

UNCLASSIFIED

**SENSITIVITY ANALYSIS FOR THE  
OPTIMAL DESIGN AND CONTROL  
OF ADVANCED GUIDANCE SYSTEMS**

**F49620-03-1-0326**

**Principal Investigator:**

Lisa G. Davis<sup>1</sup>

Montana State University  
Department of Mathematical Sciences  
PO Box 172400  
Bozeman, MT 59717-2400  
Voice: 406-994-5347, Fax: 406-994-1789  
davis@math.montana.edu

01 June 2007



UNCLASSIFIED

---

<sup>1</sup>formerly Lisa G. D. Stanley.

# REPORT DOCUMENTATION PAGE

Form Approved  
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden to Washington Headquarters Service, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington, DC 20503.

**PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.**

1. REPORT DATE (DD-MM-YYYY) 01-06-07		2. REPORT TYPE Final Technical Report		3. DATES COVERED (From - To) 01-06-03 - 30-11-07	
4. TITLE AND SUBTITLE Sensitivity Analysis for the Optimal Design and Control of Advanced Guidance Systems				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER F49620-03-1-0326	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S) Davis, Lisa G., Dr. Department of Mathematical Sciences Wilson Hall 2-214 PO Box 172400 Bozeman, MT 59717-2400 davis@math.montana.edu				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Montana State University Bozeman, MT 59717				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)  Air Force Office of Scientific Research (AFOSR)/NL 875 N. Arlington St., Rm. 3112 Arlington, VA 22203				10. SPONSOR/MONITOR'S ACRONYM(S) AFOSR	
				11. SPONSORING/MONITORING AGENCY REPORT NUMBER AFRL-SR-AR-TR-07-0037	
12. DISTRIBUTION AVAILABILITY STATEMENT  DISTRIBUTION A: Approved for public release; distribution unlimited.					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT The main goal of the research project is to use Continuous Sensitivity Equation Methods in order to design actuators and sensors for distributed parameter systems. Investigations for parameterized sensor/actuator placement indicate that computational challenges exist for certain types mathematical models. When the governing equations are partial differential equations and the sensitivity analysis is with respect to parameters that determine placement of sensors and/or actuators, there can be a loss of regularity between the model equations and that of the corresponding sensitivity equations. This issue is particularly important for accuracy and convergence of numerical sensitivity calculations that may be used within a control design framework.					
15. SUBJECT TERMS Sensitivity Equations, Partial Differential Equations, Sensors, Actuators, Linear Quadratic Regulator Control Design					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT Unclassified	18. NUMBER OF PAGES 25	19a. NAME OF RESPONSIBLE PERSON Lt. Col. Scott Wells
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified			19b. TELEPHONE NUMBER (Include area code) (703)

electronically transmitted to AFOSR, e.g., 20-12-2007 represents 20 Dec 2007.

**Block 1:** This represents the publication date of the final report. Enter the date the final report is being

## Abstract

The main goal of the research project is to use sensitivity analysis to design actuators and sensors for distributed parameter systems (DPS). The choice of sensor/actuator for a control problem, its action and placement thereof, is addressed by computing the sensitivity of the state variables with respect to small changes in a parameter characterizing placement. Investigations for parameterized sensor/actuator placement indicate that computational challenges exist for certain types of models. Parameters determining placement tend to enter the PDE models in ways that affect the differentiability of the state variables. There can be a loss of regularity between the model equations and that of the corresponding sensitivity equations. When using Continuous Sensitivity Equation Methods (CSEMs), the formulation and regularity of the mathematical model governing the underlying system behavior must be well understood. Simple models involving actuator placement contain discontinuous coefficients, and the explicit parameter location dependence of these discontinuities provides an *a priori* indication that sensitivity variables lack the same smoothness properties as those of the state. The issue is particularly important for accuracy and convergence of numerical sensitivity calculations.

## Contents

<b>1</b>	<b>Technical Summary</b>	<b>3</b>
1.1	Methodology and Theory Used in the Research . . . . .	3
1.2	Partial Differential Equation Models and Their Discretizations . . . . .	4
1.3	System Sensitivities . . . . .	4
1.3.1	The Linear Quadratic Regulator Problem . . . . .	5
1.4	Control System Sensitivities . . . . .	5
1.5	Results of Funding . . . . .	6
1.6	Investigations on Sensitivity Computations for the Heat Equation with Respect to Actuator Placement . . . . .	6
1.6.1	Numerical Experiments . . . . .	7
1.6.2	Validation of Sensitivity Calculations . . . . .	7
1.7	Computational Issues with Sensitivity Calculations for Actuator Placement on an Euler-Bernoulli Beam . . . . .	10
1.7.1	Computational Difficulties with the Sensitivity Equation . . . . .	12
1.8	Conclusions and Future Work . . . . .	14
<b>2</b>	<b>Publications</b>	<b>21</b>
<b>3</b>	<b>Technology Assists, Transitions or Transfers</b>	<b>22</b>
<b>4</b>	<b>Accomplishments and Successes</b>	<b>22</b>
<b>5</b>	<b>Professional Personnel Supported</b>	<b>23</b>
<b>6</b>	<b>Honors and Awards</b>	<b>23</b>
<b>7</b>	<b>Professional Activities</b>	<b>23</b>

# 1 Technical Summary

This work concerns the development of computational tools to incorporate into the design of control systems—control law, observers, choice and placement of sensors and actuators—for flexible wing MAVs. Such vehicles have been the focus of research efforts supported by AFOSR and have unique capabilities that can be utilized in urban missions. Capabilities such as flight control via wing shaping have been demonstrated, and it is known that these MAVs can fly at increased angle of attack without stall. However, fundamental aerodynamics of such vehicles is not well-understood, and computational studies to enhance the of control systems are now being developed. A vital contribution to that effort is the choice and placement of sensors and actuators used within the control regime.

The main goal of the research project is to use sensitivity analysis to design actuators and sensors for distributed parameter systems (DPS). The choice of sensor/actuator for a control problem, its action and placement thereof, is addressed by computing the sensitivity of the state variables with respect to small changes in a parameter characterizing placement. Preliminary investigations for parameterized sensor/actuator placement indicate that computational challenges exist for certain types of models. Parameters determining placement tend to enter the PDE models in ways that affect the differentiability of the state variables. There can be a loss of regularity between the model equations and that of the corresponding sensitivity equations. When using Continuous Sensitivity Equation Methods (CSEMs), the formulation and regularity of the mathematical model governing the underlying system behavior must be well understood. Simple models involving actuator placement contain discontinuous coefficients, and the explicit parameter location dependence of these discontinuities provides an *a priori* indication that sensitivity variables lack the same smoothness properties as those of the state variables. The issue is particularly important for accuracy and convergence of numerical sensitivity calculations.

For the last three years, my collaborators and I have investigated various aspects of the use of sensitivity analysis for locating sensors and actuators. Computational models are required for control design and sensitivity calculations (for sensor and actuator selection and placement). The focus of this research has been the mathematical analysis and development of computational tools for sensitivity approximation for models where the parameter of interest determines sensor/actuator placement. Investigations indicate that computational challenges exist for certain types of models, and one important indicator of such challenges is the manner in which the parameters appear in the underlying mathematical models. Parameters determining placement tend to enter the pde models in ways that affect the differentiability of the state variables with respect to small changes in such parameters. At the very least, there may be a loss of regularity between the model equations and that of the corresponding sensitivity equations. In Section 1.6, we discuss a proof-of-concept example which shows how sensitivity computations can be obtained efficiently and can be validated if the scale of the problem allows. In Section 1.7, an Euler-Bernoulli Beam model with patch actuators is presented in order to illustrate the computational challenges that may occur, and a corresponding steady state model problem is also examined to identify some of the mathematical issues.

## 1.1 Methodology and Theory Used in the Research

In this section, an overview of the types of models that have been examined in the research is given. Notation, methodology and theory is also outlined and will be referenced throughout

the report.

## 1.2 Partial Differential Equation Models and Their Discretizations

When we speak of a “model”, we are referring to a state equation that describes the governing physics of the problem to be controlled. Since the focus of this report examines actuator placement, we omit the measurement equation that models the sensor dynamics. Specifically, we consider the simplified underlying model considered in this research to be a linear system of partial differential equations of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad (1.1)$$

where  $x(t)$  is the state in the state space  $X$  and  $u(t)$  is the control effected by the actuators. The operator  $A$  represents the linear dynamics of the system, and  $B$  represents the physics of the actuator input to the system. Depending on the application, the state vector may be composed solely of flow velocities, or perhaps velocities and structural components if we are considering the coupling between the vehicle and flow. In a more general discussion, one would include several important terms in the model, most importantly, a nonlinear operator describing the nonlinear dynamics of the system to be modelled. Other terms such as a disturbance term that can be used to incorporate noise or unmodeled dynamics as well as a measurement term that provides a measurement of the state provided by the sensors should also appear. However, for the discussion given in this report we simplify the governing equation to a linear model and focus on the most interesting aspects of the sensitivity analysis research that has been conducted for this project.

This type of model will be used as guidance in algorithm development. For example, theory for sensitivity analysis for distributed parameter systems uses the PDE model as the starting point [17, 69, 71, 72, 74]. Similarly, design of convergent control designs relies on properties of the underlying PDE system.

Deriving a finite dimensional model, one can obtain a discretized version of (1.1) of the form

$$\dot{x}^N(t) = A^N x^N(t) + B^N u^N(t), \quad x^N(0) = x_0^N \quad (1.2)$$

where  $N$  is related to the grid used in discretization, and the state vector  $x^N(t)$  is obtained by applying an appropriate approximation scheme such as a finite element discretization.

## 1.3 System Sensitivities

In this research project, we make extensive use of the sensitivity of the state of the system, or control, to a parameter of interest,  $\alpha$ . The parameter may represent a sensor or actuator location, its length, or some other characteristic that would affect the overall closed loop system performance and or robustness. If we denote the sensitivity of the state variable

$$x_\alpha(t) = \frac{\partial x(t)}{\partial \alpha}$$

then we can derive the sensitivity equation for the system defined in (1.1) by implicitly differentiating to yield

$$\begin{aligned} \dot{x}_\alpha(t) &= Ax_\alpha(t) + A_\alpha x(t) + Bu_\alpha(t) + B_\alpha u(t), \\ x_\alpha(0) &= 0. \end{aligned} \quad (1.3)$$

Issues with respect to existence, regularity, and calculation of the sensitivities for particular systems provide fundamental questions for research. Controller sensitivities will be presented after the overview of the basic control method given in the following section.

### 1.3.1 The Linear Quadratic Regulator Problem

Given a system defined by any of the models described by equations (1.1) or by (1.2), the associated linear quadratic regulator (LQR) problem is as follows:

Determine the control  $u(t)$  in an admissible set,  $U$ , that minimizes the cost functional

$$J(u) = \int_0^\infty \langle Qx, x \rangle_X + \langle Ru, u \rangle_U dt,$$

subject to the constraint that  $x(t)$  in state space  $X$  satisfies the state equation, and  $Q$  and  $R$  are state and control weighting operators, respectively. Assuming full state feedback, optimal control theory gives the feedback control in the form

$$u(t) = -R^{-1}B^*\Pi x(t) = -Kx(t), \quad (1.4)$$

where  $K$  is the gain operator, and  $\Pi$  is the solution of the control algebraic Riccati equation

$$A^*\Pi + \Pi A - \Pi B R^{-1} B^* \Pi + Q = 0. \quad (1.5)$$

#### *Functional Gains*

For many PDE control problems, the control law (1.4) can be written as

$$u(t) = -[Kx(t)] = \int_\Omega k(s)x(t, s)ds, \quad (1.6)$$

for example see [19, 20, 48, 50]. In the case that there is more than one actuator input, e.g., more at more than one spatial location,  $\xi$ , there will be a functional gain for each location and the inputs can be represented as

$$u_\xi(t) = -[Kx(t)](\xi) = \int_\Omega k_\xi(s)w_c(t, s)ds. \quad (1.7)$$

The kernels  $k_\xi(s)$  are called functional gains, and have been utilized for reduced order controller design and sensor placement in [4, 5, 21, 41, 49, 53].

## 1.4 Control System Sensitivities

Computing sensitivities of the control law with respect to parameters is one of the primary research interests. If we define the control sensitivity

$$u_\alpha(t) = \frac{\partial u}{\partial \alpha}$$

then the associated sensitivity system is given by equation (1.3) and

$$u_\alpha(t) = -R^{-1} [B^*\Pi x_\alpha(t) + (B^*\Pi_\alpha + B_\alpha^*\Pi)x(t)],$$

where  $\Pi_\alpha$  solves the Lyapunov equation

$$\hat{A}^*\Pi_\alpha + \Pi_\alpha\hat{A} = \hat{Q}, \quad (1.8)$$

with

$$\hat{A} = A - BR^{-1}B^*\Pi \quad \text{and} \quad \hat{Q} = -A_\alpha^*\Pi - \Pi A_\alpha + \Pi(BR^{-1}B_\alpha^* + B_\alpha R^{-1}B^*)\Pi - Q_\alpha.$$

This Lyapunov equation is derived by implicitly differentiating through the Riccati equation (1.5) with respect to  $\alpha$ . Note the use of the notation  $A_\alpha = \frac{\partial A}{\partial \alpha}$  for the sensitivity of the  $A$  operator, and similarly for  $B_\alpha$  and  $Q_\alpha$ . We note that in general, the control weighting,  $R$ , does not depend on  $\alpha$ .

## 1.5 Results of Funding

### 1.6 Investigations on Sensitivity Computations for the Heat Equation with Respect to Actuator Placement

This section addresses an actuator design problem for the simple 1D heat equation. The idea is that the choice and placement of sensors and actuators for a control problem could be addressed by computing the sensitivity of the state of the system with respect to a characteristic of the actuator placement. To illustrate the ideas, we consider the sensitivity of the state variable, temperature in this case, to actuator location. This work provides the preliminary computations that demonstrate proof-of-concept for the combination of CSEMs with basic optimal control techniques. The sensitivity of the temperature with respect to actuator placement is investigated using a Continuous Sensitivity Equation Method, see [17, 43, 71, 72, 74, 75] for examples. Consider the heat equation model for the situation when an actuator location is specified by the parameter  $\alpha$ :

$$w_t(t, \xi) = \epsilon w_{\xi\xi}(t, \xi) + b(\xi; \alpha)u(t) \tag{1.9}$$

$$w(t, 0) = 0, \quad w(t, 1) = 0$$

$$w(0, \xi) = w_0(\xi) \tag{1.10}$$

with

$$b(\xi; \alpha) = e^{-(\xi-\alpha)^2}.$$

Defining the state variable  $x(t) = w(t, \cdot)$  and using standard PDE theory, one can formulate (1.9)-(1.10) in the form of (1.1) posed in an appropriate Sobolev space. We forego the presentation of these details for the sake of brevity. However, we note that the  $B$  operator above depends explicitly on the parameter of interest  $\alpha$ ; this fact is important for the sensitivity analysis that follows.

Many of the computational issues related to sensitivity analysis depend heavily on the correct mathematical formation for the problem. Here we outline the application of the Continuous Sensitivity Equation Method (CSEM) to the LQR control problem for the heat equation given above. The following discussion uses the temperature (state) sensitivity, the  $\Pi$  operator sensitivity and the controller sensitivity, respectively, defined as

$$w_\alpha(t, \xi) = \frac{\partial w(t, \xi)}{\partial \alpha}, \quad \Pi_\alpha = \frac{\partial \Pi}{\partial \alpha}, \quad u_\alpha(t) = \frac{\partial u(t)}{\partial \alpha}.$$

To derive the appropriate sensitivity equations, one implicitly differentiates equations (1.9)-(1.10) with respect to the parameter  $\alpha$ . The resulting sensitivity equation with boundary and initial conditions is given by

$$(w_\alpha)_t(t, \xi) = \epsilon(w_\alpha)_{\xi\xi}(t, \xi) + 2(\xi - \alpha)b(\xi; \alpha)u(t) + b(\xi; \alpha)u_\alpha(t) \tag{1.11}$$

$$w_\alpha(t, 0) = 0, w_\alpha(t, 1) = 0$$

$$w_\alpha(0, \xi) = 0.$$

Further, we note that this equation can be posed in a simplified form of equation (1.3) as

$$\dot{x}_\alpha(t) = Ax_\alpha(t) + Bu_\alpha(t) + B_\alpha u(t), \quad x_\alpha(0) = 0 \quad (1.12)$$

The state feedback control system is given by equations (1.1), and the optimal feedback control by (1.4) determined by the algebraic Riccati equation (1.5). The control sensitivity can be computed as described in Section 1.4. In this example, both operators,  $B$  and  $B_\alpha$ , have integral representations, and it is an easy calculation to see that the  $B$  operator varies in a differentiable manner with respect to the actuator location parameter  $\alpha$ . In the following section we outline the discretization and sensitivity approximations for both the state variable as well as the control.

### 1.6.1 Numerical Experiments

Here, a brief summary of the numerical experiments for the actuator placement model is given. The initial condition used for the numerical calculations is the piecewise constant function

$$w_0(\xi) = w(0, \xi) = \begin{cases} -1, & \xi \in [0, 0.5] \\ 1, & \xi \in (0.5, 1.0], \end{cases}$$

and the thermal diffusivity coefficient for all of the numerical results presented here is given by  $\epsilon = 0.01$ . The state equation in (1.12) is discretized using a standard finite element method in the spatial coordinate with cubic B-spline basis functions, and a time solver from Matlab's suite of ODE solvers is used for the time stepping. As this is a fairly standard treatment of the discretization and simulation, we omit details. Matlab's `lqr` and `lyap` functions are used to solve the algebraic Riccati equation to approximate  $\Pi$  and to approximate the Riccati sensitivity,  $\Pi_\alpha$ , respectively.

In Figure 1, an example of a temperature approximation using piecewise cubic basis functions is shown for the case where  $\alpha = 0.7$  and  $N = 40$  basis functions are used. Figure 2 contains the corresponding temperature sensitivity for those same parameter values. Note that the sensitivity gives the qualitative information that one would expect for the case when  $\alpha = 0.7$ . That is, the behavior of  $w_\alpha$  in Figure 2 shows that the temperature  $w$  is most sensitive in the region around  $\xi = 0.7$ , and that the temperature is most sensitive for early values of  $t$  before the diffusion and the controller have driven the temperature to be very near equilibrium. The associated control and control sensitivity computations are given in Figure 3 and 4. The behavior of these functions is also as expected. The control effort is large initially and quickly decays to 0. Similarly, the control is most sensitive to small changes in  $\alpha$  for small values of  $t$ . As the  $t$ -variable increases, the control is less sensitive to changes in  $\alpha$ . In the following section, we include a brief discussion on the effort to validate the sensitivity calculations for this simple example.

### 1.6.2 Validation of Sensitivity Calculations

This section briefly demonstrates that for the simple example of the 1D heat equation, one can validate the sensitivity calculations shown in the previous section. The following shows that this can be done for the temperature, the control variable, or the Riccati operator approximations. To clarify, the numerical approximations of the sensitivities for

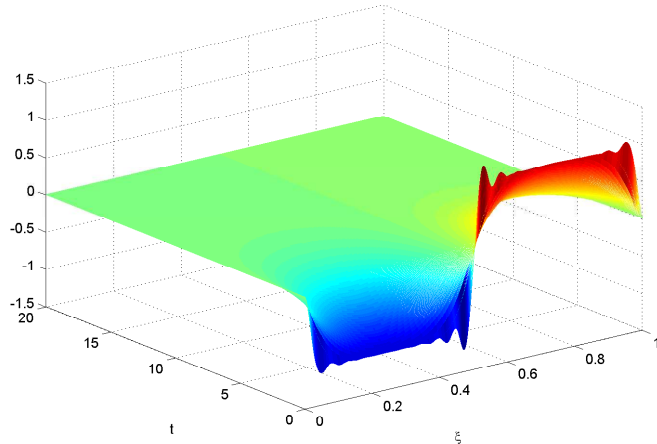


Figure 1: Temperature Approximation  $w(t, \xi)$  with  $N = 40$  and  $\alpha = 0.7$

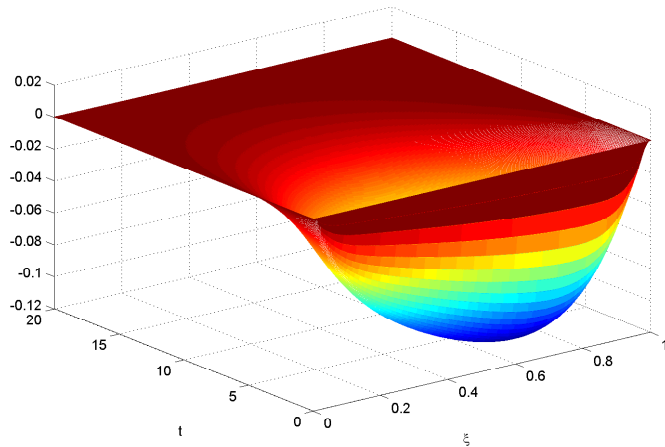


Figure 2: Sensitivity Approximation  $w_\alpha(t, \xi)$  with  $N = 40$  and  $\alpha = 0.7$

various variables are compared with finite difference calculations. For example, the control sensitivity,  $u_\alpha$  is compared with a centered finite difference approximation so that

$$u_\alpha^N(t; \alpha) \approx \frac{u^N(t; \alpha + \Delta\alpha) - u^N(t; \alpha - \Delta\alpha)}{2\Delta\alpha}.$$

The superscript notation  $u^N$  refers to the fact that the calculation is based on a finite element discretization in space, and for the results presented below, a reasonably fine mesh of  $N = 40$  is used in order to guarantee an accurate approximation for the variables of interest. Hence, the convergence that we present here refers to a convergence of the finite difference approximation as one allows  $\Delta\alpha \rightarrow 0$ . Table 1 gives results for each of the sensitivity variables  $w_\alpha$ ,  $u_\alpha$ , and  $\Pi_\alpha$  for a specific value of the  $\alpha$  parameter. Similar results were observed for a range of parameter values. Although sensitivity calculations obtained via

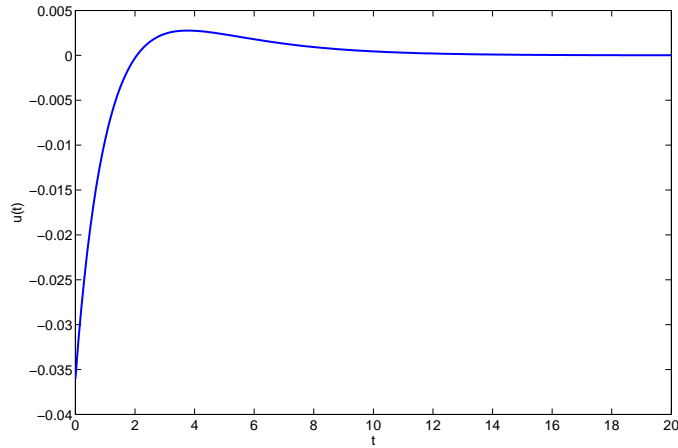


Figure 3: Control Approximation  $u(t)$  with  $N = 40$  and  $\alpha = 0.7$

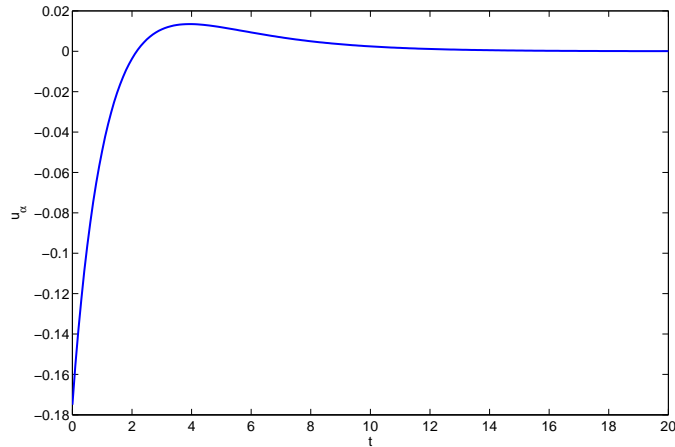


Figure 4: Control Sensitivity Approximation  $u_\alpha(t)$  with  $N = 40$  and  $\alpha = 0.7$

finite difference techniques can be difficult to obtain for large scale problems, it is appropriate to validate our sensitivity calculations whenever possible. For the case of this simple model problem, sensitivity calculations obtained by the CSEM approach are validated by comparison to centered finite difference sensitivity computations. In the following section, one will see why such a validation is not always feasible for certain types of sensitivity variables, particularly for certain types of sensitivity variables related to actuator location for some mathematical models. An investigation of actuator location for a boundary control problem with the 2D heat equation can be found in [32].

Step Size	State ( $\alpha = 0.7$ )	Control ( $\alpha = 0.7$ )	Riccati ( $\alpha = 0.2$ )
$\Delta\alpha = 10^{-1}$	$2.0600 \times 10^{-2}$	$3.1782 \times 10^{-4}$	$3.1000 \times 10^{-3}$
$\Delta\alpha = 10^{-2}$	$2.0782 \times 10^{-5}$	$3.1802 \times 10^{-7}$	$3.0808 \times 10^{-5}$
$\Delta\alpha = 10^{-3}$	$1.0914 \times 10^{-6}$	$3.1801 \times 10^{-10}$	$3.0808 \times 10^{-7}$
$\Delta\alpha = 10^{-4}$	$1.0900 \times 10^{-7}$	$3.1418 \times 10^{-13}$	$2.9993 \times 10^{-9}$
$\Delta\alpha = 10^{-5}$	$1.0900 \times 10^{-8}$	$1.4816 \times 10^{-14}$	

Table 1: Error Calculations Showing Convergence of Finite Difference Calculations for  $w_\alpha$ ,  $u_\alpha$  and  $\Pi_\alpha$ .

## 1.7 Computational Issues with Sensitivity Calculations for Actuator Placement on an Euler-Bernoulli Beam

In this section we discuss an actuator placement problem for a cantilevered Euler-Bernoulli beam with piezoceramic patch actuators as in [8, 9, 10]. We consider the sensitivity of the state—position and velocity of the beam—to patch location and to patch length. This work provides the preliminary computations which would then be used for optimal selection of patch length and location.

The framework that is currently being developed combines CSEMs, optimization and LQR control techniques to gain insight into sensor and actuator placement for control design for physical systems governed by a partial differential equations. However, this section shows that the sensitivity calculations for actuator placement can present many mathematically interesting and numerically challenging problems.

This discussion relates preliminary sensitivity calculations for a cantilevered Euler-Bernoulli beam model with a pair of piezoceramic patch actuators placed on either side of the beam at the same specified spatial location (see Figure 5). We consider the case when the patches are excited out-of-phase which results in pure bending of the beam. As in the previous example, the sensitivity of the displacement of the beam with respect to patch placement is investigated using the CSEM. We incorporate both Kelvin-Voigt and

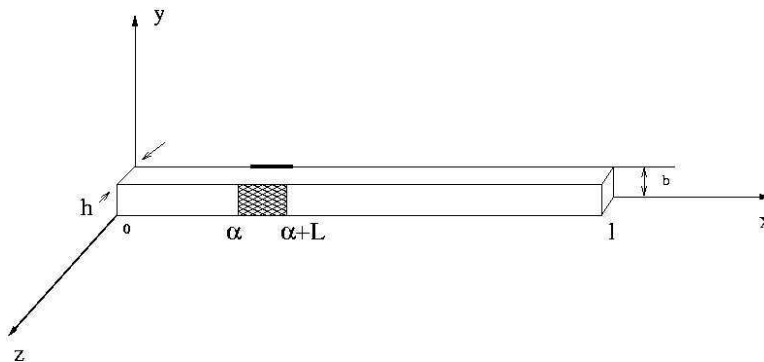


Figure 5: Diagram of beam with a pair of piezoceramic patches of length  $L$  located on either side of the beam on the interval  $x \in [\alpha, \alpha + L]$ .

linear viscous (air) damping. The length of the beam is  $\ell$ , the height is given by  $b$  and the thickness is denoted as  $h$ . The length of the patch is given by  $L$ , and the patch location is given by the interval  $[\alpha, \alpha + L]$ . Let  $w(t, x)$  denote the displacement of the beam at

time  $t$  and position  $x$ . The motion of the cantilevered beam is governed by the PDE with boundary conditions

$$\rho A w_{tt} + d_{lv} w_t + (d_{kv} I w_{txx})_{xx} + (EI w_{xx})_{xx} = g(t, x), \quad (1.13)$$

$$w(t, 0) = w_\xi(t, 0) = 0 \quad (1.14)$$

$$EI w_{xx}(t, \ell) + d_{kv} I w_{txx}(t, \ell) = 0, \quad (1.15)$$

$$(EI w_{xx})_x(t, \ell) + (d_{kv} I w_{txx})_x(t, \ell) = 0. \quad (1.16)$$

and with the initial deflection and initial velocity denoted by

$$w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x). \quad (1.17)$$

The coefficients all represent material properties of the beam at a certain spatial location  $x$ :  $\rho$  is the mass density,  $A$  is the cross-sectional area,  $I$  is the moment of inertia,  $E$  is Young's modulus, and  $d_{lv}$  and  $d_{kv}$  are the coefficients of air and Kelvin-Voigt damping, respectively.

The presence of the patches results in discontinuities in these coefficients, which can be expressed as follows

$$\rho A(x) = \rho A_1 + \rho A_2 [H_1(x) - H_2(x)], \quad (1.18)$$

$$d_{kv} I(x) = d_{kv} I_1 + d_{kv} I_2 [H_1(x) - H_2(x)], \quad (1.19)$$

$$EI(x) = EI_1 + EI_2 [H_1(x) - H_2(x)]. \quad (1.20)$$

The notation  $H_1(x)$  denotes the Heaviside function taking on the value of 0 for  $x \in [0, \alpha]$  with jump discontinuity located at  $x = \alpha$ , the left end of the patch, and  $H_2$  denotes the heaviside function with jump discontinuity located at  $x = \alpha + L$ , the right end of the patch. The constants  $\rho A_1, EI_1$ , and  $d_{kv} I_1$  correspond to the density, flexural rigidity, and Kelvin-Voigt damping properties of the beam, while the constants  $\rho A_2, EI_2$ , and  $d_{kv} I_2$  correspond to those of the patch, respectively. The patches influence the beam by exerting a moment force on that section. Therefore the spatial influence of the control is described by a difference of Heaviside functions, and the control function for the beam is given by

$$g(t, x) = \kappa [H_1(x) - H_2(x)]_{xx} u(t). \quad (1.21)$$

The constant  $\kappa$  is a parameter characterizing the patch properties, and  $u(t)$  is the voltage applied to the patch at time  $t$ . For a more thorough treatment of the model development and the specific forms of the beam and patch parameters, see [10].

One should observe that in the beam equation (1.13), the damping  $(d_{kv} I w_{txx})_{xx}$ , stiffness  $(EI w_{xx})_{xx}$ , and control term  $g(t, x)$  all contain spatial derivatives of the Heaviside functions; moreover, the actuator location parameter is embedded in the definition of these Heaviside functions. Consequently, the PDE should be mathematically interpreted using the variational (or weak) formulation. The weak form is also convenient for the numerical simulations which use the finite element method for approximation in space. The variational formulation of the beam equation is given in [10]

$$\begin{aligned} \int_0^\ell \rho A(x; \alpha) w_{tt}(t, x) \phi_{xx}(x) dx + \int_0^\ell d_{kv} I(x; \alpha) w_{txx}(t, x) \phi_{xx}(x) dx \\ + \int_0^\ell EI(x; \alpha) w_{xx}(t, x) \phi_{xx}(x) dx = \int_0^\ell \tilde{g}(t, x; \alpha) \phi_{xx}(x) dx, \end{aligned} \quad (1.22)$$

where

$$\tilde{g}(t, x; \alpha) = \kappa[H_1(x) - H_2(x)]u(t).$$

We use this formulation below to discuss computational difficulties with numerically approximating the sensitivities.

### 1.7.1 Computational Difficulties with the Sensitivity Equation

The parameter of interest for this study is the parameter determining the patch location,  $\alpha$ , and we consider both the displacement and the control term to depend explicitly on  $\alpha$ . We denote this dependence as  $w(t, x) = w(t, x; \alpha)$ . The sensitivity of the state with respect to the patch location is defined by

$$w_\alpha(t, x; \alpha) = \frac{\partial w}{\partial \alpha} w(t, x; \alpha).$$

In [73], the PI numerically approximated  $w_\alpha$  by deriving a variational sensitivity equation. This equation was obtained by differentiating through the above variational equation (1.22) for the state with respect to  $\alpha$ . This procedure causes derivatives of the Heaviside and delta functions to appear in the sensitivity equation. The regularity of the sensitivity equation can play an important role in determining the appropriate computational treatment of the problem.

To illustrate the main difficulties with the beam sensitivity equation, we remove the time dependence to focus on the issue of regularity in the spatial domain. We also remove one spatial discontinuity from the piecewise constant functions to further simplify the presentation. These modifications allow us to study a similar problem which has a closed form solution for the state and sensitivity. Therefore, instead of the variational equation (1.22), consider the following steady problem: find  $w \in V = \{v \in H^2(0, \ell) : v(0) = 0, v_x(0) = 0\}$  such that

$$\int_0^\ell EI(x; \alpha) w_{xx}(x) \phi_{xx}(x) dx = \int_0^\ell \tilde{g}(x; \alpha) \phi_{xx}(x) dx \quad (1.23)$$

for all  $\phi \in V$ . The functions  $EI(x; \alpha)$  and  $\tilde{g}(x; \alpha)$  are piecewise constant and are given by

$$EI(x; \alpha) = \begin{cases} EI_1, & 0 < x < \alpha \\ EI_2, & \alpha < x < \ell \end{cases}, \quad \tilde{g}(x; \alpha) = \begin{cases} g_1, & 0 < x < \alpha \\ g_2, & \alpha < x < \ell \end{cases}.$$

This problem can be posed in an abstract differential form; however, like the time dependent beam problem, it is best understood in weak form due to the discontinuities in the coefficient functions  $EI$  and  $\tilde{g}$ .

We assume  $EI_1, EI_2 > 0$  which implies the problem has a unique solution by the Lax-Milgram Theorem. It can be checked that the solution is given by

$$w(x; \alpha) = \begin{cases} b_1 x^2/2, & 0 < x < \alpha \\ b_2 x^2/2 + \alpha(b_1 - b_2)x - \alpha^2(b_1 - b_2)/2, & \alpha < x < \ell \end{cases}, \quad (1.24)$$

where  $b_i = g_i/EI_i$  for  $i = 1, 2$ . The solution is differentiable with respect to  $\alpha$  and the sensitivity  $w_\alpha(x; \alpha)$  is given by

$$w_\alpha(x; \alpha) = \begin{cases} 0, & 0 < x < \alpha \\ (b_1 - b_2)x - \alpha(b_1 - b_2), & \alpha < x < \ell \end{cases}. \quad (1.25)$$

The sensitivity is continuous and differentiable in  $x$  with derivative

$$(w_\alpha)_x(x; a) = \begin{cases} 0, & 0 < x < \alpha \\ b_1 - b_2, & \alpha < x < \ell \end{cases}.$$

If  $b_1 \neq b_2$ , then  $(w_\alpha)_x$  is not differentiable and therefore  $w_\alpha \notin V$ . Therefore, the sensitivity does not have the same spatial regularity as the state.

This observation has implications for computing the sensitivities. We can proceed in the same manner as described above for the beam problem and formally differentiate the steady problem (1.23) with respect to  $\alpha$  to obtain the sensitivity equation

$$\int_0^\ell EI(x; \alpha)(w_\alpha)_{xx}(x; \alpha)\phi_{xx}(x) dx + (EI_1 - EI_2)w_{xx}(\alpha)\phi_{xx}(\alpha) = (g_1 - g_2)\phi_{xx}(\alpha). \quad (1.26)$$

Recall that  $\phi$  is in  $H^2(0, \ell)$  and so the pointwise evaluations  $\phi_{xx}(\alpha)$  do not make sense in general. Also, a direct computation shows

$$w_{xx}(x) = \begin{cases} b_1/2, & 0 < x < \alpha \\ b_2/2, & \alpha < x < \ell \end{cases}$$

so that the term  $w_{xx}(\alpha)$  appearing in the sensitivity equation does not have meaning when  $b_1 \neq b_2$ . In particular, the function  $w$  possesses a second derivative only in the weak sense and not in the classical sense of differentiability.

Despite these observations, we can follow the approach taken in [73] and numerically approximate the sensitivity using the formally derived sensitivity equation (1.26). We use the finite element method for the spatial discretization with cubic B-spline basis functions. Since these basis functions have continuous second derivatives, the second derivative evaluations  $\phi_{xx}(\alpha)$  are well defined. To approximate  $w_{xx}(\alpha)$  (which does not exist in the classical sense), we use the second derivative of the finite element solution of the steady problem. The numerical solutions of the formal sensitivity equation for 32 equally spaced nodes is compared with the exact sensitivity in Figure 6.

*The numerical solution of the formal sensitivity equation is not close to the true sensitivity. Furthermore, as the mesh is refined this error does not decrease and the numerical sensitivity does not converge.* This may not be surprising since this approach uses cubic polynomial basis functions to approximate a function whose second derivative is not smooth. Hence, the accuracy of the sensitivity computations and the reliability of the Euler-Bernoulli beam system simulations is brought into question by the results of this steady problem. The discontinuous coefficients in equations (1.22) and (1.23) along with the explicit parameter dependence of these discontinuities is the first indication that such numerical issues must be confronted for this problem. This steady problem shows that approximating a sensitivity using an ill-posed sensitivity equation may lead to large errors.

**Remark:** It is interesting to note that there is a direct connection between the formal sensitivity equation and automatic differentiation (AD). In our computations, we used the same number of finite element basis functions for the state equation and the formal sensitivity equation. In this case, the discretized sensitivity equation can be derived by implicitly differentiating the discretized state equation with respect to the parameter  $\alpha$ . This is exactly how one uses AD to numerically approximate the sensitivity. Therefore, the widely used AD procedure also fails to give accurate sensitivity approximations for this problem.

In future work, the PI plans to investigate how to correctly formulate the sensitivity equation in order to obtain numerically reliable sensitivities. See Section ?? for more details.

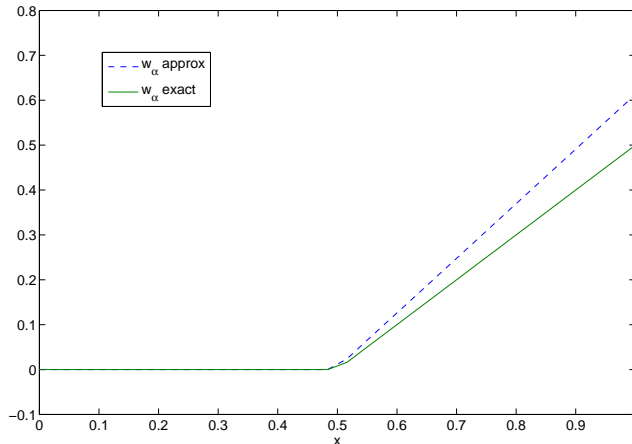


Figure 6: Finite element solution using cubic B-splines (with 32 equally spaced nodes) of the formal sensitivity equation (1.26) compared with the exact sensitivity (1.25).

## 1.8 Conclusions and Future Work

The heat equation example in Section 1.6 shows that sensitivity information has potential to provide valuable information for optimizing sensor and actuator location. However, the steady beam equation example in Section 1.7.1 shows very clearly that when using the continuous sensitivity equation method to compute sensitivities, the formulation and regularity of the mathematical model governing the underlying system behavior must be well understood. One goal of this research effort is to advance mathematical and computational tools to accurately compute parameter sensitivities for systems governed by partial differential equations.

For the steady equation in (1.23), and similarly for the beam example in (1.13)-(1.17), the discontinuous coefficients as well as the explicit parameter dependence of these discontinuities provide the researcher with an upfront indication that there may be a loss of regularity when moving from the original state equation to the derivation of the corresponding sensitivity equation. This is noted specifically in equation (1.26). Although this phenomenon occurs in a simple model problem, this issue is expected to appear in more realistic problems, such as the placement of actuators on a flexible wing. When sensor/actuator placement is parameterized in such a manner, the formulation and analysis of the corresponding sensitivity equations must be done carefully to insure that any regularity issues are accounted for. As discussed in Section 1.7.1, the regularity issue is of particular importance for accuracy and convergence of numerical sensitivity approximations.

For the beam problem with the discontinuous coefficients, research indicates that there are many existing mathematical options that can be used to help address the formulation of the sensitivity equation as well as the numerical computations. In particular, one can use a very weak formulation of the sensitivity equation to approximate the sensitivity variables that may lack the same smoothness as their original state variable counterparts. This approach has been successfully implemented in [23] to approximate sensitivities with respect to interface location in a heat conduction problem. Using this technique avoids the regularity issues encountered in the formally defined sensitivity equation presented in equation (1.26).

Another potential method to treat the regularity issue of the sensitivity equations involves smoothing the discontinuous coefficients in the original mathematical model prior to the sensitivity equation derivation and applying any discretization technique. However, preliminary research using such a technique indicates that there are computational issues involved in accurately approximating the state gradients, and the smoothness of the sensitivity is still in question. This is the subject of a paper which is currently in preparation.

In addition, one can investigate the use of the immersed finite element method [57] in order to obtain numerically reliable sensitivities. Other approaches such as discontinuous Galerkin finite elements and Niche's method are also possible. In any case, when sensor/actuator location is in question, one has to treat the derivation of the CSEM and corresponding sensitivity calculations using techniques which account for loss of smoothness in such sensitivity variables.

One other note is that if the sensitivity calculations were to be carried out using the very basic finite difference techniques, such as those used in Section 3.1.2, the lack of smoothness of the sensitivity variable would impact the accuracy and convergence rates of those computations. In addition, other sensitivity techniques such as the popular and well-advanced Automatic Differentiation (AD) software package also suffers from similar problems. Regardless of which of the popular techniques one uses for the sensitivity analysis, the mathematics of the sensor/actuator location problems must be handled carefully.

### Acknowledgment/Disclaimer

This work was sponsored by the Air Force Office of Scientific Research, USAF, under grant number F49620-03-1-0326. The views and conclusions contained herein are those of the author and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the U.S. Government.

### References

- [1] M. AMITAY AND A. GLEZER, *Role of actuation frequency in controlled flow reattachment over a stalled airfoil*, AIAA Journal, 40 (2002), pp. 209–216.
- [2] M. AMITAY, D. R. SMITH, V. KIBENS, D. E. PAREKH, AND A. GLEZER, *Modification of the aerodynamic characteristics of an unconventional airfoil using synthetic jet actuators*, AIAA Journal, 39 (2001), pp. 361–370.
- [3] E. ARIAN, M. FAHL, AND E. SACHS, *Trust-region proper orthogonal decomposition for optimal flow control*, 2000.
- [4] J. A. ATWELL, J. T. BORGGAARD, AND B. B. KING, *Reduced order controllers for Burgers' equation with a nonlinear observer*, Int. J. Appl. Math. Comput. Sci., 11 (2001), pp. 1311–1330.
- [5] J. A. ATWELL AND B. B. KING, *Reduced order controllers for spatially distributed systems via proper orthogonal decomposition*, SIAM. J. Scientific Computing, 26 (2004), pp. 128–151.
- [6] J. M. BADÍA, P. BENNER, R. MAYO, AND E. S. QUINTANA-ORTÍ, *Parallel solution of large-scale and sparse generalized algebraic Riccati equations*, in Euro-Par 2006 Par-

allel Processing: 12th International Euro-Par Conference, Lecture Notes in Computer Science, Springer-Verlag, 2006.

- [7] A. K. BANGIA, P. F. BATCHO, I. G. KEVREKIDIS, AND G. E. KARNIADAKIS, *Unsteady two-dimensional flows in complex geometries: comparative bifurcation studies with global eigenfunction expansions*, SIAM J. on Sci. Comput, 18 (1997), pp. 75–805.
- [8] H. T. BANKS, K. ITO, AND B. B. KING, *Theoretical and computational aspects of feedback in structural systems with piezoceramic controllers*, Tech. Rep. CRSC-TR93-2, Center for Research in Scientific Computation, North Carolina State University, 1993.
- [9] H. T. BANKS, R. C. SMITH, AND Y. WANG, *The modeling of piezoceramic patch interactions with shells, plates, and beams*, Quarterly of Applied Mathematics, 53 (1995), pp. 353–381.
- [10] H. T. BANKS, R. C. SMITH, AND Y. WANG, *Smart Material Structures: Modeling, Estimation and Control*, John Wiley & Sons, New York, 1996.
- [11] V. BARBU, I. LASIECKA, AND R. TRIGGIANI, *Boundary stabilization of Navier-Stokes equations*, Memoires of the AMS, to appear.
- [12] V. BARBU AND R. TRIGGIANI, *Internal stabilization of Navier-Stokes equations with finite dimensional controllers*, Indiana University Mathematics Journal, 53 (2004), pp. 1443–1494.
- [13] P. BENNER, *Solving large-scale control problems*, IEEE Control Systems Magazine, 24 (2004), pp. 44 – 59.
- [14] P. BENNER, S. GÖRNER, AND J. SAAK, *Numerical solution of optimal control problems for parabolic systems*, in Parallel Algorithms and Cluster Computing: Implementations, Algorithms and Applications, Springer-Verlag, to appear, 2006.
- [15] P. BENNER, E. S. QUINTANA-ORTÍ, AND G. QUINTANA-ORTÍ, *Solving linear-quadratic optimal control problems on parallel computers*, Tech. Rep. SFB393/05-19, Technische Universität Chemnitz, 2006.
- [16] G. BERKOOZ, P. J. HOLMES, AND J. L. LUMLEY, *The proper orthogonal decomposition in the analysis of turbulent flows*, Annu. Rev. Fluid Mech., 25 (1993), pp. 539–575.
- [17] J. BORGGAARD AND J. BURNS, *A PDE sensitivity equation method for optimal aerodynamic design*, Journal of Computational Physics, 136 (1997), pp. 366–384.
- [18] J. BORGGAARD, J. A. BURNS, E. VUGRIN, AND L. ZIETSMAN, *On strong convergence of feedback operators for non-normal distributed parameter systems*, in Proceedings of the IEEE Conference on Decision and Control, vol. 2, 2004, pp. 1526–1531.
- [19] J. A. BURNS AND B. B. KING, *A note on the regularity of solutions of infinite-dimensional Riccati equations*, Applied Mathematics Letters, 7 (1994), pp. 13–17.
- [20] ———, *Representation of feedback operators for hyperbolic systems*, in Computation and control, IV, Birkhäuser, Boston, MA, 1995, pp. 57–73.

- [21] J. A. BURNS AND B. B. KING, *A reduced basis approach to the design of low order compensators for nonlinear partial differential equation systems*, *J. Vibr. & Contr.*, 4 (1998), pp. 297–323.
- [22] J. A. BURNS, B. B. KING, AND D. RUBIO, *Feedback control of a thermal fluid using state estimation*, *Int. J. Comput. Fluid Dyn.*, 11 (1998), pp. 93–112.
- [23] J. A. BURNS, T. LIN, AND L. G. STANLEY, *A Petrov Galerkin finite-element method for interface problems arising in sensitivity computations*, *Computers & Mathematics with Applications. An International Journal*, 49 (2005), pp. 1889–1903.
- [24] K. A. E. CAMP AND B. B. KING, *A comparison of balanced truncation techniques for reduced order controllers*, *Proceedings of 2002 MTNS*, (2002).
- [25] R. C. CAMPHOUSE AND J. H. MYATT, *Feedback control for a two-dimensional Burgers equation system model*, in 2nd AIAA Flow Control Conference, 2004. AIAA-2004-2411.
- [26] D. H. CHAMBERS, R. J. ADRIAN, P. MOIN, D. S. STEWART, AND H. J. SUNG, *Karhunen-Loève expansion of Burgers' model of turbulence*, *Phys. Fluids*, 31 (1988), pp. 2573–2582.
- [27] K. COHEN, S. SIEGEL, J. SEIDEL, AND T. McLAUGHLIN, *Neural network estimator for low-dimensional modeling of a cylinder wake*, in 3rd AIAA Flow Control Conference, 2006. AIAA 2006-3491.
- [28] ———, *Reduced order modeling for closed-loop control of three dimensional wakes*, in 3rd AIAA Flow Control Conference, 2006. AIAA 2006-3356.
- [29] R. F. CURTAIN, *A new approach to model reduction in finite-dimensional control design for distributed parameter systems*. Preprint, 2001.
- [30] R. F. CURTAIN AND K. GLOVER, *Balanced realisations for infinite-dimensional systems*, *Operator Theory and Systems: Proc. Workshop Amsterdam*, (1985), pp. 87–104.
- [31] R. F. CURTAIN AND H. J. ZWART, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.
- [32] L. DAVIS AND F. PAHLEVANI, *Sensitivity calculations for actuator location for a parabolic PDE*. In preparation.
- [33] M. DEBIASI, J. LITTLE, E. CARABALLO, X. YUAN, A. SERRANI, J. H. MYATT, AND M. SAMIMY, *Influence of stochastic estimation on the control of subsonic cavity flow a preliminary study*, in 3rd AIAA Flow Control Conference, 2006. AIAA 2006-3492.
- [34] M. A. DEMETRIOU, *Numerical investigation on optimal actuator/sensor location of parabolic PDEs*, in *Proceedings of the American Control Conference*, vol. 3, 1999, pp. 1722 – 1726.
- [35] ———, *A numerical algorithm for the optimal placement of actuators and sensors for flexible structures*, in *Proceedings of the American Control Conference*, vol. 4, 2000, pp. 2290 – 2294.

- [36] M. A. DEMETRIOU AND J. BORGGAARD, *Optimization of an integrated actuator placement and robust control scheme for distributed parameter processes subject to worst-case spatial disturbance distribution*, in Proceedings of the American Control Conference, vol. 3, 2003, pp. 2114 – 2119.
- [37] ———, *Optimization of a joint sensor placement and robust estimation scheme for distributed parameter processes subject to worst case spatial disturbance distributions*, in Proceedings of the American Control Conference, vol. 3, 2004, pp. 2239 – 2244.
- [38] K. A. EVANS, *Reduced Order Controllers for Distributed Parameter Systems*, PhD thesis, Department of Mathematics, Virginia Polytechnic Institute and State University, 2003.
- [39] F. FAHROO AND M. A. DEMETRIOU, *Optimal actuator/sensor location for active noise regulator and tracking control problems*, Journal of Computational and Applied Mathematics, 114 (2000), pp. 137 – 158.
- [40] F. FAHROO AND Y. WANG, *Optimal location of piezoceramic actuators for vibration suppression of a flexible structure*, in Proceedings of the IEEE Conference on Decision and Control, vol. 2, 1997, pp. 1966 – 1971.
- [41] A. FAULDS AND B. KING, *Sensor location in feedback control of partial differential equation systems*. Anchorage, AK.
- [42] A. GLEZER AND M. AMITAY, *Synthetic jets*, Annual Review of Fluid Mechanics, 34 (2002).
- [43] A. G. GODFREY, *Using sensitivities for flow analysis*, in Computational Methods for Optimal Design and Control, Birkhäuser, Boston, MA, 1998, pp. 181–196.
- [44] L. GRASEDYCK, W. HACKBUSCH, AND B. KHOROMSKIJ, *Solution of large scale algebraic matrix Riccati equations by use of hierarchical matrices*, Computing, 70 (2003), pp. 121 – 165.
- [45] E. A. JONCKHEERE AND L. M. SILVERMAN, *A new set of invariants for linear systems—application to reduced order compensator design*, IEEE Transactions on Automatic Control, 28 (1983), pp. 953–964.
- [46] G. M. KEPLER, H. T. TRAN, AND H. T. BANKS, *Reduced order model compensator control of species transport in a CVD reactor*, Optimal Control Application & Methods, 21 (2000), pp. 143–160.
- [47] B. KING AND Y. OU, *Nonlinear dynamic compensators for control of a driven cavity*, Proc. 34th IEEE CDC, (1995), pp. 3741–3746.
- [48] B. B. KING, *Existence of functional gains for parabolic control systems*, in Computation and control, IV, Birkhäuser, Boston, MA, 1995, pp. 203–217.
- [49] B. B. KING, *Nonuniform grids for reduced basis design of low order feedback controllers for nonlinear continuous systems.*, Math. Models & Meth. in Appl. Sci., 8 (1998), pp. 1223–1241.

- [50] B. B. KING, *Representation of feedback operators for parabolic control problems*, Proceedings of the American Mathematical Society, 128 (2000), pp. 1339–1346.
- [51] B. B. KING AND N. HOVAKIMYAN, *An adaptive approach to control of distributed parameter systems*, in Proceedings of the 42nd IEEE Conference on Decision and Control, vol. 6, 2003, pp. 5715–5720.
- [52] B. B. KING, N. HOVAKIMYAN, K. A. EVANS, AND M. BUHL, *Reduced order controllers for distributed parameter systems: LQG balanced truncation and an adaptive approach*, Mathematical and Computer Modelling, 43 (2006), pp. 1136–1149.
- [53] B. B. KING AND E. W. SACHS, *Semidefinite programming techniques for reduced order systems with guaranteed stability margins*, Comput. Opt. and Appl., 17 (2000), p. 37.
- [54] M. KIRBY, J. P. BORIS, AND L. SIROVICH, *A proper orthogonal decomposition of a simulated supersonic shear layer*, Int. J. Num. Meth. Fluids, 10 (1990), pp. 411–428.
- [55] K. KUNISCH AND S. VOLKWEIN, *Control of burgers' equation by a reduced order approach using proper orthogonal decomposition*, J. Opt. Theor. & Apps., 102 (1999), pp. 345–371.
- [56] I. LASIECKA, *Finite element approximations of compensator design for analytic generators with fully unbounded controls/observations*, SIAM Journal on Control and Optimization, 33 (1995), pp. 67–88.
- [57] Z. LI AND K. ITO, *The Immersed Interface Method*, SIAM, Philadelphia, PA, 2006.
- [58] H. V. LY AND H. T. TRAN, *Proper orthogonal decomposition for flow calculations and optimal control in a horizontal CVD reactor*, Quarterly of Applied Mathematics, 60 (2002), pp. 631–656.
- [59] C. McMILLAN AND R. TRIGGIANI, *Min-max game theory and algebraic Riccati equations for boundary control problems with analytic semigroups, Part I: the stable case*, in Differential Equations, Dynamical Systems, and Control Science, Dekker, New York, 1994, pp. 757–780.
- [60] ———, *Min-max game theory and algebraic Riccati equations for boundary control problems with analytic semigroups, Part II: The general case*, Nonlinear Analysis, 22 (1994), pp. 431–465.
- [61] ———, *Min-max game theory and algebraic Riccati equations for boundary control problems with continuous input-solution map, Part I: The stable case*, in Evolution Equations, Control Theory, and Biomathematics, Dekker, New York, 1994, pp. 377–403.
- [62] ———, *Min-max game theory and algebraic Riccati equations for boundary control problems with continuous input-solution map, Part II: The general case*, Applied Mathematics and Optimization, 29 (1994), pp. 1–65.
- [63] B. C. MOORE, *Principal component analysis in linear systems: controllability, observability, and model reduction*, IEEE Transactions on Automatic Control, 26 (1981), pp. 17–32.

- [64] K. MORRIS AND C. NAVASCA, *Solution of algebraic Riccati equations arising in control of partial differential equations*, in Control and Boundary Analysis, vol. 240 of Lecture Notes in Pure and Applied Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2005, pp. 257–280.
- [65] J.-P. RAYMOND, *Feedback boundary stabilization of the two-dimensional Navier-Stokes equations*, SIAM Journal on Control and Optimization, 45 (2006), pp. 790–828.
- [66] I. RHEE AND J. SPEYER, *A game theoretic controller and its relationship to  $H_\infty$  and linear-exponential-Gaussian synthesis*, Proceedings of the 28th Conference on Decision and Control, 2 (1989), pp. 909–915.
- [67] P. J. ROACHE, *Code verification by the method of manufactured solutions*, Trans. ASME, 124 (2002), pp. 4–10.
- [68] C. ROWLEY, *Model reduction for fluids, using balanced proper orthogonal decomposition*, Int. J. on Bifurcation and Chaos, 15 (2005), pp. 997–1013.
- [69] J. SINGLER, *Sensitivity Analysis of Partial Differential Equations With Applications to Fluid Flow*, PhD thesis, Department of Mathematics, Virginia Polytechnic Institute and State University, 2005.
- [70] L. SIROVICH, *Turbulence and the dynamics of coherent structures, parts I-III*, Quarterly of Applied Mathematics, 45 (1987), pp. 561–590.
- [71] L. G. STANLEY, *Computational Methods for Sensitivity Analysis with Applications to Elliptic Boundary Value Problems*, PhD thesis, Department of Mathematics, Virginia Polytechnic Institute and State University, 1999.
- [72] ———, *Sensitivity equation methods for parameter dependent elliptic equations*, Numerical Functional Analysis and Optimization, 22 (2001), pp. 721–748.
- [73] ———, *Sensitivity analysis for actuator placement on an Euler-Bernoulli beam*, in Proceedings of the 43rd IEEE Conference on Decision and Control, 2004, pp. 1532 – 1537.
- [74] L. G. STANLEY AND D. L. STEWART, *Design Sensitivity Analysis: Computational Issues of Sensitivity Equation Methods.*, SIAM, Philadelphia, PA, 2002.
- [75] D. L. STEWART, *Numerical Methods for Accurate Computation of Design Sensitivities*, PhD thesis, Department of Mathematics, Virginia Polytechnic Institute and State University, 1998.
- [76] A. THEODOROPOULOU, R. A. ADOMAITIS, AND E. ZAFIRIOU, *Model reduction for optimization of rapid thermal chemical vapor deposition systems*, IEEE Trans. Semicond. Manuf., 11 (1998), pp. 85–98.
- [77] E. I. VERRIEST, *Low sensitivity design and optimal order reduction for the LQG problem*, in 24th Midwest Symposium on Circuits and Systems, 1981, pp. 365–369.
- [78] ———, *Suboptimal LQG-design via balanced realizations*, in 20th IEEE Conference on Decision and Control, vol. 2, 1981, pp. 686–687.
- [79] I. WYGNANSKI, *Some observations affecting the control of separation by periodic excitation*, in Proceedings of the AIAA Annual Meeting, 2000. AIAA paper 2314-2000.

- [80] K. ZHOU AND J. DOYLE, *Essentials of Robust Control*, Prentice Hall, Upper Saddle River, NJ, 1998.

## 2 Publications

### Published in Peer-Reviewed Journals and Peer-Reviewed Proceedings

1. “A Petrov-Galerkin Finite Element Method for Interface Problems Arising in Sensitivity Computations,” J. Burns, T. Lin and L. G. Stanley, *Computers and Mathematics with Applications*, Vol. 49 (2005), pages 1889-1903.
2. “Sensitivity Analysis for Actuator Placement on an Euler-Bernoulli Beam,” L. G. Stanley, 43rd IEEE Conference of Decision and Control Proceedings, (Proceedings of the 43rd IEEE Conference on Decision and Control, December 14-17, 2004, Nassau, The Bahamas.) IEEE Catalog Number 04CH37601C, CD Rom ISBN:0-7803-8683-3, Paper ISBN: 0-7803-8682-5/04, Paper Reference: 0260-WeA04.6,
3. “Sensitivity Computations of Eddy Viscosity Models with an Application in Drag Computation,” F. Pahlevani, *International Journal for Numerical Methods in Fluids*, February 2006, Vol. 52, pages 381-392.

### Published in Un-Reviewed Conference Proceedings

1. “Modeling of Bioinspired Sensors for Flow Separation Detection in Micro Air Vehicles,” B.T. Dickinson, J.R. Singler and B.A. Batten, in *Proceedings of the 3rd AIAA Flow Control Conference*, 2006, AIAA paper number 2006-3019.

### Accepted for Publication

1. “Differentiability With Respect to Parameters of Weak Solutions of Linear Parabolic Equations,” J.R. Singler, *to appear in Mathematical and Computer Modelling*.

### Submitted for Publication

1. “Estimation Methods for Resonance Electron Capture and Thermal Detachment Rates Using Pulsed High Pressure Mass Spectrometry,” L. G. Davis, M. Larkin, M. Pernarowski, submitted to the *Journal of Chemical Physics*, *in revision*.
2. “Parameter Sensitivity Analysis of Eddy Viscosity Models for Navier-Stokes Equations”, F. Pahlevani, *submitted to SIAM Journal on Numerical Analysis*, *in revision*.
3. “Semi-Implicit Schemes for Transient Navier-Stokes Equations and Eddy Viscosity Models,” F. Pahlevani and L.G. Davis, *submitted to Numerical Methods for Partial Differential Equations*, *in revision*.
4. “Transition to Turbulence, Small Disturbances, and Sensitivity Analysis I: A Motivating Problem,” J. R. Singler, *submitted to Journal of Mathematical Analysis and Applications*.
5. “Transition to Turbulence, Small Disturbances, and Sensitivity Analysis II: The Navier-Stokes Equations,” J. R. Singler, *submitted to Journal of Mathematical Analysis and Applications*.

### 3 Technology Assists, Transitions or Transfers

None

### 4 Accomplishments and Successes

The research has shown that sensitivity information has the potential to provide valuable information for optimizing sensor and actuator location for control design. However, investigations into certain types of model problems also has shown that when using the continuous sensitivity equation method to compute sensitivities, the formulation and regularity of the mathematical model governing the underlying system behavior must be well understood. Various mathematical models have been investigated with varying degrees of success. Structural models such as the outlined in the report include the Euler-Bernoulli beam model with discontinuous coefficients representing actuator placement along with some models that arose due to simplification of this model. Dr. Faranak Pahlevani investigated various aspects sensitivity analysis with respect to the eddy viscosity parameter in Eddy Viscosity Models which are a subclass of Large-Eddy Simulation models for computational fluid dynamics applications. Dr. John Singler has developed a theoretical framework for the derivation of continuous sensitivity equations for a certain class of parabolic PDE models. In addition, he has collaborated with the PI on various aspects of sensitivity computation for the Euler-Bernoulli beam model. Dr. Singler has also investigated the use sensitivity analysis to explain and predict transition to turbulence for Burgers' Equation models with the goal of future application to the study of transition to turbulence for Navier-Stokes equations.

The successes of this research effort include the advance of the mathematical and computational tools for accurate computation of parameter sensitivities for systems governed by elliptic, parabolic as well as nonlinear elliptic partial differential equations. Through the use of such available methods as the Petrov-Galerkin Finite Element Method, accurate sensitivity computation for elliptic interface problems were shown to be a viable option. Other types of methods including Discontinuous Galerkin as well as Immersed Interface methods are also very likely to prove useful for future efforts in this area. Furthermore, the research has clearly shown that relying on black-box solvers for accurate, as well as numerically convergent, sensitivity calculations may lead to problems. The research has clearly shown that for mathematical models where the parameter of interest corresponds to locations of sensors or actuators, the derivation and simulation of sensitivity equations must be handled with caution. The main successes of the project include the discovery of examples for which standard continuous sensitivity equation methods work well and may be done somewhat automatically as well as the identification of certain types of mathematical models where the casual sensitivity equation derivation will likely lead to misleading sensitivity information. This may ultimately lead to inaccurate sensitivity information used for the placement of sensors and actuators in control systems which may ultimately lead to poor performance of various control systems. Further investigations into several of these ideas is warranted, and the PI and collaborators continue to investigate many of the ideas uncovered during the life of the funding.

## 5 Professional Personnel Supported

All of the following personnel were supported in a full-time capacity.

Bowman, Michael	<b>Graduate Student</b> , Montana State University
Davis, Lisa G.	<b>Associate Professor</b> , Montana State University
Larkin, Michael	<b>Graduate Student</b> , Montana State University
Pahlevani, Faranak	<b>Postdoc</b> , Montana State University
Singler, John	<b>Postdoc</b> , Montana State University/Oregon State University
Thorenson, Jennifer	<b>Graduate Student</b> , Montana State University

## 6 Honors and Awards

None

## 7 Professional Activities

The following is a list of the professional societies for which each of the primary contributors of this research is currently a member. In addition, each of these primary contributors has given invited talks at professional society meetings over the course of the grant.

Dr. Lisa G. Davis

- Society for Industrial and Applied Mathematics (SIAM)

Dr. Faranak Pahlevani

- American Mathematical Society (AMS)
- Society for Industrial and Applied Mathematics (SIAM)

Dr. John Singler

- American Institute for Aeronautics and Astronautics (AIAA)
- Society for Industrial and Applied Mathematics (SIAM)