho_A^{-1}\mathbf{1} - \mathbf{1}\tilde{\lambda}^2 & -\mathbf{S}(\frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\omega^e) \\ -\mathbf{S}^T(\frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\omega^e) & c\mathbf{K} - \mathbf{1}\tilde{\lambda}^2 \end{bmatrix}, \tag{4.59}$$

then we can state the following theorem;

Theorem 4.6. *If the matrix $\mathbf{R} = \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e - \mathbf{Q}_e$ defined in (4.51) exists and is non-negative definite, $\mathbf{J}_e$ defined in (4.43) is positive definite. and the matrix $\mathbf{D}$ defined in equation (4.59) is positive definite, then the system described by equations (4.17)-(4.19) is nonlinearly (formally) stable at the equilibrium point (p^e, r^e, m^e).* ∎

Remark 1: This result establishes only formal stability, since it is based on the the definiteness of second variation. To establish rigorous stability of the nonlinear system one generally needs to examine convexity estimates as is done in [2].

Remark 2: Note that if $\mathbf{Q}_e^{-1}$ exists and we use the matrix inversion lemma we obtain the following

$$\begin{aligned} (\mathbf{Q}_e^{-1} + \mathbf{J})^{-1} &= \mathbf{Q}_e - \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e, \\ &= -\mathbf{R}. \end{aligned} \tag{4.60}$$

Recall that we already have an assumption of nonnegative definiteness on $\mathbf{R}$. Thus we need to specify conditions on the parameters and $\phi''(\|\boldsymbol{\alpha}^e\|^2)$ such that

$$\mathbf{J}^{-1} + \mathbf{Q}_e > 0, \tag{4.61}$$

$$(\mathbf{J} + \mathbf{Q}_e^{-1})^{-1} \leq 0, \tag{4.62}$$

which are the same conditions as $\mathbf{R} \geq 0$ and $\mathbf{J}_e > 0$. In the examples of the next section $\mathbf{Q}_e$ is singular.

Remark 3: A better result can be had by observing that $\mathbf{A}^T(s)\mathbf{A}(s)$ is frequently in the form of a block diagonal matrix

$$\mathbf{A}^T(s)\mathbf{A}(s) = \begin{bmatrix} \mathbf{A}_1^T(s)\mathbf{A}_1(s) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_k^T(s)\mathbf{A}_k(s) \end{bmatrix}, \tag{4.63}$$

where $0 \leq k \leq 6$ and because of the semidefiniteness of $\mathbf{A}^T\mathbf{A}(s)$ some of the diagonal blocks may be zero. If we let $\lambda_i^2(s)$ be the maximum eigenvalue of $\mathbf{A}_i^T(s)\mathbf{A}_i(s)$, $0 \leq i \leq k$ then we can define

$$\mathbf{D}' = \begin{bmatrix} \rho_A^{-1}\mathbf{1} & -\mathbf{S}(\frac{\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\omega}^e) \\ -\mathbf{S}^T(\frac{\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\omega}^e) & c\mathbf{K} \end{bmatrix} - \begin{bmatrix} \mathbf{1}\tilde{\lambda}_1^2 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{1}\tilde{\lambda}_k^2 \end{bmatrix}. \tag{4.64}$$

Thus, if the conditions of theorem (4.6) are satisfied and also the matrix $\mathbf{D}'$ defined in equation (4.64) is positive definite, then the system described by equations (4.17)-(4.19) is (formally) nonlinearly stable at the equilibrium point $(\boldsymbol{p}^e,\ \boldsymbol{r}^e,\ \boldsymbol{m}^e)$. In theorem (4.65) this will mean the special choice $\lambda^2(s) = \max\{\lambda_1^2(s), \ldots, \lambda_k^2(s)\}$.

4.3.3. Some Examples

In this section we apply theorem (4.6). We will assume that that the linear extensible shear beam lies along the same direction as the second principal axis of inertia of the rigid body. From geometric considerations the position of the shear beam will cause the principal axes of the rigid-body-shear-beam configuration to lie in the same directions as those of the rigid body. In this case the addition of the shear beam will have the effect of increasing the moments of inertia about the first and the third principal axes. Because the linear extensible shear beam cannot deflect laterally the principal axes of the of the configuration remain fixed for any longitudinal extension of the shear beam. Thus, for this configuration there are three axes about which the equilibria can exist. These axes will correspond to the three principal axes of the rigid body.

4.3.3.1. A Trivial Equilibrium

The simplest case to be considered is when the rotation takes place about the axis along which the linear extensible shear beam lies. In this case the equilibrium will be

$$\boldsymbol{\omega}^e = \omega_2^e\, \hat{\mathbf{e}}_2, \tag{4.66}$$

$$\boldsymbol{r}^e = (a_2 + s)\, \hat{\mathbf{e}}_2; \qquad 0 \le s \le L, \tag{4.67}$$

$$\boldsymbol{m}^e = \mathbf{0}. \tag{4.68}$$

This describes the linear-extensible-shear-beam being unstretched.

What follows is a special case of the second variation computed in *Step 1* of the previous section. In this and the following example we will assume $\phi''(\|\boldsymbol{\alpha}^e\|^2)$ is the same as in [2], thus recall from (4.37) that if this is the case then

$$\phi''(\|\boldsymbol{\alpha}^e\|^2) = \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{2\|\boldsymbol{\alpha}^e\|^4}.$$

And the two quantities, $\mathbf{J}_e^{-1}$, and $\mathbf{Q}_e$, which we define in *Step 4* are

$$\mathbf{J}_e^{-1} = \mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\left(\mathbf{1} - \frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\right), \tag{4.69}$$

$$\mathbf{Q}_e = -\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\left(\mathbf{1} - \frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\right). \tag{4.70}$$

For our example, if we first compute

$$\boldsymbol{\alpha}^e = j_{22}\omega_2\, \hat{\mathbf{e}}_2, \tag{4.71}$$

then

$$\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e = j_{22}(\omega_2^e)^2, \qquad \text{and} \qquad \boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e = j_{22}^2(\omega_2^e)^2, \tag{4.72}$$

from which we immediately compute

$$\mathbf{1} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.73}$$

and finally,

$$\mathbf{J}_e^{-1} = \begin{bmatrix} \frac{j_{11}j_{22}}{j_{22}-j_{11}} & 0 & 0 \\ 0 & j_{22} & 0 \\ 0 & 0 & \frac{j_{33}j_{22}}{j_{22}-j_{33}} \end{bmatrix}, \tag{4.74}$$

$$\mathbf{Q}_e = \begin{bmatrix} \frac{1}{j_{22}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}} \end{bmatrix}. \tag{4.75}$$

For $\mathbf{J}_e$ to be positive definite we require $j_{22} > j_{11}$, and $j_{22} > j_{33}$. This will assure positive elements along the diagonal in the inverse above.

Thus, the quantity $\mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e$ which appears in the reformulated second variation of *Step 5* will be,

$$\begin{aligned} \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e &= \left(\mathbf{J}^{-1} - \frac{\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e}(1 - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e})\right)^{-1}\left(\frac{(\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e)^2}{(\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e)^2}(1 - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e})\right) \\ &= \left(\begin{bmatrix} \frac{1}{j_{11}} & 0 & 0 \\ 0 & \frac{1}{j_{22}} & 0 \\ 0 & 0 & \frac{1}{j_{33}} \end{bmatrix} - \begin{bmatrix} \frac{1}{j_{22}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}} \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} \frac{1}{j_{22}^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}^2} \end{bmatrix}\right), \\ &= \begin{bmatrix} \frac{j_{11}j_{22}}{j_{22}-j_{11}} & 0 & 0 \\ 0 & j_{22} & 0 \\ 0 & 0 & \frac{j_{33}j_{22}}{j_{22}-j_{33}} \end{bmatrix}\begin{bmatrix} \frac{1}{j_{22}^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}^2} \end{bmatrix}, \\ &= \begin{bmatrix} \frac{j_{11}}{j_{22}(j_{22}-j_{11})} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{j_{33}}{j_{22}(j_{22}-j_{33})} \end{bmatrix}, \end{aligned} \tag{4.76}$$

where we have used equation (4.47) and the fact that $\mathbf{J}_e$ and $\mathbf{Q}_e$ are diagonal.

We also need the skew symmetric matrix which appears in *Step 7*. Thus, we compute

$$\mathbf{S}(\frac{\alpha^e\alpha^{eT}}{\alpha^{eT}\alpha^e}\omega^e) = \begin{bmatrix} 0 & 0 & \omega_2^e \\ 0 & 0 & 0 \\ -\omega_2^e & 0 & 0 \end{bmatrix}. \tag{4.77}$$

Now we compute $\mathbf{R}$, which is defined in *Step 8*.

$$\begin{aligned} \mathbf{R} &= \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e + \frac{\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e}(1 - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e}), \\ &= \begin{bmatrix} \frac{j_{11}}{j_{22}(j_{22}-j_{11})} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{j_{33}}{j_{22}(j_{22}-j_{33})} \end{bmatrix} + \begin{bmatrix} \frac{1}{j_{22}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}} \end{bmatrix}, \\ &= \begin{bmatrix} \frac{1}{j_{22}-j_{11}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}-j_{33}} \end{bmatrix}, \end{aligned} \tag{4.78}$$

which, along with the definition of $\mathbf{S}(\cdot)$ in (4.79), we can now use to compute

$$\mathbf{S}^T(\boldsymbol{r}^e)\mathbf{R}\mathbf{S}(\boldsymbol{r}^e) = \begin{bmatrix} \frac{1}{j_{22}-j_{33}} r_2^{e2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}-j_{11}} r_2^{e2} \end{bmatrix}, \tag{4.80}$$

$$\mathbf{S}^T(\boldsymbol{r}^e)\mathbf{R}\mathbf{S}(\boldsymbol{m}^e) = \mathbf{0}, \tag{4.81}$$

$$\mathbf{S}^T(\boldsymbol{m}^e)\mathbf{R}\mathbf{S}(\boldsymbol{m}^e) = \mathbf{0}. \tag{4.82}$$

These matrices are used to form the matrix $\mathbf{A}^T(s)\mathbf{A}(s)$ in (4.64), note that it has only the two nonzero elements (computed in (4.81)). These correspond to the first and second diagonal elements. Hence, $\mathbf{A}^T(s)\mathbf{A}(s)$ is a diagonal matrix and the nonzero eigenvalues are these two elements. As a consequence we will use the modified bound described in *Remark 3.* Thus, the eigenvalue inequality is easily obtained.

After using the Poincaré inequality of *Step 10* we proceed to the final step and construct the $\mathbf{D}'$ matrix in (4.64). If we define,

$$d_{11} = \rho_A^{-1} - \frac{1}{j_{22} - j_{33}} \int_0^L r_2^{e2}\, ds, \qquad d_{33} = \rho_A^{-1} - \frac{1}{j_{22} - j_{11}} \int_0^L r_2^{e2}\, ds,$$

this matrix is

$$\mathbf{D}' = \begin{bmatrix} d_{11} & 0 & 0 & 0 & 0 & -\omega_2^e \\ 0 & \rho_A^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{33} & \omega_2^e & 0 & 0 \\ 0 & 0 & \omega_2^e & (\frac{\pi}{2L})^2 k_x & 0 & 0 \\ 0 & 0 & 0 & 0 & (\frac{\pi}{2L})^2 k_y & 0 \\ -\omega_2^e & 0 & 0 & 0 & 0 & (\frac{\pi}{2L})^2 k_z \end{bmatrix}. \tag{4.83}$$

To assure that the $\mathbf{D}'$ matrix is positive definite we require

$$j_{22} - j_{11} > \rho_A \int_0^L r_2^{e2}\, ds, \tag{4.84}$$

$$j_{22} - j_{33} > \rho_A \int_0^L r_2^{e2}\, ds, \tag{4.85}$$

and also,

$$\left(\rho_A^{-1} - \frac{1}{j_{22} - j_{33}} \int_0^L r_2^{e2}\, ds\right) \left(\frac{\pi}{2L}\right)^2 k_z > (\omega_2^e)^2, \tag{4.86}$$

$$\left(\rho_A^{-1} - \frac{1}{j_{22} - j_{11}} \int_0^L r_2^{e2}\, ds\right) \left(\frac{\pi}{2L}\right)^2 k_x > (\omega_2^e)^2. \tag{4.87}$$

Physically the first two conditions are classical stability conditions on the stable axes of rotation for a rigid body. The term on the right is the additional inertia due to the flexible appendage which adds inertia about both the first and third axes. The second two inequalities are conditions on the admissible rotation rates of the configuration. They have an interesting physical interpretation.

4.3.3.2. A Non-Trivial Equilibrium

For the second example we will consider rotations of the rigid-body-shear-beam configuration about the first or third principal axes of inertia. We will examine the case when the rotation is about the first principal axis of inertia, rotations about the third axis are similar. This corresponds to the example in Krishnaprasad and Marsden [2].

$$\boldsymbol{\omega}^e = \omega_1^e \,\hat{\mathbf{e}}_1, \tag{4.88}$$

$$\boldsymbol{r}^e(s) = \left(\frac{\sin(\sqrt{\frac{\rho_A}{k_y}}\omega_1^e s)}{\sqrt{\frac{\rho_A}{k_y}}\omega_1^e} + a\frac{\cos(\sqrt{\frac{\rho_A}{k_y}}\omega_1^e (s-L))}{\cos(\sqrt{\frac{\rho_A}{k_y}}\omega_1^e L)} \right) \hat{\mathbf{e}}_2, \tag{4.89}$$

$$\boldsymbol{m}^e(s) = \rho_A \omega_1 \left(\frac{\sin(\sqrt{\frac{\rho_A}{k_y}}\omega_1^e s)}{\sqrt{\frac{\rho_A}{k_y}}\omega_1^e} + a\frac{\cos(\sqrt{\frac{\rho_A}{k_y}}\omega_1^e (s-L))}{\cos(\sqrt{\frac{\rho_A}{k_y}}\omega_1^e L)} \right) \hat{\mathbf{e}}_3. \tag{4.90}$$

In these equations we have $0 \le s \le L$. For simplicity we will denote the nonzero element of $\boldsymbol{r}$ as r_2^e, and that of $\boldsymbol{m}$ as m_3^e.

We first compute

$$\boldsymbol{\alpha}^e = j_{11}\omega_1^e + \int_0^L r_2^e m_3^e \, ds \,\hat{\mathbf{e}}_1, \tag{4.91}$$

thus

$$\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e = (j_{11}\omega_1^e + \int_0^L r_2^e m_3^e \, ds)\omega_1^e, \tag{4.92}$$

$$\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e = (j_{11}\omega_1^e + \int_0^L r_2^e m_3^e \, ds)^2. \tag{4.93}$$

Subsequently we will denote the first element of α by α_1. We now compute

$$\mathbf{1} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.94}$$

and finally, $\mathbf{J}_e^{-1}$ and $\mathbf{Q}_e$ defined in *Step 4* are

$$\mathbf{J}_e^{-1} = \begin{bmatrix} \frac{1}{j_{11}} & 0 & 0 \\ 0 & \frac{\alpha_1 - j_{22}\omega_1^e}{j_{22}\alpha_1} & 0 \\ 0 & 0 & \frac{\alpha_1 - j_{33}\omega_1^e}{j_{33}\alpha_1} \end{bmatrix}, \tag{4.95}$$

$$\mathbf{Q}_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\omega_1^e}{\alpha_1} & 0 \\ 0 & 0 & \frac{\omega_1^e}{\alpha_1} \end{bmatrix}. \tag{4.96}$$

For $\mathbf{J}_e^{-1}$ to be positive definite we require

$$\alpha_1 > j_{22}\omega_1^e, \qquad \text{and} \qquad \alpha_1 > j_{33}\omega_1^e. \tag{4.92}$$

These conditions will hold if $j_{11} > j_{22}$, and $j_{11} > j_{33}$ and will assure positive elements along the diagonal in the inverse above. These conditions are the same as (5.10) in [2].

Then from equation (4.47) we have

$$\begin{aligned}
\mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e &= \left(\mathbf{J}^{-1} - \frac{\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e}(1 - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e})\right)^{-1}\left(\frac{(\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e)^2}{(\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e)^2}(1 - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e})\right), \\
&= \left(\begin{bmatrix} \frac{1}{j_{11}} & 0 & 0 \\ 0 & \frac{1}{j_{22}} & 0 \\ 0 & 0 & \frac{1}{j_{33}} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\omega_1^e}{\alpha_1} & 0 \\ 0 & 0 & \frac{\omega_1^e}{\alpha_1} \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{(\omega_1^e)^2}{\alpha_1^2} & 0 \\ 0 & 0 & \frac{(\omega_1^e)^2}{\alpha_1^2} \end{bmatrix},\right) \\
&= \begin{bmatrix} \frac{1}{j_{11}} & 0 & 0 \\ 0 & \frac{\alpha_1 - j_{22}\omega_1}{j_{22}\alpha_1} & 0 \\ 0 & 0 & \frac{\alpha_1 - j_{33}\omega_1}{j_{33}\alpha_1} \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{(\omega_1^e)^2}{\alpha_1^2} & 0 \\ 0 & 0 & \frac{(\omega_1^e)^2}{\alpha_1^2} \end{bmatrix}. \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{(\omega_1^e)^2 j_{22}}{\alpha_1(\alpha_1 - j_{22}\omega_1^e)} & 0 \\ 0 & 0 & \frac{(\omega_1^e)^2 j_{33}}{\alpha_1(\alpha_1 - j_{33}\omega_1^e)} \end{bmatrix}.
\end{aligned} \tag{4.97}$$

The skew symmetric matrix of *Step 7* is

$$S(\tfrac{\alpha^e\alpha^{eT}}{\alpha^{eT}\alpha^e}\omega^e) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_1^e \\ 0 & \omega_1^e & 0 \end{bmatrix}. \tag{4.98}$$

Now we compute $\mathbf{R}$ as defined in *Step 8*,

$$\begin{aligned}
\mathbf{R} &= \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e + \frac{\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^{eT}}(1 - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e}), \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\omega_1^e}{\alpha_1} & 0 \\ 0 & 0 & \frac{\omega_1^e}{\alpha_1} \end{bmatrix}, \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{bmatrix},
\end{aligned} \tag{4.99}$$

where we have

$$\gamma_{22} = \frac{\omega_1^e}{\alpha_1 - j_{22}\omega_1^e}, \qquad \text{and} \qquad \gamma_{33} = \frac{\omega_1^e}{\alpha_1 - j_{33}\omega_1^e}. \tag{4.100}$$

Note that these are not the same as the γ_1, and γ_2 terms which appear in Krishnaprasad & Marsden [1987].

We can now compute

$$\mathbf{S}^T(r^e)\mathbf{R}\mathbf{S}(r^e) = \begin{bmatrix} \gamma_{33} r_2^{e2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.101}$$

$$\mathbf{S}^T(r^e)\mathbf{R}\mathbf{S}(m^e) = \mathbf{0}, \tag{4.102}$$

$$\mathbf{S}^T(m^e)\mathbf{R}\mathbf{S}(m^e) = \begin{bmatrix} \gamma_{22} m_3^{e2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.103}$$

From this we can compute the matrix $\mathbf{A}^T(s)\mathbf{A}(s)$ in (4.54), note that it has only two nonzero elements. These correspond to second and fourth diagonal elements. Hence, $\mathbf{A}^T(s)\mathbf{A}(s)$ is a diagonal matrix and the nonzero eigenvalues are these two elements. As in the previous example we will use the modified bound described in *Remark 3*.

We can construct the $\mathbf{D}'$ matrix in (4.64)

$$\mathbf{D}' = \begin{bmatrix} \rho_A^{-1} - \gamma_{33}\int_0^L r_2^{e2}\, ds & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_A^{-1} & 0 & 0 & 0 & -\omega_1^e \\ 0 & 0 & \rho_A^{-1} & 0 & \omega_1^e & 0 \\ 0 & 0 & 0 & (\frac{\pi}{2L})^2 k_x - \gamma_{22}\int_0^L m_2^{e2}\, ds & 0 & 0 \\ 0 & 0 & \omega_1^e & 0 & (\frac{\pi}{2L})^2 k_y & 0 \\ 0 & -\omega_1^e & 0 & 0 & 0 & (\frac{\pi}{2L})^2 k_z \end{bmatrix}. \tag{4.104}$$

To assure that the $\mathbf{D}'$ matrix is positive definite we require

$$\frac{1}{\gamma_{33}} > \rho_A \int_0^L r_2^{e2}\, ds, \tag{4.105}$$

$$\left(\frac{\pi}{2L}\right)^2 \frac{k_x}{\gamma_{22}} > \int_0^L m_3^{e2}\, ds, \tag{4.106}$$

and

$$\frac{k_z}{\rho_A}\left(\frac{\pi}{2L}\right)^2 > (\omega_2^e)^2, \tag{4.107}$$

$$\frac{k_y}{\rho_A}\left(\frac{\pi}{2L}\right)^2 > (\omega_2^e)^2. \tag{4.108}$$

These conditions are exactly those of (5.14) in Krishnaprasad & Marsden and they assure stability about the equilibrium which also satisfies (4.92).

Finally a remark about the difference between Krishnaprasad & Marsden and our development. If we integrate the matrix we call $\mathbf{A}^T(s)\mathbf{A}(s)$ then the elements of the integrated matrix would correspond to γ_2, and γ_1 in the paper of Krishnaprasad and Marsden. This suggests modifying the procedure in the previous section to look at the eigenvalues of the integrated matrix rather than integrating the eigenvalues.

4.4. The Energy-Momentum method

In the previous case we were able to find a Casimir and exploit the freedom it gave us to establish positive or definiteness as a test of formal stability. Suppose however we were unable to find a large enough family of Casimirs, or suppose we were unable to find any Casimirs at all. In fact, such situations exist. At the present no Casimirs are known for the convected representation of a geometrically exact rod (Simo [1985]). A similar situation exists for the case of a compressible plasma described in spatial coordinates (Marsden, et. al. [1981]). In these cases we can employ a more general method known as the energy momentum method.

The energy momentum method deals with the dynamic stability of relative equilibria. In what follow we will describe this method, give a simple example, and discuss its relationship to the energy Casimir method. Subsequently, we will apply this method to the example of the linear extensible shear beam. The reference for this section is Simo, Marsden, Posbergh, Krishnaprasad [1988]). Crucial to an understanding of this method is the idea of a momentum map which we described in detail in section 2.7 (see also Abraham and Marsden [1978], section 4.4).

As in section 2.7 we assume that G is a Lie group with associated Lie algebra $\mathcal{G}$. We assume that G act on the manifold P. We can then define the equivariant momentum map for the action

$$J\colon P \to \mathcal{G}^*.$$

From corollary 4.2.11, p.283 in Abraham & Marsden [1978] we have that if G acts on T^*Q by cotangent lifts from Q, then an equivariant map always exists and is given by

$$J(\xi)(\alpha_q) = \langle \alpha_q, \xi_Q(q) \rangle,$$

where $\alpha_q \in T_q^*Q$ and ξ_Q is the infinitesimal generator of the group action on Q (see also Abraham & Marsden, p. 285). Here J is as defined in section 2.7.

The following fact is fundamental to the energy momentum method. (for a proof see Simo, Marsden, Posbergh, Krishnaprasad [1988].

Fact. *A point $z_e \in T^*Q$ is a relative equilibrium of the dynamical system if and only if there exists a $\xi \in \mathcal{G}$ such that z_e is a critical point of*

$$H_\xi = H - J(\xi),$$

(i.e., $\delta H_\xi(z_e) = 0$).

This result, due to Souriau, Smale, and Robbin is proposition 4.3.8, p. 307 in Abraham & Marsden.

For the energy momentum method we will examine the definiteness of the second variation of H_ξ at a critical point. We will need to place certain restrictions on the variations we use to explore the definiteness of $\delta^2 H_\xi(z_e)$.

The first restriction is that for $\delta z \in T_{z_e}P$, we require $dJ(z_e) \cdot \delta z = 0$. Thus, the variations must keep us on the constraints. The purpose of this restriction is to assure that we only look at variations that lie in the submanifold corresponding to a fixed value of the momentum map.

The second restriction takes into account the symmetries of the problem. Recall that G_μ is the isotropy subgroup of $\mu \in \mathcal{G}^*$ with respect to the coadjoint action (see again section 2.8). Therefore, denoting a fixed value of the momentum as $\mu_e = J(z_e)$, each element of G_{μ_0} corresponds to those elements $g \in G$ whose coadjoint action leaves the momentum unchanged. We denote the Lie algebra of G_{μ_e} by $\mathcal{G}_{\mu_e}(z_e)$.

From these two restrictions the only variations we allow in checking the definiteness of $\delta^2 H_z$ take values in $\ker dJ(z_e)/\mathcal{G}_{\mu_e}(z_e)$. In fact under suitable conditions this space is isomorphic to the tangent space of the reduced manifold $P_{\mu_e} = J^{-1}(\mu_e)/G_\mu$ obtained by symplectic reduction. Thus, we are really examining the definiteness of a Hamiltonian H_{μ_e} induced by H on the reduced space P_{μ_e}.

The energy momentum method can be summarized as follows;

(i) Choose an element $\xi \in \mathcal{G}$ such that for an equilibrium point z_e, we have $\delta H_\xi(z_e) = 0$.

(ii) Choose a linear subspace V of $T_{z_e}P$ such that for all $\delta\boldsymbol{z} \in V$, $dJ(\boldsymbol{z}_e) \cdot \delta\boldsymbol{z} = 0$ and V complements $\mathcal{G}_{\mu_e}$ in $\ker dJ(\boldsymbol{z}_e)$.

(iii) Test the definiteness of the second variation of $\delta^2 H_\xi$ with $\delta\boldsymbol{z}$ restricted to lie in V.

We reiterate, these these three steps are used to establish *formal stability.* As with the energy-Casimir method we need to use convexity estimates in a subsequent step to establish nonlinear stability.

4.4.1. The Rigid Body Example Revisited

We can illustrate this procedure with a simple, rigid body example (compare with the earlier energy Casimir example). If we consider the rigid body again we now write the Hamiltonian in the unreduced spatial representation. (although we have reduced to the center of mass).

$$H(\boldsymbol{\pi}, \Lambda) = \tfrac{1}{2}\boldsymbol{\pi} \cdot \mathbf{I}^{-1}\boldsymbol{\pi}. \tag{4.109}$$

Here $\boldsymbol{\pi}$ is the spatial angular momentum and $\mathbf{I}$ is the time varying inertia matrix of the rigid body. We append to this a linear function of the momentum and then proceed to take the first and the second variations. Define

$$H_\xi = \tfrac{1}{2}\boldsymbol{\pi} \cdot \mathbf{I}^{-1}\boldsymbol{\pi} - \boldsymbol{\xi} \cdot \boldsymbol{\pi}. \tag{4.110}$$

For the first variation

$$\delta H_\xi = \mathbf{I}^{-1}\boldsymbol{\pi} \cdot \delta\boldsymbol{\pi} + (\mathbf{I}^{-1}\boldsymbol{\pi} \times \boldsymbol{\pi}) \cdot \delta\boldsymbol{\theta} + \boldsymbol{\xi} \cdot \boldsymbol{\pi}, \tag{4.111}$$

where we have used $\mathbf{I}^{-1} = \Lambda \mathbf{J}^{-1} \Lambda^T$, and $\delta\Lambda = \delta\hat{\boldsymbol{\theta}}\Lambda$. Thus, to satisfy $\delta H_\xi = 0$ we require that at an equilibrium;

$$\mathbf{I}_e^{-1}\boldsymbol{\pi}_e = \boldsymbol{\xi}, \tag{4.112}$$

$$\mathbf{I}_e^{-1}\boldsymbol{\pi}_e \times \boldsymbol{\pi}_e = 0. \tag{4.113}$$

Note that the second equation can be written $\mathbf{I}_e^{-1}\boldsymbol{\pi}_e = \lambda^{-1}\boldsymbol{\pi}_e$ where λ is a scalar. Those values of λ satisfying this equation correspond to the principle values of inertia while the corresponding eigenvectors are those $\boldsymbol{\pi}_e$ which satisfy (4.113). The first equation identifies $\boldsymbol{\xi}$ with the axis of rotation for an equilibrium.

From the condition $dJ(\boldsymbol{z}_e) \cdot \delta\boldsymbol{z} = 0$ we require $\delta\boldsymbol{\pi} = 0$ This restriction will assure us that the variation we use to test the definiteness of the second variation lie in the submanifold $\ker dJ(z)$, or for a fixed momentum μ, $J^{-1}(\mu)$. In addition the infinitesimal generator associated with the action of $SO(3)$ is in the same direction as $\boldsymbol{\xi}$, therefore we have the additional restriction that $\xi \cdot \delta\boldsymbol{\theta} = 0$.

For the second variation, $\delta^2 H_\xi$ we compute

$$\delta^2 H_\xi = (\mathbf{I}^{-1})\delta\pi \cdot \delta\bar{\pi} + (\mathbf{I}^{-1}\hat{\pi} - (\mathbf{I}^{-1}\pi)\hat{\ })\delta\bar{\theta} \cdot \delta\pi$$
$$+ (-\hat{\pi}\mathbf{I}^{-1} + (\mathbf{I}^{-1}\pi)\hat{\ })\delta\bar{\pi} \cdot \delta\boldsymbol{\theta} + (\hat{\pi}(\mathbf{I}^{-1}\pi)\hat{\ } - \hat{\pi}\mathbf{I}^{-1}\hat{\pi})\delta\bar{\theta} \cdot \delta\boldsymbol{\theta}. \quad (4.114)$$

However, since we require $\delta\boldsymbol{\pi} = 0$ we need only check the definiteness of the quantity

$$\delta^2 H_{\xi,\mu}\Big|_e = (\hat{\pi}_e(\mathbf{I}_e^{-1}\pi_e)\hat{\ } - \hat{\pi}_e\mathbf{I}_e^{-1}\hat{\pi}_e)\delta\bar{\theta} \cdot \delta\boldsymbol{\theta},$$

at an equilibrium subject to the restriction $\delta\boldsymbol{\theta} \cdot \xi = 0$. Using the equilibrium conditions we check conditions such that

$$(\pi_e \times \delta\boldsymbol{\theta}) \cdot \left[\frac{\mathbf{I}_e\boldsymbol{\xi} \cdot \boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}\mathbf{1} - \mathbf{I}_e\right](\pi_e \times \delta\boldsymbol{\theta}) > 0, \quad \text{or} \quad < 0, \quad (4.115)$$

subject to $\delta\boldsymbol{\theta} \cdot \xi = 0$.

Clearly this will be satisfied for the largest or smallest eigenvalues of $\mathbf{I}_e$ since these are the maximum and minimum values of the first term in the brackets. This is in complete accord with the classical result and that of the energy-Casimir method.

4.5. More Examples

In this section we will use the energy-momentum method to reproduce the the stability criteria we found for the linear extensible shear beam at the nontrivial equilibria. In contrast to the development in that section where we used convected variables, for the energy-momentum method we work in the spatial representation,

The Hamiltonian for the linear extensible shear beam in the spatial representation is

$$\mathbf{H} = K + V,$$
$$= \tfrac{1}{2}\mathbf{I}^{-1}\boldsymbol{\pi} \cdot \boldsymbol{\pi} + \tfrac{1}{2}\int_0^L \rho_A^{-1}\|\boldsymbol{p}\|^2\, dS + \tfrac{1}{2}\int_0^L \mathbf{C}\frac{\partial\phi}{\partial S} \cdot \frac{\partial\phi}{\partial S}\, dS, \quad (4.116)$$

where $\boldsymbol{\pi}$ is the spatial angular momentum associated with the rigid body; $\boldsymbol{p}$ is the momentum density of the rod, and ϕ is the configuration variable (line of centroids). We also have $\mathbf{I} = \mathbf{\Lambda J \Lambda}^T$, the time varying inertia matrix of the rod, and finally $\mathbf{C} = \mathbf{\Lambda K \Lambda}^T$, the time varying matrix of elastic coefficients. Note that for the linear extensible shear beam $\mathbf{\Lambda}$ is independent of S, thus $\Lambda(S,t) = \Lambda_0(t)$.

The momentum mapping in this case is $J(\boldsymbol{\xi}) = \boldsymbol{\xi} \cdot \boldsymbol{\alpha}$ where

$$\boldsymbol{\alpha} = \boldsymbol{\pi} + \int_0^L \boldsymbol{\phi} \times \boldsymbol{p}\, dS.$$

Thus we define

$$H_\xi = H - J(\xi),$$

and proceed to take the first and second variations.

Rigorous stability involves a convexity estimate on the second variations. In what follows we shall restrict ourselves to a *formal stability* analysis by requiring definiteness of the second variation for stability.

In general the first variation of $H - J(\boldsymbol{\xi})$ is formally computed by use of the variational derivative relative to the $L_2([0,L])$ duality pairing. The computation is performed by taking the configuration $(\boldsymbol{\phi}, \Lambda)$, and the momentum $(g, \boldsymbol{\pi})$ as the basic variables, and using the chain rule.

4.5.1. Inertial Terms: First Variations

In computing the first variation of K we need to recall that by definition $\mathbf{I} = \mathbf{\Lambda J \Lambda}^T$ where $\mathbf{J}_\rho$ is the (time independent) inertia of the cross section. To compute the variation of the rotation field, we use $\Lambda(\epsilon) = \exp(\hat{\delta\boldsymbol{\theta}})\Lambda_0$ to obtain:

$$\delta\mathbf{\Lambda} = \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} (1 + \epsilon\delta\hat{\boldsymbol{\theta}} + \ldots)\mathbf{\Lambda} = \delta\hat{\boldsymbol{\theta}}\mathbf{\Lambda}, \tag{4.117}$$

thus

$$\begin{aligned} \delta\mathbf{I}^{-1} &= \delta\mathbf{\Lambda J \Lambda}^T + \mathbf{\Lambda J}\delta\mathbf{\Lambda}^T, \\ &= \delta\hat{\boldsymbol{\theta}}\,\mathbf{I}^{-1} - \mathbf{I}^{-1}\delta\hat{\boldsymbol{\theta}}. \end{aligned} \tag{4.118}$$

Using this result, we obtain:

$$\begin{aligned} \delta K &= \mathbf{I}^{-1}\boldsymbol{\pi} \cdot \delta\boldsymbol{\pi} + \hat{\boldsymbol{\pi}}\delta\boldsymbol{\theta}_0 \cdot \mathbf{I}^{-1}\boldsymbol{\pi} + \int_0^L \rho_A^{-1}\boldsymbol{p} \cdot \delta\boldsymbol{p}\, dS, \\ &= \mathbf{I}^{-1}\boldsymbol{\pi} \cdot \delta\boldsymbol{\pi} + (\mathbf{I}^{-1}\boldsymbol{\pi} \times \boldsymbol{\pi}) \cdot \delta\boldsymbol{\theta}_0 + \int_0^L \rho_A^{-1}\boldsymbol{p} \cdot \delta\boldsymbol{p}\, dS. \end{aligned} \tag{4.119}$$

4.5.2. Potential Energy Term: First Variation

We consider constitutive equations expressed in the convected frame. For the linear extensible shear beam we need only consider the strains associated with shear and extension. In order to compute the variations with respect to the configuration variables we will need to recall that the convected strain is the pull back of the material strain,

$$\boldsymbol{\Gamma} = \boldsymbol{\Lambda}^T \frac{\partial \boldsymbol{\phi}}{\partial S}. \tag{4.120}$$

The first variation of this term can thus be expressed as

$$\delta V = \int_0^L \frac{\partial \psi}{\partial \boldsymbol{\Gamma}} \cdot \delta \boldsymbol{\Gamma} \, dS. \tag{4.121}$$

We now use our definitions for the perturbed configuration to compute $\delta\boldsymbol{\Gamma}$;

$$\begin{aligned}
\delta \boldsymbol{\Gamma} &= \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \boldsymbol{\Gamma}_\epsilon, \\
&= \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \boldsymbol{\Lambda}^T \exp(\epsilon \delta \hat{\boldsymbol{\theta}})^T (\frac{\partial \boldsymbol{\phi}}{\partial S} + \epsilon \frac{\partial \delta \boldsymbol{\phi}}{\partial S}), \\
&= \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} (\boldsymbol{\Lambda}^T + \epsilon \boldsymbol{\Lambda}^T \delta \hat{\boldsymbol{\theta}}^T + \ldots)(\frac{\partial \boldsymbol{\phi}}{\partial S} + \epsilon \frac{\partial \delta \boldsymbol{\phi}}{\partial S}), \\
&= \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} (\boldsymbol{\Lambda}^T \frac{\partial \boldsymbol{\phi}}{\partial S} + \epsilon \boldsymbol{\Lambda}^T (\delta \hat{\boldsymbol{\theta}}^T \frac{\partial \boldsymbol{\phi}}{\partial S} + \frac{\partial \delta \boldsymbol{\phi}}{\partial S}) + \ldots), \\
&= \boldsymbol{\Lambda}^T [\frac{\partial \delta \boldsymbol{\phi}}{\partial S} - \delta \boldsymbol{\theta} \times \frac{\partial \boldsymbol{\phi}}{\partial S}].
\end{aligned} \tag{4.122}$$

Since $\frac{\partial \psi}{\partial \boldsymbol{\Gamma}} = \boldsymbol{\Lambda}^T \frac{\partial \psi}{\partial \boldsymbol{\gamma}}$ the variation for the potential energy term can then be written

$$\delta V = \int_0^L (\frac{\partial \psi}{\partial \gamma} \cdot (\frac{\partial \delta \boldsymbol{\phi}}{\partial S} + \frac{\partial \boldsymbol{\phi}}{\partial S} \times \delta \boldsymbol{\theta}) \, dS. \tag{4.123}$$

Note that we can integrate by parts using the boundary conditions on $\delta\boldsymbol{\phi}$, and $\delta\boldsymbol{\theta}$ to obtain

$$\delta V = -\int_0^L \left\{ \left[\boldsymbol{\phi}' \times \frac{\partial \psi}{\partial \gamma} \right] \cdot \delta \boldsymbol{\theta} + \frac{\partial}{\partial S} (\frac{\partial \psi}{\partial \gamma}) \cdot \delta \boldsymbol{\phi} \right\} dS. \tag{4.124}$$

Recall that (in chapter 2) we defined $\boldsymbol{n} = \frac{\partial \psi}{\partial \gamma}$. For the linear extensible shear beam $\boldsymbol{n} = \mathbf{C}\boldsymbol{\phi}'$.

4.5.3. The Momentum Term: First variation

The first variation of the momentum term is computed

$$
\begin{aligned}
\delta J(\xi) &= \boldsymbol{\xi} \cdot (\delta \boldsymbol{\pi} + \int_0^L \delta \boldsymbol{\phi} \times \boldsymbol{p} + \boldsymbol{\phi} \times \delta \boldsymbol{p} \, dS), \\
&= \boldsymbol{\xi} \cdot \delta \boldsymbol{\pi} + \int_0^L (\boldsymbol{p} \times \boldsymbol{\xi}) \cdot \delta \boldsymbol{\phi} + (\boldsymbol{\xi} \times \boldsymbol{\phi}) \cdot \delta \boldsymbol{p} \, dS. \qquad (4.125)
\end{aligned}
$$

4.5.4. Summary of the First Variation

Combining the above we have for the first variation of H_ξ;

$$
\begin{aligned}
\delta H_\xi = (&\mathbf{I}^{-1}\boldsymbol{\pi} + \boldsymbol{\xi}) \cdot \delta \boldsymbol{\pi} + \mathbf{I}^{-1}\boldsymbol{\pi} \times \boldsymbol{\pi}) \cdot \delta \boldsymbol{\theta}_0 \\
&+ \int_0^L (\rho_A^{-1} \boldsymbol{p} + \boldsymbol{\xi} \times \boldsymbol{\phi}) \cdot \delta \boldsymbol{p} \, dS \\
&+ \int_0^L (-\frac{\partial^2 \psi}{\partial S \partial \gamma} + \boldsymbol{p} \times \boldsymbol{\xi}) \cdot \delta \boldsymbol{\phi} \, dS \\
&+ ((\mathbf{I}^{-1}\boldsymbol{\pi} \times \boldsymbol{\pi}) + \int_0^L (\frac{\partial \psi}{\partial \gamma} \times \frac{\partial \boldsymbol{\phi}}{\partial S} \, dS) \cdot \boldsymbol{\theta}_0. \qquad (4.126)
\end{aligned}
$$

Here we have used the fact that since $\boldsymbol{\Lambda}$ is independent of S, $\delta\boldsymbol{\theta}$ is also independent of S and consequently $\delta\boldsymbol{\theta} = \delta\boldsymbol{\theta}_0$ for all S.

From the above we obtain the four equations satisfied by an equilibrium

$$
\rho_A^{-1} \boldsymbol{p} = \boldsymbol{\xi} \times \boldsymbol{\phi}, \qquad (4.127)
$$

$$
\mathbf{I}^{-1}\boldsymbol{\pi} = \boldsymbol{\xi}, \qquad (4.128)
$$

$$
\boldsymbol{p} \times \boldsymbol{\xi} = \frac{\partial}{\partial S}(\boldsymbol{n}), \qquad (4.129)
$$

$$
\mathbf{I}^{-1}\boldsymbol{\pi} \times \boldsymbol{\pi} = -\int_0^L \boldsymbol{n} \times \frac{\partial \boldsymbol{\phi}}{\partial S}. \qquad (4.130)
$$

4.5.5. Second Variation

Next we compute the second variation of the above quantities, that is $\delta^2 K$, $\delta^2 V$, and $\delta^2 J(\boldsymbol{\xi})$. This will be the basis of our subsequent stability analysis using the Energy-momentum method. Note that in this section we denote the second variations by over bars, thus $\delta\boldsymbol{\pi}$ is the variation of $\boldsymbol{\pi}$ in taking the first variation of H_ξ while $\delta\bar{\boldsymbol{\pi}}$ is the variation that appears in taking the second variation of H_ξ.

The computation of the second variation begins with the expression for the first variation (3.3). By the directional derivative formula

$$\begin{aligned}\delta^2 K = (\mathbf{I}^{-1})\delta\pi\cdot\delta\bar{\pi} + (\mathbf{I}^{-1}\hat{\pi} - (\mathbf{I}^{-1}\pi)\hat{\ })\delta\bar{\theta}_0\cdot\delta\pi \\ + (-\hat{\pi}\mathbf{I}^{-1} + (\mathbf{I}^{-1}\pi)\hat{\ })\delta\bar{\pi}\cdot\delta\theta_0 + (\hat{\pi}(\mathbf{I}^{-1}\pi)\hat{\ } - \hat{\pi}\mathbf{I}^{-1}\hat{\pi})\delta\bar{\theta}_0\cdot\delta\theta_0 \\ + \int_0^L \rho_A^{-1}\delta p\cdot\delta\bar{p} - \xi\times\delta\phi\cdot\delta p - \xi\times\delta\phi\cdot\delta p\, dS + \delta^2 V. \end{aligned} \tag{4.131}$$

We can complete a square to eliminate p, define

$$S = \|\rho_A^{-\frac{1}{2}}\delta p - \rho_A^{\frac{1}{2}}(\xi\times\delta\phi)\|^2.$$

Then

$$\begin{aligned}\delta^2 K = (\mathbf{I}^{-1})\delta\pi\cdot\delta\bar{\pi} + (\mathbf{I}^{-1}\hat{\pi} - (\mathbf{I}^{-1}\pi)\hat{\ })\delta\bar{\theta}_0\cdot\delta\pi \\ + (-\hat{\pi}\mathbf{I}^{-1} + (\mathbf{I}^{-1}\pi)\hat{\ })\delta\bar{\pi}\cdot\delta\theta_0 + (\hat{\pi}(\mathbf{I}^{-1}\pi)\hat{\ } - \hat{\pi}\mathbf{I}^{-1}\hat{\pi})\delta\bar{\theta}_0\cdot\delta\theta_0 \\ + S + \int_0^L \rho_A\|\xi\times\delta\phi\|^2\, dS + \delta^2 V. \end{aligned} \tag{4.132}$$

4.5.6. Potential Energy Term

The computation of the Potential term begins with the equation for the first variation of this term (3.11). Thus,

$$\begin{aligned}\delta^2 V = \int_0^L (\mathbf{C}\delta\phi' + \mathbf{C}(\phi'\times\delta\theta) - \mathbf{C}\phi'\times\delta\theta)\cdot\delta\bar{\phi}' \\ + (\mathbf{C}\phi'\times\delta\phi') - (\phi'\times\mathbf{C}\delta\phi') \\ + \mathbf{C}\phi'\times(\phi'\times\delta\theta) - (\phi'\times\mathbf{C}(\phi'\times\delta\theta))\cdot\delta\bar{\theta}'\, dS. \end{aligned} \tag{4.133}$$

If we define the configuration dependent operator

$$\Xi(\Phi) = [\,\mathbf{1}\quad \hat{\phi}'\,],$$

and

$$\mathbf{B} = \begin{bmatrix} 0 & -\hat{\mathbf{n}} \\ 0 & \hat{\mathbf{n}}\hat{\phi}' \end{bmatrix}, \tag{4.134}$$

then (4.133) can be written in the form

$$\delta^2 V = \int_0^L \Xi(\Phi)\begin{bmatrix}\delta\phi' \\ \delta\theta\end{bmatrix}\cdot\mathbf{C}\Xi(\Phi)\begin{bmatrix}\delta\phi' \\ \delta\theta\end{bmatrix} dS + \int_0^L \mathbf{B}\begin{bmatrix}\delta\phi' \\ \delta\theta\end{bmatrix}\cdot\begin{bmatrix}\delta\phi' \\ \delta\theta\end{bmatrix} dS, \tag{4.135}$$

where the first integral is the material component of the tangent stiffness matrix while the second term gives rise to the geometric part. This is a special case of a more general expression which arises because of the nonlinear nature of the configuration. (see Simo and Vu-Quoc [1986].

4.5.7. Stability Analysis

In this section we will ascertain the stability for particular equilibria and relate it to the earlier energy-Casimir results. This section should be compared with the energy-Casimir analysis of the same equilibria in section 4.3.

In order to relate this to our earlier results we first pull back the spatial representation to convected coordinates. In the case of the linear extensible shear beam this will simplify our expressions, we note that *in general this is not necessarily true.* However, whether in convected or spatial coordinates the crucial fact to be exploited is that $\mathbf{\Lambda}$ does not depend on S.

Recall that in this case

$$\begin{aligned} \mathbf{N} &= \mathbf{\Lambda}^T \mathbf{n}, & \boldsymbol{\xi} &= \mathbf{\Lambda}^T \boldsymbol{\xi}, \\ \mathbf{R} &= \mathbf{\Lambda}^T \boldsymbol{\phi}, & \delta\mathbf{R} &= \mathbf{\Lambda}^T (\boldsymbol{\phi} \times \delta\boldsymbol{\theta} + \delta\boldsymbol{\phi}), \\ \mathbf{R}' &= \mathbf{\Lambda}^T \boldsymbol{\phi}', & \delta\mathbf{R}' &= \mathbf{\Lambda}^T (\boldsymbol{\phi}' \times \delta\boldsymbol{\theta} + \delta\boldsymbol{\phi}'). \end{aligned}$$

we also define $\delta\mathbf{T} = \mathbf{\Lambda}^T \delta\boldsymbol{\theta}_0$ and note that it is independent of S. Thus, in terms of convected components we can write the second variation in the form

$$\delta^2 H_J = \int_0^L \begin{bmatrix} \mathbf{K} & 0 & -\hat{\mathbf{N}} \\ 0 & \rho_A \hat{\xi}\hat{\xi} & -\rho_A \hat{\xi}\hat{\xi}\hat{\mathbf{R}} \\ \hat{\mathbf{N}} & \rho_A \hat{\mathbf{R}}\hat{\xi}\hat{\xi} & -\hat{\mathbf{R}}'\hat{\mathbf{N}} - \rho_A \hat{\mathbf{R}}\hat{\xi}\hat{\xi}\hat{\mathbf{R}} \end{bmatrix} \begin{bmatrix} \delta\mathbf{R}' \\ \delta\mathbf{R} \\ \delta\mathbf{T} \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{R}' \\ \delta\mathbf{R} \\ \delta\mathbf{T} \end{bmatrix} dS \\ + \hat{\boldsymbol{\xi}}[(\mathbf{I}\boldsymbol{\xi})\hat{} - \mathbf{I}\hat{\boldsymbol{\xi}}]\delta\mathbf{T} \cdot \delta\mathbf{T}.$$

Consider the components of the second variation which involve $\delta\mathbf{R}'$. We can obtain a lower bound for this component by a term involving variations in $\delta\mathbf{R}$, and $\delta\mathbf{T}$ by application of the Poincaré inequality and integration by parts. Thus,

$$\int_0^L \mathbf{K}\delta\mathbf{R}' \cdot \delta\mathbf{R}' + 2\hat{\mathbf{N}}\delta\mathbf{R}' \cdot \delta\mathbf{T}\, dS \geq \int_0^L (\frac{\pi}{2L})^2 \mathbf{K}\delta\mathbf{R} \cdot \delta\mathbf{R} - 2\hat{\mathbf{N}}'\delta\mathbf{R} \cdot \delta\mathbf{T}\, dS,$$

where use has been made of the boundary conditions. The second variation is now bounded below

$$\delta^2 H_J \geq \int_0^L \begin{bmatrix} (\frac{\pi}{2L})^2 \mathbf{K} + \rho_A \hat{\xi}\hat{\xi} & \hat{\mathbf{N}}' - \rho_A \hat{\xi}\hat{\xi}\hat{\mathbf{R}} \\ -\hat{\mathbf{N}}' + \rho_A \hat{\mathbf{R}}\hat{\xi}\hat{\xi} & -\hat{\mathbf{R}}'\hat{\mathbf{N}} - \rho_A \hat{\mathbf{R}}\hat{\xi}\hat{\xi}\hat{\mathbf{R}} \end{bmatrix} \begin{bmatrix} \delta\mathbf{R} \\ \delta\mathbf{T} \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{R} \\ \delta\mathbf{T} \end{bmatrix} dS \\ + \hat{\boldsymbol{\xi}}[(\mathbf{I}\boldsymbol{\xi})\hat{} - \mathbf{I}\hat{\boldsymbol{\xi}}]\delta\mathbf{T} \cdot \delta\mathbf{T}.$$

We now consider a specific equilibrium subject to appropriate constraints on the variations. For the equilibrium we are concerned with , the configuration rotates about a principal axis perpendicular to the appendage, what we referred to as a nontrivial equilibrium in an earlier section. In this case we have;

$$\xi = \begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 0 \\ 0 \\ S \end{bmatrix}. \tag{4.136}$$

In this case the we have

$$\begin{aligned} \mathbf{N}(S) &= \mathbf{N}(0) + \int_0^S \rho_A \hat{\xi}\hat{\xi}\mathbf{R}\, dS, \\ &= \mathbf{N}(0) + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\rho_A \omega^2 S^2 \end{bmatrix}, \end{aligned}$$

to satisfy the boundary condition at the tip of the appendage, $\mathbf{N}(S) = 0$, we require

$$\mathbf{N}(S) = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\rho_A \omega^2 (L^2 - S^2) \end{bmatrix}.$$

We will use the fact that integration by parts shows

$$\int_0^L \tfrac{1}{2}\rho_A \omega^2 (L^2 - S^2)\, dS = -\textstyle\int_0^L \rho_A \omega^2 s^2\, dS.$$

Thus we have the following components in the expression for the second variation

$$\rho_A \hat{\boldsymbol{\xi}}\hat{\boldsymbol{\xi}} = \begin{bmatrix} -\rho_A\omega^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho_A\omega^2 \end{bmatrix}, \qquad \rho_A \hat{\boldsymbol{\xi}}\hat{\boldsymbol{\xi}}\hat{\mathbf{R}} = -\rho_A(\hat{\mathbf{R}}\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\xi}})^T = \begin{bmatrix} 0 & \rho_A\omega^2 s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\rho_A \hat{\mathbf{R}}\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\xi}}\hat{\mathbf{R}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\rho_A\omega^2 s^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}[(\mathbf{I}\boldsymbol{\xi})\hat{} - \mathbf{I}\hat{\boldsymbol{\xi}}] = \begin{bmatrix} (j_{22} - j_{33})\omega^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (j_{22} - j_{11})\omega^2 \end{bmatrix},$$

$$\hat{\mathbf{R}}'\hat{\mathbf{N}} = \begin{bmatrix} \frac{1}{2}\rho_A\omega^2(L^2 - s^2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} 0 & -\frac{1}{2}\rho_A\omega^2(L^2 - s^2) & 0 \\ \frac{1}{2}\rho_A\omega^2(L^2 - s^2) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We will also assume that $\mathbf{K} = \mathrm{diag}(GA_1,\ GA_2,\ EA)$.

Note that our restrictions on the variations require $\delta T_2 = 0$ since this direction is the same as that of $\boldsymbol{\xi}$, the infinitesimal generator of the symmetry group.

One can now check the definiteness of the resulting quadratic form by verifying that all the submatrices along the diagonal associated with unrestricted variations are positive.

Thus for the diagonal elements we require

$$\left(\frac{\pi}{2L}\right)^2 GA_1 > \rho_A\omega^2,$$
$$\left(\frac{\pi}{2L}\right)^2 EA > \rho_A\omega^2,$$

The conditions correspond to our earlier conditions and are related to the frequency of the lowest mode. The conditions

$$j_{22} > j_{33} + \int_0^L \rho_A s^2\omega^2\, dS,$$
$$j_{22} > j_{11},$$

can be identified as the classical, rigid body conditions on the extended inertia dyadic.

In addition, we have the condition arising from the prescence of the off diagonal terms due to $\mathbf{N}$. In this case we consider the second variation arising from variations in δR_2, and δT_1. We can complete a square in this case to find conditions which assure this term is positive. Thus

$$\begin{aligned}
&\int_0^L (\frac{\pi}{2L})^2 GA_2\delta R_2^2 - 2\rho_A\omega^2 s\delta R_2\delta T_1\, dS \\
&\qquad + (j_{22} - j_{33} - \int_0^L \rho_A s^2\, dS)\omega^2\delta T_1, \\
&= - \int_0^L \|(\frac{\pi}{2L})(GA_2)^{\frac{1}{2}}\delta R_2 - (\frac{2L}{\pi})(GA_2)^{-\frac{1}{2}}\rho_A\omega^2 s\delta T_1\|^2\, dS \\
&\qquad - \int_0^L (\frac{2L}{\pi})^2 (GA_2)^{-1}\rho_A^2\omega^4 s^2\delta T_1^2\, dS + (j_{22} - j_{33} - \int_0^L \rho_A s^2\, dS)\omega^2\delta T_1^2.
\end{aligned}$$

This term is never less than zero if

$$\left(\frac{\pi}{2L}\right)^2 GA_2 + j_{22} - j_{33} - \int_0^L \rho_A s^2\, dS > \omega_2 \int_0^L \rho_A s^2\, dS.$$

This of course corresponds to equation (4.106) in our earlier example.

For the case of the linear extensible shear beam we have been able to exploit the fact that $\boldsymbol{\Lambda}$ is independent of S. In the general case of the geometrically exact case this is not longer true and the problem is significantly more complicated by this fact.

CHAPTER FIVE

CONTROL OF FLEXIBLE AND MIXED STRUCTURES

In this chapter we formulate the control problem for a particular class of mixed structures. In particular, we will address the control of models for which the control is located on the rigid body and the sensor is located at the tip of the attached flexible appendage. Several models for such configurations were developed in chapter 2, in particular we will consider linearized models in the frequency domain and approximations to them by a system of N rigid bodies.

5.1. Control by input-output linearization

Exact input-output linearization involves the use of a static feedback to modify the input-output behavior of a nonlinear system so the response appears linear. This is in contrast to approximate linearization techniques which approximate the nonlinear behavior of a system by the truncation of a power series. Generally, we would want to linearize a system in order to obtain the advantages of a linear system, particularly in regard to control system design. The structure algorithm of Silverman [1969] is the basis of the input-output linearization in Isidori and Ruberti [1984], and Isidori [1983].

The use of input-output linearization as a practical method in control system design has several potential drawbacks. Since the underlying system remains nonlinear care must be exercised in keeping the internal states from saturating. More serious is the problem of inaccurate modeling. Input-output linearization is based on exact knowledge of parameters in the model we are linearizing. In the absence of any robustness theory there exists the potential for serious problems if the model is inaccurate. Recently, techniques based on adaptive control theory have been applied to this problem in the work of Sastry [1986].

Control by exact input-output linearization is related to the more difficult problem of exact linearization. The problem of exact linearization for a state space model was first proposed and solved for the single-input case by Brockett [1978]. The solution for multi-input systems is due to Jakubczyk and Respondek [1980], while the independent work of Su [1982], and Hunt, et.al. [1983] includes a constructive algorithm for determining the static feedback. For an application of the technique to the design of automatic flight controllers see Hunt, Su, and Meyer [1983].

5.1.1. Exact Input-output Linearization, Finite Dimensional Systems

The basic idea behind exact input-output linearization is to require a linear relationship between the output measurements and a new control input. Subsequently one inverts the system between the outputs and original inputs to find the linearizing feedback. This idea will be made clear in the example below.

We consider nonlinear systems of the form.

$$\frac{dx}{dt} = f(x) + g(x)u, \qquad\qquad x\Big|_{t=0} = x_0, \tag{5.1}$$

with outputs given by the nonlinear mapping

$$y = h(x).$$

Here x takes values in some Hilbert space $\mathcal{H}$ and we assume $y \in I\!R^{\ell}$, and $u \in I\!R^{m}$.

We will consider such a system to have a linear input output behavior if the relationship between y and u can be expressed in the form

$$y(t) = y(t, x_0) + \int_0^t \mathbf{k}(t-\tau)u(\tau)\,d\tau, \tag{5.2}$$

where $\mathbf{k}(t-\tau)$ is the first order kernel of the Volterra series associated with (5.1) and (5.1) (see Brockett [1977].) In general we have

$$\mathbf{k}(t-\tau) = \mathbf{w}(t, \tau, x_0) \tag{5.3}$$

so that a necessary and sufficient condition for the input output behavior to be linear is that the first order kernel of the Volterra series is dependent only on the difference of $t-\tau$, and independent of x. For the system we consider the first order kernels can be written in terms of a Taylor series,

$$\mathbf{w}(t, \tau, x) = \sum_{k=0}^{\infty} \mathbf{T}_k (t-\tau)^k,$$

where

$$\mathbf{T}_k = L_g L_f^k h(x).$$

In this case (5.3) is satisfied if, and only if every coefficient $\mathbf{T}_k$ in the Taylor series is independent of x. Thus, given a system one first checks if it is linearizable by a simple coordinate change. This is checked by verifying that every $\mathbf{T}_k$ is independent of x. Should this not be the case we can next ask if there exists a static feedback of the form

$$u = \alpha(x) + \beta(x)v,$$

such that with this feedback the input output response can be written in the form (5.2). It is shown in Isidori [1985] that there exists a static, nonlinear feedback *if and only if* the formal power series $\mathbf{T}(s, x)$ defined as

$$\mathbf{T}(s, x) = \sum_{k=0}^{\infty} \mathbf{T}_k(x) s^{-k-1}, \tag{5.4}$$

is separable. By separable we mean that $\mathbf{T}(s, x)$ can be written

$$\mathbf{T}(s, x) = \mathbf{K}(s)\mathbf{R}(s, x), \tag{5.5}$$

where

$$\mathbf{K} = \sum_{k=0}^{\infty} \mathbf{K}_k s^{k-1}, \qquad \mathbf{R}(s, x) = \mathbf{R}_{-1} + \sum_{k=0}^{\infty} \mathbf{R}_k(s) s^{-k-1}.$$

with $\mathbf{R}_{-1}$ invertible. (see Isidori [1985], Chapter 5, Theorem (1.11)).

If the formal power series is separable then a systematic procedure for the construction of α, and β is described in Isidori [1985]. Central to this method is the so called "structure algorithm" first described by Silverman [1969] which enables us to invert the system as a first step to finding the linearizing α, β. For a complete discussion of the procedure we defer to Isidori [1985] Chapter 5, section 1. Here we recall that the linearizing α, and β are given by

$$[L_g \Gamma(x)]\alpha(x) = -L_f \Gamma(x) \tag{5.6}$$

$$[L_g \Gamma(x)]\beta(x) = [\mathbf{1}_{r_{q-1}}\ ,\ \mathbf{0}] \tag{5.7}$$

where $L_g\Gamma$ is an $r_{q-1} \times m$ matrix of rank r_{q-1}, and $L_f\Gamma$ is a r_{q-1} dimensional vector. We compute from the the algorithm in Isidori the value of r_{q-1} and the vector Γ.

5.1.2. Example: The Rigid Body

We can illustrate the technique of exact linearization with the example of a rigid body which is controlled by an applied moment. If we assume that we have available measurements of the rotation rate then the equations describing the dynamics and the output take the form

$$\dot{\boldsymbol{\Pi}} = \mathbf{J}^{-1}\boldsymbol{\Pi} \times \boldsymbol{\Pi} + \boldsymbol{M}, \tag{5.8}$$

$$\boldsymbol{Y} = \mathbf{J}^{-1}\boldsymbol{\Pi}, \tag{5.9}$$

Here $\boldsymbol{\Pi} \in \mathbb{R}^3$, $\boldsymbol{M} \in \mathbb{R}^3$, and $\boldsymbol{Y} \in \mathbb{R}^3$. Physically these correspond to the angular velocity, and external torque, and the angular velocity of the body.

This system is straight forward to linearize and we can do this in an obvious fashion as follows. If we differentiate $\boldsymbol{Y}$ we get

$$\begin{aligned} \dot{\boldsymbol{Y}} &= L_f \boldsymbol{h}, \\ &= \mathbf{J}^{-1}(\mathbf{J}^{-1}\boldsymbol{\Pi} \times \boldsymbol{\Pi} + \boldsymbol{M}). \end{aligned} \tag{5.10}$$

If one sets $\dot{\boldsymbol{Y}} = \boldsymbol{V}$, then $\boldsymbol{Y}$ is linearly related to $\boldsymbol{V}$. Now assume $\boldsymbol{V}$ is our new input. Solving for the old input $\boldsymbol{M}$, in terms of $\boldsymbol{V}$

$$\begin{aligned} \boldsymbol{M} &= -\mathbf{J}^{-1}\boldsymbol{\Pi} \times \boldsymbol{\Pi} + \mathbf{J}\boldsymbol{V}, \\ &= \boldsymbol{\alpha}(\boldsymbol{x}) + \boldsymbol{\beta}(\boldsymbol{x})\boldsymbol{V}. \end{aligned}$$

Clearly, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the elements of the static feedback which linearize our system.

Of course this problem could also be solved by the Isidori's method using the structure algorithm. In this case we note that,

$$\begin{aligned} \mathbf{T}_0 &= L_g \boldsymbol{h} = \mathbf{J}^{-1}, \\ \mathbf{T}_1 &= L_g L_f \boldsymbol{h} = -(\mathbf{J}^{-1}\boldsymbol{\Pi})\hat{} + \mathbf{J}^{-1}\hat{\boldsymbol{\Pi}}, \end{aligned}$$

so that the system is *not* linearizable by a change in coordinates. Consequently, we proceed to construct the linearizing feedback using the structure algorithm. The matrices below correspond to the notation in Isidori [1985]. In this case the construction is almost trivial. Since $\operatorname{rank}(\mathbf{T}_0) = \ell$, we terminate with the initial conditions. We set $\mathbf{P}_1 = \mathbf{1}$, $\boldsymbol{V}_1 = \mathbf{1}$, and $\gamma_1 = L_f \boldsymbol{h}$. In this case the matrix $\boldsymbol{\Gamma}(\boldsymbol{\Pi}) = \gamma_1$ and we have

$$
\begin{aligned}
L_g \boldsymbol{\Gamma} &= \mathbf{J}^{-1}, \\
L_f \boldsymbol{\Gamma} &= \mathbf{J}^{-1}(\mathbf{J}^{-1}\boldsymbol{\Pi} \times \boldsymbol{\Pi}).
\end{aligned}
$$

The linearizing feedback can then be computed from (5.6) and (5.7) which in this case are

$$
\begin{aligned}
\mathbf{J}^{-1}\alpha(\boldsymbol{\Pi}) &= -\mathbf{J}^{-1}(\mathbf{J}^{-1}\boldsymbol{\Pi} \times \boldsymbol{\Pi}), \\
\mathbf{J}^{-1}\beta(\boldsymbol{\Pi}) &= -\mathbf{1},
\end{aligned}
$$

from which

$$
\begin{aligned}
\boldsymbol{M} &= \boldsymbol{\alpha}(\boldsymbol{x}) + \boldsymbol{\beta}(\boldsymbol{x})\boldsymbol{V}, \\
&= -\mathbf{J}^{-1}\boldsymbol{\Pi} \times \boldsymbol{\Pi} + \mathbf{J}\boldsymbol{V},
\end{aligned}
$$

in agreement with our earlier result.

The technique of linearizing equation (5.8) by canceling the nonlinear terms arising from the cross product has been suggested for the control of large angle multi axis slewing of spacecraft. In Lindberg, Fisher, Posbergh [1985] equation (5.8) was used as an idealized model of a (assumed) rigid body spacecraft. In this case the applied moment was provided by momentum wheels on board the spacecraft while angular rate measurements were provided by gyroscopes. To obtain a linear model for a subsequent control design the angular rate measurements were used to construct a linearizing feedback exactly as in this example. Of course for a real design numerous other issues, such as robustness, need to be addressed.

5.1.3. Exact Linearization of N-body approximations

In this section we apply the algorithm of Isidori to compute the input-output linearizing feedback for particular examples of the N-body approximation of section 2.5. The exact input-output linearization problem for an N-body chain formulated in a Hamiltonian framework was solved by Sreenath [1987].

Two Body Approximation

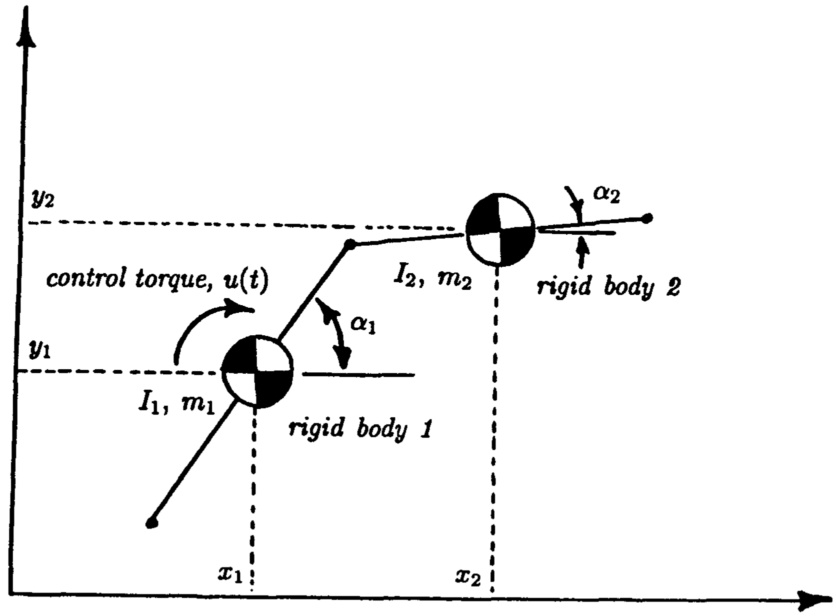

Figure 5.1. Two Body Approximation.

We next proceed to use the structure algorithm described in Isidori to find the exact, linearizing control for the N body approximation of section 2.5. We will first consider the case for $N = 2$, this will give us a great deal of insight into what happens in cases for larger N.

For the two body approximation we consider the configuration shown in figure 5.1. Our notation will be that introduced in section 2.5. In this case we have two rigid bodies connected by a single elastic joint. The centers of mass are located at (x_1, y_1) and (x_2, y_2) respectively.

To find the dynamics we can use (2.91), (2.94), and (2.96) or proceed to rederive these equations. For the sake of completeness we will outline the derivation which the interested reader can compare with the more general case in chapter 2.

For this two body system we have a Lagrangian which is;

$$\begin{aligned} L = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}I_1\dot{\phi}_1 \\ + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \tfrac{1}{2}I_2\dot{\phi}_1 \\ - \tfrac{1}{2}k_1(\phi_2 - \phi_1)^2, \end{aligned} \tag{5.11}$$

as well as the two constraints

$$x_2 = r_1(1-\epsilon_1)\cos(\phi_1) + r_2\epsilon_2\cos(\phi_2) + x_1, \tag{5.12}$$

$$y_2 = r_1(1-\epsilon_1)\sin(\phi_1) + r_2\epsilon_2\sin(\phi_2) + y_1. \tag{5.13}$$

Using (5.12) and (5.13) we can write the Lagrangian in terms of the four variables x_1, y_1, ϕ_1, and ϕ_2 and their time derivatives. The equations of motion are then give by Lagranges equation

$$0 = \frac{d}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \tag{5.14}$$

For our two body system this gives the four equations

$$\begin{aligned} 0 = (m_1+m_2)\ddot{x}_1 + m_2\{&-r_1(1-\epsilon_1)\cos(\phi_1)\dot{\phi}_1^2 - r_1(1-\epsilon_1)\sin(\phi_1)\ddot{\phi}_1 \\ &- r_2\epsilon_2\cos(\phi_2)\dot{\phi}_2^2 - r_2\epsilon_2\sin(\phi_2)\ddot{\phi}_2\}, \end{aligned} \tag{5.15}$$

$$\begin{aligned} 0 = (m_1+m_2)\ddot{y}_1 + m_2\{&-r_1(1-\epsilon_1)\sin(\phi_1)\dot{\phi}_1^2 + r_1(1-\epsilon_1)\cos(\phi_1)\ddot{\phi}_1 \\ &- r_2\epsilon_2\sin(\phi_2)\dot{\phi}_2^2 + r_2\epsilon_2\cos(\phi_2)\ddot{\phi}_2\}, \end{aligned} \tag{5.16}$$

$$\begin{aligned} 0 = m_2\{&r_1^2(1-\epsilon_1)^2\ddot{\phi}_1 + r_1r_2(1-\epsilon_1)\epsilon_2\sin(\phi_1-\phi_2)\dot{\phi}_2^2 \\ &+ r_1r_2(1-\epsilon_1)\epsilon_2\cos(\phi_1-\phi_2)\ddot{\phi}_2\} \\ &+ I_1\ddot{\phi}_1 - k_1(\phi_2-\phi_1) + m_2r_1(1-\epsilon_1)(\ddot{x}_1\sin(\phi_1) + \ddot{y}_1\cos(\phi_1)), \end{aligned} \tag{5.17}$$

$$\begin{aligned} 0 = m_2\{&r_1r_2(1-\epsilon_1)\epsilon_2\sin(\phi_2-\phi_1)\dot{\phi}_1^2 \\ &+ r_1r_2(1-\epsilon_1)\epsilon_2\cos(\phi_1-\phi_2)\ddot{\phi}_1\} + r_2^2\epsilon_2^2\ddot{\phi}_2 \\ &+ I_2\ddot{\phi}_2 + k_1(\phi_2-\phi_1) + m_2r_2\epsilon_2(-\ddot{x}_1\sin(\phi_1) + \ddot{y}_1\cos(\phi_1)). \end{aligned} \tag{5.18}$$

In what follows we will assume that $m_1 \gg m_2$ so that we can consider the center of mass of the entire configuration to coincide with the center of mass of the first rigid body. In this case we can assume $\ddot{x}_1 = 0$, and $\ddot{y}_1 = 0$ (divide (5.15) and (5.16) by m_1 and let $m_1 \to \infty$.) We are left with (5.17) and (5.18) in which we have set $\ddot{x}_1 = 0$, $\ddot{y}_1 = 0$ describing the dynamics.

These equations can be written as the (nonlinear) matrix differential equation

$$0 = \mathbf{I}(\phi_1,\phi_2)\begin{bmatrix}\ddot{\phi}_1 \\ \ddot{\phi}_2\end{bmatrix} + \mathbf{\Omega}(\phi_1,\phi_2)\begin{bmatrix}\dot{\phi}_1^2 \\ \dot{\phi}_2^2\end{bmatrix} + \mathbf{K}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}, \tag{5.19}$$

where we define

$$\mathbf{I}(\phi_1,\phi_2) = \begin{bmatrix} r_1^2(1-\epsilon_1)^2 m_2 + I_1 & r_1 r_2(1-\epsilon_1)\epsilon_2\cos(\phi_1-\phi_2)m_2 \\ r_1 r_2(1-\epsilon_1)\epsilon_2\cos(\phi_1-\phi_2)m_2 & r_2^2\epsilon_2^2 m_2 + I_2 \end{bmatrix},$$

$$\mathbf{\Omega}(\phi_1,\phi_2) = \begin{bmatrix} 0 & r_1 r_2(1-\epsilon_1)\epsilon_2\sin(\phi_1-\phi_2)m_2 \\ r_1 r_2(1-\epsilon_1)\epsilon_2\sin(\phi_2-\phi_1)m_2 & 0 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}.$$

We note that $\mathbf{I}(\phi_1,\phi_2)$, is a symmetric, positive definite matrix being the sum of a diagonal matrix of inertia terms and a positive semidefinite matrix. Furthermore, $\mathbf{\Omega}(\phi_1,\phi_2)$ is skew symmetric and $\mathbf{K}$ is positive semidefinite.

The control problem associated with this system can be formulated as follows; we assume a control torque $u(t)$ is applied to the first rigid body. Then, defining $\omega_1 = \dot{\phi}_1$, $\omega_2 = \dot{\phi}_2$ (5.19) can be written as the nonlinear first order equation

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ f_1(\phi_1,\phi_2,\omega_1,\omega_2) \\ f_2(\phi_1,\phi_2,\omega_1,\omega_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \\ 0 \end{bmatrix} u(t), \tag{5.20}$$

where we define

$$\begin{bmatrix} f_1(\phi_1,\phi_2,\omega_1,\omega_2) \\ f_2(\phi_1,\phi_2,\omega_1,\omega_2) \end{bmatrix} = -\mathbf{I}^{-1}(\phi_1,\phi_2)\mathbf{\Omega}(\phi_1,\phi_2)\begin{bmatrix} \dot{\phi}_1^2 \\ \dot{\phi}_2^2 \end{bmatrix} - \mathbf{I}^{-1}(\phi_1,\phi_2)\mathbf{K}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{5.21}$$

We consider two outputs, the first z_1, is base rotation rate. The second z_2 is the acceleration at the center of mass of the second rigid body in a direction perpendicular to the centerline. Thus,

$$z_1 = \omega_1, \tag{5.22}$$

$$\begin{aligned} z_2 = {} & \begin{bmatrix} r_1(1-\epsilon_1)\sin(\phi_2-\phi_1) & 0 \end{bmatrix}\begin{bmatrix} \dot{\phi}_1^2 \\ \dot{\phi}_2^2 \end{bmatrix} \\ & + \begin{bmatrix} r_1\epsilon_2\cos(\phi_2-\phi_1) & r_2\epsilon_2 \end{bmatrix}\begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix}. \end{aligned} \tag{5.23}$$

For the first case we consider the construction of the linearizing static feedback by the method described in Isidori. We first need to check to see if the system is linearizable by a coordinate transformation. Thus we first compute,

$$
\begin{aligned}
T_0 &= L_g \boldsymbol{h}, \\
&= b_1, \\
T_1 &= L_g L_f \boldsymbol{h}, \\
&= \frac{\partial f_1}{\partial \omega_1} b_1, \\
&= \frac{-r_1^2 r_2^2 \epsilon_1^2 (1-\epsilon_2)^2 \cos(\phi_1 - \phi_2) m_2^2 \sin(\phi_1 - \phi_2)\omega_1}{(I_1 + r_1^2 \epsilon_1^2 m_2)(I_2 + r_2^2 (1-\epsilon_2)^2 m_2) - r_1^2 r_2^2 \epsilon_1^2 (1-\epsilon_2)^2 \cos(\phi_1 - \phi_2) m_2} b_1.
\end{aligned}
$$

Since the partial derivative in the the last expression is a function of the state variables we conclude that the system is not linearizable by a coordinate transformation.

We now proceed to apply the structure algorithm. Since $\text{rank}(\mathbf{T}_0) = 1$ the rank of $\boldsymbol{h}(\phi_1, \phi_2, \omega_1, \omega_2)$, we terminate with the initial conditions. We set $\mathbf{P}_1 = \mathbf{1}$, $\mathbf{V}_1 = \mathbf{1}$, and $\gamma_1 = L_f \boldsymbol{h}$. In this case

$$
\begin{aligned}
L_g \gamma_1 &= b_1, \\
L_f \gamma_1 &= f_1(\phi_1, \phi_2, \omega_1, \omega_2),
\end{aligned}
$$

and the linearizing feedback is computed from

$$
\begin{aligned}
[L_g \gamma_1] \alpha(\phi_1, \phi_2, \omega_1, \omega_2) &= -L_f \gamma_1, \\
[L_g \gamma_1] \beta(\phi_1, \phi_2, \omega_1, \omega_2) &= 1,
\end{aligned}
$$

from which

$$
\begin{aligned}
u(t) &= \alpha(\phi_1, \phi_2, \omega_1, \omega_2) + \beta(\phi_1, \phi_2, \omega_1, \omega_2) v(t), \\
&= -b_1^{-1} f_1(\phi_1, \phi_2, \omega_1, \omega_2) - b_1^{-1} v(t).
\end{aligned}
$$

Three Body Approximation

For the three body approximation we again assume that the center of mass of the first link is fixed in inertial space. Therefore, as in the previous example we will set $\ddot{x} = 0$, $\ddot{y} = 0$ and obtain from (2.91) three equations of motion.

$$
0 = (m_2 + m_3)(1-\epsilon_1)^2 r_1^2 \ddot{\phi}_1 + (m_2 \epsilon_2 + m_3)(1-\epsilon_1) r_1 r_2 \cos(\phi_2 - \phi_1) \ddot{\phi}_2
$$

$$+ m_3(1-\epsilon_1)\epsilon_3 r_1 r_3 \cos(\phi_3 - \phi_1)\ddot{\phi}_3$$

$$+ (m_2\epsilon_2 + m_3)(1-\epsilon_1) r_1 r_2 \sin(\phi_2 - \phi_1)\dot{\phi}_2^2 + m_3(1-\epsilon_1)\epsilon_3 r_1 r_3 \sin(\phi_3 - \phi_1)\dot{\phi}_3^2$$

$$+ I_1\ddot{\phi}_1 - k_1(\phi_2 - \phi_1), \tag{5.24}$$

$$0 = (m_2\epsilon_2 + m_3)(1-\epsilon_1) r_1 r_2 \cos(\phi_2 - \phi_1)\ddot{\phi}_1 + (m_2\epsilon_2^2 + m_3) r_2^2 \ddot{\phi}_2$$

$$+ m_3\epsilon_3 r_1 r_3 \cos(\phi_3 - \phi_1)\ddot{\phi}_3$$

$$+ (m_2\epsilon_2 + m_3)(1-\epsilon_1) r_1 r_2 \sin(\phi_1 - \phi_2)\dot{\phi}_1^2 + m_3\epsilon_3 r_1 r_3 \sin(\phi_3 - \phi_1)\dot{\phi}_3^2$$

$$+ I_2\ddot{\phi}_2 + k_1(\phi_2 - \phi_1) - k_2(\phi_3 - \phi_2), \tag{5.25}$$

$$0 = m_3(1-\epsilon_1)\epsilon_3 r_1 r_3 \cos(\phi_3 - \phi_1)\ddot{\phi}_1 + m_3\epsilon_3 r_1 r_3 \cos(\phi_3 - \phi_2)\ddot{\phi}_2$$

$$+ m_3\epsilon_3^2 r_3^2 \ddot{\phi}_3$$

$$+ m_3(1-\epsilon_1)\epsilon_3 r_1 r_3 \sin(\phi_1 - \phi_3)\dot{\phi}_1^2 + m_3\epsilon_3 r_2 r_3 \sin(\phi_2 - \phi_3)\dot{\phi}_2^2$$

$$+ I_3\ddot{\phi}_3 - k_2(\phi_2 - \phi_3). \tag{5.26}$$

As before we can rewrite this in the form of a nonlinear, matrix differential equation. If we define the symmetric matrix,

$$\mathbf{I}(\phi_1, \phi_2, \phi_3) =$$

$$\begin{bmatrix} (m_2+m_3)r_1^2(1-\epsilon_1)^2+I_1 & (m_2\epsilon_2+m_3)(1-\epsilon_1)r_1 r_2\cos(\phi_2-\phi_1) & m_3(1-\epsilon_1)\epsilon_3 r_1 r_3\cos(\phi_3-\phi_1) \\ \cdot & (m_2\epsilon_2^2+m_3)r_2^2+I_2 & m_3\epsilon_3 r_1 r_3\cos(\phi_3-\phi_2) \\ \cdot & \cdot & m_3\epsilon_3^2 r_3^2+I_3 \end{bmatrix},$$

and the skew symmetric matrix,

$$\mathbf{\Omega}(\phi_1, \phi_2, \phi_3) =$$

$$\begin{bmatrix} 0 & (m_2\epsilon_2+m_3)(1-\epsilon_1)r_1 r_2\sin(\phi_2-\phi_1) & m_3(1-\epsilon_1)\epsilon_3 r_1 r_3\sin(\phi_3-\phi_1) \\ \cdot & 0 & m_3\epsilon_3 r_2 r_3\sin(\phi_3-\phi_2) \\ \cdot & \cdot & 0 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}.$$

Equations (5.24), (5.25), and (5.26) can now be expressed

$$0 = \mathbf{I}(\phi_1, \phi_2, \phi_3)\begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \end{bmatrix} + \mathbf{\Omega}(\phi_1, \phi_2, \phi_3)\begin{bmatrix} \dot{\phi}_1^2 \\ \dot{\phi}_2^2 \\ \dot{\phi}_3^2 \end{bmatrix} + \mathbf{K}\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}. \tag{5.27}$$

Again we note that $\mathbf{I}(\phi_1, \phi_2)$, is a symmetric, positive definite matrix, $\mathbf{\Omega}(\phi_1, \phi_2)$ is skew symmetric and is $\mathbf{K}$ positive semidefinite.

Again we consider a control torque $u(t)$ which is applied to the first link with a gain of b_1. Measurements are either of base rotation rate or tangential tip acceleration of the third link. These measurements will be

$$
\begin{aligned}
z_1 &= \dot{\phi}_1, \\
z_2 &= -\sin(\phi_3)\ddot{x}_{tip} + \cos(\phi_3)\ddot{y}_{tip}, \\
&= r_2 \sin(\phi_3 - \phi_2)\dot{\phi}_2^2 + \epsilon_1 r_1 \sin(\phi_3 - \phi_1)\dot{\phi}_1^2 \\
&\qquad + r_3\ddot{\phi}_3 + r_2 \cos(\phi_3 - \phi_2)\ddot{\phi}_2 + \epsilon_1 r_1 \cos(\phi_3 - \phi_1)\ddot{\phi}_1.
\end{aligned}
$$

For the first case we consider the construction of the linearizing static We can recast this in the form a system of first order differential equations;

$$
\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ f_1(\phi_1, \phi_2, \omega_1, \omega_2) \\ f_2(\phi_1, \phi_2, \omega_1, \omega_2) \\ f_3(\phi_1, \phi_2, \omega_1, \omega_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} u(t), \tag{5.28}
$$

where we define

$$
\begin{bmatrix} f_1(\phi_1, \phi_2, \omega_1, \omega_2) \\ f_2(\phi_1, \phi_2, \omega_1, \omega_2) \\ f_3(\phi_1, \phi_2, \omega_1, \omega_2) \end{bmatrix} = -\mathbf{I}^{-1}(\phi_1, \phi_2, \phi_3)\mathbf{\Omega}(\phi_1, \phi_2, \phi_3) \begin{bmatrix} \dot{\phi}_1^2 \\ \dot{\phi}_2^2 \\ \dot{\phi}_3^2 \end{bmatrix} - \mathbf{I}(\phi_1, \phi_2, \phi_3)\mathbf{K} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}.
$$

We consider exact linearization by static feedback when the output is base rotation rate. We first need to check to see if the system is linearizable by a coordinate transformation. Thus we first compute,

$$
\begin{aligned}
T_0 &= L_g \boldsymbol{h}, \\
&= b_1.
\end{aligned}
$$

$$
\begin{aligned}
T_1 &= L_g L_f \boldsymbol{h}, \\
&= \frac{\partial f_1}{\partial \omega_1} b_3.
\end{aligned}
$$

Since the partial derivative in the the last expression is a function of (ϕ_1, ϕ_2, ϕ_3) we conclude that the system is not linearizable by a coordinate transformation.

We now proceed to apply the structure algorithm. Since $\mathrm{rank}(\mathbf{T}_0) = 1$ the rank of $\boldsymbol{h}(\boldsymbol{x})$, we terminate with the initial conditions. We set $\mathbf{P}_1 = \mathbf{1}$, $\mathbf{V}_1 = \mathbf{1}$, and $\gamma_1 = L_f\boldsymbol{h}$. In this case

$$
\begin{aligned}
L_g\gamma_1 &= b_1, \\
L_f\gamma_1 &= f_1(\phi_1, \phi_2, \omega_1, \omega_2),
\end{aligned}
$$

and the linearizing feedback is computed from case are

$$
\begin{aligned}
[L_g\gamma_1]\alpha(\boldsymbol{x}) &= -L_f\gamma_1, \\
[L_g\gamma_1]\beta(\boldsymbol{x}) &= 1,
\end{aligned}
$$

from which

$$
\begin{aligned}
u(t) &= \alpha(\boldsymbol{x}) + \beta(\boldsymbol{x})v(t), \\
&= -b_1^{-1} f_1(\phi_1, \phi_2, \omega_1, \omega_2) - b_1^{-1} v(t).
\end{aligned}
$$

Note that as in the previous example, while we can find a linearizing feedback *it depends on knowledge of all the joint angular positions and velocities.* Thus, this feedback requires a complete state vector to compute the feedback.

5.2. Control by frequency domain methods

In chapter 3 we derived some transfer functions associated with several models of a rigid body attached to the base of a nonshearable, inextensible beam. In this section we will use frequency domain techniques to design controllers for them.

We will restrict our attention to the methods of control system design related to H^∞ theory (Francis [1986]). This methodology rigorizes the more classical techniques which were used extensively in the decade and a half after the second world war (Horowitz [1963], Truxel [1955]). More recently Zames [1981] and others succeeded in placing much of this earlier methodology on a rigorous mathematical foundation. This rigorous mathematical foundation together with modern computer aided design tools has brought about the current renaissance of frequency domain methods.

While much of the current literature deals with finite dimensional systems we are concerned with the more challenging problem of infinite dimensional systems. One approach to this problem is described in Flamm [1986] where the problem addressed is the minimization of the maximum weighted sensitivity in the frequency domain of a single input, single output linear time invariant feedback compensator. The plant considered is a cascade of a rational transfer function and a pure delay. They were successful in designing a rational finite dimensional compensator to control the plant.

Another approach, and the one we shall employ, is that of the L_∞ methods pioneered by Curtain [1987], and Curtain and Glover [1986]. They consider the transfer function describing the plant to be the sum of a finite and an infinite dimensional part. They then employ Hankel norm approximations (Glover [1984]) to design for a prespecified robustness margin, a finite dimensional compensator which stabilizes the plant.

5.2.1. The Methodology

The fundamental idea underlying the methodology of control system design for infinite dimensional linear systems as proposed by Curtain and Glover [1986] is to approximate the an infinite dimensional system by a finite dimensional system and guarantee that the subsequent control design is robust with respect to the modeling error arising from the approximation. Fundamental to this methodology is the idea of a Hankel-norm approximation for linear multivariable system and the associated L_∞ error bounds.

First we define several of the quantities we will need subsequently. We will start with the definition of a Hankel operator for a linear system. In this we will restrict our attention to the class of spectral systems introduced by Curtain [1984]. Let $\mathcal{H}$ be a real, separable Hilbert space and consider systems defined on this space of the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}\ ; \qquad\qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \boldsymbol{y} &= \mathbf{C}\boldsymbol{x}, \end{aligned}$$

with $\mathbf{A}: \mathcal{D}(\mathbf{A}) \longrightarrow \mathcal{H}$ a closed linear operator with compact resolvent whose normalized eigenvectors form a basis in $\mathcal{H}$. The operators $\mathbf{B}: \mathbb{R}^m \longrightarrow \mathcal{H}$, and $\mathbf{C}: \mathcal{H} \longrightarrow \mathbb{R}^p$ are assumed to be bounded.

Definition. *The Hankel operator* $\mathbf{\Gamma}: L_2(0,\infty;\mathbb{R}^m) \longrightarrow L_2(0,\infty;\mathbb{R}^m)$ *for the spectral system* $(\mathbf{A},\ \mathbf{B},\ \mathbf{C})$ *is given by*

$$(\mathbf{\Gamma} u)(t) = \int_0^\infty \mathbf{C}\exp(\mathbf{A}(t-\sigma)\mathbf{B}\boldsymbol{u}(\sigma)\, d\sigma. \tag{5.29}$$

If we assume that $\sup(\mathrm{Re}\lambda_i) < 0$ then $\exp At$ is exponentially stable and $\mathbf{\Gamma}$ is a bounded map. It can also be shown that $\mathbf{\Gamma}$ is compact with the consequence that it has countably many *Hankel singular values* which are the square roots of the eigenvalues of $\mathbf{\Gamma}^*\mathbf{\Gamma}$†.

We will be interested in bounds on the L_∞ norm. This norm is defined by

$$||\mathbf{M}||_\infty = \sup_{\omega\in(-\infty,\infty)} \mu_{\max}^{\frac{1}{2}}(\mathbf{M}^*(j\omega)\mathbf{M}(j\omega)) \tag{5.30}$$

where $\mu_{\max}(\mathbf{M})$ denotes the largest eigenvalue of the square matrix $\mathbf{M}$.

We now turn our attention to finite dimensional approximations of the transfer function, $\mathbf{G}(s)$, and the Hankel operator $\mathbf{\Gamma}$. Recall that the n^{th} order model approximation for the transfer function will be

$$\mathbf{G}_n(s) = \sum_{i=1}^{n} \frac{\mathbf{C}\phi_i\mathbf{B}^*\tilde{\phi}_i}{s-\lambda_i}, \tag{5.31}$$

where $\phi_i,\ i = 1,\ldots$ are the eigenvectors of the operator $\mathbf{A}$ and $\lambda_i,\ i = 1,\ldots$ are the corresponding eigenvalues. It is straight forward to show this truncation converges (see Curtain and Glover [1986]). In this case we have an L_∞ error estimate on the approximation given by (see Glover [1984])

$$||\mathbf{G}(s) - \mathbf{G}_n(s)||_\infty \leq \sum_{i=n+1}^{\infty} 2\mu_i, \tag{5.32}$$

where

$$\mu_i = \frac{|\mathbf{C}\phi_i||\mathbf{B}^*\tilde{\phi}_i|}{-2\mathrm{Re}(\lambda_i)}. \tag{5.33}$$

For a Hankel norm approximation we can do better, in this case it can be shown that

$$||\mathbf{G}(s) - \hat{\mathbf{G}}_n(s)||_\infty \leq \sum_{i=n+1}^{\infty} \mu_i. \tag{5.34}$$

Again the reference is Glover [1984].

Model Approximation

We assume that we have an infinite dimensional model whose transfer function is of the form

$$\mathbf{G}(s) = \mathbf{G}_u(s) + \mathbf{G}_s(s). \tag{5.35}$$

† We use * to denote Hermitian transpose

Here $\mathbf{G}_u(s)$ is the finite dimensional, possibly unstable component of the transfer function we want to control, $\mathbf{G}_s(s)$ is a stable, infinite dimensional component of the transfer function.

For the models we have developed in the previous chapters we have no poles strictly in the right half plane. However, for undamped systems our poles lie on the imaginary axis. In what follows we will restrict our attention to models with damping introduced in the constitutive relations (see section 3.5.5). For these models we may want to ensure that there are no poles with $\mathrm{Re}(\lambda_i) > -\alpha$, for $\alpha > 0$ (i.e. all the poles in the left half plane are no closer than α to the imaginary axis. Thus, in this case we might choose $\mathbf{G}_u(s)$ such that it contains all λ_i with $\lambda_i > -\alpha$.

The strategy for the model reduction is as follows: First we determine the robustness margin associated with $\mathbf{G}_u(s)$. This will be the smallest Hankel singular value for this system, i.e. $\sigma_{\min}(\mathbf{G}_u)$. It is known that there exists a controller which will control $\mathbf{K}$ such that for all $\mathbf{G}_u + \Delta$, with $||\Delta||_\infty < \sigma_{\min}(\mathbf{G}_u)$ the system is stable. The next step is to compute the optimal Hankel norm approximation such that $||\mathbf{G} - \hat{\mathbf{G}}||_\infty < \sigma_{\min}(\mathbf{G}_u)$. In order that we might employ the algorithm described in Glover [1984] one frequently first obtains a truncated realization of $\mathbf{G}_s$, which we will denote as $\mathbf{G}_s^R$, and such that $||\mathbf{G}_s^R - \mathbf{G}||_\infty < \epsilon_1$ with ϵ_1 a very small value. The Hankel norm approximation $\hat{\mathbf{G}}_s$ of $\mathbf{G}_s^R$ is then computed by Glover's algorithm. In this case a suitable McMillan degree of $\hat{\mathbf{G}}_s$ is chosen such that $||\mathbf{G}_s^R - \hat{\mathbf{G}}_s||_\infty < \epsilon_2$ and $\epsilon_1 + \epsilon_2 < \sigma_{\min}(\mathbf{G}_u)$. Figure 5.1 gives the block diagram of the control system we describe.

5.2.2. Discussion

As we have remarked above, the L_∞ technique is applicable to our model of a rigid body with an attached, inextensible rod with with rate damping, discussed in section 3.5.5. As can be seen from figure 3.2, all of the eigenvalues in the case of the example in that chapter lie in the left half plane, except for the zero at the origin. A typical control design for this model would be to move the zero at the origin into the left half plane. If we recall that this zero physically corresponds to the rigid body mode of the system, then our feedback controller would be designed to keep the angular position of the beam fixed.

For this example we can identify $G_u(s)$, a single input, single output transfer func-

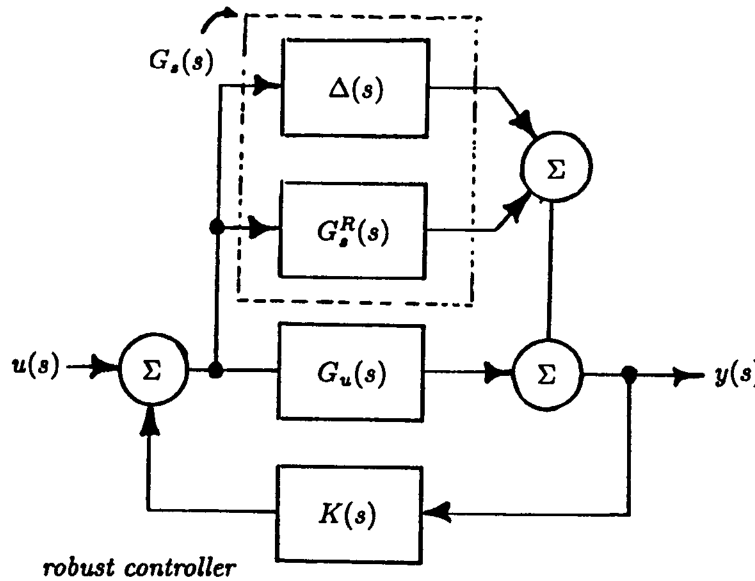

Figure 5.1. Block Diagram of L_∞ Design.

tion in this case as consisting of the pole at the origin and a finite number of the remaining terms in the model expansion. The remaining terms define $G_s(s)$. In (5.35) we therefore have

$$G_u(s) = \frac{a_0}{s} + \sum_{n=1}^{N} \frac{a_n + b_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.36}$$

and

$$G_u(s) = \frac{a_0}{s} + \sum_{n=N+1}^{\infty} \frac{a_n + b_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.37}$$

where a_0, and $a_n\, b_n\, n = \ldots$ are constants computed from the model parameters.

For the first attempt at the design we can eliminate the Hankel norm approximation used by Curtain & Glover [1986], and consider an L_∞ design based on a finite dimensional model obtained by simple modal truncation. We note that control design based on modal truncation has been used frequently in the past, see for example Hughes & Skelton [1981].

Analytically the computation of the bound can be difficult, the relevant equations for our model are (3.130), (3.131) and (3.132). While one can carry out the computations in principle, in general the usual practice is to resort to numerical computation and choose the number of terms to retain on a heuristic basis.

CHAPTER SIX

CONCLUSIONS AND FUTURE RESEARCH

6.1. Conclusion

In this dissertation we have explored some of the issues associated with the modeling and control of mixed and flexible structures. In particular, we have dwelt at some length on the problem of modeling and control of a rigid body plus attached, flexible appendage where the motion is restricted to a plane. This problem has been motivated by actual hardware configurations and we have attempted to relate our work to one such configuration.

In chapter two we introduced the geometrically exact nonlinear rod models which are the basic models used in our work. We showed that for the special case of a planar, inextensible, nonshearable rod attached to a rigid body we get an integro-differential equation which describes the dynamics. This equation is in fact a special case of Euler's elastica and we showed that by linearizing it in a nonrotating configuration we recover the classical, Euler-Bernoulli beam equation.

We then showed that this rod model can be obtained from standard variational methods. We also showed that this rod model represents in certain regards the limiting case of a chain of rigid bodies.

In chapter three we turned our attention to the linearized models of the rigid body plus flexible appendage. For this model, we found equilibria associated with simple rotating and nonrotating configurations. For the equilibria associated with the rotating configuration we showed that the linearization of our model correctly accounts for effects such as stiffening which are frequently neglected in more ad hoc methods.

We next proceeded to obtain transfer functions associated with our model. The transfer functions included such often neglected effects such as rotary inertia and damping. We showed that the transfer function associated with the rotating configuration could be viewed as one associated to a nonrotating system together with some perturbation terms coming from the effects of the rotation.

In chapter four we turned our attention to the relative stability of a rigid body with attached, flexible appendage. In the case of a linear extensible shear beam we used the energy-Casimir method to find conditions assuring the relative (formal) stability of the configuration. In particular, for any equilibrium we showed that relative stability is determined by the positive definiteness of a certain quadratic form. Motivated by issues associated with more general models we then introduced the method of energy-momentum. Using this method we reproduced the stability results obtained from the energy-Casimir method.

In the last chapter we turned our attention to the control problem. We first showed how exact linearization could be used to linearize finite dimensional systems based on the N-body approximations of the geometrically exact rod. The results were related to an existing piece of hardware at the Intelligent Servomechanisms laboratory at the University of Maryland. We also noted the several issues related to the exact linearization of infinite dimensional systems such as the nonlinear model of the geometrically exact rod. In the second half of the chapter we outlined the application of L_∞ control methodology to our models. Both approaches to control were based on finite dimensional approximations to infinite dimensional models, an important consideration from an implementation point of view.

6.2. Future Work

There are several areas of investigation suggested by this dissertation which need to be further explored.

Other equilibria for geometrically exact models need to be explored. Nonlinear effects play a crucial role in obtaining physically meaningful linearized models and also for investigating stability. Furthermore, equilibria associated with geometrically exact rods may be significantly different from those associated with nonexact models for large deformations. Graphics and symbolic manipulation on modern AI and graphics workstations may have a role to play here.

In chapter three we treated the model of the rotating configuration as a perturbation of the nonrotating configuration. This approach needs to be explored more, both analytically and numerically.

Finally, we note that the approximation of infinite dimensional systems by finite dimensional systems for control purposes is very common. In our case we used the N-body approximation the basis of an exact linearization of an infinite dimensional system. An intersting problem which this suggests is the formulation and solution to the exact linearization problem for an infinite dimensional system. Is it the same as the limit of the linearized systems based on the N-body approximations.

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