

The Conditional Adjoint Process

by

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The Conditional Adjoint Process

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Abstract

The adjoint and minimum principle for a partially observed diffusion can be obtained by differentiating the statement that a control u^* is optimal. Using stochastic flows the variation in the cost resulting from a change in an optimal control can be computed explicitly. The technical difficulty is to justify the differentiation.

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1. INTRODUCTION.

Using stochastic flows we calculate below the change in the cost due to a 'strong' variation of an optimal control. Differentiating this quantity enables us to identify the adjoint, or co-state variable, and give a partially observed minimum principle. If the drift coefficient is differentiable in the control variable the related result of Bensoussan [2] follows from our theorem. Full details will appear in [1]. The method appears simpler than that employed in Haussman [4].

2. DYNAMICAL EQUATIONS.

Suppose the state of a stochastic system is described by the equation

$$\begin{aligned} d\xi_t &= f(t, \xi_t, u)dt + g(t, \xi_t)dw_t, \\ \xi_t &\in R^d, \quad \xi_0 = x_0, \quad 0 \leq t \leq T. \end{aligned} \tag{2.1}$$

The control variable u will take values in a compact subset U of some Euclidean space R^k .

We shall assume

A_1 : $x_0 \in R^d$ is given.

A_2 : $f : [0, T] \times R^d \times U \rightarrow R^d$ is Borel measurable, continuous in u for each (t, x) , continuously differentiable in x for each (t, u) and

$$(1 + |x|)^{-1} |f(t, x, u)| + |f_x(t, x, u)| \leq K_1.$$

A_3 : $g : [0, T] \times R^d \rightarrow R^d \otimes R^n$ is a matrix valued function, Borel measurable, continuously differentiable in x , and for some K_2 :

$$|g(t, x)| + |g_x(t, x)| \leq K_2.$$

The observation process is defined by

$$dy_t = h(\xi_t)dt + d\nu_t \tag{2.2}$$

$$y_t \in R^m, \quad y_0 = 0, \quad 0 \leq t \leq T.$$

In (2.1) and (2.2) $w = (w^1, \dots, w^n)$ and $\nu = (\nu^1, \dots, \nu^m)$ are independent Brownian motions defined on a probability space (Ω, F, P) .

Furthermore, we assume

A_4 : $h : R^d \rightarrow R^m$ is Borel measurable, continuously differentiable in x and

$$|h(t, x)| + |h_x(t, x)| \leq K_3.$$

REMARK 2.1. These hypotheses can be weakened to those discussed by Haussman [4]. See [1].

Write $\hat{P}$ for the Wiener measure on $C([0, T], R^n)$ and μ for the Wiener measure on $C([0, T], R^m)$.

$$\Omega = C([0, T], R^n) \times C([0, T], R^m)$$

and the coordinate functions in Ω will be denoted (x_t, y_t) . Wiener measure P on Ω is

$$P(dx, dy) = \hat{P}(dx)\mu(dy).$$

DEFINITION 2.2. $Y = \{Y_t\}$ will be the right continuous, complete filtration on $C([0, T], R^m)$ generated by

$$Y_t^0 = \sigma\{y_s : s \leq t\}.$$

The set of admissible control functions $\underline{U}$ will be the Y -predictable functions defined on $[0, T] \times C([0, T], R^m)$ with values in U .

For $u \in \underline{U}$ and $x \in R^d$, $\xi_{s,t}^u(x)$ will denote the strong solution of (2.1) corresponding to u with $\xi_{s,s}^u = x$.

Define

$$Z_{s,t}^u(x) = \exp \left(\int_s^t h(\xi_{s,r}^u(x))' dy_r - \frac{1}{2} \int_s^t h(\xi_{s,r}^u(x))^2 dr \right). \quad (2.3)$$

Note a version of Z defined for every trajectory y can be obtained by integrating the stochastic integral in the exponential by parts.

If a new probability measure P^u defined on Ω by putting

$$\frac{dP^u}{dP} = Z_{0,T}^u(x_0),$$

under P^u $(\xi_{0,t}^u(x_0), y_t)$ is a solution of the system (2.1) and (2.2). That is, under P^u , $\xi_{0,t}^u(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion ν such that y_t satisfies (2.2).

Because of hypothesis A_4 , for $0 \leq t \leq T$ easy applications of Burkholder's and Gronwall's inequalities show that

$$E[(Z_{0,t}^u(x_0))^p] < \infty \quad (2.4)$$

for all $u \in \underline{U}$ and all p , $1 \leq p < \infty$.

COST 2.3. We shall suppose the cost is purely terminal and equals

$$c(\xi_{0,T}^u(x_0))$$

where c is a bounded, differentiable function. If control $u \in \underline{U}$ is used the expected cost is

$$J(u) = E_u[c(\xi_{0,T}^u(x_0))].$$

With respect to P , under which y_t is a Brownian motion

$$J(u) = E[Z_{0,T}^u(x_0)c(\xi_{0,T}^u(x_0))]. \quad (2.5)$$

A control $u^* \in \underline{U}$ is optimal if

$$J(u^*) \leq J(u)$$

for all $u \in \underline{U}$. We shall suppose there is an optimal control u^* .

3. FLOWS.

For $u \in \underline{U}$ and $x \in R^d$ consider the strong solution

$$\xi_{s,t}^u(x) = x + \int_s^t f(r, \xi_{s,r}^u(x), u_r) dr + \int_s^t g(r, \xi_{s,r}^u(x)) dw_r. \quad (3.1)$$

We wish to consider the behaviour of $\xi_{s,t}^u(x)$ for each trajectory y of the observation process. In fact the results of Bismut [3] and Kunita [6] extend and show the map

$$\xi_{s,t}^u : R^d \rightarrow R^d$$

is, almost surely, a diffeomorphism for each $y \in C([0, T], R^m)$.

Write

$$\|\xi^u(x_0)\|_t = \sup_{0 \leq s \leq t} |\xi_{0,s}^u(x_0)|.$$

Then, using Gronwall's and Jensen's inequalities, for any p , $1 \leq p < \infty$

$$\|\xi^u(x_0)\|_T^p \leq C \left(1 + |x_0|^p + \left| \int_0^T g(r, \xi_{0,r}^u(x_0)) dw_r \right|^p \right)$$

almost surely, for some constant C .

Using A_3 and Burkholder's inequality

$$\|\xi^u(x_0)\|_T \in L^p \quad \text{for } 1 \leq p < \infty.$$

Suppose u^* is an optimal control, and write

$$\xi_{s,t}^*(\cdot) \quad \text{for } \xi_{s,t}^{u^*}(\cdot).$$

The Jacobian $\frac{\partial \xi_{s,t}^*}{\partial x}$ is the matrix solution C_t of the equation

$$dC_t = f_x(t, \xi_{s,t}^*(x), u^*) C_t dt + \sum_{i=1}^n g_x^{(i)}(t, \xi_{s,t}^*(x)) C_t dw_t^i. \quad (3.2)$$

with $C_s = I$.

Here $g^{(i)}$ is the i^{th} column of g and I is the $n \times n$ identity matrix. Writing $\|C\|_T = \sup_{0 \leq s \leq t} |C_s|$ and using Burkholder's, Jensen's and Gronwall's inequalities we see $\|C\|_T \in L^p$, $1 \leq p < \infty$.

Consider the matrix valued process D defined by

$$D_t = I - \int_s^t D_r f_x(r, \xi_{s,r}^*(x), u_r^*) dr - \sum_{i=1}^n \int_s^t D_r g_x^{(i)}(r, \xi_{s,r}^*(x)) dw_r^i + \sum_{i=1}^n \int_s^t D_r (g_x^{(i)}(r, \xi_{s,r}^*(x)))^2 dr \quad (3.3)$$

Then as in [5] or [6] $d(D_t C_t) = 0$ and $D_s C_s = I$ so

$$D_t = C_t^{-1} = \left(\frac{\partial \xi_{s,t}^*}{\partial x} \right)^{-1}.$$

Furthermore, $\|D\|_t \in L^p$, $1 \leq p < \infty$.

Suppose $z_t = z_s + A_t + \sum_{i=1}^n \int_s^t H_i dw_r^i$ is a d -dimensional semimartingale. Bismut [3] shows one can consider the process $\xi_{s,t}^*(z_t)$ and in fact:

$$\begin{aligned} \xi_{s,t}^*(z_t) &= z_s + \int_s^t \left(f(r, \xi_{s,r}^*(z_r), u_r^*) \right. \\ &\quad + \sum_{i=1}^n g_x^{(i)}(r, \xi_{s,r}^*(z_r), u_r^*) \frac{\partial \xi_{s,r}^*}{\partial x} H_i \\ &\quad + \frac{1}{2} \sum_{i=2}^n \frac{\partial^2 \xi_{s,r}^*}{\partial x^2} (H_i, H_i) \Big) dr \\ &\quad + \int_s^t \frac{\partial \xi_{s,r}^*}{\partial x}(z_r) dA_r + \sum_{i=1}^n \int_s^t \left(g^{(i)}(r, \xi_{s,r}^*(z_r)) + \frac{\partial \xi_{s,r}^*}{\partial x}(z_r) H_i \right) dw_r^i. \end{aligned} \quad (3.4)$$

DEFINITION 3.1. For $s \in [0, T]$, $h > 0$ such that $0 \leq s < s+h \leq T$, for any $\tilde{u} \in U$, and $A \in Y_s$ consider a 'strong' variation u of u^* defined by

$$u(t, w) = \begin{cases} u^*(t, w) & \text{if } (t, w) \notin [s, s+h] \times A \\ \tilde{u}(t, w) & \text{if } (t, w) \in [s, s+h] \times A. \end{cases}$$

THEOREM 3.2. For any strong variation u of u^* consider the process

$$z_t = x + \int_s^t \left(\frac{\partial \xi_{s,r}^*}{\partial x}(z_r) \right)^{-1} (f(r, \xi_{s,r}^*(z_r), u_r) - f(r, \xi_{s,r}^*(z_r), u_r^*)) dr. \quad (3.5)$$

Then the process $\xi_{s,t}^*(z_t)$ is indistinguishable from $\xi_{s,t}^u(x)$.

PROOF We shall substitute in (3.4), (noting $H_i = 0$ for all i). Therefore,

$$\begin{aligned} \xi_{s,t}^*(z_t) &= x + \int_s^t f(r, \xi_{s,r}^*(z_r), u_r^*) dr \\ &\quad + \int_s^t \left(\frac{\partial \xi_{s,r}^*}{\partial x}(z_r) \right) \left(\frac{\partial \xi_{s,r}^*}{\partial x}(z_r) \right)^{-1} (f(r, \xi_{s,r}^*(z_r), u_r) - f(r, \xi_{s,r}^*(z_r), u_r^*)) dr \\ &\quad + \int_s^t g(r, \xi_{s,r}^*(z_r)) dw_r. \end{aligned}$$

The solution of (3.1) is unique, so $\xi_{s,t}^*(z_t) = \xi_{s,t}^u(x)$. Note $u(t) = u^*(t)$ if $t > s + h$ so $z_t = z_{s+h}$ if $t > s + h$ and

$$\begin{aligned} \xi_{s,t}^*(z_t) &= \xi_{s,t}^*(z_{s+h}) \\ &= \xi_{s+h,t}^*(\xi_{s,s+h}^u(x)). \end{aligned} \quad (3.6)$$

4. THE EXPONENTIAL DENSITY.

Consider the $(d+1)$ -dimensional system

$$\begin{aligned} \xi_{s,t}^*(x) &= x + \int_s^t f(r, \xi_{s,r}^*(x), u_r^*) dr + \int_s^t g(r, \xi_{s,r}^*(x)) dw_r \\ Z_{s,t}^*(x, z) &= z + \int_s^t Z_{s,r}^*(x, z) h(\xi_{s,r}^*(x))' dy_r. \end{aligned} \quad (4.1)$$

That is, we are considering an augmented flow (ξ, Z) in R^{d+1} in which Z^* has a variable initial condition $z \in R$. Note:

$$Z_{s,t}^*(x, z) = z Z_{s,t}^*(x).$$

The map $(x, z) \rightarrow (\xi_{s,t}^*(x), Z_{s,t}^*(x, z))$ is, almost surely, a diffeomorphism of R^{d+1} . Clearly,

$$\frac{\partial \xi_{s,t}^*}{\partial z} = 0, \quad \frac{\partial f}{\partial z} = 0 \quad \text{and} \quad \frac{\partial g}{\partial z} = 0.$$

The Jacobian of this augmented map is represented by the matrix

$$\tilde{C}_t = \begin{pmatrix} \frac{\partial \xi_{s,t}^*}{\partial x} & 0 \\ \frac{\partial Z_{s,t}^*}{\partial x} & \frac{\partial Z_{s,t}^*}{\partial z} \end{pmatrix}.$$

In particular, from (4.1), for $1 \leq i \leq d$

$$\frac{\partial Z_{s,t}^*}{\partial x_i} = \sum_{j=1}^m \int_s^t (Z_{s,r}^*(x, z) \sum_{k=1}^n \frac{\partial h^j}{\partial \xi_k} \cdot \frac{\partial \xi_{k,s,r}^*}{\partial x_i} + h^j(\xi_{s,r}^*(x)) \frac{\partial Z_{s,r}^*}{\partial x_i}) dy_r^j. \quad (4.2)$$

We are interested in solutions of (4.1) and (4.2) only when $z = 1$, so as above we write

$$Z_{s,t}^*(x) \quad \text{for} \quad Z_{s,t}^*(x, 1) \quad \text{etc.}$$

LEMMA 4.1.

$$\frac{\partial Z_{s,t}^*}{\partial x} = Z_{s,t}^*(x) \left(\int_s^t h_x(\xi_{s,t}^*(x)) \cdot \frac{\partial \xi_{s,r}^*}{\partial x} d\nu_r \right)$$

where, as in (2.2), $d\nu_t = dy_t - h(\xi_{s,t}^*(x))dt$.

PROOF From (4.2)

$$\frac{\partial Z_{s,t}^*}{\partial x} = \int_s^t \left(\frac{\partial Z_{s,r}^*}{\partial x} h'(\xi_{s,r}^*(x)) + Z_{s,r}^*(x) h_x(\xi_{s,r}^*(x)) \frac{\partial \xi_{s,r}^*}{\partial x} \right) dy_r. \quad (4.3)$$

Write

$$L_{s,t}(x) = Z_{s,t}^*(x) \left(\int_s^t h_x \cdot \frac{\partial \xi_{s,r}^*}{\partial x} d\nu_r \right).$$

Then

$$Z_{s,t}^*(x) = 1 + \int_s^t Z_{s,r}^*(x) h'(\xi_{s,r}^*(x)) dy_r$$

and the product rule gives

$$\begin{aligned} L_{s,t}(x) &= \int_s^t L_{s,r}(x) h'(\xi_{s,r}^*(x)) dy_r \\ &\quad + \int_s^t Z_{s,r}^*(x) h_x \cdot \frac{\partial \xi_{s,r}^*}{\partial x} dy_r. \end{aligned}$$

The minimum cost is

$$\begin{aligned} J(u^*) &= E[Z_{0,T}^*(x_0)c(\xi_{0,T}^*(x_0))] \\ &= E[Z_{0,s}^*(x_0)Z_{s,T}^*(x)c(\xi_{s,T}^*(x))]. \end{aligned}$$

Also,

$$\begin{aligned} J(u) &= E[Z_{0,s}^*(x_0)Z_{s,T}^u(x)c(\xi_{s,T}^u(x))] \\ &= E[Z_{0,s}^*(x_0)Z_{s,T}^*(z_{s+h})c(\xi_{s,T}^*(z_{s+h}))] \end{aligned}$$

by (3.6) and (4.5). Recall $Z_{s,T}^*(\cdot)$ and $c(\xi_{s,T}^*(\cdot))$ are differentiable almost surely, with continuous and uniformly integrable derivatives. Consequently, writing

$$\begin{aligned} \Gamma(s, z_r) &= Z_{0,s}^*(x_0)Z_{s,T}^*(z_r) \left\{ c_\xi(\xi_{s,T}^*(z_r)) \frac{\partial \xi_{s,T}^*}{\partial x}(z_r) \right. \\ &\quad \left. + c(\xi_{s,T}^*(z_r)) \left(\int_s^T h_\xi(\xi_{s,\sigma}^*(z_r)) \frac{\partial \xi_{s,\sigma}^*}{\partial x}(z_r) d\nu_\sigma \right) \right\} \left(\frac{\partial \xi_{s,r}^*}{\partial x}(z_r) \right)^{-1} \end{aligned}$$

for $s \leq r \leq s+h$, we have

$$\begin{aligned} J(u) - J(u^*) &= E[Z_{0,s}^*(x_0) \{ Z_{s,t}^*(z_{s+h})c(\xi_{s,t}^*(z_{s+h})) - Z_{s,T}^*(x)c(\xi_{s,T}^*(x)) \}] \\ &= E \left[\int_s^{s+h} \Gamma(s, z_r) (f(r, \xi_{s,r}^*(z_r), u_r) - f(r, \xi_{s,r}^*(x), u_r^*)) dr \right]. \end{aligned} \tag{5.1}$$

This formula describes the change in the expected cost arising from the perturbation u of the optimal control. However, $J(u) \geq J(u^*)$ for all $u \in \underline{U}$ so the right hand side of (5.1) is non-negative for all $h > 0$. We wish to divide by $h > 0$ and let $h \rightarrow 0$. This requires some careful arguments using the uniform boundedness of the random variables and the monotone class theorem. It can be shown that there is a set $S \subset [0, T]$ of zero Lebesgue measure such that if $s \notin S$

$$E[\Gamma(s, x)(f(s, \xi_{0,s}^*(x_0), u) - f(s, \xi_{0,s}^*(x_0), u_s^*))I_A] \geq 0 \tag{5.2}$$

for any $u \in U$ and $A \in Y_s$.

Details of this argument can be found in [1]. Define

$$p_s(x) = E^* \left[c_\xi(\xi_{0,T}^*(x_0)) \frac{\partial \xi_{s,T}^*}{\partial x}(x) \right. \\ \left. + c(\xi_{0,T}^*(x_0)) \left(\int_s^T h_\xi(\xi_{0,\sigma}^*(x_0)) \frac{\partial \xi_{s,\sigma}^*}{\partial x}(x) d\nu_\sigma \right) \middle| Y_{s \vee \{x\}} \right]$$

where $x = \xi_{0,s}^*(x_0)$ and E^* is the expectation under $P^* = P^{u^*}$.

In (5.2) we have established the following:

THEOREM 5.1. $p_s(x)$ is the adjoint process for the partially observed optimal control problem. That is, if $u^* \in \underline{U}$ is optimal there is a set $S \subset [0, T]$ of zero Lebesgue measure such that for $s \notin S$

$$E^* [p_s(x) f(s, x, u^*) \mid Y_s] \geq E^* [p_s(x) f(s, x, u) \mid Y_s] \quad \text{a.s.} \quad (5.3)$$

so the optimal control u^* almost surely minimizes the conditional Hamiltonian.

If $x = \xi_{0,s}^*(x_0)$ has a conditional density $q_s(x)$ under P^* , and if f is differentiable in u , (5.3) implies

$$\sum_{i=1}^k (u_i(s) - u_i^*(s)) \int_{R^d} \Gamma(s, x) \frac{\partial f}{\partial u_i}(s, x, u^*) q_s(x) dx \geq 0.$$

This is the result of Bensoussan [2].

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