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Distorted Risk Measures with Application to Military Capability Shortfalls

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Presentation Outline

- Motivation
- Distributions and distortion
- Examination of distortion functions
- Numerical example
- Conclusions



Motivation

- Suppose a risk's associated severity is described by a distribution
- Risk *measures* summarize distribution (e.g., mean-variance methods, VaR, conditional VaR, distorted expectation)
- Expectation dampens catastrophic outcomes – right tail may require further emphasis (risk aversion)
- Questions
 - How do distortions interact with distributions?
 - Which distortion function and parameters to select?



Distributions

- Risk (R) distribution

$$R(x) = P(X > x \mid Y = 1) \cdot (1 - p)$$

where $X \equiv$ severity given undesirable outcome

$Y \equiv$ binary RV of occurrence (1=yes, 0=no)

$p \equiv$ prob of no undesirable outcome

- Severity (S) distribution ($p = 0$)

$$S(x) = P(X > x \mid Y = 1).$$



Distortion Effects

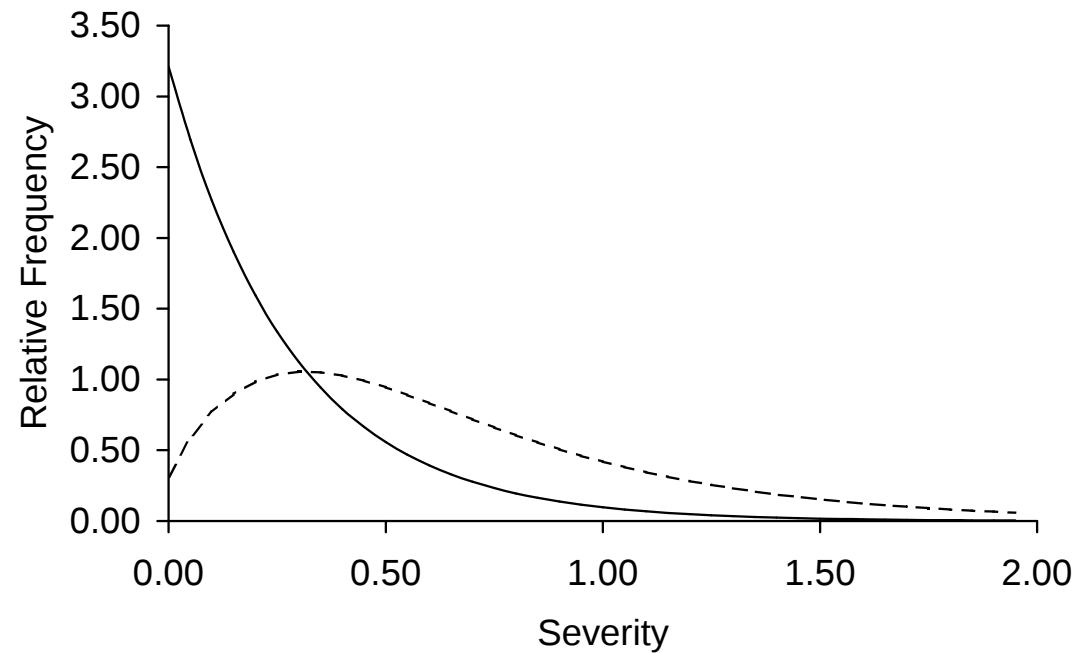


Figure 1. Distortion effects on exponential distribution.



Distortion

- Distortion function, g
 - Emphasizes worst outcomes (“pushes” density right)
 - Forms a composition, $g(S(x)) \equiv (g \circ S)(x)$
 - Is a transformation, $g : [0, 1] \rightarrow [0, 1]$
- Gamma-beta distortion (McLeish & Reesor, 2003)

$$g_{GB}(u) = \int_0^u K t^{a-1} (1-t)^{b-1} \exp(-t/c) dt,$$

where

$$K^{-1} = \int_0^1 t^{a-1} (1-t)^{b-1} \exp(-t/c) dt \quad \text{and} \quad u = S(x)$$



Literature Review: Distortion

- GB family – six distortions, selected parameters:
gamma-beta (GB) (a, b, c), beta ($c \rightarrow \infty$), proportional hazard (PH) ($b = 1, c \rightarrow \infty$), dual power (DP) ($a = 1, c \rightarrow \infty$), gamma ($b = 1$), exponential (EX) ($a = 1, b = 1$) (McLeish & Reesor, 2003)
- Parameter ranges: $0 \leq a \leq 1$, $b \geq 1$, and $c \geq 0$ are sufficient to ensure *coherency* (i.e., that risks behave “reasonably”) (Artzner, et al., 1997)
- Apparently no published works on appropriate choice for a distortion function or selection of associated parameters



Distribution Selections

- Unbounded distributions
 - Exponential
 - Weibull
- Bounded distributions
 - Triangular
 - Uniform



Distortion Function Selections and Effects

Selected distortion functions: PH, DP, EX, [GB]

Table 1. General distortion effects for survivor function $S(x)$.

Distortion	Parameter	$(g \circ S)(x)$
Proportional Hazard (g_{PH})	$0 < a \leq 1$	$S^a(x)$
Dual Power (g_{DP})	$b \geq 1$	$1 - (1 - S(x))^b$
Exponential (g_{EX})	$0 < c < \infty$	$\frac{1 - \exp(-S(x)/c)}{1 - \exp(-1/c)}$
Gamma-Beta (g_{GB})	a, b, c (as above)	$\frac{\int_0^{S(x)} t^{a-1}(1-t)^{b-1}e^{-t/c} dt}{\int_0^1 t^{a-1}(1-t)^{b-1}e^{-t/c} dt}$



Distortion Parameters for Experimentation

- Background
 - New distortion measure for GB req'd (expectation N/A)
 - If ψ is undistorted median, $R_g = \frac{(g \circ S)(\psi)}{S(\psi)}$
- “Region of sensitivity” for $R_g \Rightarrow 1 \leq R_g \leq 2$ (loses track of “distance pushed”)
- 3^k -factorial design for GB – “fair” analysis required each parameter have equal influence over R_g measure
- Face-centered cube: three values, equally spaced



Selected Distortion Parameter Treatments

Table 2. Selected distortion parameter treatments.

Distortion (Parameter)	Selected Value(s)		R_g (% density shift)
Proportional Hazard (a)	High	0.9	1.07 (7%)
	Mid	0.75	1.19 (19%)
	Low	0.6	1.32 (32%)
Dual Power (b)	Low	1.1	1.07 (7%)
	Mid	1.3	1.19 (19%)
	High	1.5	1.29 (29%)
Exponential (c)	High	3.6	1.07 (7%)
	Mid	2.2	1.11 (11%)
	Low	0.8	1.30 (30%)



Analytical Expectation Results

- Explicit expressions for distorted expectation risk measure
- For single-parameter distortions, numerical results attainable even when analytical expectation intractable

Table 3. Summary of risk measures, $X \sim \exp(\lambda)$.

Distortion	$\hat{S}(x)$	$\hat{E}[X]$
g_{PH}	$e^{-\lambda ax}$	$(\lambda a)^{-1}$
g_{DP}	$1 - (1 - e^{-\lambda x})^b$	$\int_0^\infty [1 - (1 - e^{-\lambda x})^b] dx$
g_{EX}	$\frac{1 - \exp(-e^{-\lambda x}/c)}{1 - \exp(-1/c)}$	$\int_0^\infty \frac{1 - \exp(-e^{-\lambda x}/c)}{1 - \exp(-1/c)} dx$



Weibull Distribution: Analytical Results

Table 4. Summary of risk measures, $X \sim \text{Weib}(\beta, \theta)$.

Distortion	$\hat{S}(x)$	$\hat{E}[X]$
g_{PH}	$e^{a(-x/\theta)^\beta}$	$\frac{\theta}{\beta \sqrt[\beta]{a}} \Gamma(\frac{1}{\beta})$
g_{DP}	$1 - (1 - e^{(-x/\theta)^\beta})^b$	$\int_0^\infty [1 - (1 - e^{(-x/\theta)^\beta})^b] dx$
g_{EX}	$\frac{1 - \exp(-e^{(-x/\theta)^\beta}/c)}{1 - \exp(-1/c)}$	$\int_0^\infty \frac{1 - \exp(-e^{(-x/\theta)^\beta}/c)}{1 - \exp(-1/c)} dx$



Triangular Distribution: Analytical Results

Table 5. Summary of risk measures, $X \sim \text{tria}(\theta_1, \theta_2, m)$.

Distortion	$\hat{S}(x)$	$\hat{E}[X]$
g_{PH}	$\left(1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^a, \theta_1 \leq x \leq m$ $\left(\frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^a, m < x \leq \theta_2$	$\int_{\theta_1}^m \left(1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^a dx$ $\frac{(\theta_2-m)^{a+1}}{(2a+1)(\theta_2-\theta_1)^a}$
g_{DP}	$1 - \left(\frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^b, \theta_1 \leq x \leq m$ $1 - \left(1 - \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^b, m < x \leq \theta_2$	$m - \theta_1 - \frac{(m-\theta_1)^{b+1}}{(\theta_2-\theta_1)^b(2b+1)}$ $\int_m^{\theta_2} \left[1 - \left(1 - \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^b\right] dx$
g_{EX}	$\frac{1 - \exp\left(\frac{-1}{c} + \frac{(x-\theta_1)^2}{c(\theta_2-\theta_1)(m-\theta_1)}\right)}{1 - \exp(-1/c)}, \theta_1 \leq x \leq m$ $\frac{1 - \exp\left(\frac{-(\theta_2-x)^2}{c(\theta_2-\theta_1)(\theta_2-m)}\right)}{1 - \exp(-1/c)}, m < x \leq \theta_2$	$\int_{\theta_1}^m \frac{1 - \exp\left(\frac{-1}{c} + \frac{(x-\theta_1)^2}{c(\theta_2-\theta_1)(m-\theta_1)}\right)}{1 - \exp(-1/c)} dx$ $\int_m^{\theta_2} \frac{1 - \exp\left(\frac{-(\theta_2-x)^2}{c(\theta_2-\theta_1)(\theta_2-m)}\right)}{1 - \exp(-1/c)} dx$



Uniform Distribution: Analytical Results

Table 6. Summary of risk measures, $X \sim \text{unif}(\theta_1, \theta_2)$.

Distortion	$\hat{S}(x)$	$\hat{E}[X]$
g_{PH}	$(1 - \frac{x-\theta_1}{\theta_2-\theta_1})^a$	$(\theta_2 - \theta_1)(\frac{1}{a+1})$
g_{DP}	$1 - (\frac{x-\theta_1}{\theta_2-\theta_1})^b$	$(\theta_2 - \theta_1)(\frac{b}{b+1})$
g_{EX}	$\frac{1 - \exp(-(1 - \frac{x-\theta_1}{\theta_2-\theta_1})/c)}{1 - \exp(-1/c)}$	$(\theta_2 - \theta_1) \left(\frac{1-c+ce^{-1/c}}{1-e^{-1/c}} \right)$



Effectiveness and Efficiency

- *Effectiveness*: $K = \mu_g / \mu_0 \Rightarrow K \geq 1$
- *Efficiency*: $\frac{K}{R_g} = \frac{\text{change in } \mu}{\text{change in density}} = \frac{\Delta\mu}{\Delta\text{density}}$
- Importance of combined measure
 - Without it, no ability to distinguish between pairings with identical *effectiveness* (“many-to-one” mapping)
 - Every increase in R_g = additional “step” from SME recommendations → undesirable consequence



Effectiveness and Efficiency Measures

Table 7. Summary of effectiveness and efficiency measures.

Distortion →	PH			DP			EX		
Measure ↓	$a = 0.9$	$a = 0.75$	$a = 0.6$	$b = 1.1$	$b = 1.3$	$b = 1.5$	$c = 3.6$	$c = 2.2$	$c = 0.8$
Exponential(3.5), $\mu_0 = 0.285714$									
μ_g	0.318	0.382	0.476	0.304	0.336	0.366	0.306	0.319	0.379
R_g	1.072	1.189	1.319	1.067	1.188	1.293	1.069	1.113	1.303
K	1.111	1.333	1.667	1.063	1.177	1.280	1.070	1.116	1.327
K/R_g	1.037	1.121	1.263	0.996	0.991	0.990	1.001	1.003	1.019
Weibull(2,2), $\mu_0 = 1.772454$									
μ_g	1.868	2.047	2.288	1.845	1.971	2.079	1.845	1.891	2.097
R_g	1.072	1.189	1.319	1.067	1.188	1.293	1.069	1.113	1.303
K	1.054	1.155	1.291	1.041	1.112	1.173	1.041	1.067	1.183
K/R_g	0.983	0.971	0.978	0.976	0.936	0.907	0.973	0.958	0.908
Triangular(1,7,4), $\mu_0 = 4.000$									
μ_g	4.116	4.322	4.578	4.103	4.279	4.425	4.097	4.159	4.428
R_g	1.072	1.189	1.319	1.067	1.188	1.293	1.069	1.113	1.303
K	1.029	1.080	1.144	1.026	1.069	1.106	1.024	1.039	1.107
K/R_g	0.960	0.909	0.868	0.961	0.901	0.856	0.958	0.934	0.849
Uniform(1,7), $\mu_0 = 4.000$									
μ_g	4.158	4.429	4.750	4.143	4.391	4.600	4.139	4.227	4.609
R_g	1.072	1.189	1.319	1.067	1.188	1.293	1.069	1.113	1.303
K	1.039	1.107	1.188	1.036	1.098	1.150	1.035	1.057	1.152
K/R_g	0.969	0.931	0.900	0.971	0.924	0.889	0.968	0.949	0.885



Preferred Distortion Functions

Table 8. Preferred distortion functions.

Distribution	Low Distortion (0-10%)	Moderate Distortion (11-20%)	Heavy Distortion (21-30+%)
Exponential(3.5)	PH	PH	PH
Weibull(2,2)	PH	PH	PH
Triangular(1,7,4)	DP	EX	PH
Uniform(1,7)	DP	EX	PH

- Achieve largest possible increase in mean given a specified maximum shift in density
- Shift density by smallest amount required to achieve a specified increase in mean



Guidelines for Distortion Selection

- GB: Inability to analytically compute distorted expectation $\Rightarrow$ less appealing choice
- If SMEs suggest exponential, Weibull: PH distortion most *efficient*
- If SMEs suggest triangular, uniform: not as clear
- If additional moments are obtained from distorted distribution (e.g., σ^2), DP and EX may be preferred over PH



Results: Decision Maker Policies

Decision maker's assigned weights capture priorities

Table 9. Notional point allocations.

Capability	Assigned Weight
A	20.0
B	30.0
C	19.0
D	13.0
E	6.0
F	6.0
G	6.0
H	0.0
J	0.0



Notional Distributions

Table 10. Notional distributions.

Capability	Distribution
A	Weibull(3.5,1101)
B	Tria(1,46773,1585)
C	Unif(1,10 ⁴)
D	Tria(1,10 ⁴ ,100)
E	Weib(2.04,24.73)
F	Weib(3.08,359.1)
G	Unif(1,100)
H	Exp(0.0063)
J	Tria(1,75,3.16)



Application of Distortion

- Proposed methodology applies distortion on distribution-by-distribution basis
- Specific distortions applied objectively in accordance with guidelines previously discussed

Table 11. Selected distortion function application results.

Capability	Distribution	$(1 - p)\mu_0$	R_g	Combination	$(1 - p)\mu_g$
A	Weib(3.5,1101)	24.766	1.20	PH, $a = 0.735$	27.043
B	Tria(1,46773,1585)	32.239	1.30	PH, $a = 0.62$	42.641
C	Unif(1,10 ⁴)	37.504	1.19	EX, $c = 1.3$	42.264
D	Tria(1,10 ⁴ ,100)	33.670	1.13	EX, $c = 1.9$	37.189
E	Weib(2.04,24.73)	7.887	1.06	PH, $a = 0.915$	8.238
F	Weib(3.08,359.1)	9.631	1.06	PH, $a = 0.915$	9.913
G	Unif(1,100)	18.938	1.06	DP, $b = 1.09$	19.737
H	Exp(0.0063)	38.095	1.0	N/A	38.095
J	Tria(1,75,3.16)	13.193	1.0	N/A	13.193



Linear Programming Formulation

$$\text{Maximize} \quad \sum_{i=1}^9 \sum_{j=1}^6 S_i m_{i,j} x_j$$

$$\text{subject to} \quad \sum_{j=1}^6 k_j x_j = 25$$

$$0 \leq x_j \leq 1, \quad j = 1, 2, \dots, 6,$$

where

S_i is risk expectation accompanying Capability i

$m_{i,j}$ denotes mitigation to Capability i by system j

x_j is “amount” of each mitigator to be purchased

k_j is cost of any *complete* mitigator j (25 unit budget)



Optimal Purchase Plan (LP Solution)

Table 12. Notional acquisition recommendations.

Risk Measure	Mitigator					
	1	2	3	4	5	6
None	1.0	1.0	0.2	0.0	0.0	1.0
Undistorted Expectation	1.0	1.0	0.0	0.0	0.25	1.0
Distorted Expectation	1.0	1.0	0.0	0.25	0.0	1.0



Areas for Further Study

- Computing expectation of multi-parameter distortions
- Measures other than R_g should be considered
- Study of correlation between Pearson's skewness coefficient and normalized mean
- Application of distortion functions to other distributions
- Effects of distortion on variance



Open Forum

- Comments
- Questions



Coherency

Artzner, et al. (1997)

- i. Sub-additivity. $\rho(X + Y) \leq \rho(X) + \rho(Y)$
- ii. Homogeneity. $\rho(t \cdot X) = t \cdot \rho(X)$
- iii. Monotonicity. $\rho(X) \geq \rho(Y)$, if $X \leq Y$
- iv. Risk-free condition. $\rho(X + r \cdot n) = \rho(X) - n$



Graphical Effects: Weibull

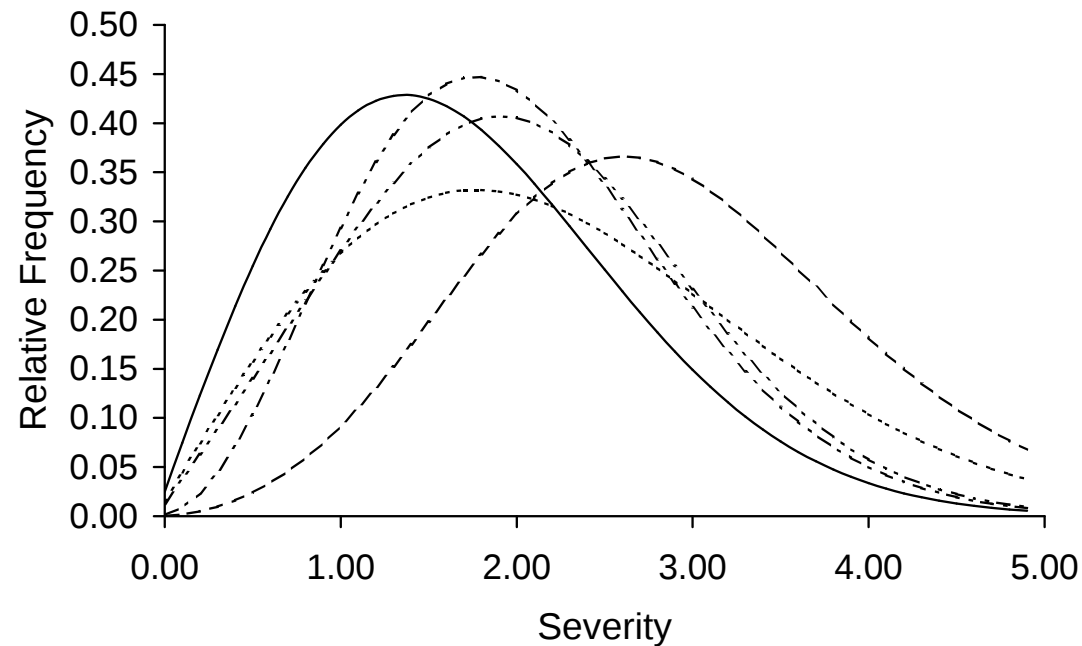


Figure 3. Relative freq density for severity, Weib(2,2) distribution, given distortion parameters $a = 0.6$, $b = 1.5$, $c = 0.8$ (solid is no distortion, — — — GB, ···· PH, — · — · DP, — ··· — EX).



Typical μ Effects Plot (Weibull)

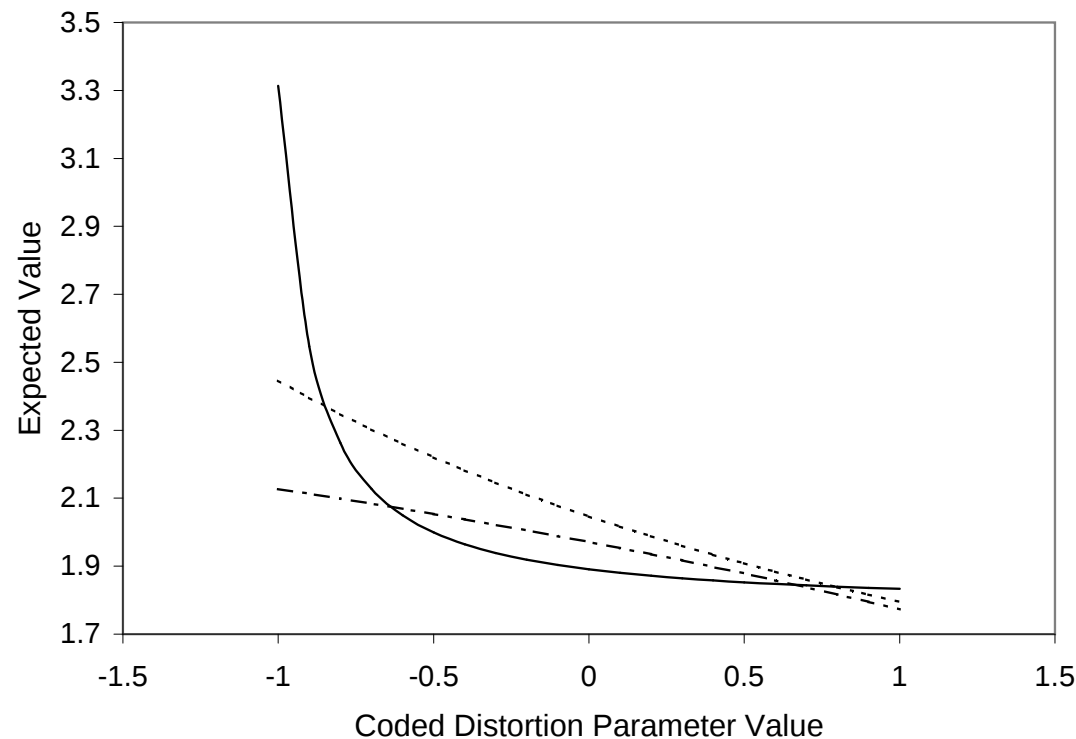


Figure 4. Expected value versus coded distortion parameters, Weib(2,2) distribution, given distortion parameter ranges $a = [0.525, 0.975]$, $b = [1, 1.6]$, $c = [0.1, 4.3]$.



Graphical Effects: Triangular

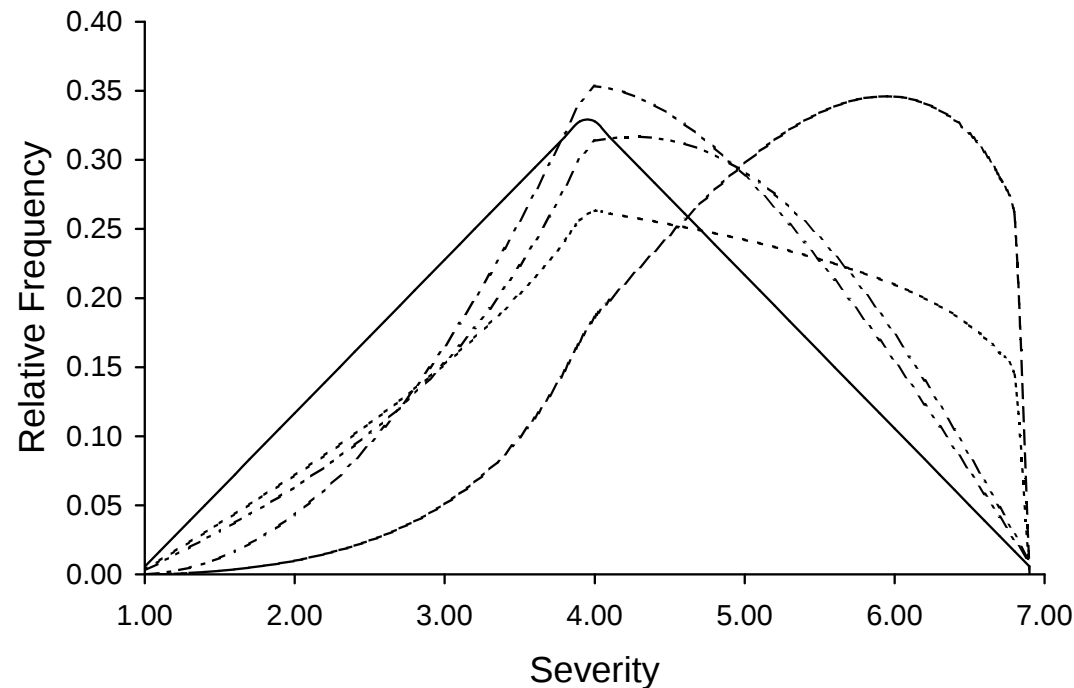


Figure 5. Relative freq density for severity, $\text{tria}(1,7,4)$ distribution, given distortion parameters $a = 0.6$, $b = 1.5$, $c = 0.8$ (solid is no distortion, — — — GB, PH, — . — . DP, — . . — EX).



Graphical Effects: Uniform

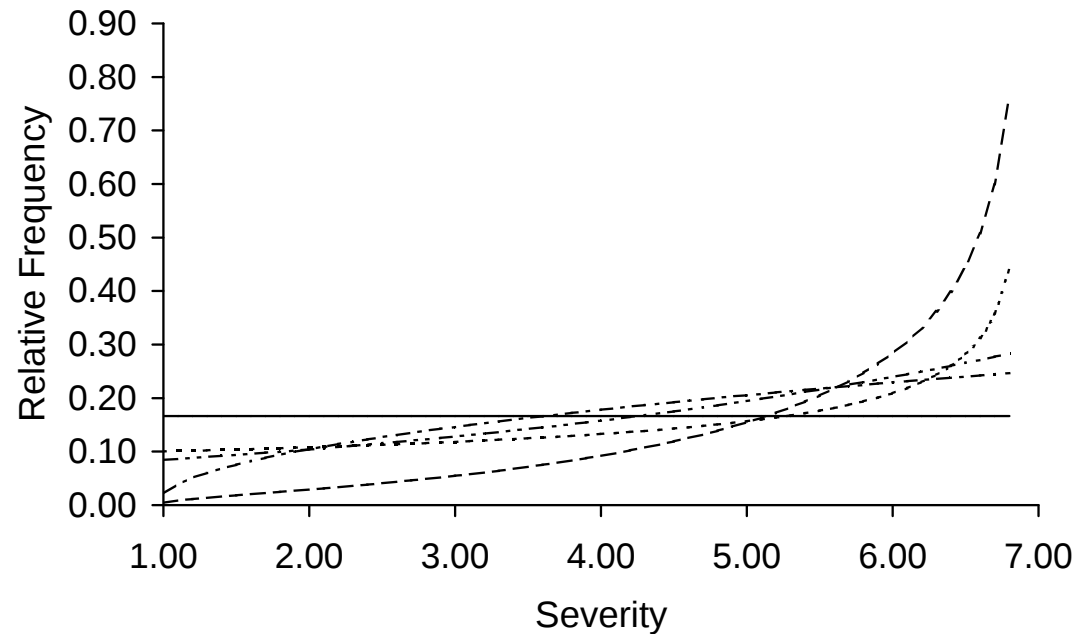


Figure 7. Relative freq density for severity, $\text{unif}(1,7)$ distribution, given distortion parameters $a = 0.6$, $b = 1.5$, $c = 0.8$ (solid is no distortion, — — — GB, . . . PH, — · — DP, — · — EX).