ARMY RESEARCH LABORATORY

Classical Method for Deriving the Electromagnetic Propagation Equations for Double Negative Materials With Application for Antenna Design

by Ira Kohlberg

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prepared by

Kohlberg Associates, Inc.
11308 South Shore Road
Reston, VA 20190

under contract

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14. ABSTRACT

We derive a system of propagation equations in a Double Negative (DN) material in a way that appears to differ from previous derivations—although the end result is the same. Our derivation assumes the Poynting vector theorem applies, real materials always have some loss, $\varepsilon(\omega)$ and $\mu(\omega)$ are obtained from real materials, and wave energy traveling in a specified direction must always be accompanied by a loss of energy in that direction. Additional mathematics beyond Maxwell’s equation is not required. Energy losses per unit length of travel are finite, and can be extremely small. Propagation in a lossless DN media is found as the mathematical limiting solution of an extremely small energy loss per unit length. When developed along these principles, the equations developed for designing leaky antennas are straightforward.

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1. Introduction

Understanding and using Double Negative (DN) materials—negative $\varepsilon(\omega)$ and $\mu(\omega)$ for some frequencies, in the design of devices using electromagnetic waves remains a challenge. This topic has become of great interest since these man-made materials became popular this decade. Many of us had difficulties understanding propagation because we had not experienced DN materials before and we had to except the idea of a negative phase velocity as a consequence of our equations. The concept of a positive phase velocity appears to have always been sacrosanct. It's interesting to note, however, that scientists could accept the idea of a phase velocity going faster then the speed of light because invariably a real physical situation associated with this occurrence could be found.

Here we derive the propagation equations in DN materials in a way that differs from previous derivations. We propose that, for any real material, the energy of the wave must decrease as it moves from its source of energy. Energy losses per unit length of travel are finite, and can be extremely small. Propagation in a lossless DN media is found as the mathematical limiting solution of an extremely small energy loss per unit length for real $\varepsilon(\omega)$ and $\mu(\omega)$. We end up with the same conclusions as the other authors using nothing more than an algebraic solution to Maxwell's equations without the need to even mention phase velocity; negative phase velocity is a consequence of our solution.

Why was it necessary to develop this derivation when the result was already available? We've gone through the trouble because we plan to examine complex multi-dimensional structures and do not want to confront potential ambiguities about which signs to choose in taking square roots, etc. Having a guiding principle to go by is a big help. In this report we apply the basic principles to reflection and refraction of an electromagnetic waves between a material having positive $\varepsilon(\omega)$ and $\mu(\omega)$, and one having negative $\varepsilon(\omega)$ and $\mu(\omega)$.

In the next section we develop the propagations using the principles just mentioned. Use of the equations to solve the aforementioned reflection/refraction problem is rendered in section 3. Concluding remarks are rendered in section 4 and related references are listed in section 5.

2. Electromagnetic Propagation Model for DN Materials

For a DN material Maxwell's equations are

$$\nabla \bullet \vec{B} = 0 \tag{1}$$

$$\nabla \bullet \vec{D} = 0 \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{4}$$

Using the formulas

$$\nabla \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \vec{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \vec{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \vec{a}_z \tag{5}$$

$$\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \vec{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \vec{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \vec{a}_z \tag{6}$$

we write equations 5 and 6 in component form

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) = -\frac{\partial B_x}{\partial t} \tag{5a}$$

$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) = -\frac{\partial B_y}{\partial t} \tag{5b}$$

$$\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = -\frac{\partial B_z}{\partial t} \tag{5c}$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) = J_x + \frac{\partial D_x}{\partial t} \tag{6a}$$

$$\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) = J_y + \frac{\partial D_y}{\partial t} \tag{6b}$$

$$\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) = J_z + \frac{\partial D_z}{\partial t} \tag{6c}$$

The constitutive equations are

$$\vec{D}(t) = \varepsilon_0 \vec{E}(t) + \vec{P}(t) \, , \tag{7}$$

$$\vec{B} = \mu_0 (\vec{H}(t) + \vec{M}(t)) = \mu_0 \vec{H}(t) + \mu_0 \vec{M}(t) \, , \tag{8}$$

A dynamic relationship is assumed between the polarization, $\vec{P}(t)$, and $\vec{E}(t)$, and between the magnetization, $\vec{M}(t)$, and $\vec{H}(t)$. An example of dynamic equations for the polarization and magnetization is the set provided by Smith (*16*):

$$\frac{d^2\vec{P}}{dt^2} + \Gamma_E \frac{d\vec{P}}{dt} + \omega_{e0}^2 \vec{P} = \varepsilon_0 \omega_{ep}^2 \vec{E} \tag{9}$$

$$\frac{d^2\vec{M}}{dt^2} + \Gamma_H \frac{d\vec{M}}{dt} + \omega_{m0}^2 \vec{M} = \omega_{mp}^2 \vec{H} \tag{10}$$

The system of equations is solved with the formula $(/\partial t) = j\omega$ and using the transform pair

$$\vec{F}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(j\omega t) \tilde{\vec{F}}(\omega) d\omega \tag{11}$$

$$\tilde{\vec{F}}(\omega) = \int_{-\infty}^{+\infty} \exp(-j\omega t) \vec{F}(t) dt \tag{12}$$

The function $\vec{F}(t)$ represents either $\vec{P}(t)$, $\vec{E}(t)$, $\vec{M}(t)$, $\vec{H}(t)$, or $\vec{J}$. We have

$$\tilde{\vec{P}} = \varepsilon_0 \chi_E \tilde{\vec{E}} \tag{13}$$

$$\tilde{\vec{D}} = \varepsilon_0 \tilde{\vec{E}} + \tilde{\vec{P}} = \varepsilon_0 (1 + \chi(\omega)) \tilde{\vec{E}} = \varepsilon_0 \varepsilon_r(\omega) \tilde{\vec{E}} = \varepsilon(\omega) \tilde{\vec{E}} \tag{14}$$

$$\varepsilon_r(\omega) = 1 + \chi(\omega) \tag{15}$$

$$\varepsilon(\omega) = \varepsilon_0 \varepsilon_r \tag{16}$$

For illustrative purposes we use the following form for $\chi_E(\omega)$, although this is never necessary in the calculations.

$$\chi_E(\omega) = \frac{\omega_{ep}^2}{\omega_{eo}^2 - \omega^2 + j\omega\Gamma_E} = \frac{\omega_{ep}^2 \left(\omega_{eo}^2 - \omega^2 - j\omega\Gamma_E\right)}{\left(\omega_{eo}^2 - \omega^2\right)^2 + \omega^2 \Gamma_E^2} \tag{17}$$

$$\chi_E(\omega) = \chi_E'(\omega) + j\chi_E''(\omega) \tag{18}$$

$$\chi_E'(\omega) = \frac{\omega_{ep}^2 \left(\omega_{eo}^2 - \omega^2\right)}{\left(\omega_{eo}^2 - \omega^2\right)^2 + \omega^2 \Gamma_E^2} \tag{19}$$

$$\chi_E''(\omega) = -\frac{\omega_{ep}^2 \omega \Gamma_E}{\left(\omega_{eo}^2 - \omega^2\right)^2 + \omega^2 \Gamma_E^2} \tag{20}$$

$$\tilde{\vec{M}} = \chi_H \tilde{\vec{H}} \tag{21}$$

$$\tilde{\vec{B}}(\omega) = \mu_0(1+\chi(\omega))\tilde{\vec{H}}(\omega) = \mu_0\mu_r(\omega)\tilde{\vec{H}}(\omega) = \mu(\omega)\tilde{\vec{H}}(\omega) \tag{22}$$

$$\mu_r = (1+\chi_H(\omega)) \tag{23}$$

$$\mu(\omega) = \mu_0\mu_r \tag{24}$$

$$\chi_H(\omega) = \frac{\omega_{mp}^2}{\omega_{mo}^2 - \omega^2 + j\omega\Gamma_H} = \frac{\omega_{mp}^2\left(\omega_{mo}^2 - \omega^2 - j\omega\Gamma_H\right)}{\left(\omega_{mo}^2 - \omega^2\right)^2 + j\omega^2\Gamma_H^2} \tag{25}$$

$$\chi_H(\omega) = \chi_H'(\omega) + j\chi_H''(\omega) \tag{26}$$

As done for $\chi_E(\omega)$, we use an illustrative expression for $\chi_H(\omega)$

$$\chi_H'(\omega) = \frac{\omega_{mp}^2\left(\omega_{mo}^2 - \omega^2\right)}{\left(\omega_{mo}^2 - \omega^2\right)^2 + \omega^2\Gamma_H^2} \tag{27}$$

$$\chi_H''(\omega) = -\frac{\omega_{mp}^2\omega\Gamma_H}{\left(\omega_{mo}^2 - \omega^2\right)^2 + \omega^2\Gamma_H^2} \tag{28}$$

In Fourier transform space equations 5 and 6 now become

$$\left(\frac{\partial\tilde{E}_z}{\partial y} - \frac{\partial\tilde{E}_y}{\partial z}\right) = -j\omega\mu(\omega)\tilde{H}_x \tag{29a}$$

$$\left(\frac{\partial\tilde{E}_x}{\partial z} - \frac{\partial\tilde{E}_z}{\partial x}\right) = -j\omega\mu(\omega)\tilde{H}_y \tag{29b}$$

$$\left(\frac{\partial\tilde{E}_y}{\partial x} - \frac{\partial\tilde{E}_x}{\partial y}\right) = -j\omega\mu(\omega)\tilde{H}_z \tag{29c}$$

$$\left(\frac{\partial\tilde{H}_z}{\partial y} - \frac{\partial\tilde{H}_y}{\partial z}\right) = \tilde{J}_x + j\omega\varepsilon(\omega)\tilde{E}_x \tag{30a}$$

$$\left(\frac{\partial\tilde{H}_x}{\partial z} - \frac{\partial\tilde{H}_z}{\partial x}\right) = \tilde{J}_y + j\omega\varepsilon(\omega)\tilde{E}_y \tag{30b}$$

$$\left(\frac{\partial\tilde{H}_y}{\partial x} - \frac{\partial\tilde{H}_x}{\partial y}\right) = \tilde{J}_z + j\omega\varepsilon(\omega)\tilde{E}_z \tag{30c}$$

Now let us consider propagation in the positive *z*-direction where there are no current sources. Let $\tilde{\vec{J}} = 0$ for $z > 0$. Thus for $z > 0$ the magnitudes of the fields must decrease as *z* increase for a lossy medium because all the sources of energy lie in the region $z \leq 0$.

For brevity we choose a TEM wave with $\tilde{E}_x$ and $\tilde{H}_y$. We also assume $\partial/\partial x = 0$ and $\partial/\partial y = 0$. Equations 29 and 30 then become

$$\frac{\partial \tilde{E}_x}{\partial z} = -j\omega\mu(\omega)\tilde{H}_y \quad (31)$$

$$-\frac{\partial \tilde{H}_y}{\partial z} = j\omega\varepsilon(\omega)\tilde{E}_x \quad (32)$$

$$\varepsilon(\omega) = \varepsilon_0\varepsilon_r = \varepsilon_0(1+\chi_E) = \varepsilon_0(\alpha_E(\omega) - j\beta_E(\omega)) \quad (33a)$$

$$\alpha_E(\omega) = 1 + \chi'_E(\omega) \quad (33b)$$

$$\beta_E(\omega) = -\chi''_E(\omega) \quad (33c)$$

$$\mu(\omega) = \mu_0\mu_r = \mu_0(1+\chi_H) = \mu_0(\alpha_H(\omega) - j\beta_H(\omega)) \quad (34a)$$

$$\alpha_H(\omega) = 1 + \chi'_H(\omega) \quad (34b)$$

$$\beta_H(\omega) = -\chi''_H(\omega) \quad (34c)$$

We seek solutions of the form

$$\tilde{E}_x = \tilde{E}\exp(-jkz) \quad (35)$$

$$\tilde{H}_y = \tilde{H}\exp(-jkz) \quad (36)$$

and substitute these expressions into equations 31 and 32. We get $\partial/\partial z = -jk$ and

$$-jk\tilde{E} = -j\omega\mu(\omega)\tilde{H} \quad (37)$$

$$jk\tilde{H} = j\omega\varepsilon(\omega)\tilde{E} \quad (38)$$

A solution occurs only when

$$k^2 = \omega^2\varepsilon(\omega)\mu(\omega) \quad (39)$$

All of the equations presented so far have been derived before in various forms in most books in electromagnetic wave propagation. Solutions for k have been addressed for three cases: ε and μ are both positive, ε is positive and μ is negative, and ε is negative and μ is positive. We are now considering the case where both ε and μ are both negative. We show that the solution is a straightforward extension of the three previous cases to this new regime, and needs no additional mathematical or physical assumptions other than using the fact that propagation material is lossy, albeit extremely minute.

Before going further, a few comments about using the foregoing set of equations is appropriate. First, we are dealing with casual functions, whose mathematical properties are well known. Being casual we need only deal with the $\omega > 0$ part of the complex plane. From equations 20, 28, 33c, and 34c we then see that $\beta_E(\omega)$ and $\beta_H(\omega)$ are both positive. In addition, we also notice that $\alpha_E(\omega)$ and $\alpha_H(\omega)$ are even functions of ω, namely they are functions of ω^2. These are important properties in the analysis.

Also, even though our analysis uses specific forms for $\alpha_E(\omega)$, $\alpha_H(\omega)$, $\beta_E(\omega)$, and $\beta_H(\omega)$, the aforementioned symmetries of the functions are general. They apply to any casual function.

Using equations 33a and 34a in equation 39 we get

$$k^2 = k_0^2(\alpha_E - j\beta_E)(\alpha_H - j\beta_H) = k_0^2(\alpha_E\alpha_H - \beta_E\beta_H) - jk_0^2(\alpha_E\beta_H + \alpha_H\beta_E) \qquad (40)$$

$$k_0^2 = \omega^2\varepsilon_0\mu_0 = \frac{\omega^2}{c^2} \qquad (41)$$

where $c = (1/\sqrt{\varepsilon_0\mu_0})$ is the velocity of light in vacuum. The right hand side of equation 40 is a complex number and therefore k will be a complex number. Let us write k as

$$k = k_0(\eta + j\xi) \qquad (42)$$

where

$$\eta + j\xi = \pm\left((\alpha_E\alpha_H - \beta_E\beta_H) - j(\alpha_E\beta_H + \alpha_H\beta_E)\right)^{1/2} \qquad (43)$$

When equation 42 is used in equations 35 and 36 we get

$$\tilde{E}_x = \tilde{E}\exp(-jkz) = \tilde{E}\exp(-jk_0\eta\, z)\exp(k_0\xi\, z) \qquad (44)$$

$$\tilde{H}_y = \tilde{H}\exp(-jkz) = \tilde{H}\exp(-jk_0\eta\, z)\exp(k_0\xi\, z) \qquad (45)$$

If the fields are to diminish as z increases, then ξ must be negative. This is how the + or – sign is chosen in equation 43.

For orientation let us consider the traditional case first. This is the one where the α-terms are *positive* and are much larger than the β terms. Equation 43 is then approximated by the equation

$$\eta + j\xi \cong \pm\left(\alpha_E\alpha_H - j(\alpha_E\beta_H + \alpha_H\beta_E)\right)^{1/2} = \pm(\alpha_E\alpha_H)^{1/2}\left(1 - j\left(\frac{\beta_H}{\alpha_H} + \frac{\beta_E}{\alpha_E}\right)\right)^{1/2} \qquad (46)$$

Since the (β/α)-terms are each <<1 we can use the approximation

$$\left(1 - j\left(\frac{\beta_H}{\alpha_H} + \frac{\beta_E}{\alpha_E}\right)\right)^{1/2} \cong 1 - \frac{j}{2}\left(\frac{\beta_H}{\alpha_H} + \frac{\beta_E}{\alpha_E}\right) \qquad (47)$$

Equation 46 becomes

$$\eta + j\xi \cong \pm(\alpha_E\alpha_H)^{1/2}\left(1-\frac{j}{2}\left(\frac{\beta_H}{\alpha_H}+\frac{\beta_E}{\alpha_E}\right)\right) = \pm\left((\alpha_E\alpha_H)^{1/2}-\frac{j}{2}\left(\frac{\alpha_E^{1/2}\beta_H}{\alpha_H^{1/2}}+\frac{\alpha_H^{1/2}\beta_E}{\alpha_E^{1/2}}\right)\right) \quad (48)$$

If the "+" is used for the right hand side of equation 48, then ξ will be negative and satisfy the conditions of equations 44 and 45. This is therefore the correct decision. We then get

$$\eta^{(+)} = (\alpha_E\alpha_H)^{1/2} \quad (49)$$

$$\xi^{(+)} = -\frac{1}{2}\left(\frac{\alpha_E^{1/2}\beta_H}{\alpha_H^{1/2}}+\frac{\alpha_H^{1/2}\beta_E}{\alpha_E^{1/2}}\right) \quad (50)$$

Because $\alpha_E(\omega)$, $\alpha_H(\omega)$, $\beta_E(\omega)$, and $\beta_H(\omega)$ are all positive, $\eta^{(+)}$ is positive and $\xi^{(+)}$ is negative. Equations 49 and 50 are what we're used to seeing. As we shall show, this result leads to the notion of a positive phase velocity in the same direction of energy flow.

When equations 49 and 50 are inserted into equations 44 and 45 we immediately notice that the phase part, $\exp(-jk_0\eta z)$, is

$$\exp(-jk_0\eta^{(+)}z) = \exp\left(-jk_0(\alpha_E\alpha_H)^{1/2}z\right) = \exp\left(\frac{-j\omega z}{v_{ph}^{(+)}}\right) \quad (51)$$

Since the time dependence is $\exp(j\omega t)$ we see that equation 51 defines a frequency dependent phase velocity,

$$v_{ph}^{(+)} = \frac{c}{\eta^{(+)}} = \frac{c}{(\alpha_E\alpha_H)^{1/2}} \quad (52)$$

The attenuation is given by $\exp(k_0\xi z)$ which is

$$\exp(k_0\xi^{(+)}z) = \exp\left[-\frac{k_0 z}{2}\left(\frac{\alpha_E^{1/2}\beta_H}{\alpha_H^{1/2}}+\frac{\alpha_H^{1/2}\beta_E}{\alpha_E^{1/2}}\right)\right] \quad (53)$$

The foregoing system of equations that we have just analyzed is called a right-hand (RH) system. In our construct it is comprised of the following three vectors:

$$\vec{\tilde{E}} = \tilde{E}\vec{a}_x \quad (54a)$$

$$\vec{\tilde{H}} = \tilde{H}\vec{a}_y \quad (54b)$$

$$\vec{\tilde{K}} = k_0\eta^{(+)}\vec{a}_z \quad (54c)$$

In the foregoing equation $\vec{a}_x, \vec{a}_y$, and $\vec{a}_z$ are the unit vectors in the *x*-, *y*-, and *z*-directions. Since $\eta^{(+)}$ is positive the three vectors in equation 54 follow the RH rule, that is

$$\vec{\tilde{E}} \times \vec{\tilde{H}} = \tilde{E}\tilde{H}\vec{a}_z \tag{55a}$$

$$\vec{\tilde{K}} \times \vec{\tilde{E}} = k_0 \eta^{(+)} \tilde{E}\vec{a}_y \tag{55b}$$

$$\vec{\tilde{H}} \times \vec{\tilde{K}} = \tilde{H} k_0 \eta^{(+)} \vec{a}_x \tag{55c}$$

The RH rule is natural for cases where the phase velocity points in the positive *z*-direction. This is also physically comforting since the root mean square Poynting vector, $\vec{S}_{rms}$ also point in the positive *z*-direction. Using equations 45 and 46 we have

$$\vec{S}_{rms} = \frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}^*) = \frac{1}{2}\vec{a}_z\ \mathrm{Re}(\tilde{E}\tilde{H}^*)\exp(2k_0\xi^{(+)}z) \tag{56}$$

Using equation 50 we write

$$\vec{S}_{rms} = \frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}^*) = \frac{1}{2}\vec{a}_z\ \mathrm{Re}(\tilde{E}\tilde{H}^*)\exp(-k_0\gamma^{(+)}z) \tag{57}$$

$$\xi^{(+)} = -\frac{1}{2}\gamma^{(+)} \tag{58a}$$

$$\gamma^{(+)} = \left(\frac{\alpha_E^{1/2}\beta_H}{\alpha_H^{1/2}} + \frac{\alpha_H^{1/2}\beta_E}{\alpha_E^{1/2}} \right) \tag{58b}$$

The last remaining step to calculate $\vec{S}_{rms}$ is to compute $\mathrm{Re}(\tilde{E}\tilde{H}^*)$ from either equation 37 and 38. Equation 38 gives

$$\tilde{H} = \frac{\omega\varepsilon(\omega)}{k}\tilde{E} \tag{59}$$

Using equation 33a for $\varepsilon(\omega)$ and equation 42 for k(ω) we get

$$\tilde{H} = \frac{\omega\varepsilon_0\left(\alpha_E(\omega) - j\beta_E(\omega)\right)}{k_0\left(\eta^{(+)}(\omega) + j\xi^{(+)}(\omega)\right)}\tilde{E} = \frac{\left(\alpha_E - j\beta_E\right)}{Z_0\left((\alpha_E\alpha_H)^{1/2} - \frac{1}{2}j\gamma^{(+)}\right)}\tilde{E} = \frac{\tilde{E}}{gZ_0} = \frac{\tilde{E}}{Z} \tag{60}$$

In the foregoing equation $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the impedance of free space,

$$g^{(+)} = \frac{(\alpha_E\alpha_H)^{1/2} - \frac{1}{2}j\gamma^{(+)}}{\alpha_E - j\beta_E} \equiv P^{(+)}\exp(j\Psi^{(+)}), \tag{61}$$

$P^{(+)}$ and $\Psi^{(+)}$ are the amplitude and phase of $g^{(+)}$ respectively, and $Z^{(+)} = g^{(+)}Z_0$ is the wave impedance. Since all the parameters in g are positive the phase, $\Psi^{(+)}$, is less than $\pi/2$. The real part of $g^{(+)}$ is positive.

Thus,

$$\mathrm{Re}(\tilde{E}\tilde{H}^*) = \frac{\left|\tilde{E}\right|^2 \cos\Psi^{(+)}}{P^{(+)} Z_0} \tag{62}$$

$$\vec{S}_{rms} = \vec{a}_z \frac{\left[\tilde{E}\right]^2 \cos\Psi^{(+)}}{2P^{(+)} Z_0} \exp(-k_0 \gamma^{(+)} z) \tag{63}$$

As we see from equation 55 the three vectors: $\vec{E}, \vec{H}$ and $\vec{S}_{rms}$ form a RH system since $\vec{S}_{rms}$ points in the positive z direction.

Now, let us see what happens at the other extreme, when both α_E and α_H are negative. This happens for the set of radian frequencies, $\{\omega\}$, that satisfies the conditions

$$\alpha_E(\omega) = 1 + \chi'_E(\hat{\omega}) < 0 \tag{64}$$

$$\alpha_H(\omega) = 1 + \chi'_H(\hat{\omega}) < 0 \tag{65}$$

We write

$$\alpha_E = -\left|\alpha_E\right| \tag{66}$$

$$\alpha_H = -\left|\alpha_H\right| \tag{67}$$

$$\eta + j\xi = \pm\left((\left|\alpha_E\right|\left|\alpha_H\right| - \beta_E\beta_H) + j(\left|\alpha_E\right|\beta_H + \left|\alpha_H\right|\beta_E)\right)^{1/2} \tag{68}$$

Let's first evaluate equation 68 when the β-terms are much smaller than the α-terms. This is what we did before and is the usual case of interest. We now get

$$\eta + j\xi \cong \pm\left((\left|\alpha_E\right|^{1/2}\left|\alpha_H\right|^{1/2}) + \frac{j}{2}\left(\frac{\left|\alpha_E\right|^{1/2}\beta_H}{\left|\alpha_H\right|^{1/2}} + \frac{\left|\alpha_H\right|^{1/2}\beta_E}{\left|\alpha_E\right|^{1/2}}\right)\right) \tag{69}$$

Again, imposing the condition that ξ be negative to ensure energy decay in the positive z, we now require that we take the negative sign of the right hand sign of equation 69. In lieu of equations 49 and 50 we now have

$$\eta^{(-)} = -(\left|\alpha_E\right|\left|\alpha_H\right|)^{1/2} \tag{70}$$

$$\xi^{(-)} = -\frac{1}{2}\left(\frac{\left[\alpha_E\right]^{1/2}\beta_H}{\left[\alpha_H\right]^{1/2}} + \frac{\left[\alpha_H\right]^{1/2}\beta_E}{\left[\alpha_E\right]^{1/2}}\right) = -\frac{1}{2}\gamma^{(-)} \tag{71a}$$

$$\gamma^{(-)} = \left(\frac{\left[\alpha_E\right]^{1/2}\beta_H}{\left[\alpha_H\right]^{1/2}} + \frac{\left[\alpha_H\right]^{1/2}\beta_E}{\left[\alpha_E\right]^{1/2}}\right) \tag{71b}$$

Notice that $\xi^{(-)}$ and $\xi^{(+)}$ are both negative: compare equation 71 with 58.

When equations 70 and 71 are inserted into equations 44 and 45 we immediately notice that the phase part, $\exp(-jk_0\eta z)$, is

$$\exp(-jk_0\eta^{(-)}z) = \exp\left(jk_0(|\alpha_E||\alpha_H|)^{1/2}z\right) = \exp\left(\frac{-j\omega z}{v_{ph}^{(-)}}\right) \tag{72}$$

Since the time dependence is $\exp(j\omega t)$ we see that equation 72 defines a negative frequency dependent phase velocity,

$$v_{ph}^{(-)} = -\frac{c}{(|\alpha_E||\alpha_H|)^{1/2}} \tag{73}$$

The attenuation is given by

$$\exp(k_0\xi^{(-)}z) = \exp\left[-\frac{k_0 z}{2}\left(\frac{|\alpha_E|^{1/2}\beta_H}{|\alpha_H|^{1/2}} + \frac{|\alpha_H|^{1/2}\beta_E}{|\alpha_E|^{1/2}}\right)\right] = \exp\left[-\frac{k_0 z}{2}\gamma^{(-)}\right] \tag{74}$$

As before we have

$$\tilde{H} = \frac{\omega\varepsilon(\omega)}{k}\tilde{E} = \frac{\omega\varepsilon(\omega)}{k_0(\eta^{(-)} + j\xi^{(-)})}\tilde{E}, \tag{75}$$

but because of equation 64 this time we need to use the expression

$$\varepsilon(\omega) = \varepsilon_0(\alpha_E(\omega) - j\beta_E(\omega)) = -\varepsilon_0(|\alpha_E| + j\beta_E) \tag{76}$$

From equations 70 and 71 we have

$$\eta^{(-)} + j\xi^{(-)} = -\left((|\alpha_E||\alpha_H|)^{1/2} + \frac{j}{2}\gamma^{(-)}\right) \tag{77}$$

Inserting equations 76 and 77 into equation 75 and using previous definitions gives

$$\tilde{H} = \frac{\tilde{E}}{Z^{(-)}} \tag{78}$$

$$Z^{(-)} = g^{(-)}Z_0 \tag{79}$$

$$g^{(-)} = \frac{(|\alpha_E||\alpha_H|)^{1/2} + \frac{1}{2}j\gamma^{(-)}}{|\alpha_E| + j\beta_E} \equiv P^{(-)}\exp(j\Psi^{(-)}) \tag{80}$$

$$\vec{S}_{rms} = \vec{a}_z\frac{[\tilde{E}]^2\cos\Psi^{(-)}}{2Z_0P^{(-)}}\exp(-k_0\gamma^{(-)}z) \tag{81}$$

Again we see from equation 81 that the Poynting vector points in the positive z-direction, and therefore, $\vec{S}_{rms}, \vec{E}$, and $\vec{H}$ form a right handed system. The fact that this situation is accompanied with a negative phase does not violate any law. Both cases, the double positive and double negative α_E and α_H are described within the same mathematical framework for a plane wave traveling in any direction.

Let $\vec{\Omega}_E$ be the direction of the electric field and $\vec{\Omega}_H$ be the direction of the magnetic field. We have

$$\vec{\Omega}_E \bullet \vec{\Omega}_H = 0 \tag{82}$$

$$\vec{\Omega}_S = \vec{\Omega}_E \times \vec{\Omega}_H \tag{83}$$

$$\vec{\Omega}_S \bullet \vec{\Omega}_E = \vec{\Omega}_S \bullet \vec{\Omega}_H = 0 \tag{84}$$

$$\tilde{\vec{E}} = \tilde{E}_0 \vec{\Omega}_E \exp(-jk_0 \eta \vec{\Omega}_S \bullet \vec{r}) \exp(-k_0 \gamma \vec{\Omega}_S \bullet \vec{r}) \tag{85}$$

$$\tilde{\vec{H}} = \frac{\tilde{E}_0}{Z} \vec{\Omega}_H \exp(-jk_0 \eta \vec{\Omega}_S \bullet \vec{r}) \exp(-k_0 \gamma \vec{\Omega}_S \bullet \vec{r}) \tag{86}$$

For double positive materials use

$$\eta^{(+)} = (\alpha_E \alpha_H)^{1/2} \ , \quad \gamma^{(+)} = \left(\frac{\alpha_E^{1/2} \beta_H}{\alpha_H^{1/2}} + \frac{\alpha_H^{1/2} \beta_E}{\alpha_E^{1/2}} \right), \quad Z = Z^{(+)} \tag{87}$$

and for double negative materials use

$$\eta^{(-)} = -(|\alpha_E||\alpha_H|)^{1/2}, \quad \gamma^{(-)} = \left(\frac{[\alpha_E]^{1/2} \beta_H}{[\alpha_H]^{1/2}} + \frac{[\alpha_H]^{1/2} \beta_E}{[\alpha_E]^{1/2}} \right), \qquad Z = Z^{(-)} \tag{88}$$

---

3. Reflection and Refraction Involving a DN Material

In this section we demonstrate the unique features of reflection and refraction involving a DN material. In the three cases considered in figures 1 through 3 the electric field is in the plane of incidence. The behavior of figure 1 is well known; we include this for the reader’s orientation. We demonstrate the results shown in figures 2 and 3 from the solution of the equations.

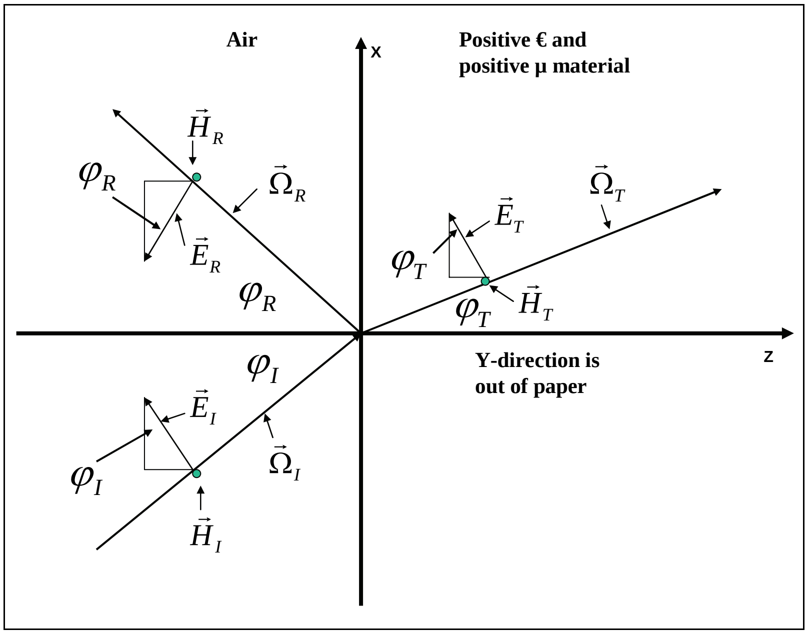

Figure 1. Reflection in air off a positive ε and positive μ material.

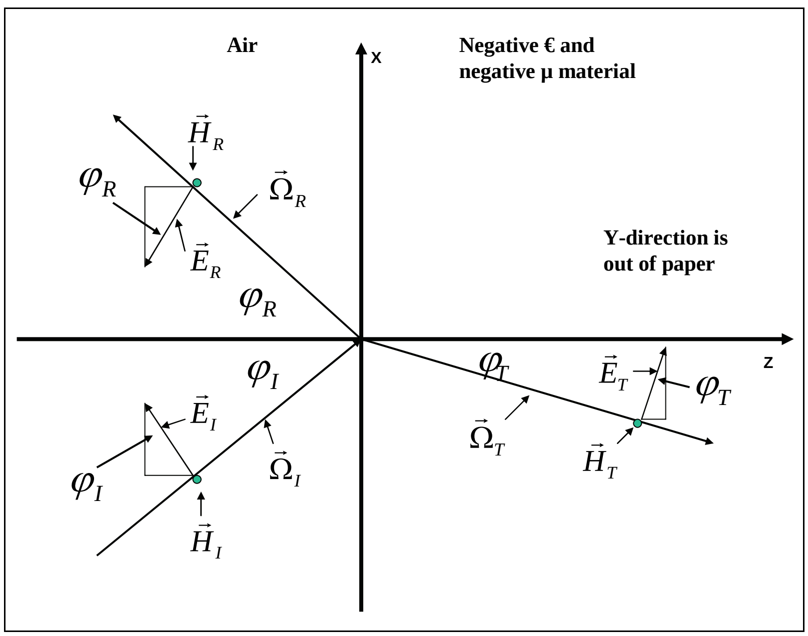

Figure 2. Reflection in air off a negative ε and negative μ material.

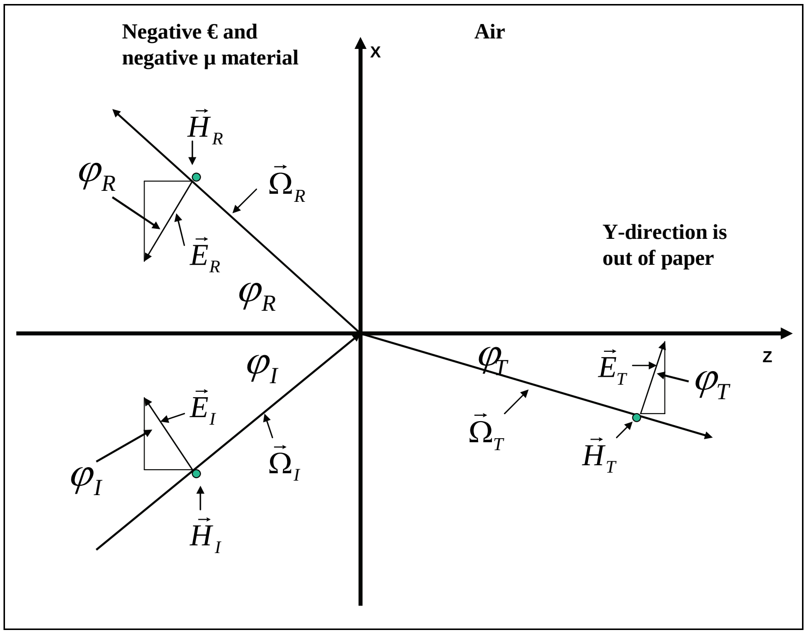

Figure 3. Reflection in a negative ε and negative μ material off an air interface.

In all cases considered in figures 1 through 3 equations 29 and 30 apply. They are simplified according to equation 89.

$$\left(\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}\right) = -j\omega\mu(\omega)\tilde{H}_x = 0 \tag{89a}$$

$$\left(\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x}\right) = -j\omega\mu(\omega)\tilde{H}_y \tag{89b}$$

$$\left(\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}\right) = -j\omega\mu(\omega)\tilde{H}_z = 0 \tag{89c}$$

$$\left(\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z}\right) = -\frac{\partial \tilde{H}_y}{\partial z} = j\omega\varepsilon(\omega)\tilde{E}_x \tag{89d}$$

$$\left(\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x}\right) = j\omega\varepsilon(\omega)\tilde{E}_y = 0 \tag{89e}$$

$$\left(\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y}\right) = \frac{\partial \tilde{H}_y}{\partial x} = j\omega\varepsilon(\omega)\tilde{E}_z \tag{89f}$$

$$-\frac{\partial^2 \tilde{H}_y}{\partial z^2} = j\omega\varepsilon(\omega)\frac{\partial \tilde{E}_x}{\partial z} \tag{90a}$$

$$\frac{\partial^2 \tilde{H}_y}{\partial x^2} = j\omega\varepsilon(\omega)\frac{\partial \tilde{E}_z}{\partial x} \tag{90b}$$

Inserting equations 90a and 90b into equation 89b gives

$$\frac{\partial \tilde{H}_y}{\partial z^2} + \frac{\partial \tilde{H}_y}{\partial x^2} = -\omega^2 \mu(\omega)\varepsilon(\omega)\tilde{H}_y \tag{90c}$$

For the case where $\mu(\omega)$ and $\varepsilon(\omega)$ we get the familiar results

$$\tilde{H}_y = A\exp\left[-j\left(k_x x + k_z z\right)\right] = A\exp-\frac{\omega}{v}(\vec{\Omega}\bullet\vec{r}) \tag{91}$$

$$\tilde{E}_x = \frac{k_z}{\omega\varepsilon}\tilde{H}_y \tag{92}$$

$$\tilde{E}_z = -\frac{k_x}{\omega\varepsilon}\tilde{H}_y \tag{93}$$

$$\vec{r} = x\vec{a}_x + z\vec{a}_z \tag{94}$$

$$\vec{\Omega} = \sin\varphi\vec{a}_x + \cos\varphi\vec{a}_z \tag{95}$$

$$v = \frac{1}{\sqrt{\varepsilon\mu}} \tag{96}$$

In the foregoing equations $\vec{a}_x$ and $\vec{a}_z$ are unit vectors in the *x*- and *z*- directions respectively, $\vec{\Omega}$ is the direction of propagation, $\sin\varphi = \vec{\Omega}\bullet\vec{a}_x$, and $\cos\varphi = \vec{\Omega}\bullet\vec{a}_z$. For a wave traveling in the positive *x*- and positive *z*-direction we have

$$k_x = \frac{\omega}{v}\sin\varphi \tag{97}$$

$$k_z = \frac{\omega}{v}\cos\varphi \tag{98}$$

$$\tilde{E}_x = \frac{k_z}{\omega\varepsilon}\tilde{H}_y = Z\cos\varphi\tilde{H}_y \tag{99}$$

$$\tilde{E}_z = -\frac{k_x}{\omega\varepsilon}\tilde{H}_y = -Z\sin\varphi\tilde{H}_y \tag{100}$$

$$Z = \sqrt{\frac{\mu}{\varepsilon}} \tag{101}$$

Solution for reflection in air off a positive ε and positive μ material

From figure 1 we have the following

$$\vec{\Omega}_I \bullet \vec{r} = x\sin\varphi_I + z\cos\varphi_I \tag{102a}$$

$$\vec{\Omega}_R \bullet \vec{r} = x\sin\varphi_R - z\cos\varphi_R \tag{102b}$$

$$\vec{\Omega}_T \bullet \vec{r} = x\sin\varphi_T + z\cos\varphi_T \tag{102c}$$

Using

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon}\frac{\partial\tilde{H}_y}{\partial x} \tag{103a}$$

$$\tilde{E}_x = \frac{-1}{j\omega\varepsilon}\frac{\partial\tilde{H}_y}{\partial z} \tag{103b}$$

$$\tilde{H}_{y,I} = H\exp-\frac{\omega}{v_0}j(\vec{\Omega}_I \bullet \vec{r}) = H\exp-\frac{\omega}{v_0}j(x\sin\varphi_I + z\cos\varphi_I) \tag{104a}$$

$$\tilde{E}_{x,I} = Z_0\cos\varphi_I H\exp-\frac{\omega}{v_0}j(x\sin\varphi_I + z\cos\varphi_I) \tag{104b}$$

$$\tilde{E}_{z,I} = -Z_0\sin\varphi_I H\exp-\frac{\omega}{v_0}j(x\sin\varphi_I + z\cos\varphi_I) \tag{104c}$$

$$\tilde{H}_{y,R} = A\exp-\frac{\omega}{v_0}j(\vec{\Omega}_R \bullet \vec{r}) = A\exp-\frac{\omega}{v_0}j(x\sin\varphi_R - z\cos\varphi_R) \tag{105a}$$

$$\tilde{E}_{x,R} = -Z_0\cos\varphi_R A\exp-\frac{\omega}{v_0}j(x\sin\varphi_R - z\cos\varphi_R) \tag{105b}$$

$$\tilde{E}_{z,R} = -Z_0\sin\varphi_R A\exp-\frac{\omega}{v_0}j(x\sin\varphi_R - z\cos\varphi_R) \tag{105c}$$

$$\tilde{H}_{y,T} = B\exp-\frac{\omega}{v_m}j(\vec{\Omega}_T \bullet \vec{r}) = B\exp-\frac{\omega}{v_m}j(x\sin\varphi_T + z\cos\varphi_T) \tag{106a}$$

$$\tilde{E}_{x,T} = Z_m\cos\varphi_T B\exp-\frac{\omega}{v_m}j(x\sin\varphi_T + z\cos\varphi_T) \tag{106b}$$

$$\tilde{E}_{z,T} = -Z_m\sin\varphi_T B\exp-\frac{\omega}{v_m}j(x\sin\varphi_T + z\cos\varphi_T) \tag{106c}$$

$$v_0 = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \quad \text{(a)} \qquad v_m = \frac{1}{\sqrt{\varepsilon_m\mu_m}} \quad \text{(b)} \tag{107}$$

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \quad \text{(a)} \qquad Z_m = \sqrt{\frac{\mu_m}{\varepsilon_m}} \quad \text{(b)} \tag{108}$$

In the foregoing expression, H is the magnitude of the incident magnetic field, A is the magnitude of the reflection coefficient and B is the magnitude of the transmitted wave. These quantities are determined by matching boundary conditions at $z = 0$

$$\tilde{H}_{y,I}(x, z=0) + \tilde{H}_{y,R}(x, z=0) = \tilde{H}_{y,T}(x, z=0) \tag{109a}$$

$$H \exp\left[-\frac{\omega}{v_0} j(x \sin\varphi_I)\right] + A \exp\left[-\frac{\omega}{v_0} j(x \sin\varphi_R)\right] = B \exp\left[-\frac{\omega}{v_m} j(x \sin\varphi_T)\right] \tag{109b}$$

$$\tilde{E}_{x,I}(x, z=0) + \tilde{E}_{x,R}(x, z=0) = \tilde{E}_{x,T}(x, z=0) \tag{110a}$$

$$Z_0 \cos\varphi_I H \exp\left[-\frac{\omega}{v_0} j(x \sin\varphi_I)\right] - Z_0 \cos\varphi_R A \exp\left[-\frac{\omega}{v_0} j(x \sin\varphi_R)\right] = Z_m \cos\varphi_T B \exp\left[-\frac{\omega}{v_m} j(x \sin\varphi_T)\right] \tag{110b}$$

The solution of equations 109 and 110 is available in standard electromagnetic texts. We have

$$\varphi_R = \varphi_I \tag{111a}$$

$$\sqrt{\varepsilon_0 \mu_0} \sin\varphi_I = \sqrt{\varepsilon_m \mu_m} \sin\varphi_T \tag{111b}$$

$$H + A = B \tag{111c}$$

$$Z_0 \cos\varphi_I (H - A) = Z_m \cos\varphi_T B = Z_m \cos\varphi_T (H + A) \tag{111d}$$

$$A = \frac{Z_0 \cos\varphi_I - Z_m \cos\varphi_T}{Z_0 \cos\varphi_I + Z_m \cos\varphi_T} H \tag{111e}$$

$$B = \frac{2 Z_0 \cos\varphi_I}{Z_0 \cos\varphi_I + Z_m \cos\varphi_T} H \tag{111f}$$

Solution for reflection in air off a negative ε and negative μ material

The solution for this case is readily found using the discussion for the double case combined with the basic conclusions deduced in section 2. The basic behavior for propagation for a DN material is summarized from equations 84 to 88 applied to the loss-free model considered here. Restating them in the notation of this section we have

$$\vec{\Omega}_S \bullet \vec{\Omega}_E = \vec{\Omega}_S \bullet \vec{\Omega}_H = 0 \tag{112}$$

$$\tilde{\vec{E}} = \tilde{E}_0 \vec{\Omega}_E \exp(j \frac{\omega}{v_m} \vec{\Omega}_S \bullet \vec{r}) \tag{113}$$

$$\tilde{\vec{H}} = \frac{\tilde{E}_0}{Z_m}\vec{\Omega}_H \exp(j\frac{\omega}{v_m}\vec{\Omega}_S \bullet \vec{r}) \tag{114}$$

Equations 113 and 114 have a negative phase velocity.

Suppose, for example, that we assume the transmitted wave looks like the one for the double positive case just considered—that is, figure 1 is valid. What would the solution for the system of equations look like? The behavior for the incident (equation 104) and reflected terms (equation 105) remains unchanged. However, the equation for the transmitted terms is now

$$\tilde{H}_{y,T} = B\exp\frac{\omega}{v_m} j(x\sin\varphi_T + z\cos\varphi_T) \tag{115a}$$

$$\tilde{E}_{x,T} = Z_m \cos\varphi_T B\exp\frac{\omega}{v_m} j(x\sin\varphi_T + z\cos\varphi_T) \tag{115b}$$

$$\tilde{E}_{z,T} = -Z_m \sin\varphi_T B\exp\frac{\omega}{v_m} j(x\sin\varphi_T + z\cos\varphi_T) \tag{115c}$$

At the boundary, $z = 0$, the spatial variation of equation 115 goes like

$$\tilde{H}_{y,T} \to \exp\frac{\omega}{v_m} j(x\sin\varphi_T) \tag{116a}$$

$$\tilde{E}_{x,T} \to \exp\frac{\omega}{v_m} j(x\sin\varphi_T) \tag{116b}$$

By comparing the foregoing behavior with incident and reflected behavior at $z = 0$, which go as $\exp\left[-\frac{\omega}{v_0} j(x\sin\varphi_R)\right]$, we see that there is no way a transmitted wave traveling in the direction shown in figure 1 can satisfy the boundary conditions.

Now let's consider the transmitted wave shown in figure 2. For this case we have

$$\vec{\Omega}_T \bullet \vec{r} = -x\sin\varphi_T + z\cos\varphi_T \tag{117}$$

Again, using the concept of negative phase velocity we now have

$$\tilde{H}_{y,T} = B\exp\frac{\omega}{v_m} j(-x\sin\varphi_T + z\cos\varphi_T) \tag{118a}$$

$$\tilde{E}_{x,T} = Z_m \cos\varphi_T B\exp\frac{\omega}{v_m} j(-x\sin\varphi_T + z\cos\varphi_T) \tag{118b}$$

$$\tilde{E}_{z,T} = Z_m \sin\varphi_T B\exp\frac{\omega}{v_m} j(-x\sin\varphi_T + z\cos\varphi_T) \tag{118c}$$

Repeating the same procedure as in equations 109 and 110 we have

$$H\exp\left[-\frac{\omega}{v_0}j(x\sin\varphi_I)\right]+A\exp\left[-\frac{\omega}{v_0}j(x\sin\varphi_R)\right]=B\exp\left[-\frac{\omega}{v_m}j(x\sin\varphi_T)\right] \qquad (119a)$$

$$Z_0\cos\varphi_I H\exp\left[-\frac{\omega}{v_0}j(x\sin\varphi_I)\right]-Z_0\cos\varphi_R A\exp\left[-\frac{\omega}{v_0}j(x\sin\varphi_R)\right]=Z_m\cos\varphi_T B\exp\left[-\frac{\omega}{v_m}j(x\sin\varphi_T)\right] \qquad (119b)$$

The foregoing equations are identical to equations and hence we have for negative refraction.

Solution for reflection in a negative ε and negative μ material off an air interface

Based on what we learned in the previous cases we can readily write down the equations for figure 3.

$$\tilde{H}_{y,I}=H\exp\frac{\omega}{v_m}j(x\sin\varphi_I+z\cos\varphi_I) \qquad (120a)$$

$$\tilde{E}_{x,I}=Z_m\cos\varphi_I H\exp\frac{\omega}{v_m}j(x\sin\varphi_I+z\cos\varphi_I) \qquad (120b)$$

$$\tilde{E}_{z,I}=-Z_m\sin\varphi_I H\exp\frac{\omega}{v_m}j(x\sin\varphi_I+z\cos\varphi_I) \qquad (120c)$$

$$\tilde{H}_{y,R}=A\exp\frac{\omega}{v_m}j(x\sin\varphi_R-z\cos\varphi_R) \qquad (121a)$$

$$\tilde{E}_{x,R}=-Z_m\cos\varphi_R A\exp\frac{\omega}{v_m}j(x\sin\varphi_R-z\cos\varphi_R) \qquad (121b)$$

$$\tilde{E}_{z,R}=-Z_m\sin\varphi_R A\exp\frac{\omega}{v_m}j(x\sin\varphi_R-z\cos\varphi_R) \qquad (121c)$$

$$\tilde{H}_{y,T}=B\exp-\frac{\omega}{v_0}j(-x\sin\varphi_T+z\cos\varphi_T) \qquad (122a)$$

$$\tilde{E}_{x,T}=Z_0\cos\varphi_T B\exp-\frac{\omega}{v_0}j(-x\sin\varphi_T+z\cos\varphi_T) \qquad (122b)$$

$$\tilde{E}_{z,T}=Z_0\sin\varphi_T B\exp-\frac{\omega}{v_0}j(-x\sin\varphi_T+z\cos\varphi_T) \qquad (122c)$$

Matching boundary conditions at $z = 0$ gives

$$H \exp \frac{\omega}{v_m} j(x \sin \varphi_I) + A \exp \frac{\omega}{v_m} j(x \sin \varphi_R) = B \exp \frac{\omega}{v_0} j(x \sin \varphi_T) \tag{123a}$$

$$Z_m \cos \varphi_I H \exp \frac{\omega}{v_m} j(x \sin \varphi_I) - Z_m \cos \varphi_R A \exp \frac{\omega}{v_m} j(x \sin \varphi_R) = Z_0 \cos \varphi_T B \exp \frac{\omega}{v_0} j(x \sin \varphi_T) \tag{123b}$$

The solution is

$$\varphi_R = \varphi_I \tag{124a}$$

$$\sqrt{\varepsilon_m \mu_m} \sin \varphi_I = \sqrt{\varepsilon_0 \mu_0} \sin \varphi_T \tag{124b}$$

$$H + A = B \tag{124c}$$

$$Z_m \cos \varphi_I (H - A) = Z_0 \cos \varphi_T B = Z_0 \cos \varphi_T (H + A) \tag{124d}$$

$$A = \frac{Z_m \cos \varphi_I - Z_0 \cos \varphi_T}{Z_m \cos \varphi_I + Z_0 \cos \varphi_T} H \tag{124e}$$

$$B = \frac{2 Z_m \cos \varphi_I}{Z_m \cos \varphi_I + Z_0 \cos \varphi_T} H \tag{124f}$$

---

4. Conclusion

We derive a system of propagation equations and model in a Double Negative material in a way that differs from previous derivations. Our derivation is based entirely of on the idea that real materials always have some loss, and because of this, wave energy traveling in a certain direction must always be accompanied by a loss of energy in that direction. Additional mathematics is not required. Energy losses per unit length of travel are finite, and can be extremely small. Propagation in a lossless DN media is found as the mathematical limiting solution of an extremely small energy loss per unit length. Our system of equations and methodology is used to derive the reflection and refraction equations between a positive $\varepsilon(\omega)$ and $\mu(\omega)$ and a DN material. These results are in agreement with other predictions.

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