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Expansions for determinants and for characteristic
polynomials of stochastic matrices¹

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Abstract: An expansion of the determinant of any matrix in terms of row sums and off-diagonal entries is given and used to obtain expressions for the coefficients of the characteristic polynomial of stochastic matrices.

Key words: Determinants, eigenvalues, stochastic matrices.

AMS subject classification: 15A15, 15A18, 15A51.

Abbreviated title: Expansions for determinants.

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We found in [1,page 208 last paragraph] a quite interesting result (stated here in theorem2) which is actually neither clearly stated nor proved. This result provides an expression for each coefficient of the polynomial $\det(P-I-\lambda I)$ (P is a stochastic matrix) involving sums of products of p_{ij} 's (without any change in sign, as for the usual expansion which is a sum of minors). The aim of this paper is to prove it as a consequence of a particular expansion of determinants which is given in theorem1; this expansion is a sum of products of off-diagonal entries and negated row sums of the matrix (with no sign added).

The main tool used here is the W -graphs introduced in [1].

Definition [1]: *Let L be a finite set and let a subset W be selected in L . A graph on L is called a W -graph if it satisfies the following conditions:*

- (1) every point $m \in L \setminus W$ is the initial point of exactly one arrow, and any arrow has its initial point in $L \setminus W$.*
- (2) there are no closed cycles in the graph.*

Note that (2) may be replaced by

- (2') every point $m \in L \setminus W$ is the initial point of a sequence of arrows leading to some point $n \in W$.*

These W -graphs may be seen as disjoint unions of directed trees on L with roots in W .

Notations:

The set of W -graphs will be denoted by $G(W)$.

Suppose that we are given a set of numbers p_{ij} ($i, j \in L$), then for any graph g on L we define the number $\pi(g)$ by:

$$(1) \quad \pi(g) = \prod_{(m \rightarrow n) \in g} p_{m \ n}$$

$$\pi(\text{empty graph}) = 1.$$

For any subset W of L , we put;

$$(2) \quad \sigma(W) = \sum_{g \in G(W)} \pi(g)$$

In particular, $\sigma(L) = 1$.

Theorem1: Consider a $n \times n$ matrix $A=(a_{ij})$ with row sums $r_i = \sum_{j=1}^n a_{ij}$ and

define

$L=\{1,2,\dots,n+1\}$ and

$$(3) \quad \begin{aligned} p_{ij} &= a_{ij} & 1 \leq i,j \leq n \\ p_{i,n+1} &= -r_i & 1 \leq i \leq n. \end{aligned}$$

Then

$$(4) \quad \det(M) = (-1)^n \sigma(\{n+1\})$$

where σ is defined as above.

An easy consequence will be

theorem2: Consider a $n \times n$ matrix $P=(p_{ij})$ with constant row sums $r_i=r$, then its characteristic polynomial has the form:

$$(5) \quad P(\lambda) = \sum_{i=1}^n \sigma_i (\lambda-r)^i$$

where

$$(6) \quad \sigma_i = \sum_{|W|=i} \sigma(W).$$

Remark: Note that this last result applies also to matrices M with different row sums, by considering the matrix M' obtained by adding to M one column containing the negated row sums and one zero row.

Proof of theorem1;

Consider the function Δ defined by

$$\Delta(A) = (-1)^n \sigma(\{n+1\}).$$

We have to show that $\Delta(A) = \det(A)$. This will be done by proving some properties of the function Δ .

Property1: If A has a zero column, then $\Delta(A) = 0$.

Denote by m the index of the zero column and put

$$L = \{1, 2, \dots, n+1\}, W = \{n+1\}, \text{ and } L' = L \setminus \{m\}$$

G = set of W -graphs on L

G' = set of W -graphs on L' .

Note that, because of the zero-column property of A , any graph g of G satisfying $\pi(g) \neq 0$ will not have any arrow leading to m , so that g can be described as a graph g' of G' to which has been added an arrow leading from m to any other point $i \in L'$; we call this graph $g(g', i)$. Because all these graphs are distinct (for distinct g' or i) we get:

$$\Delta(A) = \sum_{g \in G(W)} \pi(g) = \sum_{\substack{g' \in G'(W) \\ i \in L'}} \pi(g(g', i)) = \sum_{g' \in G'(W)} \pi(g') \sum_{i \in L'} p_{mi} = 0.$$

The last equality follows from the definition of the p_{ij} 's.

This ends The proof of property1.

Property2: The application Δ is invariant under permutation of indices of the matrix (that is by successive permutation of rows and corresponding columns).

This property is obvious.

Property3: If A is a block-diagonal matrix $A = \text{diag}(A_1, \dots, A_p)$, then $\Delta(A) = \Delta(A_1) \dots \Delta(A_p)$.

This has only to be proved for $p=2$ (for larger p , one can use a recursion). Denote by m the size of the matrix A_1 and let

$$L' = \{1, 2, \dots, m, n+1\}, L'' = \{m+1, \dots, n, n+1\}, W = \{n+1\},$$

G' = set of W -graphs on L' ,

G'' = set of W -graphs on L'' .

Then, by the same reasoning as in property1, any graph g of G such that $\pi(g) \neq 0$ is constructed as the union of two graphs $g' \in G'$ and $g'' \in G''$ with the point $n+1$ in common and we obtain

$$\Delta(A) = (-1)^n \sum_{g \in G(W)} \pi(g) = (-1)^m (-1)^{n-m} \sum_{\substack{g' \in G'(W) \\ g'' \in G''(W)}} \pi(g') \pi(g'') = \Delta(A_1)$$

$\Delta(A_2)$.

Property4: If two matrices A_1 and A_2 are the same, except for one row, then $\Delta(A_1 + A_2) = \Delta(A_1) + \Delta(A_2)$.

This comes from the fact that, for any W -graph g , this additivity property is satisfied by $\pi(g)$.

End of the proof of theorem2:

Property4 implies that if we want to prove that $\Delta(A) = \det(A)$ for any matrix M , we have only to check this for matrices having one non-zero entry on each row.

By virtue of property1, this is true if there exists a zero-column. If there is not any zero-column, then there is exactly one non-zero entry in each row and in each column, and there exists a permutation of indices which transform A into $\text{diag}(A_1, \dots, A_p)$ for some matrices $A_1, \dots, A_p$, where the non-zero entries of A_i occur only

in positions immediately above the diagonal and in the lower-left corner, or A_i is a 1×1 matrix; i.e., A_i has the form (we give the picture for a 4×4 matrix):

$$\begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{pmatrix}$$

$a \neq 0, b \neq 0, c \neq 0, d \neq 0$.

If $p > 1$, we get the result by induction. The problem is then reduced to the case $p=1$ and $n > 1$, i.e. to the study of $\Delta(A)$ when the non-zero entries of A are above the diagonal and in the lower-left corner. In that case, the W -graphs g for which $\pi(g) \neq 0$ are described by the following property:

for any $1 \leq i \leq n$, the arrow starting from i leads to $i+1$ (1 if $i=n$) or $n+1$.

Such a graph is exactly determined by the arrows leading to $n+1$. For any $1 \leq k \leq n$, there exist exactly $\binom{n}{k}$ W -graphs having k arrows leading to $n+1$, and all these graphs g satisfy $\pi(g) = (-1)^k \pi_0$, where π_0 is the product of the non-zero entries of A . Finally we obtain

$$\Delta(A) = (-1)^n \sum_{k=1}^n \binom{n}{k} (-1)^k \pi_0 = (-1)^{n+1} \pi_0 = \det(A).$$

This ends the proof of theorem1.

Proof of theorem2: Apply theorem1 to $P - \lambda I$. The row sums of this matrix are all equal to $r - \lambda$. Note that there exists a one-to-one map between the set of $\{n+1\}$ -graphs on $\{1, 2, \dots, n+1\}$ and the set of all W -graphs, W non-empty, on $\{1, 2, \dots, n\}$. This map associates with any $\{n+1\}$ -graph g on the set $\{1, 2, \dots, n+1\}$ the graph g' obtained by deleting the point $n+1$ and all arrows leading to it. Note that if g has k arrows leading to $n+1$, then, using (1) and (3), we obtain:

$$\pi(g) = (\lambda-r)^k \pi(g').$$

This equality, inserted in (2) and (4) leads to the result.

References

- [1] M.I.Friedlin, A.D.Wentzell,"Random Pertubations of Dynamical Systems", Springer-Verlag, 1984.