

Effects of Dynamic Forcing on Hillslope Water Balance Models

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ABSTRACT

Recently there has been much interest in scaling water flow and species transport at the continuum level to the watershed. A particularly simple and, therefore, appealing approach is based on the water mass balance at the hillslope scale. Such models require parameterization of closure relations (flux–storage relations) based on field data. In several recent studies, this data was instead generated by steady-state numerical simulations of the hillslope. In this work we focus specifically on closure relations for hillslope water balance models as in previous work, but we use transient numerical solutions of continuum-scale models of the subdomain to generate the data. Our goal is to study the effects of non-equilibrium behavior on time-averaged flux–storage relationships for the hillslope. We show that for simulated hillslopes, the flux–storage relations are multivalued for a wide range of time scales, discuss how this situation arises, and provide some alternative parameterizations of the flux–storage relations.

1. INTRODUCTION

Continuum mechanical models of water flow form the basis of the standard physical description of watershed hydrology. In principle, at least, the physical, mathematical, and computational components of a continuum scale model of large catchments and watershed have been available for three decades [20]. In practice, general continuum models have not been capable of making accurate predictions for systems at the scale of watersheds [3]. Limitations of the continuum modeling approach include theoretical difficulties in consistently formulating and closing continuum models for all significant flow processes, mathematical and computational difficulties in solving the resulting three-dimensional partial differential equations, and perhaps most importantly difficulty obtaining accurate model parameters at an appropriate scale.

Our focus in this work is on the simplified representation of hillslope subsurface flow for the purposes of watershed modeling. As accurate continuum models of variably saturated flow at the scale of hillslopes are computationally demanding and require extensive characterization of the subsurface, researchers have pursued several alternative strategies to modeling

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hillslopes. One approach is to use approximate or exact analytical solutions of continuum models for the hillslope based on simplifying assumptions such as soil homogeneity or simplified geometry [39]. Parameter estimation can be used to fit such models to real hillslopes where the physical parameters in the model are then interpreted as effective parameters. A yet simpler approach is to parameterize integral mass balance equations for a hillslope region directly, based on physical data or numerical simulations [17, 18, 15, 31, 33].

We will review some of the predominant watershed modeling approaches in more detail in section 2 to provide a context for our work on hillslope models. As a broad range of techniques have been applied to the watershed modeling problem, we direct the interested reader to several recent journal issues devoted to modeling issues at the watershed scale [4, 41].

Briefly stated, our objective in this work is to study the effect of system transience on hillslope water balance models. In particular, their effect on the flux-storage relations that such models require. A logical approach for an initial study is to use detailed numerical simulations of idealized hillslopes, in which case the data for the water-balance flux-storage relations is easily obtainable [15].

In section 3 we present our derivation of the hillslope water balance model, which combines ideas from [15] and [31]. In section 4 we present our model of the hillslope based on macroscopic continuum equations and incorporating the geometry of the hillslope and nonlinear submodels of unsaturated flow processes. We study the behavior of this continuum model to guide the parameterization of the exchange terms required for closure of the hillslope model. There are natural limitations to using continuum models to obtain closure relations for the water balance models, and there are many open questions about how to account properly for capillary forces and heterogeneous soil properties at an appropriate scale. Nevertheless, the continuum approach is capable of approximating the dynamics of simple systems and should provide a useful benchmark and starting point for watershed scale models. Finally in section 5 we present the resulting flux-storage relationships for the hillslope water balance model based on fully transient macroscale data.

In summary, our objectives are

- (1) to construct a simulator for a single hillslope based on macroscale continuum models of subsurface flow,
- (2) to obtain flux-storage relations for the hillslope using data from the continuum model, and
- (3) to explore the feasibility of constructing a low-dimensional model from these relations.

2. BACKGROUND

The long term objective of watershed-scale hydrological modeling is to predict the response of watersheds to input of precipitation from the atmosphere, extraction of moisture via evapotranspiration and discharge via regional groundwater flow and channel/overland flow. There are a number of different modeling approaches. First, one could draw on the large body of work on continuum models for subsurface and surface fluid flow and transport models to obtain a coupled, spatially distributed continuum model for the watershed system. This approach was outlined and implemented as far back as 1969 [20]. Second, one could formulate a finite-dimensional model of the watershed using various techniques to yield a model appropriate for a significantly larger scale (possibly in time and space) to yield a description of the averaged water balance [17]. We have in mind in the latter case simple

water balance models as well as complex linkages of analytical solutions of the continuum equations and other approximations [2]. Over the last decades widely used models have evolved as some mixture of both approaches, but we nevertheless divide recent and historical approaches into two categories: continuum models and process models.

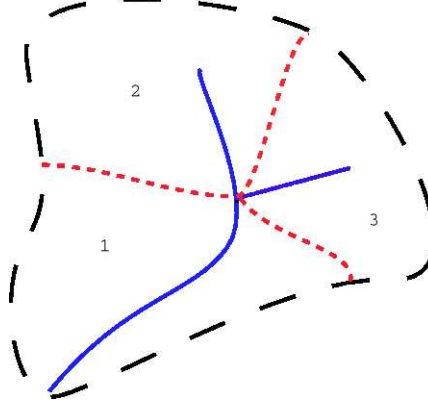
The continuum modeling approach for watersheds is based on coupling three dimensional continuum model equations governing all relevant processes participating in watershed hydrodynamics. Thus, at a minimum models of flow in porous media must be coupled to open channel flow and boundary conditions reflecting evapotranspiration and precipitation. This approach was formulated in [20], though given computational and mathematical limitations at the time the subsurface sub-models were one- and two-dimensional approximations to the full three-dimensional system. Advances in computing power and numerical analysis have led to many recent attempts at modeling catchments and hillslopes using three-dimensional subsurface flow models most notably [29, 9, 27]. These more recent models address some of the shortcomings of continuum models cited by [20] by incorporating digital elevation data for defining the spatial domain, and employing models of canopy interception, evapotranspiration, and soil heterogeneity.

In spite of recent advancements in continuum modeling of the various watershed flow processes, several of the shortcomings cited in [20] and in more recent critiques of watershed modeling [5, 7, 8, 3, 30] remain. We break down these shortcomings into two groups: 1) the inability to determine the physical parameters of the models at the continuum scale for the entire watershed from either in situ measurements or parameter estimation, assuming we have correct physical model of the processes, and 2) inability to characterize all the relevant processes with a rigorous physical model. Both barriers to progress are rooted in the variability of the domain and boundary conditions, and both may yield partially to continued developments in modeling and measurement. On the other hand, severe limitations on modeling predictions due to the propagation of uncertainty may be a fundamental limitation for continuum modeling at this scale. Given that in many applications finely detailed knowledge of the hydrodynamics is not even required of the models, many researchers have pursued simplifications of the full three-dimensional continuum approach.

Quasi-three-dimensional models, where one or two dimensions have been integrated to yield a simplified continuum model, are widely used for describing saturated and unsaturated flow in porous media [10, 11]. A number of other model simplifications still strongly tied to the continuum approach, including an array of upscaling approaches such as volume averaging and homogenization, will not be discussed further here. The simplifications that we have termed process-based models make a more complete break with continuum models and yield directly a model whose solution is itself finite dimensional. The most widely used example of such models is TOPMODEL [2], which discretizes the watershed based on a topographic similarity index related to the surface slope and drainage area. TOPMODEL makes use of analytical solutions of continuum equations to piece together a description for each relevant process.

A number of other process-based models have appeared over the last fifteen years [17, 18, 38, 23, 44, 43, 12, 13, 24, 25, 26]. Agreement between process-based models, field data, and continuum models has been the subject of a several studies [40, 37, 36, 21]. The results of these comparisons have been mixed. However, given that the much simpler process-based models are quite good at predicting at least a subset of the dynamics recorded in the field and simulated by detailed continuum models, improving process-based models might be a worthy

FIGURE 1. A watershed partitioned into 3 hillslopes. Streams are shown in blue (solid lines), drainage divides in red (dotted lines), and the watershed boundary in black (dashed lines).



avenue of research. Significant progress has been made on analytical approximations to continuum-based subsurface flow models with the express purpose of incorporating dominant topographic effects [39] into process models.

Due to the widespread use of process-based models and the realistic dynamics produced by continuum models, some recent research has focused on using continuum models to improve process models. Duffy and colleagues studied equilibrium solutions of a continuum model of unsaturated flow in a simplified hillslope [15, 16, 6] in order to extract a simplified model of a single hillslope for both flow and species transport. Numerical solutions of continuum equations on hillslopes have also been used to study the dependence of saturated area formation on topographic factors, soil properties, and rainfall intensity [28].

A comprehensive watershed modeling framework, was presented in [31, 33] and applied in [32, 34]. This formal framework for finite dimensional models of watersheds could conceivably unite many of the useful features of process models into a more rigorous physical and mathematical theory based on volume and time averaging as well as thermodynamically constrained closure of the balance equations. Our approach will take the time-averaging approach used in the framework and examples presented in [31, 33, 32] to derive models quite similar to the water balance models in [15, 16, 6].

3. HILLSLOPE WATER BALANCE

To formulate watershed models based on integrated continuum scale quantities we must specify an integration volume. In this work the integration volume is a single hillslope, which is a subdomain of a watershed bounded by a stream reach, the land surface, and a set of drainage divides. Any watershed can be decomposed into hillslopes as follows: A watershed has an associated stream network consisting of all streams in the watershed as shown for a simple watershed composed of three hillslopes in figure 1. The stream network can be decomposed into segments (channel reaches). Each channel reach has an associated drainage area determined by the topography, and each drainage area can itself be decomposed into a surface (overland) and subsurface flow subdomains. Thus the organization inherent in the watershed furnishes a decomposition of the watershed into channel reaches, subsurface, and overland flow [31].

In this work we focus simply on the hillslope water balance, in particular, on the flow of water out of the hillslope as either exfiltration to the land surface or base flow to the stream reach. We will, however, make strong limiting assumptions about the channel reach and land surface bounding the hillslope in order to constrain the scope of this work to subsurface processes as was done in [15]. These assumptions are as follows

- (1) The water level in the channel is constant.
- (2) The water level in the overland flow region is negligible.
- (3) The exchange of water between the atmosphere and the unsaturated portion of the surface is determined by the precipitation rate.
- (4) The exchange of water between the subsurface subdomain and neighboring subdomains is negligible.

Thus we will not maintain the capability to simulate any feedback to the subsurface due to significant surface ponding or water level fluctuations in the channel, nor will we be able to simulate so-called infiltration excess overland flow or regional groundwater flow. While the assumptions are likely valid for some hillslopes, such as a steep hillslope bordering a large channel reach, these assumptions are only invoked to allow us to focus on the subsurface dynamics and could be relaxed if the continuum scale model described in the next section was coupled to continuum models for open channel and overland flow. In the context of macroscale continuum models of the subsurface our hillslope is bounded by a low permeability material except where the subdomain intersects the channel reach and the land surface.

3.1. Integration in Time. Following [31] we introduce a characteristic time scale Δt . The hillslope water balance we consider will be a time-integrated mass balance calculated in terms of continuum variables over a hillslope with volume, $\mathcal{V}_s$, and time interval $\mathcal{T} = [t - \Delta t, t + \Delta t]$ of length $T = 2\Delta t$. This approach is different from that used in [15] where only steady-state solutions of the underlying continuum model were considered, and allows us to investigate the effects of system transience on our ability to generate unique low order flux-storage relations.

3.2. Mass Conservation Equations. Continuum scale quantities for water in the subsurface are density $\hat{\rho}$, saturation $\hat{S}$, and porosity $\hat{\epsilon}$ where $\hat{\cdot}$ denotes a macroscale quantity. The time-averaged water mass in the hillslope is then

$$(1) \quad \mathcal{M} = \frac{1}{T} \int_{\mathcal{T}} \int_{\mathcal{V}_s} \hat{\epsilon} \hat{S} \hat{\rho} d\mathcal{V} d\tau$$

where $\mathcal{V}_s$ is the subsurface domain of the hillslope. We now factor $\mathcal{M}$ into physically relevant quantities. First we define the hillslope volume, porosity, and saturation

$$(2) \quad V = \frac{1}{T} \int_{\mathcal{T}} \int_{\mathcal{V}_s} d\mathcal{V} d\tau$$

$$(3) \quad \epsilon = \frac{1}{TV} \int_{\mathcal{T}} \int_{\mathcal{V}_s} \hat{\epsilon} d\mathcal{V} d\tau$$

$$(4) \quad S = \frac{1}{TV\epsilon} \int_{\mathcal{T}} \int_{\mathcal{V}_s} \hat{\epsilon} \hat{S} d\mathcal{V} d\tau$$

Note that with this definition $S = 1$ if and only if $\hat{S} = 1$ throughout the entire hillslope. The hillslope water density is

$$(5) \quad \rho = \frac{1}{TV\epsilon S} \int_{\mathcal{T}} \int_{\mathcal{V}_s} \hat{\epsilon} \hat{S} \hat{\rho} d\mathcal{V} d\tau$$

Thus we have finally

$$(6) \quad \mathcal{M} = \rho S \epsilon V = \frac{1}{T} \int_{\mathcal{T}} \int_{\mathcal{V}_s} \hat{\epsilon} \hat{S} \hat{\rho} d\mathcal{V} d\tau$$

Note that if $\hat{\epsilon}$ is constant in time and $\hat{\rho}$ is constant in time and space then ϵ and $\rho = \hat{\rho}$ are both constants. That is, if the medium and water are assumed to be incompressible then the analogous assumptions hold for the hillslope. We will derive mass conservation on a mass per unit volume basis so we define

$$(7) \quad m = \rho S \epsilon$$

If there are no internal sources of mass in the subdomain then

$$(8) \quad \frac{d}{dt} \int_{\mathcal{V}_s} \hat{\epsilon} \hat{\rho} \hat{S} d\mathcal{V} + \int_{\mathcal{A}_s} \hat{\mathbf{n}} \cdot \hat{\epsilon} \hat{\rho} \hat{S} \hat{\mathbf{v}} d\mathcal{A} = 0$$

where $\mathcal{A}_s$ is the boundary of $\mathcal{V}_s$, $\hat{\mathbf{n}}$ is the unit outward normal on $\mathcal{A}_s$, and $\hat{\mathbf{v}}$ is the macroscopic velocity of the water phase (i.e. the filtration velocity or Darcy velocity). Taking the time average and dividing by the subsurface volume V

$$(9) \quad \frac{1}{TV} \int_{\mathcal{T}} \frac{d}{dt} \int_{\mathcal{V}_s} \hat{m} d\mathcal{V} d\tau + \frac{1}{TV} \int_{\mathcal{T}} \int_{\mathcal{A}_s} \hat{\mathbf{n}} \cdot \hat{m} \hat{\mathbf{v}} d\mathcal{A} d\tau = 0$$

We can interchange differentiation and integration to obtain

$$(10) \quad \frac{1}{TV} \frac{d}{dt} \int_{\mathcal{T}} \int_{\mathcal{V}_s} \hat{m} d\mathcal{V} d\tau + \frac{1}{TV} \int_{\mathcal{T}} \int_{\mathcal{A}_s} \hat{\mathbf{n}} \cdot \hat{m} \hat{\mathbf{v}} d\mathcal{A} d\tau = 0$$

or simply

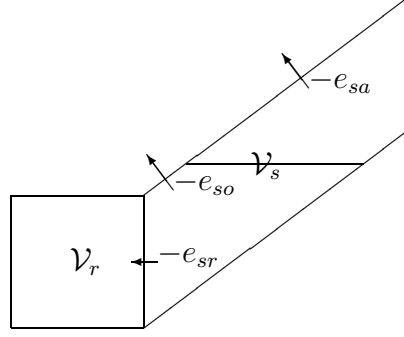
$$(11) \quad \frac{\partial m}{\partial t} = e^A$$

Using the variables and simplifying assumptions above we partition the exchange term e^A into terms from the subsurface region (s) into the overland flow (o), channel reach (r), and atmospheric (a) bounding regions

$$(12) \quad \frac{\partial(\rho S \epsilon)}{\partial t} = e_{so} + e_{sr} + e_{sa}$$

The simple hillslope with the exchange terms labeled is shown in figure 2. We have formulated a hillslope equation for mass conservation, which now needs to be closed by finding additional equations describing the exchange terms. If ρ and ϵ are constant, we need only find a relation between the fluxes e and the storage S . Our closure approach will be to generate data using macroscale numerical simulations. We next describe the macroscale model.

FIGURE 2. 2D Hillslope. The region is divided into stream reach ($\mathcal{V}_r$) and subsurface ($\mathcal{V}_s$) subdomains. The water table divides $\mathcal{V}_s$ roughly in half, and the fluxes out of each boundary segment are labeled. For our sign convention, fluxes out of the domain are negative.



4. CONTINUUM MODELING APPROACH

We will follow the standard macroscale continuum approach for modeling flow in the subsurface [1]. The application of continuum models of the subsurface to hillslope and watershed systems was apparently first outlined in [20]. Notable recent efforts are described in [42, 7, 29].

4.1. Assumptions. We use the following assumptions, which are consistent with those given above, in formulating the continuum model equations

- (1) The geometry and pressure distribution of the channel is constant, in particular we assume a static equilibrium pressure distribution.
- (2) The water level in the overland flow subdomain is negligible and constant, in particular the pressure is atmospheric for the thin sheet of overland flow.
- (3) The exchange of water between the atmosphere and the unsaturated surface of the hillslope is determined by atmospheric conditions.
- (4) The hillslope is symmetric about a long channel reach. We will consider a 2D slice of hillslope by symmetry.
- (5) Air in the subsurface is at constant atmospheric pressure and infinitely mobile.

4.2. Boundary and Initial Conditions. The assumptions above allow us to ignore simulating overland flow and channel flow and to simulate the hillslope using Richards' equation [35]. The effects of the channel and overland flow subdomain enter only through the boundary conditions that we use for Richards' equation. The boundary conditions are

$$(13) \quad \mathcal{A}_{sr} \quad -- \quad p = \rho g(z_{top} - z)$$

$$(14) \quad \mathcal{A}_{so} \quad -- \quad \mathbf{v} \cdot \hat{\mathbf{n}} = \max(k \cdot p - q_p, -q_p)$$

$$(15) \quad \mathcal{A}_{s\partial\mathcal{V}} \quad -- \quad \mathbf{v} \cdot \hat{\mathbf{n}} = 0$$

where z_{top} is the depth of the channel reach, q_p is the precipitation rate and k is a parameter representing surface conductivity (10m/d in this work). The initial conditions were static equilibrium conditions corresponding to $q_p = 0$. We use a mass conservative formulation

TABLE 1. Homogeneous Hillslope Parameters. These values correspond to a sandy soil.

ω	3.010×10^{-1}
S_r	2.799×10^{-2}
K_s (m/day)	5.040×10^0
α (m ⁻¹)	5.470×10^0
n	4.264×10^0
ρ_0 (kg/m ³)	9.982×10^2
β (m · day ² / kg)	6.564×10^{-20}

of Richards' equation and numerical methods that are described in [22] along with closure relations of Mualem and Van Genuchten. The mass balance and closure relations are given by

$$(16) \quad \frac{\partial \rho \omega S}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$(17) \quad \mathbf{v} = -K (\nabla p - \rho \mathbf{g})$$

$$(18) \quad \rho = \rho_0 e^{\beta p}$$

$$(19) \quad S = S_e(1 - S_r) + S_r$$

$$(20) \quad S_e = [1 + (\alpha \max(-p, 0))^n]^{-m}$$

$$(21) \quad K = K_s \sqrt{S_e} [1 - (1 - S_e^{1/m})^m]^2$$

where ρ_0 , β , ω , $\mathbf{g}$, K_s , S_r , α , n and $m = 1 - 1/n$ are constants.

4.3. Hillslope Simulator. For this work the hillslope domain $\mathcal{V}_s$ is a $10m \times 100m$ rectangular region with the long side tilted at an angle $\pi/4$ with the horizontal. The channel reach boundary is applied at the left hand side so that $z_{top} = 10 \sin \pi/4$. The fluid and soil parameters, which correspond to a homogeneous sand, are given in table 1. For completeness we also tested slopes of $\pi/5$ and $\pi/6$, correlated random fields for the soil parameters based on the Miller-similar approach with a variance of 1.0, and a block heterogeneous slope described in detail in [19]. Since the features we describe in what follows are apparent in the simple homogeneous slope described above and across the range of hillslopes we studied, we will henceforth deal only with the simple homogeneous slope with slope $\pi/4$.

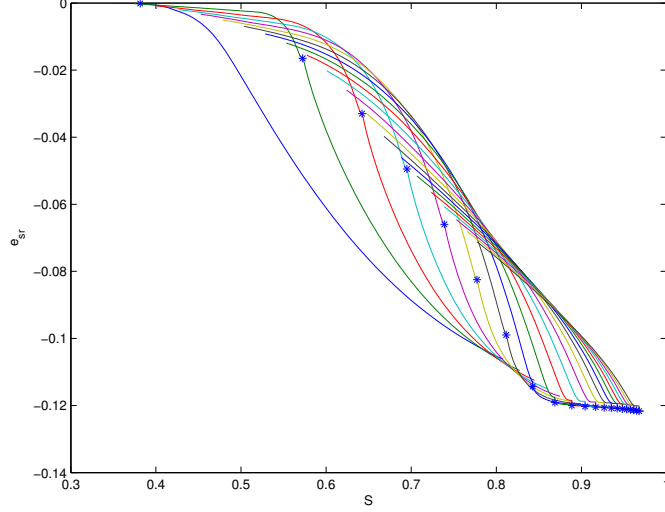
5. WATERSHED-SCALE CLOSURE RELATIONS

The simplest approach to closing the equation for the subsurface would be to assume a constant watershed-scale density and porosity and then determine the exchange terms as functions of the watershed-scale saturation. For instance, given suitably defined watershed scale pressure p we might assume that a “watershed-scale Darcy’s law” holds [33] and that furthermore p can be determined from S . Since the pressure in the channel reach is constant this would yield the parameterization

$$(22) \quad e_{sr} = e_{sr}(S)$$

In order to test the hypothesis that $e_{sr} = e_{sr}(S)$ for our simple system we ran a series of simulations to equilibrium starting from both fully saturated and fully dry (equilibrium for $P=0.0$) initial conditions for precipitation rates $P = 0.0, 0.05, 0.1, \dots, 1.0$ (m/day). We

FIGURE 3. e_{sr} vs. S for a watershed time scale of $T = 3.2$ days, $\theta = \pi/4$, homogeneous slope. Equilibrium values are denoted by *. Each line represents drainage and infiltration to equilibrium under constant precipitation.



collected the spatially integrated fluxes e_{sr} and e_{so} as well as watershed scale saturation S at 0.1 day increments for 100 days, which was approximately enough time for the hillslope to reach equilibrium with the precipitation from both sets of initial conditions. To obtain the watershed-scale variables we approximated time integrals of these quantities for several time scales, $T = 0.1, 0.2, 0.4, \dots, 102.4$ days, using the midpoint rule. Note that equilibrium values are not affected by the temporal upscaling since the system is assumed to be at equilibrium for $t > 100$ days. That is, the equilibrium state dominates the time-averaging for T large enough. In order to preserve this property in our discrete integral approximations it is necessary to add sufficiently many copies of the equilibrium value (T days worth of equilibrium values) of each variable to the 100 day sequence. A plot of e_{sr} versus S is given in figure 3 for a watershed time scale of $T = 3.2$ days. The result is that e_{sr} cannot be parameterized simply as a function of the watershed saturation since $e_{sr}(S)$ is multivalued. If we were to use only equilibrium data (an asterisk denotes the equilibrium value of e_{sr} in figure 3), our data reproduces the result in [15] where the exchange term was parameterized as a low order polynomial of the saturated storage. Furthermore, it was noted in [15] that for moderate storage the equilibrium flux to the channel reach is roughly linear in the saturated storage, and our results demonstrate the same behavior if we partition the hillslope storage S into unsaturated, S_u , and saturated, S_s , storage as in [15]. In our model a low order polynomial in the hillslope saturation S would likely suffice for a moderate range of precipitation rates. Regardless of whether the hillslope is modeled as a single-state or two-state model, however, the flux terms remain multivalued. In the latter case the (S_u, S_s, e_{sr}) data form a multivalued surface (a surface that overturns). As should be clear from the fact that the equilibrium states dominate the time integration for large enough time scales, increasing the watershed time-scale T tends to collapse the multivalued flux functions onto the equilibrium curve. The source of the non-uniqueness above in fluxes with respect to hillslope storage is that disturbances in pressure propagate with infinite speed due to the non-degenerate parabolic form of the governing equations (Richards' equation), and, therefore, for any given watershed

FIGURE 4. e_{sr} vs. (S, P) for $T = 3.2$ days. The height of the surface gives flow into the subsurface from the stream. The equilibrium contour is superimposed in blue. The surface is single valued for all (S, P) .

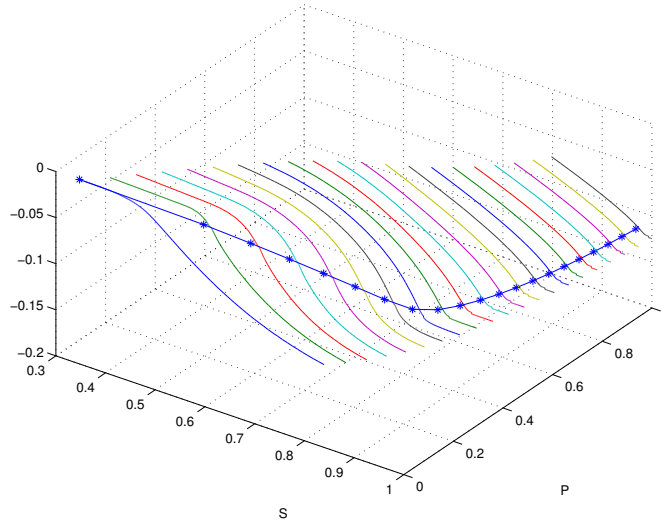
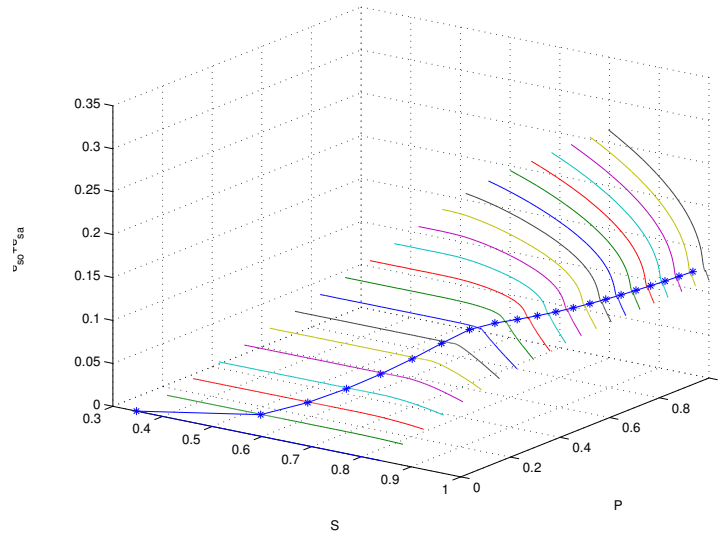


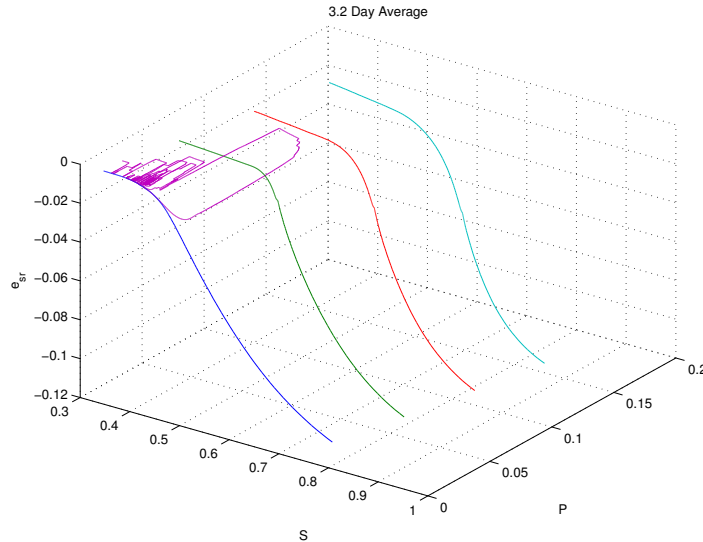
FIGURE 5. $e_{so} + e_{sa}$ vs. (S, P) for $T = 3.2$ days. The height of the surface gives flow into the subsurface from the surface, including both atmospheric fluxes into the domain and exfiltration from the saturated subsurface. The equilibrium contour is superimposed in blue.



scale saturation S (or likewise given values of S_s and S_u) infinitely many values of the fluxes at the r and o boundaries can be generated simply by varying the atmospheric flux e_{sa} .

One route to parameterizing e_{sr} is then to add another dependency that reflects the boundary conditions. Figure 4 plots the same data versus both S and the precipitation rate P , which shows a single valued surface for e_{sr} . Figure 5 gives the total flux at the hillslope surface as a function of (S, P) as well.

FIGURE 6. e_{sr} vs. (S, P) for $T = 3.2$ days. The baseflow driven by the dynamic precipitation time series is shown in purple. The fluxes often do not lie on the surface, particularly during drainage.



5.1. Realistic Atmospheric Forcing. The data set for figures 3, 4, and 5 consists of monotonic drainage and wetting experiments, and a more realistic case would be relatively short precipitation events interspersed with long drying periods. To this end we obtained a year of precipitation data from a site in North Carolina consisting of 15 minute cumulative precipitation measurements (Station: Clinton 2 NE, COOP: 311881, year 1999, <http://www.ncdc.noaa.gov>). If we use this data to drive the hillslope then the flux becomes significantly more complicated. We plot the values from the variable precipitation run as the purple trajectory superimposed on the the previous data in figure 6 and 7. The exchange term is yet again multivalued for both the simple storage parameterization as well as the more complex parameterization of flux with respect to (S, P) . Either approach would be biased under some conditions toward more base flow than actually occurs. The amount of over-prediction relaxes to zero over time. Choosing larger watershed time scales damps some of this behavior, but even for the largest time scale in this study ($T = 102.5$) days the flux-storage behavior is still quite complex (figure 8). More importantly, the time-integrated flux-storage data for the hillslope under dynamic forcing does not collapse onto the equilibrium curve. In other words, the average state of the dynamically forced hillslope is not an equilibrium state, which is well known (c.f. [14]).

5.2. Watershed Model Closure and Simulations. To eliminate as many sources of error as possible we simply used piecewise bilinear splines to generate closure relations from the flux- (S, P) surface data. For simplicity we lumped the surface fluxes e_{so} and e_{sa} into a single term, $\bar{e}_{so} = e_{sr} + e_{sa}$ (figure 5). Lastly, to compensate for the fact, discussed in the previous section, that the dynamically forced hillslope drains more slowly than the numerical experiments used to generate the flux-storage data, we shifted the input to the bilinear spline for e_{sr} by a constant, S^* . In summary, the watershed-scale model for the simple hillslope is

$$(23) \quad \frac{dS}{dt} = e_{sr}(S - S^*) + \bar{e}_{so}(S)$$

FIGURE 7. e_{sr} vs. (S) for $T = 3.2$ days; The surface driven by the dynamic precipitation time series is shown in purple. Over time, the base flow appears to relax to the trajectory corresponding to zero precipitation.

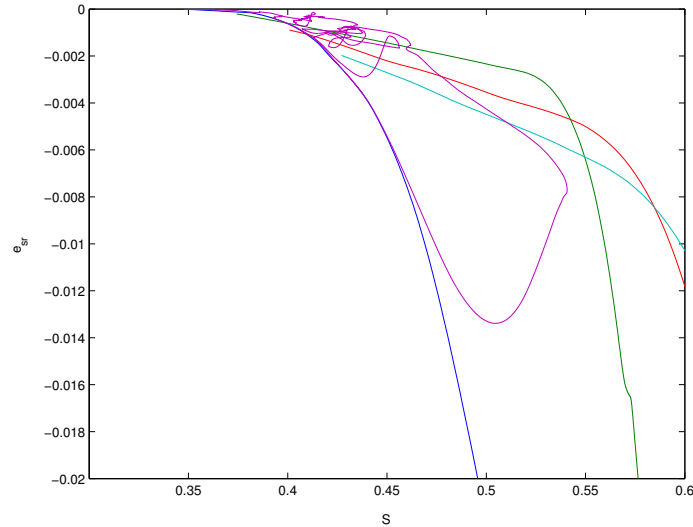
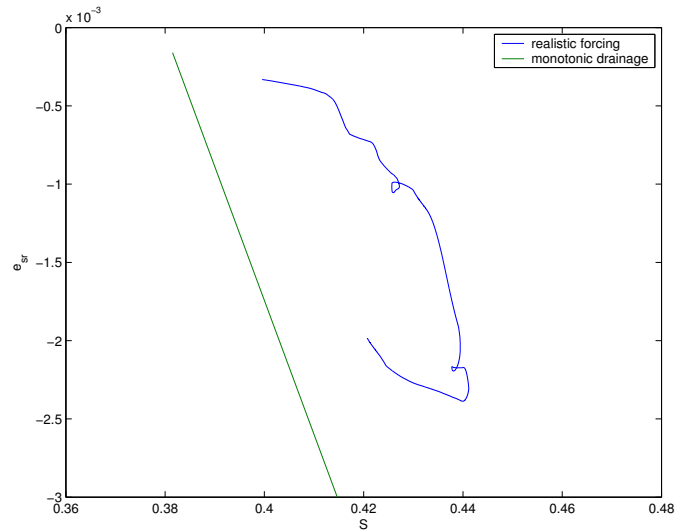


FIGURE 8. e_{sr} vs. (S) for $T = 102.5$ days; The response driven by the dynamic precipitation time series is shown in blue. For large time scales the dynamic data moves away from the zero precipitation trajectory.



For this work we simply set $S^* = \text{mean}(\tilde{S} - S)$ where $\tilde{S}$ is the output of the model with $S^* = 0$, and S is the data generated by the continuum model. We solved the ordinary differential equation above using the MATLAB routine ode23 with a maximum time step of $\min(T/2, 1)$ days. The output of the model and the data generated by the continuum model are presented for two time-scales, $T = 0.1$ and $T = 6.4$ days, in figures 9 and 10. In table 2 we present a measure of error, $\|S^\dagger(t) - S(t)\|_\infty$, where $S^\dagger$ is the output of the watershed-scale model as well as the parameter S^* .

FIGURE 9. Model comparison, $T = 0.1$ days. The over-prediction of base flow is particularly apparent after large precipitation events.

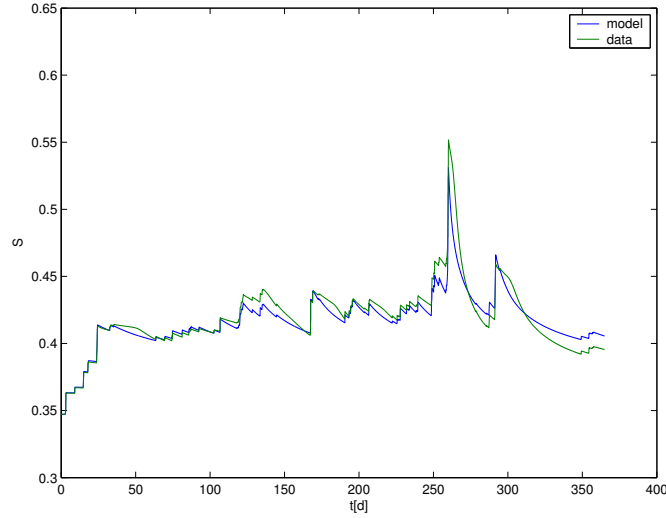


FIGURE 10. Model comparison, $T = 6.4$ days. The time averaging damps out small errors but has no significant effect on errors due to large precipitation events and long dry periods.

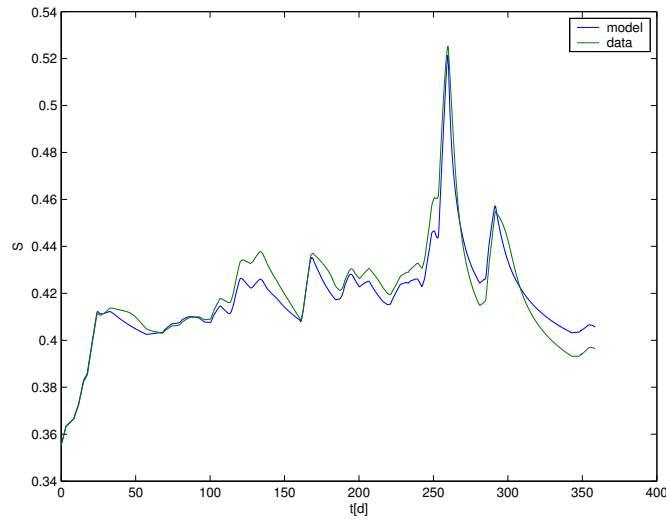


TABLE 2. Watershed-scale model errors and coefficient, S^*

T	0.1	0.2	0.4	0.8	1.6	3.2
$\ S^\dagger - S\ _{\text{inf}}$	0.0474	0.0469	0.0461	0.0448	0.0409	0.0325
S^*	0.0146	0.0146	0.0146	0.0145	0.0143	0.0138
T	6.4	12.8	25.6	51.2	102.4	
$\ S^\dagger - S\ _{\text{inf}}$	0.0174	0.0173	0.0240	0.0231	0.0285	
S^*	0.0137	0.0150	0.0246	0.0333	0.0410	

6. DISCUSSION

In response to a change in forcing conditions, two processes occur in the subsurface 1) the redistribution of moisture and 2) the propagation of pressure disturbances. The former is associated with a very slow time scale—hours to months (depending on soil properties, initial and boundary conditions), while the latter occurs nearly instantaneously. As modeled by Richards’ equation, the hillslope boundary fluxes are strongly dependent on the pressure gradient and the moisture distribution due to the highly nonlinear form of the macroscale closure relations. The fast time scale of pressure signals produces a kind of non-locality or rate-dependence in the behavior of the watershed-scale system; the behavior of the subsurface as a buffer in the hydrology of the watershed depends not only on the water stored in the subsurface but also on the rate at which water is being supplied to the subsurface since that rate affects the pressure gradient across the hillslope. The rate-dependent effect can be incorporated into the water balance model by parameterizing the fluxes as functions of both S and P . The slow (and variable) time scale of moisture redistribution produces a lag or memory effect that cannot be fully quantified with the simple approaches we investigated. This effect is particularly apparent under drying conditions.

7. CONCLUSIONS

We investigated several approaches to parameterizing flux–storage closure relations for a time-averaged, integrated hillslope water balance. The approaches are not able to reproduce all of the complex dynamics of a simulated hillslope driven by a year long sequence of natural precipitation. The complex dynamics for the simulated hillslope derive from the nonlinear parabolic form of the governing equations at the continuum scale. While some dynamic effects related to variable precipitation are incorporated, those particularly related to the slow redistribution of moisture within the hillslope are not. The time-averaging over large time scales may be useful in recovering some accuracy if only long term averages are required of integrated hillslope model. The hillslope simulator in this work was quite simple. Phenomena in natural hillslopes such as hysteresis, heterogeneity, macropores, and fractures, could conceivably dampen or exacerbate some of the effects we noted. Further work is needed in constructing more complex continuum models of the hillslopes as well as in formulating more rigorous upscaling techniques for this problem.

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