A MINIMIZATION METHOD FOR THE SOLUTION OF THE EIGENPROBLEM ARISING IN STRUCTURAL DYNAMICS

R. L. Fox**

Case Western Reserve University

Cleveland, Ohio

M. P. Kapoor***

Indian Institute of Technology

Kanpur, India

This paper presents an iterative method to obtain a partial or complete solution of the general eigenproblem which does not require any preliminary modification to put it into the form of a special eigenvalue problem. The Rayleigh Quotient is minimized by the use of the conjugate gradient method to obtain the lowest eigenvalue and the associated eigenvector. The approach, originally proposed by Bradbury and Fletcher (Reference 1), is extended to permit the intermediate eigenvalues and eigenvectors to be obtained by adapting a projection scheme which is akin to Rosen's Gradient Projection Method (Reference 2). This technique constrains the minimization search to the subspace M-orthogonal to the previously determined eigenvectors. A Theoretical justification is presented that the quadratic convergence of the conjugate gradient method is preserved. The important computer storage advantages of the conjugate gradient method are extended by eliminating the need for assembled stiffness and mass matrices. A number of structural examples are presented to demonstrate the effectiveness, generality and stability of the method.

* This research was supported in part by NASA Research Grant NSG-110-61.

** Associate Professor of Engineering, Member AIAA.

*** Assistant Professor, Department of Civil Engineering

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
OCT 1968		**00-00-1968 to 00-00-1968**

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
A Minimization Method for the Solution of the Eigenproblem Arising in Structural Dynamics	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Air Force Flight Dynamics Laboratory,Wright Patterson AFB,OH,45433**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES
See also AD0703685, Proceedings of the Conference on Matrix Methods in Structural Mechanics (2nd) Held at Wright-Patterson Air Force Base, Ohio, on 15-17 October 1968.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			
unclassified	**unclassified**	**unclassified**		**36**	

SECTION I

INTRODUCTION

A frequent intermediate step in the linear dynamic analysis of complex structures is the solution of the eigenproblem:

$$\mathbf{K}\mathbf{X} = \lambda \mathbf{M}\mathbf{X} \tag{1}$$

This requires the determination of the scalar quantities λ (eigenvalues) and the corresponding non-trivial solutions $\mathbf{X}$ (eigenvectors) for the given n x n matrices $\mathbf{K}$ and $\mathbf{M}$. In the common structural application, $\mathbf{K}$ and $\mathbf{M}$ are respectively, the master stiffness and mass matrices of the structure, and their order, n, corresponds to the elastic degrees of freedom of the system. In this paper, the $\mathbf{K}$ and $\mathbf{M}$ matrices are assumed to result from a finite element idealization of the actual structure.

Frequently, in applying the formulation of Equation 1 to the study of the vibration characteristics of a structure, the order of the $\mathbf{K}$ and $\mathbf{M}$ matrices is so high that it is impractical or prohibitively expensive to obtain the complete eigensolution. On the other hand, to carry out a reasonably accurate dynamic analysis of the structure, it is possible to get along with only a partial eigensolution. It is this class of problems for which the method described in this paper is especially useful.

There are two general types of methods for the eigensolution of Equation 1: transformation methods and iterative methods. The transformation methods such as the Jacobi, Givens, and Householder schemes (Reference3) are almost always preferable when a complete eigensolution is required. On the other hand, the labor saving involved in obtaining only a partial solution by these methods can be a small fraction of the total. Furthermore, the transformation methods accomplish the solution by operating on the matrices of the system which necessitates the storage of large matrices. With the emergence of the "consistent mass matrix," (Reference 4) another difficulty develops because of the necessity to transform Equation 1, the general eigenproblem, into a special eigenproblem:

$$\mathbf{D}\mathbf{Y} = \lambda \mathbf{Y} \tag{2}$$

as required by the transformation methods. If $\mathbf{K}$ or $\mathbf{M}$ happen to be sparse or banded, this step generally produces a dynamic matrix $\mathbf{D}$ with more extensive storage requirements than either $\mathbf{K}$ or $\mathbf{M}$.

The direct iterative methods, on the other hand, can avoid the necessity of storing the entire matrix by using modern matrix interpretive methods, yet they are disadvantaged in the general eigenproblem, Equation 1, and generally require preliminary modification similar to the transformation methods. In addition, these methods are plagued by convergence difficulties and are computationally expensive for the intermediate eigenvalues and eigenvectors.

This paper describes an iterative method which can be applied directly to the eigenproblem, Equation 1, without preliminary modification. It uses the well known property of the Rayleigh quotient,

$$R(\mathbf{X}) = \frac{\mathbf{X}^T \mathbf{K} \mathbf{X}}{\mathbf{X}^T \mathbf{M} \mathbf{X}} \tag{3}$$

that it equals the eigenvalue when the eigenvector is substituted into it and that it is stationary in the neighborhood of an eigenvector.

The basic algorithm is simple: The Rayleigh quotient is minimized to obtain the lowest eigenvalue and the associated eigenvector. This minimization is done numerically using the conjugate gradient method. Next, for the second eigenvalue and eigenvector, the Rayleigh quotient is again minimized, only this time in a subspace which is M-orthogonal to the first eigenvector. This process can be repeated as many times as desired to obtain as many of the eigenvalues and eigenvectors as are desired up to the complete eigensolution. This approach of obtaining the lowest (or highest) eigenvalue and the associated eigenvector was originally proposed by Bradbury and Fletcher (Reference 1).

The contribution of the present paper is: (1) that it extends the approach so that it is practical to obtain the intermediate eigenvalues and eigenvectors without a lessening of the storage and efficiency advantages, and (2) it explores it in application to the special characteristics of finite element structural dynamics problems.

The extension to intermediate eigenvalues is accomplished by using a gradient projection scheme (Reference 2) for constraining the minimization search to the subspace M-orthogonal to the previously determined eigenvectors.

The advantages in structural problems of the formulation of Equation 3 are that both the numerator and denominator, as well as all of the other quantities required by the iteration procedure for all of the eigenvalues desired, can be computed without having the assembled $\mathbf{K}$ and $\mathbf{M}$ matrices at hand. This is accomplished by noting that the numerator is twice the strain energy for a given $\mathbf{X}$ (the generalized displacements), and that the denominator is

twice the maximum kinetic energy of the structure and that these can be computed by summing the potential and kinetic energies of the individual elements of the discretized structural model. In this sense, the approach is an extension of the energy search method documented in the literature (References 5 and 6).

Because the method is iterative and converges quite rapidly when reasonable initial estimates of the eigenvector are available, it lends itself well to embedment within structural optimization procedures where dynamic behavior is to be considered. This is because as the optimum structural design is evolved, the eigensolution generally is expected to change only incrementally from design to design. Hence, the previous solution provides good initial estimates of the eigenvectors.

SECTION II

FORMULATION OF THE PROBLEM

The eigenproblem, Equation 1, can be written as

$$\left[\mathbf{K} - \lambda \mathbf{M}\right] \mathbf{X} = \mathbf{0} \tag{4}$$

If $\mathbf{X}$ is a solution to Equation 4, then $b\mathbf{X}$ is also a solution for any nonzero value of the scalar b, thus the eigenvector corresponding to any eigenvalue λ is arbitrary to the extent of a scalar multiplier. In other words, the Rayleigh quotient defined in Equation 3 has no unique minimum, but takes on the same value at every point along any line in the n-dimensional space passing through the origin. Furthermore, the quotient is not defined at the origin. Consequently, the minimization of the Rayleigh quotient is not quite as simple as that of a function with a well defined minimum.

The redundant degree of freedom, which prevents us from determining the absolute magnitudes of the components of the eigenvector, can be eliminated by an arbitrary normalization. The simplest normalization for the present purpose is to set any non-zero component of the eigenvector equal to one.

The Rayleigh quotient, Equation 3, equals the eigenvalue when the eigenvector is substituted in it. Moreover, it is stationary in the neighborhood of an eigenvector and its value is

bounded by the lowest and highest eigenvalues of the physical system. Thus, the minimization of the Rayleigh quotient will yield the lowest eigenvalue.

The minimization problem to find the lowest eigenvalue can thus be stated as:

Find $X = X_1$ such that

$$R(X_1) \equiv \frac{X_1^T K X_1}{X_1^T M X_1} \tag{5}$$

is minimum, subject to

$$x_{q1} \equiv X_1^T e_q = 1 \tag{6}$$

where x_{q1} is the normalizing or reference component and e_q is a vector with its qth component as one and zero elsewhere (i.e. e_q is a unit coordinate vector for the qth coordinate).

An illustrative example with a geometrical interpretation might be convenient to elucidate the underlying idea. Consider the three degrees of freedom system depicted in Figure 1. The tubular member AB of mean diameter D = 0.8 in. and the wall thickness t = 0.2 in. is held fixed at the end A and is hinged at the end B. A model consisting of two standard beam elements was used and the vertical displacement of the middle point C of the beam and the rotation at the points C and B are taken to be the three degrees of freedom.

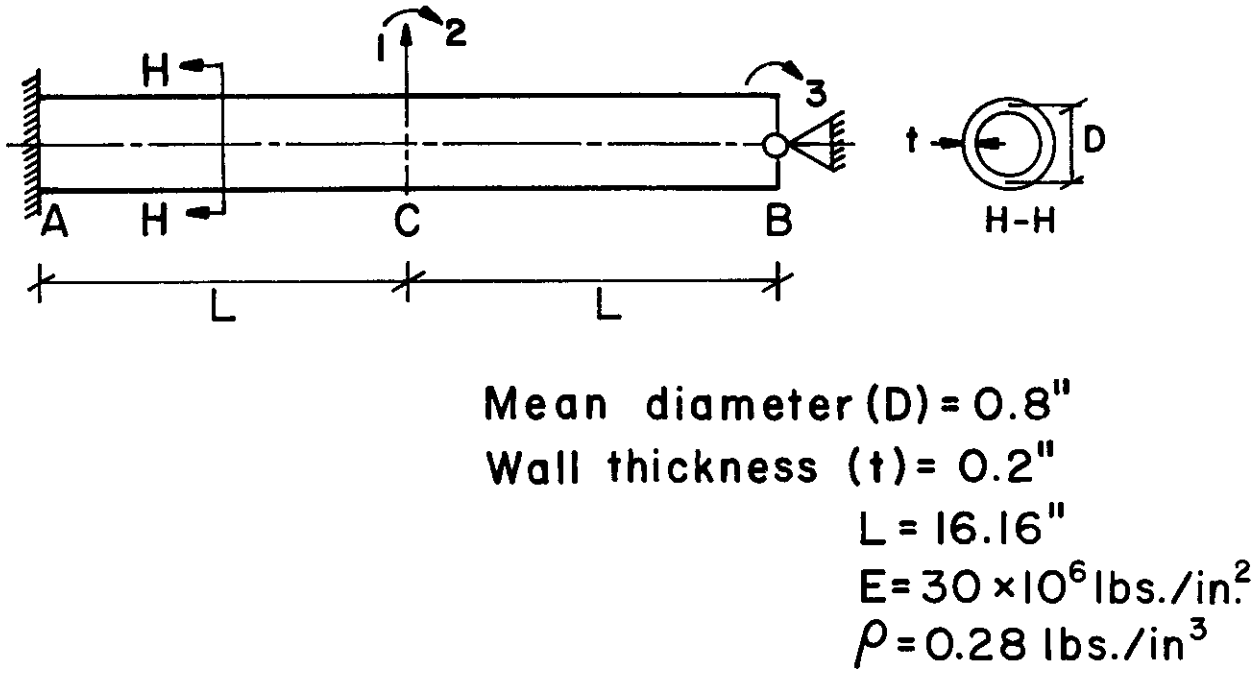


Figure 1. Tubular Beam Fixed at the End A and Hinged at the End B.

The stiffness and the mass matrices of the structure are given as:

$$\mathbf{K} = \begin{bmatrix} 0.729 \times 10^{4} & 0.0 & -0.295 \times 10^{5} \\ 0.0 & 0.635 \times 10^{6} & 0.159 \times 10^{6} \\ -0.295 \times 10^{5} & 0.159 \times 10^{6} & 0.317 \times 10^{6} \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 0.438 \times 10^{-2} & 0.0 & 0.295 \times 10^{-2} \\ 0.0 & 0.293 \times 10^{-1} & -0.110 \times 10^{-1} \\ 0.295 \times 10^{-2} & -0.110 \times 10^{-1} & 0.147 \times 10^{-1} \end{bmatrix}$$

Let the vector space, comprised of the degrees of freedom be represented as

$$\mathbf{X} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and if we pick the normalizing component as x_1, then the normalized vector has only two unknown components:

$$\mathbf{X} \equiv \begin{pmatrix} 1 \\ x_2/x_1 \\ x_3/x_1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$$

where $a = \dfrac{x_2}{x_1}$ and $b = \dfrac{x_3}{x_1}$

Note that in this example we have chosen $\mathbf{e}_q$ vector as

$$\mathbf{e}_q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

From the expressions of $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{X}$ given, we can write the Rayleigh quotient, R, explicitly as:

$$R(\mathbf{X}) = \frac{0.73 - 6.0b + 63.5a^2 + 31.7ab + 31.7b^2}{0.44 + 0.6b + 2.9a^2 - 2.2ab + 1.5b^2} \times 10^{6}$$

A plot of $R(\mathbf{X})$ for different values of a and b is shown in Figure 2, which represents the contour map of the values of the Rayleigh quotient corresponding to the normalized modes of the system. As is seen in Figure 2, the Rayleigh quotient takes on the minimum value at the point 1, a maximum value at the point 3 and an intermediate value at the saddle point 2.

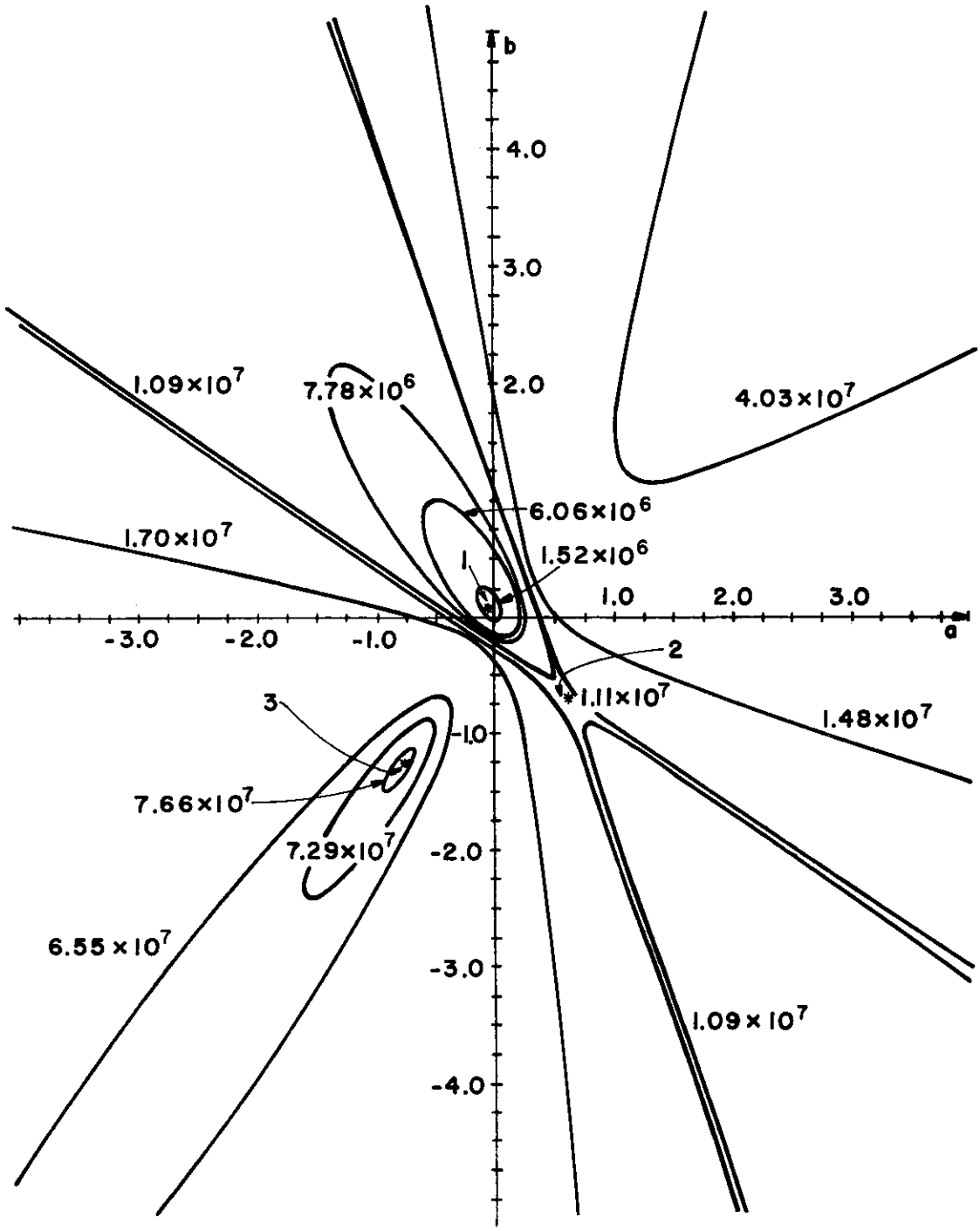


Figure 2. Contour Map of Rayleigh Quotient.

The first three eigenvalues λ_1, λ_2, and λ_3 and their associated eigenvectors $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ are given by

$$\lambda_1 = 0.780 \times 10^6 \quad , \quad \lambda_2 = 0.1099 \times 10^8 \quad , \quad \lambda_3 = 0.780 \times 10^8 \text{ (radians/sec.)}^2$$

and

$$\mathbf{X}_1 = \begin{pmatrix} 1.0 \\ -0.033 \\ 0.122 \end{pmatrix} \quad , \quad \mathbf{X}_2 = \begin{pmatrix} 1.0 \\ 0.592 \\ -0.662 \end{pmatrix} \quad , \quad \mathbf{X}_3 = \begin{pmatrix} 1.0 \\ -0.793 \\ -1.29 \end{pmatrix}$$

The minimization problem posed in Equations 5, 6 will yeild λ_1 and $\mathbf{X}_1$.

In problems of structural dynamics, the eigenvectors represent the mode shapes and the choice of a non-zero component ordinarily presents no serious problem. The mechanization of this aspect of the method is briefly described in the discussion of Example 1, Section VII Numerical Examples.

Once the lowest eigenvalue $\lambda_1 \equiv R(\mathbf{X}_1)$ is known, the next higher or (second) eigenvalue and its associated eigenvector can be determined by posing a new minimization problem:

Find $\mathbf{X} = \mathbf{X}_2$ such that

$$R(\mathbf{X}_2) = \frac{\mathbf{X}_2^T \mathbf{K} \mathbf{X}_2}{\mathbf{X}_2^T \mathbf{M} \mathbf{X}_2} \tag{7}$$

is minimum, subject to

$$\mathbf{X}_2^T \mathbf{e}_r = 1 \tag{8}$$

and

$$\mathbf{X}_2^T \mathbf{M} \mathbf{X}_1 = 0 \tag{9}$$

In the subspace defined by the constraints, Equations 8 and 9, the Rayleigh quotient takes on a unique minimum (assuming distinct eigenvalues) at the eigenvector associated with the second lowest eigenvalue. The constraint Equation 8 is of the type already discussed and 9 represents the imposition of the M-orthogonality condition between $\mathbf{X}_1$ and $\mathbf{X}_2$. Geometrically speaking, these constraints merely restrict the portion of vector space in which the search for the second eigenvector is carried out and in this restricted space R has a minimum corresponding to λ_2.

The determination of the third and subsequent eigenvalues together with their associated eigenvectors up to the complete eigensolution is accomplished by solving a sequence of problems similar to the one presented by Equations 7 to 8. The only change is that each time one additional equation of constraint has to be imposed on the minimization problem to satisfy the condition that the eigenvector currently being sought is M-orthogonal to all of the previously determined eigenvectors. The problem of determining the ℓth eigenvalue ($2 \leq \ell \leq n$) can thus be written

Find $\mathbf{X} = \mathbf{X}_\ell$ such that

$$R(\mathbf{X}_\ell) = \frac{\mathbf{X}_\ell^T \mathbf{K} \mathbf{X}_\ell}{\mathbf{X}_\ell^T \mathbf{M} \mathbf{X}_\ell} \tag{10}$$

subject to

$$\mathbf{X}_\ell^T \mathbf{e}_j = 1 \tag{11}$$

and

$$\mathbf{X}_\ell^T \mathbf{M} \mathbf{X}_i = 0 \quad , \quad i = 1, 2, \cdots, \ell - 1 \tag{12}$$

where $\mathbf{X}_i$, $i = 1,2,\ldots,\ell-1$ are assumed to be known when the ℓth eigenvector is being sought.

Denoting

$$\mathbf{M} \mathbf{X}_i \equiv \mathbf{V}_i \tag{13}$$

the constraint Equation 12 can be written as

$$\mathbf{X}_\ell^T \mathbf{V}_i = 0, \quad i = 1, 2, \cdots, \ell - 1 \tag{14}$$

SECTION III

MINIMIZATION ALGORITHM

The methods of minimizing a function of several variables can, in general, be classified as gradient methods and non-gradient methods. The gradient methods use the local information about the rate of change of the function with respect to the changes in the variables and require the evaluation of the gradient vector, in this case ∇R. These methods are inherently the more powerful as more information about the function is used and are preferred over the nongradient methods.

The Rayleigh quotient as a function of the n variables $(x_1, x_2, \ldots, x_n) = \mathbf{X}$ defined in Equation 3 is differentiable, and its gradient vector

$$\nabla R \equiv \mathbf{g} = \frac{2\,\mathbf{KX}}{\mathbf{X}^T\,\mathbf{MX}} - \frac{(\mathbf{X}^T\,\mathbf{KX})}{(\mathbf{X}^T\,\mathbf{MX})^2}\,2\,\mathbf{MX}$$

$$= \frac{2(\mathbf{KX} - R\,\mathbf{MX})}{(\mathbf{X}^T\,\mathbf{MX})} \tag{15}$$

is easily computed. Therefore, it is logical to carry out the minimization of the Rayleigh quotient by one of the gradient methods.

Through the middle 1950's, one of the most popular gradient methods was the method of steepest descent. This method chooses each direction of search to be the negative of the gradient vector. Though used moderate success on a variety of problems, it often turns out to be hopelessly slow because of the fact that successive moves are perpendicular to each other and the method gradually settles into a steady n-dimensional zig-zag for functions having any significant eccentricity. The convergence difficulties of the steepest descent method have been largely eliminated by a modification of the basic iteration which has been called the conjugate gradient method (Reference 7). This method has the property that, for a quadratic function of n variables, it will converge in n steps, apart from round off errors. For general functions, as the iterate approaches the minimum, the function is usually more nearly approximated by a quadratic and so convergence accelerates toward the solution.

The method of Davidon (Reference 8) (1959) which was amended by Fletcher and Powell (Reference 9) (1963) is also a gradient technique which has the property of quadratic convergence. However, the simplicity of the conjugate gradient method and its modest demands on storage, compounded by the successful experience of Fletcher and Bradbury (Reference 1). dictates the use of conjugate gradient method to minimize the Rayleigh quotient.

The algorithm can be written as:

$$\mathbf{X}_0 = \text{arbitrary} \qquad (a)$$

$$\mathbf{G}_0 = \nabla R(\mathbf{X}_0) \qquad (b)$$

$$\mathbf{S}_0 = -\mathbf{G}_0 \qquad (c)$$

$$\mathbf{X}_{i+1} = \mathbf{X}_i + \alpha_i^* \mathbf{S}_i \qquad (d) \qquad (16)$$

$$\mathbf{G}_{i+1} = \nabla R(\mathbf{X}_{i+1}) \qquad (e)$$

$$\beta_i = \mathbf{G}_{i+1}\Big|^2 / \Big|\mathbf{G}_i\Big|^2 \qquad (f)$$

$$\mathbf{S}_{i+1} = -\mathbf{G}_{i+1} + \beta_i \mathbf{S}_i \qquad (g)$$

where the step length α_i^* is the value of α which minimizes $R(\mathbf{X}_i + \alpha \mathbf{S}_i)$. From Equation 16g we note that $\mathbf{S}_{i+1}$ is a linear combination of $\mathbf{G}_{i+1}$ and $\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_i$ and hence it is a linear combination of $\mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_{i+1}$. The algorithm is based on a Gram-Schmidt orthogonalization of the $\mathbf{G}_i$ and the derivation is documented in the literature (Reference 10).

The method described in Equations 16 is applicable in principle to any unconstrained minimization problem. It will be noted that the constraint Equation 6 is trivially satisfied by setting the qth component of the starting point to be unity and setting the corresponding component of the gradient vector to be zero throughout the search space. Thus, the problem of minimizing the Rayleigh quotient function to find the lowest eigenvalue is similar to an unconstrained minimization problem and the Fletcher Reeves algorithm can be directly applied.

However, the use of conjugate gradient method for finding the intermediate eigenvalue is possible only when the minimization of the Rayleigh quotient is restricted to a subspace of $\mathbf{X}$ in which the constraint Equations 11 and 12 are continuously satisfied. In order to insure that the search is carried out in the desired subspace of $\mathbf{X}$, it is necessary: (1) to start the iteration with a point in that subspace; and (2) to project the gradient vector $\mathbf{g}$, Equation 15 on to that subspace. Both of these requirements necessitate the use of some matrices which project the arbitrary starting point and the gradient vector $\mathbf{g}$ on to the subspace of constraints. Henceforth, such a matrix will be called the "projection" matrix and the way it is generated is discussed in the subsequent section.

SECTION IV

PROJECTION MATRIX

Let $\mathbf{P}$ be a matrix which has the property that for any vector $\mathbf{W}$, the vector

$$\mathbf{W}_p \equiv \mathbf{P}\,\mathbf{W} \tag{17}$$

satisfies

$$\mathbf{W}_p^T \mathbf{Z}_i = 0 \quad , i = 1, 2, \cdots, q \tag{18}$$

where $\mathbf{Z}_i$, i = 1,2,...,q are q linearly independent vectors.

Note that Equation 18 can also be written in matrix form as:

$$\underset{(q\times n)}{\mathbf{N}^T}\ \underset{(n\times 1)}{\mathbf{W}_p} = \mathbf{O} \tag{19}$$

where

$$\underset{(n\times q)}{\mathbf{N}} \equiv \left[\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_q\right] \tag{20}$$

In other words, the matrix operator $\mathbf{P}$ eliminates from the vector $\mathbf{W}$ the non-orthogonal components, thus giving the vector $\mathbf{W}_p$ which is orthogonal to the subspace spanned by the vectors $\mathbf{Z}_i$, i = 1,2,...,q. This idea can thus be expressed differently as:

$$\mathbf{W}_p = \mathbf{W} - \sum_{i=1}^{q} u_i\, \mathbf{Z}_i \tag{21}$$

or, in the matrix form as:

$$\mathbf{W}_p = \mathbf{W} - \underset{(n\times q)}{\mathbf{N}}\ \underset{(q\times 1)}{\mathbf{U}} \tag{22}$$

where the components of vector $\mathbf{U}$ are u_i, i = 1,2,...,q. Premultiplying Equation 22 by $\mathbf{N}^T$ we obtain from Equation 19

$$\mathbf{N}^T \mathbf{W}_p = \mathbf{N}^T \mathbf{W} - (\mathbf{N}^T \mathbf{N})\mathbf{U} = \mathbf{O} \tag{23}$$

Therefore

$$\mathbf{U} = (\mathbf{N}^T \mathbf{N})^{-1} \mathbf{N}^T \mathbf{W} \tag{24}$$

From Equations 24 and 22 we obtain

$$\begin{aligned} \mathbf{W}_p &= \mathbf{W} - \mathbf{N}\ (\mathbf{N}^T \mathbf{N})^{-1}\ \mathbf{N}^T \mathbf{W} \\ &= \left\{ \mathbf{I} - \mathbf{N}\,(\mathbf{N}^T \mathbf{N})^{-1}\ \mathbf{N}^T \right\} \mathbf{W} \end{aligned} \tag{25}$$

where $\mathbf{I}$ is the identity matrix.

Note that ($\mathbf{N}^T\mathbf{N}$) will be a (qxq) symmetric matrix, and is nonsingular since $\mathbf{N}$ is a (nxq) matrix composed of q linearly independent columns. Hence ($\mathbf{N}^T\mathbf{N}$)$^{-1}$ exists.

Comparing Equations 17 and 25 we obtain the projection matrix as:

$$\mathbf{P} = \left\{ \mathbf{I} \quad \mathbf{N} \ (\mathbf{N}^T \mathbf{N})^{-1} \ \mathbf{N}^T \right\} \tag{26}$$

In particular, for determining the ℓth eigensolution, a projection matrix

$$\mathbf{P}_\ell = \left\{ \mathbf{I} \ - \mathbf{N}_\ell \ (\mathbf{N}_\ell^T \mathbf{N}_\ell)^{-1} \ \mathbf{N}_\ell^T \right\} \tag{27}$$

where

$$\mathbf{N}_\ell = \left[\mathbf{e}_j \ , \ \mathbf{V}_1 \ , \mathbf{V}_2 \ , \cdots , \mathbf{V}_{\ell-1} \right] \tag{28}$$

will project the gradient vector $\mathbf{g}$, Equation 15 on to the subspace of constraints defined by Equations 11 and 12. Note that the column vectors of the matrix $\mathbf{N}_\ell$ are linearly independent. Thus

$$\mathbf{g}_p = \mathbf{P}_\ell \ \mathbf{g} \tag{29}$$

A question which immediately warrants attention is, since we use $\mathbf{g}_p$ instead of $\mathbf{g}$, will the conjugate gradient method actually converge? In other words, is there a function of which $\mathbf{g}_p$ is the gradient and whose function value equals R in the subspace defined by Equations 11 and 12. To see that the answer is affirmative, it is only necessary to construct the Lagrange function for the constrained minimization problem, Equations 10 to 12. Define a "Lagrange-Rayleigh" function for the ℓth eigensolution as:

$$L_\ell = R(\mathbf{X}_\ell) - u_1 (\mathbf{X}_\ell^T \mathbf{e}_j - 1) \ - \sum_{i=2}^{\ell} u_i \mathbf{X}_\ell^T \mathbf{V}_{i-1} \tag{30}$$

where u_i, i = 1,2,..., ℓ are the Lagrange Multipliers. Note that the stationary condition over the variables $\mathbf{X}$ is

$$\nabla L_\ell = \nabla R - u_1 \mathbf{e}_j - \sum_{i=2}^{\ell} u_i \mathbf{V}_{i-1} = \mathbf{0} \tag{31}$$

or in matrix form

$$\nabla L_\ell = \mathbf{g} - \mathbf{N}_\ell \mathbf{U} = \mathbf{0} \tag{32}$$

and if the Lagrange multiplier vector $\mathbf{U}$ is defined by Equation 24 as

$$\mathbf{U} = (\mathbf{N}_\ell^T \mathbf{N}_\ell)^{-1} \ \mathbf{N}_\ell^T \mathbf{g} \tag{33}$$

we obtain

$$\nabla L_\ell = g - N_\ell (N_\ell^T N_\ell)^{-1} N_\ell^T g \equiv P_\ell g = g_p \tag{34}$$

where P_ℓ is given by Equation 27. Furthermore, L = R in the proper subspace, because the second and third terms of Equation 30 are identically zero by virtue of constraint Equations 11 and 12.

Thus, the "Lagrange-Rayleigh" function has the same value as the Rayleigh quotient in the proper subspace and the projection matrix P_ℓ projects the gradient vector g on to this subspace, giving thereby the gradient of the "Lagrange-Rayleigh" function.

SECTION V

RECURSION PROCEDURE

It is always possible to form the $(\ell \times \ell)$ matrix $(N_\ell^T N_\ell)$ and then invert it to obtain the projection matrix P_ℓ, Equation 27, when the search for the ℓth eigenvalue and its corresponding eigenvector is made. However, it is desirable to avoid this computation, as the value of ℓ will be large when higher eigenvalues are searched.

It will be noticed from Equation 28 that the size of the rectangular matrix N_ℓ increases by one column, every time a new eigenvalue and its associated eigenvector is searched. This suggests that some sort of a recursion procedure should be used which permits the insertion of the vector $V_{\ell-1}$ on to the set

$$N_{\ell-1} \equiv \left[e_j, V_1, V_2, \cdots, V_{\ell-2} \right] \tag{35}$$

and uses $(N_{\ell-1}^T N_{\ell-1})^{-1}$, which is presumed to be known. Such a recursion procedure to obtain $(N_\ell^T N_\ell)^{-1}$ from $(N_{\ell-1}^T N_{\ell-1})^{-1}$ is described in Reference 2. It is based on

the method for the inverse of a matrix in terms of the inverses of certain of its partitions. For purposes of the present work, let

$$Q_{\ell-1} \equiv \left(N_{\ell-1}^T \quad N_{\ell-1} \right) = \begin{bmatrix} e_j^T e_j & & & \\ e_j^T V_1 & V_1 V_1 & \text{SYM} & \\ e_j^T V_2 & V_1^T V_2 & V_2^T V_2 & \\ \vdots & \vdots & \vdots & \\ e_j^T V_{\ell-2} & V_1^T V_{\ell-2} & \cdots & V_{\ell-2}^T V_{\ell-2} \end{bmatrix} \tag{36}$$

Therefore,

$$Q_\ell = \left(N_\ell^T \quad N_\ell \right) = \left[\begin{array}{ccc:c} & & & e_j^T V_{\ell-1} \\ & Q_{\ell-1} & & V_1^T V_{\ell-1} \\ & & & \vdots \\ \hdashline e_j^T V_{\ell-1} & V_1^T V_{\ell-1} & \cdots & V_{\ell-1}^T V_{\ell-1} \end{array}\right]$$

$$\equiv \left[\begin{array}{c:c} A_1 & A_2 \\ \hdashline A_2^T & A_3 \end{array}\right] \tag{37}$$

where

$$A_1 = Q_{\ell-1}, \; A_3 = V_{\ell-1}^T V_{\ell-1} \text{ and } \quad A_2^T = \left[e_j^T V_{\ell-1}, V_1^T V_{\ell-1}, \cdots, V_{\ell-2}^T V_{\ell-1} \right].$$

We presume that A_1^{-1} is known, then

$$Q_\ell^{-1} = \left(N_\ell^T N_\ell \right)^{-1} \equiv \left[\begin{array}{c:c} B_1 & B_2 \\ \hdashline B_2^T & B_3 \end{array}\right] \tag{38}$$

can be obtained from

$$B_1 = A_1^{-1} + \frac{1}{s}\left(A_1^{-1} A_2 A_2^T A_1^{-1}\right) \tag{39}$$

$$B_2 = -\frac{1}{s}\left(A_1^{-1} A_2\right)$$

and

$$B_3 = \frac{1}{s}$$

where the scalar quantity s is given by

$$s = A_3 - A_2^T A_1^{-1} A_2 \tag{40}$$

A procedure to obtain B_1, B_2, and B_3 which is even more efficient than the direct formulas shown above and which uses only the old projection matrix, $P_{\ell-1}$, the old inverse, $Q_{\ell-1}^{-1}$ and the vector $V_{\ell-1}$ is described in Appendix I.

SECTION VI

STEP BY STEP PROCEDURE

CHOICE OF A STARTING POINT

The iterative methods should have a good starting point, otherwise unnecessary time is wasted inside the minimization procedure to minimize the function. However, there seems to be no simple method for finding a good starting value for any iterative method. In Reference 1, a choice of the starting point is made on the basis that the first step of the minimization procedure makes the fastest descent towards the minimum. It is a unit vector $\mathbf{e}_i$ (the jth element of $\mathbf{e}_i$ is δ_{ij}) lying along the coordinate axes. Such a starting point gives a badly distorted mode shape and was not found to be the best choice for solving the structural dynamics problem.

Contrary to expectation, a set of random vectors proved to be superior to the selected unit vectors and in this work they were used as starting points. The method showed reasonably good convergence from these points. Once the solutions converged, a knowledge of the mode shapes of the structural system was obtained. Experiments were conducted in which some of the design variables of the structure were changed and the eigensolution of the modified system was obtained by using the mode shapes of the original design as the starting points. This showed rapid convergence.

The starting point for the search of the lowest eigenvalue needs to satisfy only one constraint, Equation 6, which is trivially satisfied by dividing through by the qth component. However, the minimization algorithm, Equations 16, generates a sequence of vectors which, in the limit, tend directionally to the minimum eigenvalue on the search space. Thus the qth component of the vectors so generated have to be maintained as unity, so as to satisfy the constraint, Equation 6, continuously in the space. This is achieved by setting the qth component of the gradient vector at the particular point equal to zero. In other words, no "move" is made in the qth direction of the search space.

The starting point for the search of second eigenvalue has to satisfy an additional constraint, Equation 9, and this can be easily satisfied by Schmidt orthogonalization.

Let $\tilde{\mathbf{x}}^{(0)}$ be some initial estimate. Therefore,

$$\mathbf{x}_2^{(0)} = \tilde{\mathbf{x}}_2^{(0)} - (\tilde{\mathbf{x}}_2^{(0)^T} \mathbf{v}_1)\mathbf{v}_1 \qquad (41)$$

satisfies Equation 9 if $V_1^T V_1 = 1$. Note that $V_1 = MX_1$, Equation 13 where X_1 is the eigenvector corresponding to the first (lowest) eigenvalue.

In order to get the starting point for the search of subsequent eigenvalues, the initial estimate is passed through a projection matrix as discussed in Section IV, Projection Matrix, where the matrix is given by Equation 27 but the matrix $\tilde{N}_\ell$ where

$$\tilde{N}_\ell \equiv \left[V_1, V_2, \cdots, V_{\ell-1} \right] \tag{42}$$

is to be used instead of matrix N_ℓ, Equation 28.

FUNCTION EVALUATION

The value of the function to be minimized is required at the end of each iteration cycle for almost all the iteration methods since the convergence criterion is based on the function value. Moreover, for the particular problem of minimizing the Rayleigh quotient, the function value is required at each cycle in order to evaluate the gradient vector, Equation 15. It is, therefore, necessary to have an efficient routine for function evaluation, in order to avoid the time which would be otherwise wasted inside the minimization procedure. The value of the Rayleigh quotient can be computed without having the assembled K and M matrices at hand. This is accomplished by noting that the numerator is twice the strain energy for a given X (the generalized displacements), and that the denominator is twice the maximum kinetic energy of the structure and these can be computed by summing the potential and kinetic energies of the individual elements of the discretized structural model. Thus $R(X)$ can be written as

$$R(X) = \frac{\sum_{i=1}^{r} Y_i^T k_i Y_i}{\sum_{i=1}^{r} Y_i^T m_i Y_i} \tag{43}$$

where r is the number of discrete elements, k_i and m_i are, respectively, the stiffness and mass matrices of the ith element of the discretized structure and Y_i is the displacement vector of the ith element corresponding to the generalized displacement vector X. To obtain the vector Y_i, i = 1,2,...,r from the vector X is rather easy and is a matter of logical operations. A variety of methods exist for such logical operations. One such scheme is described in Reference 11 and an equivalent technique is discussed in detail in Appendix II.

For the purposes of the present work, a separate subroutine was written in Fortran IV to decompose the vector $\mathbf{X}$ to the vectors $\mathbf{Y}_i$, to evaluate the vectors $\mathbf{k}_i \mathbf{Y}_i$ and $\mathbf{m}_i \mathbf{Y}_i$ and then to assemble back the product $\mathbf{KX} \equiv \mathbf{A}$ and $\mathbf{MX} \equiv \mathbf{B}$. The assembly of the vectors $\mathbf{A}$ and $\mathbf{B}$ is accomplished through a logical operation on the vectors $\mathbf{k}_i \mathbf{Y}_i$ and $\mathbf{m}_i \mathbf{Y}_i$ which is merely the inverse of the operation described in Appendix II.

As will be seen later, this subroutine was used over the over again to evaluate certain other quantities required in the iteration process, other than the function evaluation which is now obtained by two vector multiplications $\mathbf{X}^T \mathbf{A}$ and $\mathbf{X}^T \mathbf{B}$ and one division in order to get:

$$R(\mathbf{X}) = \frac{\mathbf{X}^T \mathbf{A}}{\mathbf{X}^T \mathbf{B}}$$

Needless to emphasize the advantage gained by getting along without the assembly of $\mathbf{K}$ and $\mathbf{M}$ matrices. Frequently, the order of $\mathbf{K}$ and $\mathbf{M}$ matrices encountered for large structures is so high that it is impractical or prohibitively expensive to study their vibration characteristics without making approximations. The size of element stiffness and mass matrices are relatively much smaller than the size of the assembled stiffness and mass matrices of large complex structures. Furthermore, advantage can also be taken of similar elements. For example, in a structural system of a large number of elements, if only three types of elements are used, then we need store only the stiffness and mass matrices corresponding to these three elements, rather than for all of the r elements.

GRADIENT EVALUATION

The minimization algorithm, Equation 16, requires the evaluation of the gradient vector at each cycle of the iteration. The gradient vector, $\mathbf{g}$, of the Rayleigh quotient function is given by Equation 15. For determining the lowest (first) eigensolution, the component of the gradient vector corresponding to the normalizing component of the eigenvector is set equal to zero, in order to continuously satisfy the constraint imposed due to the normalization of the eigenvector. While determining the intermediate eigensolution, the projection matrix $\mathbf{P}_\ell$, Equation 27 is used to project this gradient vector, $\mathbf{g}$, on to the proper subspace of search, M-orthogonal to the previously determined eigenvectors. The component of the gradient vector, $\mathbf{g}_p$, corresponding to the normalizing component of the eigenvector turns out to be zero automatically, but a small number often appears due to roundoff errors and this is simply removed by setting that component equal to zero.

EVALUATION OF STEP LENGTH

Once a direction of move $\mathbf{S}_i$ has been chosen, we must determine α_i^* so that the function is minimized in that direction. Thus the problem of finding the step length is essentially the linear search problem which requires the determination of the α_i^* along $\mathbf{S}_i$ through $\mathbf{X}_i$ at which the value of the Rayleigh quotient function

$$R(\mathbf{X}_i + \alpha_i \mathbf{S}_i) = \frac{(\mathbf{X}_i + \alpha_i \mathbf{S}_i)^T \mathbf{K} (\mathbf{X}_i + \alpha_i \mathbf{S}_i)}{(\mathbf{X}_i + \alpha_i \mathbf{S}_i)^T \mathbf{M} (\mathbf{X}_i + \alpha_i \mathbf{S}_i)} \tag{44}$$

is a minimum, i.e.,

$$\left.\frac{dR}{d\alpha_i}\right|_{\alpha_i = \alpha_i^*} \equiv R'(\alpha_i^*) = 0 \tag{45}$$

In the general problem, no expression is available to determine α_i^*, so an interpolation approach (References 7, 8, and 9) is adopted. In the particular problem of Rayleigh quotient, however, an explicit expression in α_i can be generated (Reference 1) from Equations 44 and 45 which has the form

$$u\alpha_i^2 + v\alpha_i + w = 0 \tag{46}$$

where

$$\begin{aligned} u &= (\mathbf{S}_i^T \mathbf{K} \mathbf{S}_i)(\mathbf{X}_i^T \mathbf{M} \mathbf{S}_i) - (\mathbf{X}_i^T \mathbf{K} \mathbf{S}_i)(\mathbf{S}_i^T \mathbf{M} \mathbf{S}_i) \\ v &= (\mathbf{S}_i^T \mathbf{K} \mathbf{S}_i)(\mathbf{X}_i^T \mathbf{M} \mathbf{X}_i) - (\mathbf{X}_i^T \mathbf{K} \mathbf{X}_i)(\mathbf{S}_i^T \mathbf{M} \mathbf{S}_i) \\ w &= (\mathbf{X}_i^T \mathbf{K} \mathbf{S}_i)(\mathbf{X}_i^T \mathbf{M} \mathbf{X}_i) - (\mathbf{X}_i^T \mathbf{K} \mathbf{X}_i)(\mathbf{X}_i^T \mathbf{M} \mathbf{S}_i) \end{aligned} \tag{47}$$

The two roots of Equation 46 correspond to the maximal and minimal points of the Rayleigh quotient in the direction $\mathbf{S}_i$ through $\mathbf{X}_i$ as shown in Figure 3. The minimal function value corresponds to α_i^*.

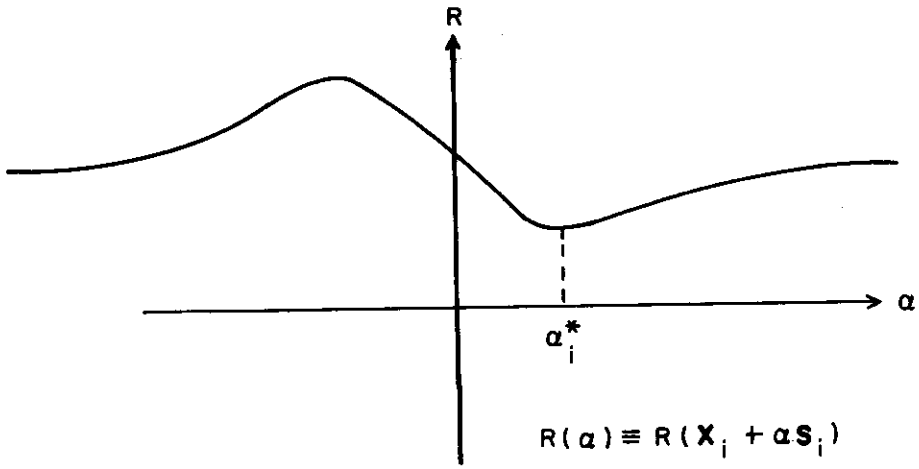


Figure 3. Representative Variation of Rayleigh Quotient Along a Line

Note that the matrix products required in Equation 47 to evaluate the coefficients of the quadratic, Equation 46, are easily obtained through the subroutine discussed above in subsection "Functional Evaluation."

SECTION VII
NUMERICAL EXAMPLES

In all the illustrative examples presented in this section, (1) a general planar beam element having six degrees of freedom was used to model the various structures, (2) the distributed mass of the system was used to evaluate the components of the mass matrix (the "consistent" mass matrix) (Reference 4), (3) the Univac 1108 digital computer was used to obtain the numerical results and (4) except where noted, random vectors were taken as starting points for the minimization algorithm to obtain the eigensolution.

EXAMPLE 1

As a simple application, the cylindrical cantilever rod shown in Figure 4a is considered. An attempt was made to obtain the complete eigensolution of this simple structure by the successive minimization of the Rayleigh quotient in the appropriate subspace.

Based on the surmise that the lowest eigenvalue would correspond to the first cantilever mode, the degree of freedom associated with the transverse deflection of the cantilever at the free and (x_8) was made to equal one and the others were taken as zero in order to start the search for the lowest eigenvalue. (This obviously makes x_8 the normalizing component). A capability was built into the computer program to change the normalizing component whenever the magnitude of any other component of the vector in the appropriate subspace exceeded five times the magnitude of the current normalizing component (which in any case is one). Every change of the normalizing component necessitated the restart of the minimization algorithm due to the change in the subspace of search.

Curiously enough the complete eigensolution so obtained gave the first six eigenvalues and their associated eigenvectors and did not pick up the remaining three eigenvalues. This is because the three axial degrees of freedom are uncoupled from the six translational and rotational degrees of freedom of the cantilever beam and with the particular choice of starting point, the method could not enter into the subspace in which the eigenvalues associated with the axial modes of vibration are located. If instead of $\mathbf{e}_8$, the starting points are taken as $\mathbf{e}_7$, only the eigenvalues associated with the three axial modes of vibration are obtained.

Furthermore, if the same structure is oriented differently as shown in Figure 4b where the translational, rotational and axial degrees of freedom are all coupled, the complete eigensolution, i.e. all the nine eigenvalues and the corresponding eigenvectors, were obtained from the starting point of $\mathbf{e}_8$.

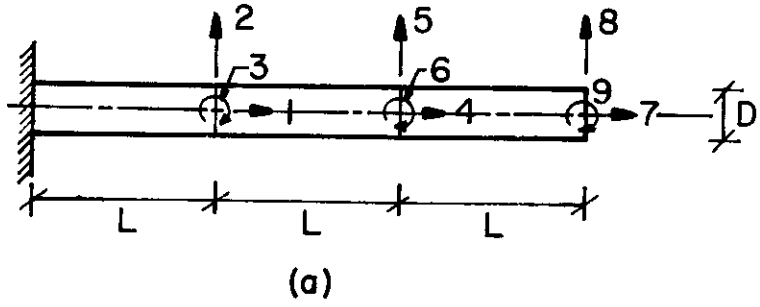
The first three eigenvalues of the model are: 0.2466×10^8, 0.1161×10^9 and 0.9747×10^9 which correspond to the frequencies of 0.4966×10^4, 0.1079×10^5 and 0.3122×10^5 radians/sec. These compare favorably with the exact values of 0.4945×10^4, 0.1065×10^5 and 0.3100×10^5 radians/sec., and respectively correspond to the first cantilever beam mode, the first axial mode and the second cantilever beam mode.

EXAMPLE 2

Consider as another illustrative example the planar frame shown in Figure 5a consisting of tubular members pinned together at the nodes. Each member was modeled with a single beam element. Thus, the structure has fourteen degrees of freedom. A complete eigensolution (i.e. all the 14 eigenvalues and eigenvectors) of the system was obtained by successive minimization of the Rayleigh quotient. The time taken was 4.17 seconds.

EXAMPLE 3

The partial eigensolution of a 56 degrees of freedom system shown in Figure 5b was obtained by minimizing the Rayleigh quotient. Each member of the four bay planar frame, pinned at the nodes, was modeled with a single beam element. The determination of ten eigenvalues and eigenvectors took 77.91 seconds.



L = 1.0"
D = 0.25"
E = 30×10^6 lbs./in.^{2}
ρ = 0.28 lbs./in.^{3}

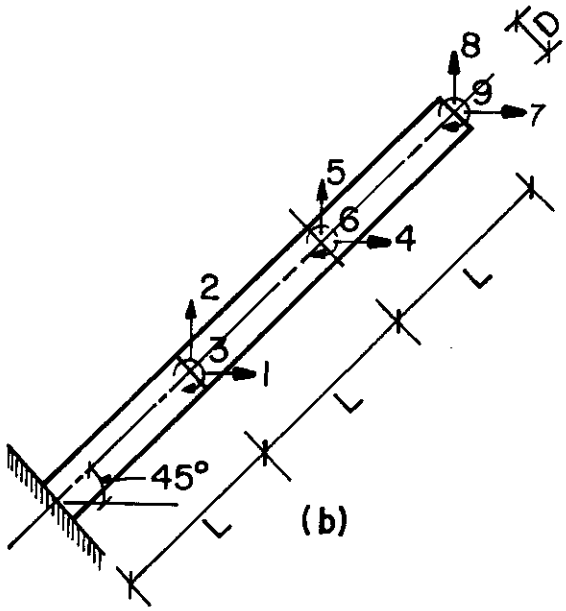
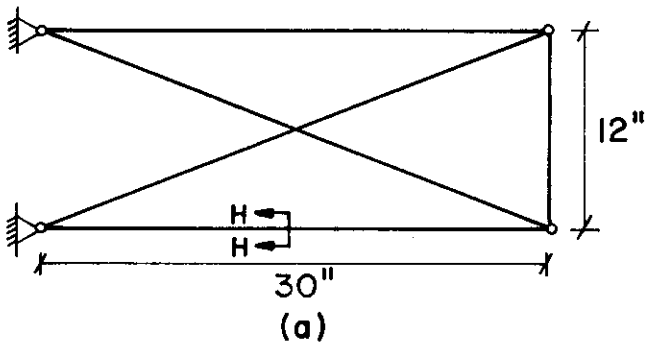
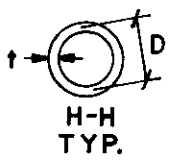


Figure 4. Cantilever Rod





Mean Dia. (D) = 1.6"

Wall Thickness (t) = 0.1"

$E = 30 \times 10^6$ lbs./in^2

$\rho = 0.28$ lbs./in^2

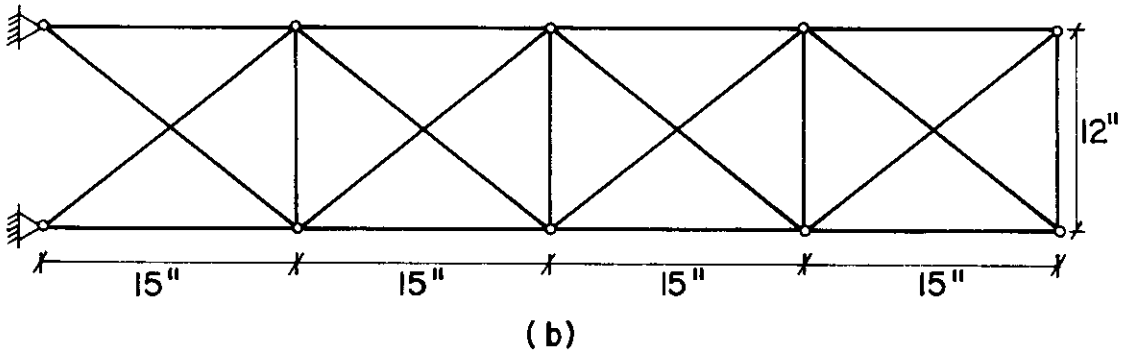


Figure 5. Planar Frames: (a) Fourteen Degrees of Freedom: (b) Fifty-six Degrees of Freedom.

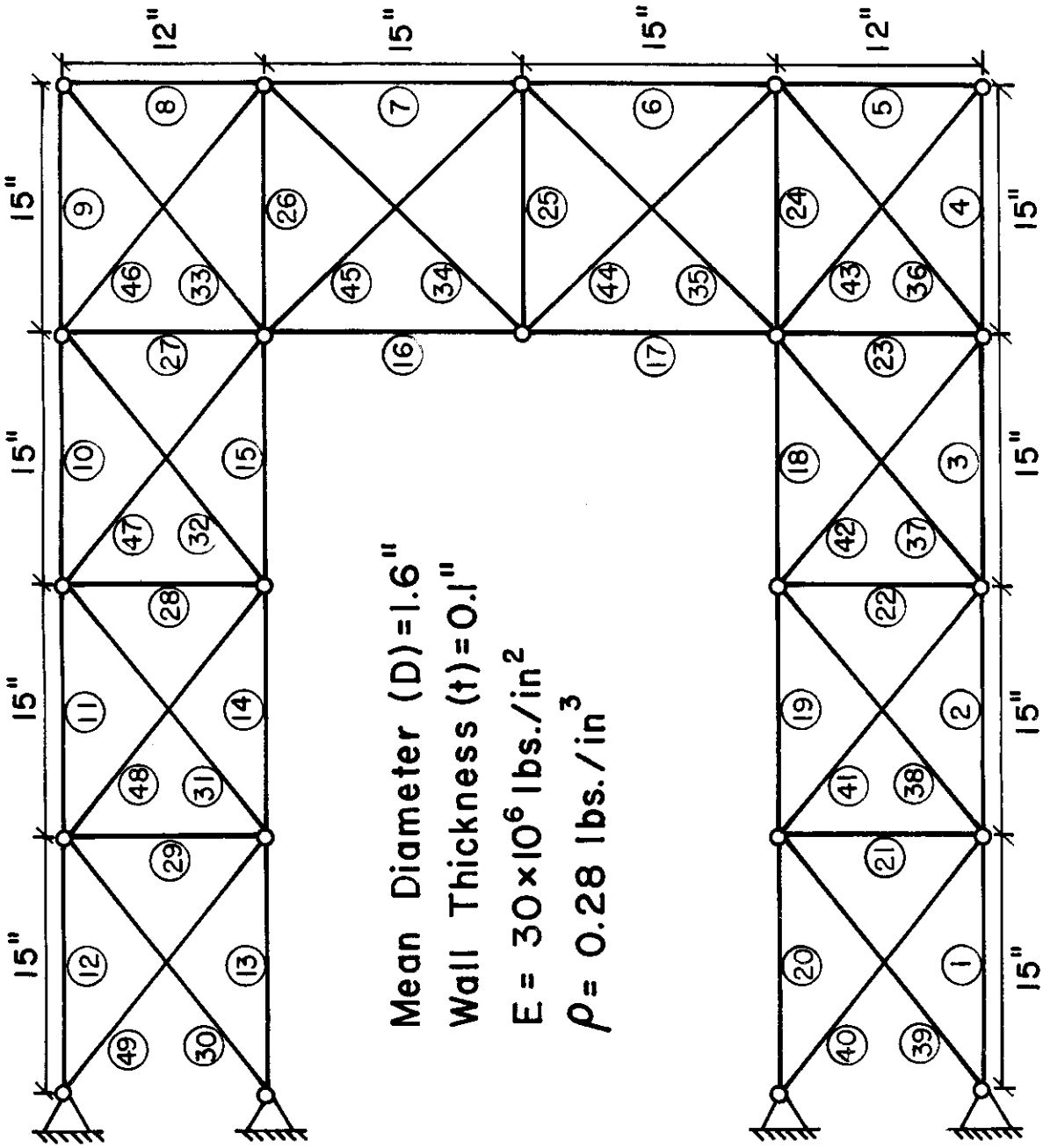


Figure 6. Planar Frame (134 Degrees of Freedom, If Each Member is Modeled of One Element).

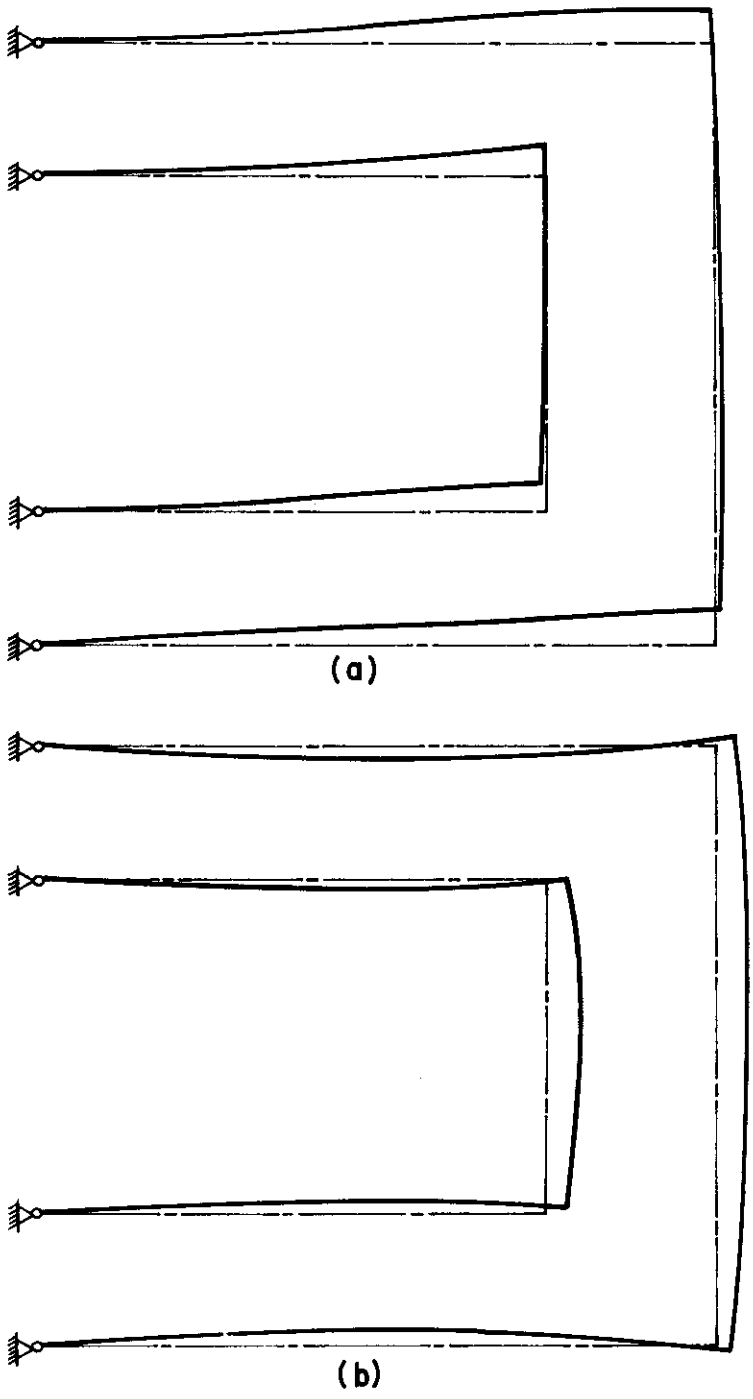


Figure 7. Schematic Representation of Mode Shape of 134 Degrees of Freedom System Corresponding to: (a) Lowest Natural Frequency; (b) Next Higher Natural Frequency

EXAMPLE 4

A frame structure having 134 degrees of freedom shown in Figure 6 was analyzed to obtain the first five eigenvalues and the associated eigenvectors. Each member was pinned at its nodes and modeled with only one element. The number of iterations required to converge to the solution up to the eighth decimal place, the time taken to obtain each eigenvalue and the numerical result obtained are given in Table I. The eigenvectors so obtained were used as the starting points for the minimization algorithm to obtain the eigensolution of a changed design of the same structure. The change in the original design was brought about by increasing the mean diameter of members 1 to 20 by 10% and by decreasing the mean diameter of members 21 to 29 by 10%. The results for the changed design are also given in Table I. The total time taken to obtain the partial (first five) eigensolution of the original design was 46.95 seconds while that for the changed design was 20.30 seconds. Thus, the method converges rapidly for the changed design. The reason is that the eigenvectors of the original design provide the reasonable initial estimates of the eigenvectors for the changed design and thus are good starting points for the minimization algorithm.

It would be observed from Table I that the first two eigenvalues are well-separated while the next three are closely spaced. A study of the associated eigenvectors explains this behavior. The first two eigenvalues, respectively, correspond to the mode shapes schematically represented in Figures 7a and b. The next three closely spaced eigenvalues correspond to the mode shapes where the overall structure frequency is dominated by the individual frequency of the long members of the structure marked 34, 35, 44, and 45. This interaction of the individual member frequency with the overall structure frequency is responsible for the three closely spaced eigenvalues.

TABLE I

FIRST FIVE EIGENVALUES OF THE ORIGINAL AND THE CHANGED DESIGN. EIGENVALUES REQUIRED TO CONVERGE UP TO THE EIGHTH DECIMAL PLACE

No. of Eigen-value	Original Design			Changed Design		
	Iterations	Time (secs.)	Value	Iterations	Time (secs.)	Value
1	59	5.3	$.62309556 \times 10^{6}$	26	2.22	$.64067844 \times 10^{6}$
2	56	4.27	$.58762266 \times 10^{7}$	40	3.04	$.60087324 \times 10^{7}$
3	205	19.53	$.75223340 \times 10^{7}$	73	6.11	$.75745767 \times 10^{7}$
4	100	8.36	$.77330454 \times 10^{7}$	46	4.50	$.75729213 \times 10^{7}$
5	105	9.49	$.79875840 \times 10^{7}$	43	4.43	$.75753798 \times 10^{7}$

The same example was rerun with a relaxed convergence criterion (this time the solution was required to converge to only the sixth decimal place) and the results obtained are given in Table II. The total time taken for the partial eigensolution of the original design is 24.34 seconds while that for the changed design is 7.97 seconds. Thus, there is a considerable saving of time in choosing a less stringent convergence criterion. However, although the error does not seem to propagate badly from eigenvalue to eigenvalue, the eigenvectors so obtained are not as accurate as those obtained by the more stringent convergence criterion.

TABLE II

FIRST FIVE EIGENVALUES OF THE ORIGINAL AND THE CHANGED DESIGN. EIGENVALUES REQUIRED TO CONVERGE UP TO THE SIXTH DECIMAL PLACE

No. of Eigne-value	Original Design			Changed Design		
	Iterations	Time (secs.)	Value	Iterations	Time (secs.)	Value
1	45	3.49	$.623109 \times 10^6$	6	0.56	$.640686 \times 10^6$
2	45	3.51	$.587632 \times 10^7$	9	1.14	$.600891 \times 10^7$
3	50	4.15	$.752796 \times 10^7$	22	2.13	$.757763 \times 10^7$
4	55	5.55	$.773182 \times 10^7$	27	2.11	$.757269 \times 10^7$
5	76	7.64	$.798727 \times 10^7$	28	2.03	$.757271 \times 10^7$

EXAMPLE 5

When each of its members is modeled by two beam elements the number of degrees of freedom of the system shown in Figure 6 is 281. The results for the first five eigenvalues of this refined model are given in:

TABLE III

RESULTS OF 281 DEGREE-OF-FREEDOM MODEL OF FIGURE 6.

No.	Iterations	Time (secs.)	Value
1	417	71.78	$.6207 \times 10^6$
2	301	55.57	$.5291 \times 10^7$
3	614	117.22	$.6295 \times 10^7$
4	222	42.48	$.6371 \times 10^7$
5	297	59.12	$.6643 \times 10^7$

As anticipated, the refined modeling gave a better correlation between the closely spaced eigenvalues of the overall structure and the eigenvalue corresponding to the lowest natural frequency of the individual members having a length of 21.21 in. (any of the ones marked 34, 35, 44 and 45 on Figure 6). The lowest natural frequency of the simply supported tubular beam of mean diameter 1.6 in., wall thickness 0.1 in. and a length of 21.21 in.is 0.251×10^4 radians/sec., which correspond to an eigenvalue of 0.6300×10^7. Thus, the closely spaced eigenvalues are the result of the interaction between the individual member frequency and the overall structure frequency.

SECTION VIII

CONCLUSION

An iterative method to obtain the partial/complete eigensolution of a general eigenproblem arising in structural dynamics is described. The method does not require preliminary modification to put the general eigenproblem into any special form. It has been found to be effective for the partial eigensolution of complex structures and thus is useful for the dynamic analysis of complex structures.

Since the method converges rapidly when reasonable initial estimates of the eigenvector are available, it lends itself well to embedment within structural optimization procedures where dynamic behavior is to be considered.

SECTION IX

REFERENCES

1. Bradbury, W. W., and Fletcher, R., "New Iterative Methods for Solution of the Eigenproblem," Numerische Mathematik 9, pp. 259-267, 1966.

2. Rosen, J. B., "The Gradient Projection Method for Nonlinear Programming Part I - Linear Constraints," Journal Soc. Indust. Appl. Math., Vol. 8, No. 1, March 1960.

3. Stanton, E. L., The Modal Analysis of Structure Idealized as an Assembly of Discrete Elements, Reports DAC-59062, Missile & Space Division, Douglas Aircraft Company, Inc., September, 1966.

4. Archer, John S., "Consistent Mass Matrix for Distributed Mass Systems," Journal of the Structural Division, Proceedings of A.S.C.E., August, 1963.

5. Bogner, F. K., Mallet, R. H., Minich, M. D., and Schmit, L. A., Development and Evaluation of Energy Search Methods of Nonlinear Structural Analysis, Case Institute of Technology, Cleveland, Ohio, Technical Report No. AFFDL-TR-65-113, October, 1965.

6. Fox, R. L. and Stanton, E. L., "Developments in Structural Analysis by Direct Energy Minimization," AIAA Journal, Vol. 6, pp. 1036-1042, 1968.

7. Fletcher, R., and Reeves, C. M., "Function Minimization by Conjugate Gradients," The Computer Journal, Vol. 7, pp. 149-154, 1964.

8. Davidon, W. C., Variable Metric Method for Minimization, A.E.C. Research and Development Report, Argonne Natl. Lab. ANL-5990 (rev.), 1959.

9. Fletcher, R. and Powell, M. J. D., "A Rapidly Convergent Descent Method for Minimization", the Computer Journal, Vol. 6, pp. 163-168, 1963.

10. Kowalik, J., "Iterative Methods for Linear Equations in Matrix Structural Analysis," International Ship Building Progress, Vol. 13, pp. 59-68, 1966.

11. Rubinstein, M. F., "Matrix Computer Analysis of Structures," Prentice Hall, Chapter 12, 1966.

APPENDIX I

PROCEDURE TO OBTAIN Q_ℓ^{-1} FROM $Q_{\ell-1}^{-1}$:

As discussed in the text of the paper, Q_ℓ^{-1} is given by:

$$Q_\ell^{-1} = (N_\ell^T N_\ell)^{-1} \equiv \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_2^T & B_3 \end{array}\right] \tag{48}$$

where

$$\begin{aligned} B_1 &= A_1^{-1} + \frac{1}{s}(A_1^{-1} A_2 A_2^T A_1^{-1}) \\ B_2 &= -\frac{1}{s}(A_1^{-1} A_2) \\ B_3 &= \frac{1}{s} \end{aligned} \tag{49}$$

and

$$s = A_3 - A_2^T A_1^{-1} A_2 \tag{50}$$

Furthermore,

$$A_1 = Q_{\ell-1} \equiv N_{\ell-1} N_{\ell-1}^T$$

$$A_2^T = \left[e_j^T V_{\ell-1}, V_1^T V_{\ell-1}, \cdots, V_{\ell-2}^T V_{\ell-1}\right]$$

and

$$A_3 = V_{\ell-1}^T V_{\ell-1} \tag{51}$$

It is presumed that

$$A_1^{-1} = Q_{\ell-1}^{-1} \equiv (N_{\ell-1}^T N_{\ell-1})^{-1} \tag{52}$$

is known.

We can write

$$A_2 = N_{\ell-1}^T V_{\ell-1} \tag{53}$$

where

$$N_{\ell-1} = \left[e_j, V_1, V_2, \cdots, V_{\ell-2}\right]$$

Therefore, from Equations 48, 49, 50, and 51, we obtain

$$s = V_{\ell-1}^T V_{\ell-1} - V_{\ell-1}^T N_{\ell-1} (N_{\ell-1}^T N_{\ell-1})^{-1} N_{\ell-1}^T V_{\ell-1}$$

$$= V_{\ell-1}^T P_{\ell-1} V_{\ell-1} \tag{54}$$

where

$$P_{\ell-1} \equiv \left\{ I - N_{\ell-1} (N_{\ell-1}^T N_{\ell-1})^{-1} N_{\ell-1}^T \right\} \tag{55}$$

is the "projection" matrix.

The projection matrix, Equation 55, has a neat property that the vectors $P_{\ell-1} W$ and $\left\{ I - P_{\ell-1} \right\} W$ are orthogonal. This can be seen by mere multiplication of the two vectors:

$$W^T P_{\ell-1} \left\{ I - P_{\ell-1} \right\} W = W^T \left\{ I - N_{\ell-1} (N_{\ell-1}^T N_{\ell-1})^{-1} N_{\ell-1}^T \right\} \times$$

$$\left\{ N_{\ell-1} (N_{\ell-1}^T N_{\ell-1})^{-1} N_{\ell-1}^T \right\} W =$$

$$W^T \left\{ N_{\ell-1} (N_{\ell-1}^T N_{\ell-1})^{-1} N_{\ell-1}^T - \right.$$

$$\left. N_{\ell-1} (N_{\ell-1}^T N_{\ell-1})^{-1} N_{\ell-1}^T \right\} W \tag{56}$$

since $(N_{\ell-1}^T N_{\ell-1})^{-1} (N_{\ell-1}^T N_{\ell-1}) \equiv I$

The term within the brackets $\{\}$ on the right hand side of Equation 56 is identically zero and hence we obtain

$$W^T P_{\ell-1} \left\{ I - P_{\ell-1} \right\} W = 0 \tag{57}$$

Equation 57 can be rewritten as

$$W^T P_{\ell-1} W = W^T P_{\ell-1} P_{\ell-1} W = \left| P_{\ell-1} W \right|^2 \tag{58}$$

Therefore, we obtain from Equation 54

$$s = \left| P_{\ell-1} V_{\ell-1} \right|^2 \tag{59}$$

Note that the projection matrix $P_{\ell-1}$ is already known and thus to obtain s from Equation 59 is computationally more efficient than that from Equation 50.

If we denote

$$r_{\ell-1} \equiv A_1^{-1} A_2 = (N_{\ell-1}^T N_{\ell-1})^{-1} N_{\ell-1}^T V_{\ell-1} \tag{60}$$

then from Equations 55 and 60 we obtain

$$P_{\ell-1} V_{\ell-1} = V_{\ell-1} - N_{\ell-1} r_{\ell-1} \tag{61}$$

Furthermore, from Equations 52, 60, 61, and 49 we obtain

$$B_1 = (N_{\ell-1}^T N_{\ell-1})^{-1} + \frac{1}{s} r_{\ell-1} r_{\ell-1}^T$$

$$B_2 = -\frac{1}{s} r_{\ell-1} \tag{62}$$

and

$$B_3 = \frac{1}{s}$$

Thus the procedure to obtain Q_ℓ^{-1} from $Q_{\ell-1}^{-1}$ can be summarized as:

1. Compute the vector $r_{\ell-1}$ from Equation 60 as two matrix vector multiplications.
2. Compute the scalar, s, from Equation 61 and 59.
3. Compute the matrix B_1, the vector B_2 and the scalar B_3 from Equation 62.
4. Form the desired Q_ℓ^{-1} from Equation 48

APPENDIX II

TRANSFORMATION BETWEEN REFERENCE AND LOCAL COORDINATES

Let SR be an rx(t+1) table whose first column contains the element numbers arranged from 1 to r and whose rows contain in the remaining t columns the corresponding local consistent coordinates of the elements, replaced one to one by the number of the corresponding degree of freedom of the structure in the reference coordinate system. The ith row of the SR table is used to obtain the (txl) vector $\mathbf{Y}_i$, i = 1,2,...,r from the (nxl) vector $\mathbf{X}$. $\mathbf{Y}_i$ is the displacement vector of the ith element corresponding to the generalized displacement vector $\mathbf{X}$ of the system. As an illustrative example, consider the simple cantilever beam shown in Figure 4, which is modeled with three general planar beam elements. Thus, the number of elements, r, is 3, the local degrees of freedom of an element in the reference coordinate system, t, is 6, and the number of degrees of freedom of the system, n, is 9. The SR table for this system is shown below:

Element No.	Degrees of Freedom					
	Axial Displacement At End		Transverse Displacement At End		Rotation of End	
	P	Q	P	Q	P	Q
1	0	1	0	2	0	3
2	1	4	2	5	3	6
3	4	7	5	8	6	9

Let the generalized displacement vector be:

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix}$$

Then

$$\mathbf{Y}_1 = \begin{pmatrix} 0 \\ x_1 \\ 0 \\ x_2 \\ 0 \\ x_3 \end{pmatrix}, \quad \mathbf{Y}_2 = \begin{pmatrix} x_1 \\ x_4 \\ x_2 \\ x_5 \\ x_3 \\ x_6 \end{pmatrix} \text{ and } \quad \mathbf{Y}_3 = \begin{pmatrix} x_4 \\ x_7 \\ x_5 \\ x_8 \\ x_6 \\ x_9 \end{pmatrix}$$

Thus we can say that

$$\mathbf{Y}_i \xleftarrow{SR_i} \mathbf{X}$$

i.e., the vector $\mathbf{X}$ is transformed to the vector $\mathbf{Y}_i$ through the ith row of the SR table, where the non-zero components of the vector $\mathbf{Y}_i$ are the components of the vector $\mathbf{X}$ associated with the ith element.