

Estimation of the Rate of a Doubly-
Stochastic Time-Space Poisson Process

by

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ESTIMATION OF THE RATE OF A
DOUBLY-STOCHASTIC TIME-SPACE POISSON PROCESS

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Abstract

We consider the problem of estimating the rate of a doubly-stochastic, time-space Poisson process when the observations are restricted to a region $D \subseteq \mathbb{R}^2$. In the general case, we obtain a representation of the minimum mean-square-error (MMSE) estimate in terms of the conditional characteristic function of an underlying state process. In the case $D = \mathbb{R}^2$, we extend a known result to compute the MMSE estimate explicitly. For a special form of the rate process, a well-defined integral equation is presented which defines the *linear* MMSE estimate of the rate.

Key Words: doubly-stochastic, time-space Poisson process, MMSE estimate, linear MMSE estimate, likelihood ratio.

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I. Introduction

We consider a doubly-stochastic, time-space Poisson process N^0 with intensity function $\lambda(t, r) = f(t, r - H(t)x_t)$, where $t > 0$ and $r \in \mathbb{R}^2$. Here, f is a known, deterministic function; $x_t \in \mathbb{R}^n$ is the solution of an Ito stochastic differential equation, and $H(t)$ is a known, deterministic, $\mathbb{R}^{2 \times n}$ -valued function. The process N^0 under consideration counts events which occur in all of $\mathbb{R}^2$; however, suppose that only those events which occur within a region $D \subseteq \mathbb{R}^2$ can be observed. We wish to compute minimum mean-square-error (MMSE) estimates of $\lambda(t, r)$, given our limited observations. In the general case, $D \neq \mathbb{R}^2$, we obtain a representation of these estimates in terms of the conditional characteristic function of x_t . When $D = \mathbb{R}^2$, and $f(t, r) = e^{-\frac{1}{2}r'R(t)^{-1}r}$, for some deterministic matrix $R(t)$, we extend a result of Rhodes and Snyder [1] to compute the MMSE estimate of $\lambda(t, r)$ explicitly. We also consider *linear* estimates of $\lambda(t, r)$ for the same choice of f when $D \neq \mathbb{R}^2$. These filtering problems are frequently encountered in optical communication systems [2, 3], particularly in the context of hypothesis-testing; this issue is discussed in Section V.

II. Probabilistic Setting

Let $\mathcal{B}^2$ denote the Borel subsets of $\mathbb{R}^2$. Next, if I is any interval of $\mathbb{R}$, let $\mathcal{B}(I)$ denote the Borel subsets of I . We define $\mathcal{B}(I) \otimes \mathcal{B}^2$ to be the smallest σ -field containing all sets of the form $E \times A$, such that $E \in \mathcal{B}(I)$ and $A \in \mathcal{B}^2$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which we let

$$N^0 = \{ N(B) : B \in \mathcal{B}(0, \infty) \otimes \mathcal{B}^2 \},$$

be a time-space point process. Sometimes, N^0 is called a random point field or a random measure. Here, this means that with each $B \in \mathcal{B}(0, \infty) \otimes \mathcal{B}^2$, we associate a nonnegative, integer-valued random variable, $N(B) = N(\omega, B)$; in addition, for each $\omega \in \Omega$, $N(\omega, \cdot)$ is assumed to be an integer-valued measure on $\mathcal{B}(0, \infty) \otimes \mathcal{B}^2$. We let F_t represent the times and locations at which points have occurred up to and including time t . More precisely, let

$\mathcal{F}_0$ denote the trivial σ -field, and for $t > 0$, set

$$\mathcal{F}_t = \sigma\{ N(B) : B \in \mathcal{B}(0, t] \otimes \mathcal{B}^2 \}.$$

Now, let D be a Borel subset of $\mathbb{R}^2$. We take $\mathcal{G}_0$ to be the trivial σ -field, and for $t > 0$, we set

$$\mathcal{G}_t = \sigma\{ N(B \cap \{(0, \infty) \times D\}) : B \in \mathcal{B}(0, t] \otimes \mathcal{B}^2 \}.$$

Note that $\mathcal{G}_t$ represents the history of the point process restricted to the region D , up to time t . We shall refer to $\mathcal{G}_t$ as our "observations up to time t ." On the same probability space, $(\Omega, \mathcal{F}, \mathbf{P})$, let X be an n -dimensional Gaussian random vector with known mean, m , and known, positive-definite covariance, S . Let $\{v_t, t \geq 0\}$ be a standard Wiener process independent of X . We let the n -dimensional process $\{x_t, t \geq 0\}$ be the solution to the Ito stochastic differential equation

$$dx_t = F(t)x_t dt + V(t)dv_t; \quad x_0 = X. \quad (1)$$

Here F and V are known matrices with appropriate dimensions. We also assume that F and V are piecewise-continuous so that a unique solution of (1) exists (see Davis [4], pp. 108-111). Let

$$\mathbf{X}_0 \triangleq \sigma\{x_s, 0 \leq s < \infty\}.$$

For $t > 0$, let $\mathbf{X}_t$ denote the smallest σ -field containing $\mathcal{F}_t \cup \mathbf{X}_0$. We write this symbolically as

$$\mathbf{X}_t \triangleq \mathcal{F}_t \vee \mathbf{X}_0; \quad t > 0.$$

We shall assume that N^0 is an $\{\mathbf{X}_t\}$ -doubly-stochastic, time-space Poisson process, with $\mathbf{X}_0$ -measurable intensity (see Bremaud [5], pp. 21-23 and 233-238)

$$\lambda(t, r) = f(t, r - H(t)x_t),$$

where $t \in (0, \infty)$, $r \in \mathbb{R}^2$, and x_t is defined by (1). Assume that $H: (0, \infty) \rightarrow \mathbb{R}^{2 \times n}$ and $f: (0, \infty) \times \mathbb{R}^2 \rightarrow (0, \infty)$ are deterministic and known. We further assume that the function

$$\mu(t) \triangleq \int_{\mathbb{R}^2} f(t, r) dr \quad (2)$$

is finite for all $t < \infty$. This means that for each $t \geq 0$, the process

$$\mathbf{N}^t \triangleq \{ N(B) : B \in \mathcal{B}(t, \infty) \otimes \mathcal{B}^2 \}$$

is a Poisson random field under the measure $\mathbf{P}(\bullet | \mathbf{X}_t)$, with rate $\lambda(s, r)$, where $s \in (t, \infty)$, and $r \in \mathbb{R}^2$. This implies the following. First, for $B \in \mathcal{B}(0, \infty) \otimes \mathcal{B}^2$, let $\Lambda(B) \triangleq \int_B \lambda(s, r) dr ds$; then if $B \in \mathcal{B}(t, \infty) \otimes \mathcal{B}^2$ and n is an arbitrary, nonnegative integer,

$$\mathbf{P}(N(B) = n | \mathbf{X}_t) = \frac{\Lambda(B)^n}{n!} e^{-\Lambda(B)}, \quad (3)$$

and hence, for $\theta \in \mathbb{R}$,

$$\mathbf{E}[e^{j\theta N(B)} | \mathbf{X}_t] = \exp[(e^{j\theta} - 1) \Lambda(B)]. \quad (4)$$

The second implication is that if B_1 and B_2 are disjoint sets in $\mathcal{B}(t, \infty) \otimes \mathcal{B}^2$, then the random variables $N(B_1)$ and $N(B_2)$ are independent under the measure $\mathbf{P}(\bullet | \mathbf{X}_t)$.

Notation. We let $N_0 \equiv 0$ and for $t > 0$, $N_t \triangleq N((0, t] \times D)$.

III. Nonlinear Filtering Results

We first establish some notation in order to state our results more compactly. Let $P_t(x)$, $x \in \mathbb{R}^n$, denote the (regular) conditional probability of $\mathbf{x}_t$ given $\mathcal{G}_t$. Let $\psi_t(\eta)$, $\eta \in \mathbb{R}^n$, denote the conditional characteristic function of $\mathbf{x}_t$ given $\mathcal{G}_t$:

$$\psi_t(\eta) \triangleq \mathbf{E}[e^{j\eta' \mathbf{x}_t} | \mathcal{G}_t] = \int_{\mathbb{R}^n} e^{j\eta' x} dP_t(x); \quad \eta \in \mathbb{R}^n.$$

Next, let

$$\hat{\lambda}(t, r) \triangleq \mathbf{E}[\lambda(t, r) | \mathcal{G}_t] = \mathbf{E}[f(t, r - H(t)\mathbf{x}_t) | \mathcal{G}_t],$$

and

$$\hat{l}(t, \theta) \triangleq \int_{\mathbb{R}^2} \hat{\lambda}(t, r) e^{j\theta' r} dr ; \quad \theta \in \mathbb{R}^2.$$

We also set

$$F(t, \theta) \triangleq \int_{\mathbb{R}^2} f(t, r) e^{j\theta' r} dr .$$

Theorem 1. *Under the foregoing assumptions,*

$$\hat{l}(t, \theta) = F(t, \theta) \psi_t(H(t)' \theta) .$$

Proof. Observe that

$$\begin{aligned} \hat{l}(t, \theta) &= \int_{\mathbb{R}^2} \mathbf{E} [f(t, r - H(t)x_t) \mid \mathcal{G}_t] e^{j\theta' r} dr \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^n} f(t, r - H(t)x) dP_t(x) e^{j\theta' r} dr . \end{aligned}$$

By Fubini's Theorem,

$$\begin{aligned} \hat{l}(t, \theta) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^2} f(t, r - H(t)x) e^{j\theta' r} dr dP_t(x) \\ &= F(t, \theta) \int_{\mathbb{R}^n} e^{j\theta' H(t)x} dP_t(x) \\ &= F(t, \theta) \int_{\mathbb{R}^n} e^{j(H(t)' \theta)' x} dP_t(x) \\ &= F(t, \theta) \psi_t(H(t)' \theta) . \end{aligned}$$

QED

Theorem 2. *If $D = \mathbb{R}^2$, and if*

$$f(t, r) = e^{-\frac{1}{2} r' R(t)^{-1} r} , \tag{5}$$

for some deterministic, positive-definite matrix $R(t)$, then

$$\begin{aligned}
\hat{\lambda}(t, r) &\triangleq \mathbf{E} [\lambda(t, r) \mid \mathcal{G}_t] \\
&= \mathbf{E} [f(t, r - H(t)x_t) \mid \mathcal{G}_t] \\
&= \frac{\sqrt{\det R(t)}}{\sqrt{\det Q_t}} \exp \left[-\frac{1}{2} (r - H(t)\hat{x}_t)' Q_t^{-1} (r - H(t)\hat{x}_t) \right],
\end{aligned}$$

where

$$\begin{aligned}
\hat{x}_t &\triangleq \mathbf{E} [x_t \mid \mathcal{G}_t], \\
\hat{\Sigma}_t &\triangleq \mathbf{E} [(x_t - \hat{x}_t)(x_t - \hat{x}_t)' \mid \mathcal{G}_t] > 0, \quad \mathbf{P} - \text{a.s.}, \\
Q_t &\triangleq H(t)\hat{\Sigma}_t H(t)' + R(t),
\end{aligned}$$

and

$$\begin{aligned}
d\hat{x}_t &= F(t)\hat{x}_t dt \\
&+ \int_{\mathbb{R}^2} \hat{\Sigma}_t H(t)' Q_t^{-1} (r - H(t)\hat{x}_t) N(dt \times dr); \quad \hat{x}_0 = m,
\end{aligned} \tag{6}$$

$$\begin{aligned}
d\hat{\Sigma}_t &= F(t)\hat{\Sigma}_t dt + \hat{\Sigma}_t F(t)' dt + V(t)V(t)' dt \\
&- \hat{\Sigma}_t H(t)' Q_t^{-1} H(t)\hat{\Sigma}_t N(dt \times \mathbb{R}^2); \quad \hat{\Sigma}_0 = S.
\end{aligned} \tag{7}$$

Proof. First, since $D = \mathbb{R}^2$, $\mathcal{G}_t = \mathcal{F}_t$. Next, in [1] it is proved that the conditional density of x_t given $\mathcal{F}_t$ is *Gaussian* with conditional mean $\hat{x}_t$ and conditional covariance $\hat{\Sigma}_t$ (which is positive definite almost surely because of the assumption that S is positive definite) satisfying (6) and (7) above. So,

$$\psi_t(\eta) = e^{j\eta'\hat{x}_t - \frac{1}{2}\eta'\hat{\Sigma}_t\eta}.$$

Next, from equation (5), it follows that

$$F(t, \theta) = 2\pi \sqrt{\det R(t)} e^{-\frac{1}{2}\theta'R(t)\theta}.$$

Hence, by Theorem 1,

$$\hat{l}(t, \theta) = 2\pi \sqrt{\det R(t)} e^{j\theta'H(t)\hat{x}_t - \theta'Q_t\theta}.$$

Taking inverse Fourier transforms, we see by inspection that

$$\hat{\lambda}(t, r) = \frac{\sqrt{\det \bar{R}(t)}}{\sqrt{\det Q_t}} \exp \left[-\frac{1}{2} (r - H(t)\hat{x}_t)' Q_t^{-1} (r - H(t)\hat{x}_t) \right].$$

QED

When $D \neq \mathbb{R}^2$, or equation (5) does not hold, $\psi_t(\eta)$ is, in general, not known. This has led us to consider *linear* estimates of $\lambda(t, r)$. We discuss this in the next section.

IV. Linear Filtering Results

We call $\hat{\lambda}_L(t, r)$ a *linear* estimate of $\lambda(t, r)$ given G_t , if $\hat{\lambda}_L$ can be written in the form

$$\hat{\lambda}_L(t, r) = \int_0^t \int_D h(t, r; \tau, \rho) [N(d\tau \times d\rho) - \bar{\lambda}(\tau, \rho) d\tau d\rho] + h_0(t, r), \quad (8)$$

where h and h_0 are deterministic, and $\bar{\lambda}(t, r) \triangleq \mathbf{E}[\lambda(t, r)]$. We wish to choose h and h_0 to minimize

$$\mathbf{E} [|\lambda(t, r) - \hat{\lambda}_L(t, r)|^2]. \quad (9)$$

Lemma 1. (Grandell [6]). *Let $\hat{\lambda}_L(t, r)$ be given by (8). Under the conditions outlined in Section II, the quantity in (9) will be minimized if $h_0(t, r) = \bar{\lambda}(t, r)$, and if h satisfies*

$$\Gamma(t, r; \tau, \rho) = \int_0^t \int_D h(t, r; \sigma, \zeta) \Gamma(\sigma, \zeta; \tau, \rho) d\zeta d\sigma + h(t, r; \tau, \rho) \bar{\lambda}(\tau, \rho), \quad (10)$$

where

$$\Gamma(t, r; \tau, \rho) \triangleq \mathbf{cov} [\lambda(t, r), \lambda(\tau, \rho)].$$

With Lemma 1 in mind, we state our Theorem 3.

Theorem 3. *If $f(t, r)$ is given by (5), and the conditions outlined in Section II hold, then*

$$\bar{\lambda}(t, r) = \frac{\sqrt{\det R(t)}}{\sqrt{\det Q(t)}} \exp\left[-\frac{1}{2}(r-H(t)\bar{x}(t))' Q(t)^{-1}(r-H(t)\bar{x}(t))\right], \quad (11)$$

where

$$\bar{x}(t) \triangleq \mathbf{E}[x_t],$$

$$\Sigma(t) \triangleq \mathbf{cov}[x_t],$$

$$Q(t) \triangleq H(t)\Sigma(t)H(t)' + R(t).$$

Furthermore,

$$\begin{aligned} \Gamma(t, r; \tau, \rho) + \bar{\lambda}(t, r)\bar{\lambda}(\tau, \rho) &= \sqrt{\frac{\det R(t)}{\det Q(t)} \frac{\det R(\tau)}{\det Q(\tau)}} \times \\ &\exp\left[-\frac{1}{2}\left(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{x}(\tau) \end{bmatrix}\right)' Q(t, \tau)^{-1} \left(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{x}(\tau) \end{bmatrix}\right)\right], \end{aligned} \quad (12)$$

where

$$\Sigma(t, \tau) \triangleq \mathbf{cov}[x_t, x_\tau],$$

and

$$Q(t, \tau) \triangleq \begin{bmatrix} Q(t) & H(t)\Sigma(t, \tau)H(\tau)' \\ H(\tau)\Sigma(\tau, t)H(t)' & Q(\tau) \end{bmatrix}.$$

Proof. For completeness, we make the following observations. Recall that

$$dx_t = F(t)x_t dt + V(t)dv_t; \quad x_0 = X. \quad (13)$$

Let $\Phi(t_2, t_1)$ be the transition matrix corresponding to $F(t)$. Then

$$\bar{x}(t) = \Phi(t, 0)m, \quad (14)$$

and

$$\Sigma(t, \tau) = \Phi(t, 0)S\Phi(\tau, 0)' + \int_0^{\min(t, \tau)} \Phi(t, s)V(s)V(s)'\Phi(\tau, s)'ds.$$

Note that $\Sigma(t) = \Sigma(t, t)$.

To compute $\bar{\lambda}(t, \tau) = \mathbf{E}[\lambda(t, \tau)]$, observe that x_t is *Gaussian* with mean $\bar{x}(t)$ and covariance $\Sigma(t)$. By considering the proofs of Theorem 1 and Theorem 2, equation (11) is immediate.

The computation of (12) is similar, but requires some judicious preliminary arithmetic. First, observe that $\Gamma(t, \tau; \tau, \rho) + \bar{\lambda}(t, \tau)\bar{\lambda}(\tau, \rho)$ is just another way of writing $\mathbf{E}[\lambda(t, \tau)\lambda(\tau, \rho)]$. Next, rewrite $\lambda(t, \tau)\lambda(\tau, \rho)$ as

$$\exp\left[-\frac{1}{2}\left(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix} \begin{bmatrix} x_t \\ x_\tau \end{bmatrix}\right)' \begin{bmatrix} R(t)^{-1} & 0 \\ 0 & R(\tau)^{-1} \end{bmatrix} \left(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix} \begin{bmatrix} x_t \\ x_\tau \end{bmatrix}\right)\right],$$

which is equal to

$$\exp\left[-\frac{1}{2}\left(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix} \begin{bmatrix} x_t \\ x_\tau \end{bmatrix}\right)' \begin{bmatrix} R(t) & 0 \\ 0 & R(\tau) \end{bmatrix}^{-1} \left(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix} \begin{bmatrix} x_t \\ x_\tau \end{bmatrix}\right)\right]. \quad (15)$$

Because $\{x_t, t \geq 0\}$ is a Gaussian process, $\begin{bmatrix} x_t \\ x_\tau \end{bmatrix}$ is a Gaussian random vector with mean,

$\begin{bmatrix} \bar{x}(t) \\ \bar{x}(\tau) \end{bmatrix}$, and covariance $\begin{bmatrix} \Sigma(t) & \Sigma(t, \tau) \\ \Sigma(\tau, t) & \Sigma(\tau) \end{bmatrix}$. By the same reasoning used to deduce (11), (12)

also follows.

QED

Remark. In equation (10), if we regard t and τ as fixed, and divide through by $\bar{\lambda}(\tau, \rho)$, then the result has the form of the Fredholm equation

$$g = Bh + h,$$

for known function g , known operator B , and unknown function h .

V. Discussion

The filtering problems considered above often arise in the design and implementation of receivers for optical communication systems. Typically, a binary message source is used by a transmitter to select the modulation of the intensity of a laser beam in accordance with whether a “0” or a “1” is to be sent. The laser beam travels to a receiver and strikes its photodetector. We assume that the laser beam has an intensity profile of the form

$$\nu_i(t)f(t, r); \quad i = 0, 1.$$

Here, $\nu_i(t)$ is a known, deterministic function, where $i = 0$ or 1 has been selected by the transmitter.

We model the surface of the receiver’s photodetector as $\mathbb{R}^2$. If the receiver, for example, is subject to vibrations, the center of the spot of laser light may wander randomly over the photodetector surface [2]. We assume, as in [2], that the center of the spot of laser light is given by $H(t)x_t \in \mathbb{R}^2$. The output of photoelectrons from the photodetector is modeled by the process N^0 , with stochastic intensity now given by

$$\lambda_i(t, r) = \nu_i(t)f(t, r - H(t)x_t). \quad (16)$$

Of course, an actual photodetector does not have an infinite photosensitive surface. We account for this fact by assuming that only those photoelectrons which occur in a region $D \subseteq \mathbb{R}^2$ are observed. For example, in this setting, D might be a square or a circle centered at the origin. After observing photoelectrons occurring in D during some time interval $[0, T]$, a decision as to whether a “0” or a “1” was sent has to be made based on one of the estimates $\hat{\lambda}_i(t, r)$ or $\hat{\lambda}_{i,L}(t, r)$. As an example of a decoding scheme, we could use the likelihood ratio test

$$L_T \begin{matrix} & H_1 \\ & > \\ & < \\ & H_0 \end{matrix} 1,$$

to make the decision, using the minimum probability of error cost criterion and assuming equiprobable hypotheses (see Snyder [3], section 2.5). The likelihood ratio, L_T , is given by (see Snyder [3], pp. 471-476)

$$L_T = \frac{\prod_{j=1}^{N_T} \hat{\lambda}_1(t_j, r_j) \exp[-\int_0^T \int_D \hat{\lambda}_1(s, r) dr ds]}{\prod_{j=1}^{N_T} \hat{\lambda}_0(t_j, r_j) \exp[-\int_0^T \int_D \hat{\lambda}_0(s, r) dr ds]}, \quad (17)$$

where t_j and r_j are respectively the time and the location of the j th photoevent in the region D , and we adopt the convention that when $N_T = 0$, the factors preceeding $\exp$ in equation (17) are taken to be unity. Here, of course,

$$\hat{\lambda}_i(t, r) \triangleq \mathbf{E} [\lambda_i(t, r) \mid \mathcal{G}_t]; \quad i = 0, 1.$$

Now, using (16), (17) simplifies to

$$L_T = \prod_{j=1}^{N_T} \frac{\nu_1(t_j)}{\nu_0(t_j)} \exp[-\int_0^T \int_D \hat{\lambda}_1(s, r) - \hat{\lambda}_0(s, r) dr ds]. \quad (19)$$

In the general case, $D \neq \mathbb{R}^2$, $\hat{\lambda}_i(t, r)$ is not known, and hence, L_T cannot be computed. However, when $D = \mathbb{R}^2$, it turns out that we do not need to know $\hat{\lambda}_i(t, r)$ in order to compute L_T . Observe that if $D = \mathbb{R}^2$, then

$$\begin{aligned} \int_D \hat{\lambda}_1(s, r) - \hat{\lambda}_0(s, r) dr &= \mathbf{E} [\int_{\mathbb{R}^2} \lambda_1(s, r) - \lambda_0(s, r) dr \mid \mathcal{G}_s] \\ &= \mathbf{E} [(\nu_1(s) - \nu_0(s)) \int_{\mathbb{R}^2} f(s, r - H(s)x_e) dr \mid \mathcal{G}_s] \\ &= \mathbf{E} [(\nu_1(s) - \nu_0(s)) \mu(s) \mid \mathcal{G}_s] \\ &= \mu(s) [\nu_1(s) - \nu_0(s)]. \end{aligned} \quad (20)$$

In equation (20) we used the fact that for all $r_0 \in \mathbb{R}^2$,

$$\mu(s) \triangleq \int_{\mathbb{R}^2} f(s, r) dr = \int_{\mathbb{R}^2} f(s, r - r_0) dr.$$

Thus, when $D = \mathbb{R}^2$, (19) becomes

$$L_T = \prod_{j=1}^{N_T} \frac{\nu_1(t_j)}{\nu_0(t_j)} \exp\left[-\int_0^T \mu(s) [\nu_1(s) - \nu_0(s)] ds\right]. \quad (21)$$

With (21) in mind, consider the following theorem.

Theorem 4. *The random field*

$$\mathbf{M}^t \triangleq \{ N(E \times \mathbb{R}^2) : E \in \mathcal{B}(t, \infty) \},$$

is independent of the σ -field $\mathbf{X}_t$.

Proof. To prove that $\mathbf{M}^t$ is independent of $\mathbf{X}_t$, it is sufficient to show that the conditional characteristic function of $N(E \times \mathbb{R}^2)$ is deterministic for $E \in \mathcal{B}(t, \infty)$. Now, it follows immediately from the assumption that $\mathbf{N}^0$ is an $\{\mathbf{X}_t\}$ -doubly-stochastic, time-space Poisson process, that for $\theta \in \mathbb{R}$,

$$\begin{aligned} \mathbf{E} [e^{j\theta N(E \times \mathbb{R}^2)} \mid \mathbf{X}_t] &= \exp[(e^{j\theta} - 1) \int_E \int_{\mathbb{R}^2} \lambda_i(s, \mathbf{r}) d\mathbf{r} ds] \\ &= \exp[(e^{j\theta} - 1) \int_E \nu_i(s) \int_{\mathbb{R}^2} f(s, \mathbf{r} - H(s)x_s) d\mathbf{r} ds] \\ &= \exp[(e^{j\theta} - 1) \int_E \nu_i(s) \mu(s) ds]. \end{aligned}$$

Hence $\mathbf{M}^t$ is independent of $\mathbf{X}_t$.

QED

It follows from equation (21) and Theorem 4 that for all $t \geq 0$, the random variable L_t is independent of the σ -field $\mathbf{X}_t$.

If we replace equation (1) by

$$dx_t = F(t)x_t dt + G(t)u_t dt + V(t)dv_t; \quad x_0 = X, \quad (22)$$

where $\{u_t, t \geq 0\}$ is predictable with respect to $\{\mathcal{G}_t, t \geq 0\}$ and $G(t)$ is a known matrix with appropriate dimensions, then most of the above results hold with only minor

modifications. The term $G(t)u_t$ in (22) is interpreted as a control signal driven by the output of the photodetector. Since $H(t)x_t$ represents the center of the spot of laser light striking the receiver, one might try to use $G(t)u_t$ to drive x_t to the origin. This problem is addressed in [1]. If (1) is replaced by (22), Theorem 1 is unchanged. Theorem 2 still holds except that equation (6) must be replaced by

$$\begin{aligned} d\hat{x}_t &= F(t)\hat{x}_t dt + G(t)u_t dt \\ &+ \int_{\mathbb{R}^2} \hat{\Sigma}_t H(t)' Q_t^{-1} (r - H(t)\hat{x}_t) N(dt \times dr); \quad \hat{x}_0 = m. \end{aligned}$$

Lemma 1 is unchanged, and if $u_t = u(t)$ for some deterministic control $\{u(t), t \geq 0\}$, then Theorem 3 holds; of course, (13) becomes (22) and (14) is replaced by

$$\bar{x}(t) = \Phi(t, 0)m + \int_0^t \Phi(t, s)G(s)u(s) ds.$$

In addition, the results of the preceding paragraphs of Section V, including Theorem 4, are unchanged by substituting equation (22) for equation (1). Note also that since $G_t \subseteq \mathbf{X}_t$, and L_t is independent of $\mathbf{X}_t$ when $D = \mathbb{R}^2$, it follows that L_T is independent of the control law $\{u_t, 0 \leq t \leq T\}$ when $D = \mathbb{R}^2$. This implies that the probability of a decoding error corresponding to the likelihood ratio test preceding equation (17) is not a function of the control law $\{u_t, 0 \leq t \leq T\}$ when $D = \mathbb{R}^2$. In this sense, *all controls are optimal*, when $D = \mathbb{R}^2$. In general, when $D \neq \mathbb{R}^2$, this is not to be expected.

REFERENCES

- [1] I.B. Rhodes and D.L. Snyder, "Estimation and Control Performance for Space-Time Point Process Observations," *IEEE Transactions on Automatic Control* vol. AC-22, No.3, pp. 338-346 (June 1977).
- [2] D.L. Snyder, "Applications of Stochastic Calculus for Point Process Models Arising in Optical Communications," pp. 789-804 in *Communication Systems and Random Process Theory*, ed. J.K. Skwirzynski, Sijthoff and Noordhoff, Alphen aan der Rijn, The Netherlands (1978).
- [3] D.L. Snyder, *Random Point Processes, Ch.7*, John Wiley and Sons, (1975).
- [4] M.H.A. Davis, in *Linear Estimation and Stochastic Control*, Chapman and Hall, London (1977).

- [5] P. Bremaud, in *Point Process and Queues, Martingale Dynamics*, Springer-Verlag, New York (1981).
- [6] J. Grandell, "A Note on Linear Estimation of the Intensity of a Doubly Stochastic Poisson Field," *Journal of Applied Probability* vol. 8, pp. 612-614 (1971).