

A fractional step θ -method for viscoelastic fluid flow using a SUPG approximation

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Abstract

In this article a fractional step θ -method is described and studied for the approximation of time dependent viscoelastic fluid flow equations, using the Johnson-Segalman constitutive model. The θ -method implementation allows the velocity and pressure approximations to be decoupled from the stress, reducing the number of unknowns resolved at each step of the method. The constitutive equation is stabilized using a Streamline Upwinded Petrov-Galerkin (SUPG)-method. A priori error estimates are given for the approximation scheme. Numerical computations supporting the theoretical results and demonstrating the θ -method are also presented.

Key words. θ -method; splitting method; viscoelastic flow

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1 Introduction

Numerical methods for modeling viscoelastic fluid flow are difficult for a variety of reasons. The modeling equations (assuming *slow* flow) represent a “Stokes system” for the conservation of mass and momentum equations, coupled with a non-linear hyperbolic equation describing the constitutive equation for the stress. The numerical approximation requires the determination of the fluid’s velocity, pressure and stress (a symmetric tensor). For an accurate approximation a direct approximation technique requires the solution of a very large non-linear system of equations at each time step.

The fractional step θ -method [25, 23, 24] decouples the approximation of velocity and pressure from the approximation of the stress, thereby reducing the size of the algebraic systems which have to be solved at each sub-step. An added benefit of the θ -method [25] is that the algebraic systems to be solved at each sub-step are linear.

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The fractional step θ -method was introduced, and its temporal approximation accuracy studied for a symmetric, positive definite spatial operator, by Glowinski and Pirreanu in [11]. The method is widely used for the accurate approximation of the Navier-Stokes equations (NSE) [26, 27, 14]. In [15], Kloucek and Rys showed, assuming a unique solution existed, that the θ -method approximation converged to the solution of the NSE as the spatial and mesh parameters went to zero ($h, \Delta t \rightarrow 0_+$). The temporal discretization error for the θ -method for the NSE was studied by Müller-Urbaniak in [18] and shown to be second order. In [7] the θ -method applied to convection-diffusion equations was shown to be second order in time.

The implementation of the fractional step θ -method in [25] for viscoelasticity differs significantly from that for the NSE. For the NSE at each sub-step the discretization contains the *stabilizing* operator $-\Delta \mathbf{u}$. For the viscoelasticity problem the middle-substep is a pure convection (transport) problem that requires stabilization in order to control the creation of spurious oscillations in the numerical approximation. Marchal and Crochet [17] were the first to use streamline upwinding to stabilize the hyperbolic constitutive equation in viscoelastic flow. A second common approach to stabilizing the convective transport problem is to use a discontinuous Galerkin (DG) approximation for the stress [2, 1].

Error analysis of finite element approximations to steady state viscoelastic flow was first done by Baranger and Sandri in [2] using a DG formulation of the constitutive equation. In [22] Sandri presented analysis of the steady state problem using a streamline upwind Petrov-Galerkin (SUPG) method of stabilization. The time-dependent problem was first analyzed by Baranger and Wardi in [3], using an implicit Euler temporal discretization, and DG approximation for the hyperbolic constitutive equation. Ervin and Miles analyzed the problem using an implicit Euler time discretization, and a SUPG discretization for the stress in [9]. Analysis of a modified Euler-SUPG approximation to the transient viscoelastic flow problem was presented by Bensaada and Esselaoui in [4]. The temporal accuracy of the approximation schemes studied in [3, 4, 9] are all $O(\Delta t)$. Ervin and Heuer proposed a Crank-Nicolson time discretization method [8] which they showed was $O((\Delta t^2))$. Their method uses a three level scheme to approximate the non-linear terms in the equations. Consequently their approximation algorithm only requires linear systems of equations to be solved.

In this article we analyze a fractional step θ -method for the approximation of viscoelastic fluid flows. Advantages of the method are three fold: (i) second order accuracy with respect to the temporal discretization, (ii) only linear systems of equations need to be solved, (iii) the linear systems to be solved only involve the velocity-pressure or the stress unknowns (resulting in smaller linear systems).

This paper is organized as follows. In the next section we specify the problem and describe the fractional step θ -method for the viscoelastic modeling equations. In Section 3 the mathematical notation used is given. In Section 4 we show computability of the algorithm, and present the a priori error estimates that support the method. Two numerical examples demonstrating the method are presented in Section 7.

2 The Mathematical Model and θ -Method Approximation

In this section we present the modeling equations for viscoelastic fluid flow as well as a fractional step θ -method approximation scheme. Following the description of the θ -method, several definitions

used to formulate the problem in an appropriate mathematical setting are given.

2.1 Modeling Equations

The non-dimensional modeling equations for a viscoelastic fluid in a given domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) using a Johnson-Segalman constitutive equation are written as:

$$\boldsymbol{\sigma} + \lambda \left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}) \right) - 2\alpha \mathbf{d}(\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega, \quad (2.1)$$

$$Re \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p - 2(1 - \alpha) \nabla \cdot \mathbf{d}(\mathbf{u}) - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f} \quad \text{in } \Omega, \quad (2.2)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (2.3)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \quad (2.4)$$

$$\mathbf{u}(0, x) = \mathbf{u}_0(x) \quad \text{in } \Omega, \quad (2.5)$$

$$\boldsymbol{\sigma}(0, x) = \boldsymbol{\sigma}_0(x) \quad \text{in } \Omega. \quad (2.6)$$

Here (2.1) is the constitutive equation relating the fluids velocity $\mathbf{u}$ to the stress $\boldsymbol{\sigma}$, and (2.2) and (2.3) are the conservation of momentum and conservation of mass equations. The fluid pressure is denoted by p . The Weissenberg number λ is a dimensionless constant defined as the product of a characteristic strain rate and the relaxation time of the fluid [5]. Re denotes the fluids Reynolds number, $\mathbf{f}$ are the body forces acting on the fluid, and $\alpha \in (0, 1)$ denotes the fraction of the total viscosity that is viscoelastic.

The term g_a and deformation tensor $\mathbf{d}(\mathbf{u})$ are defined as:

$$g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}) := \frac{1-a}{2} \left(\boldsymbol{\sigma} \nabla \mathbf{u} + (\nabla \mathbf{u})^T \boldsymbol{\sigma} \right) - \frac{1+a}{2} \left(\nabla \mathbf{u} \boldsymbol{\sigma} + \boldsymbol{\sigma} (\nabla \mathbf{u})^T \right)$$

and

$$\mathbf{d}(\mathbf{u}) = \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right).$$

The gradient of $\mathbf{u}$ is defined such that $(\nabla \mathbf{u})_{i,j} = \partial u_i / \partial x_j$. For the remainder of the paper slow or inertialess flow is assumed, allowing the term $\mathbf{u} \cdot \nabla \mathbf{u}$ in (2.2) to be ignored. Note that when the parameter $a = 1$ in $g_a(\boldsymbol{\sigma}, \nabla \mathbf{u})$ an Oldroyd B constitutive model is obtained. For existence and uniqueness of solutions to (2.1)-(2.6) see [21, 12, 10].

2.2 θ -method

In order to implement a fractional step θ -method for the viscoelastic flow equations, an additive decomposition is used for equations (2.1) and (2.2). Here we introduce a splitting parameter $\omega \in (0, 1)$, and define:

Constitutive equation:

$${}_1G\boldsymbol{\sigma} := \omega \boldsymbol{\sigma}, \quad (2.7)$$

$${}_2G\boldsymbol{\sigma} := (1 - \omega) \boldsymbol{\sigma} + \lambda (\mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \mathbf{u})) - 2\alpha \mathbf{d}(\mathbf{u}). \quad (2.8)$$

Conservation of Momentum:

$${}_1F\mathbf{u} := -2(1 - \alpha)\nabla \cdot \mathbf{d}(\mathbf{u}) - \nabla \cdot \boldsymbol{\sigma} - \mathbf{f}, \quad (2.9)$$

$${}_2F\mathbf{u} := 0. \quad (2.10)$$

Let Δt denote the temporal increment between times t^n and t^{n+1} , and for $c \in \{\theta, \omega, a, \alpha\}$ let $\tilde{c} := 1 - c$. Also, let $f^{(n)} := f(\cdot, n\Delta t)$.

The θ -method approximation for viscoelasticity may then be described as follows. (See also [25, 23]).

θ -Method Algorithm for Viscoelasticity

Step 1a: (Update the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta \Delta t} + {}_1G\boldsymbol{\sigma}^{(n+\theta)} = -{}_2G\boldsymbol{\sigma}^{(n)}.$$

Step 1b: (Solve for velocity and pressure.)

$$\begin{aligned} Re \frac{\mathbf{u}^{(n+\theta)} - \mathbf{u}^{(n)}}{\theta \Delta t} + \nabla p^{(n+\theta)} + {}_1F\mathbf{u}^{(n+\theta)} &= -{}_2F\mathbf{u}^{(n)}, \\ \nabla \cdot \mathbf{u}^{(n+\theta)} &= 0. \end{aligned}$$

Step 2a: (Update the velocity and pressure.)

$$\begin{aligned} Re \frac{\mathbf{u}^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\theta)}}{(1 - 2\theta) \Delta t} + \nabla p^{(n+\tilde{\theta})} + {}_2F\mathbf{u}^{(n+\tilde{\theta})} &= -{}_1F\mathbf{u}^{(n+\theta)}, \\ \nabla \cdot \mathbf{u}^{(n+\tilde{\theta})} &= 0. \end{aligned}$$

Step 2b: (Solve for the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\theta)}}{(1 - 2\theta) \Delta t} + {}_2G\boldsymbol{\sigma}^{(n+\tilde{\theta})} = -{}_1G\boldsymbol{\sigma}^{(n+\theta)}.$$

Step 3a and Step 3b: In order to complete the temporal advancement to time t_{n+1} , Step 1a, and Step 1b are repeated with (n) and $(n + \theta)$ replaced by $(n + \tilde{\theta})$ and $(n + 1)$ respectively.

Note: Due to the chosen decomposition of the constitutive and conservation of momentum equations, (2.7)-(2.10), the approximation of the nonlinear system (2.1)-(2.6) using the θ -method only requires linear systems of equations to be solved at each step in the process.

3 Mathematical Notation

The $L^2(\Omega)$ inner product and norm are denoted by $(\cdot, \cdot)$, and $\|u\|$ respectively. The Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{W_p^k}$. We use H^k to represent the Sobolev space W_2^k , and $\|\cdot\|_k$ to denote

the norm in H^k . Function spaces used in the analysis are:

$$\begin{aligned}
X &:= H_0^1(\Omega) := \{\mathbf{u} \in H^1(\Omega) : \mathbf{u} = 0 \text{ on } \partial\Omega\}, \\
S &:= \left\{ \boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji}; \sigma_{ij} \in L^2(\Omega); 1 \leq i, j \leq \acute{d} \right\} \\
&\quad \cap \left\{ \boldsymbol{\sigma} = (\sigma_{ij}) : \mathbf{u} \cdot \nabla \boldsymbol{\sigma} \in L^2(\Omega), \forall \mathbf{u} \in X \right\}, \\
Q &:= L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \right\}, \\
Z &:= \left\{ \mathbf{v} \in X : \int_{\Omega} q(\nabla \cdot \mathbf{v}) \, dx = 0, \forall q \in Q \right\}.
\end{aligned}$$

Recall that the spaces X and Q satisfy the *inf-sup* condition

$$\inf_{q \in Q} \sup_{\mathbf{v} \in X} \frac{(q, \nabla \cdot \mathbf{v})}{\|q\| \|\mathbf{v}\|_1} \geq \beta > 0. \quad (3.1)$$

A variational formulation of (2.1)-(2.3), found by multiplication of the modeling equations by test functions and integrating over Ω , is: *Find* $(\mathbf{u}, \boldsymbol{\sigma}, p) : (0, T] \rightarrow X \times S \times Q$ *such that*

$$\lambda \left(\frac{\partial \boldsymbol{\sigma}}{\partial t}, \boldsymbol{\tau} \right) + (\boldsymbol{\sigma}, \boldsymbol{\tau}) - 2\alpha (\mathbf{d}(\mathbf{u}), \boldsymbol{\tau}) + \lambda (\mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}), \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in S, \quad (3.2)$$

$$Re \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) - (p, \nabla \cdot \mathbf{v}) + 2(1 - \alpha) (\mathbf{d}(\mathbf{u}), \mathbf{d}(\mathbf{v})) + (\boldsymbol{\sigma}, \mathbf{d}(\mathbf{v})) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in X, \quad (3.3)$$

$$(\nabla \cdot \mathbf{u}, q) = 0, \quad \forall q \in Q, \quad (3.4)$$

$$\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad (3.5)$$

$$\boldsymbol{\sigma}(0, x) = \boldsymbol{\sigma}_0(x). \quad (3.6)$$

As the velocity and pressure spaces X and Q satisfy the inf-sup condition (3.1), an equivalent variational formulation to (3.2)-(3.4) is given by: *Find* $(\mathbf{u}, \boldsymbol{\sigma}) : (0, T] \rightarrow Z \times S$ *such that*

$$\lambda \left(\frac{\partial \boldsymbol{\sigma}}{\partial t}, \boldsymbol{\tau} \right) + (\boldsymbol{\sigma}, \boldsymbol{\tau}) - 2\alpha (\mathbf{d}(\mathbf{u}), \boldsymbol{\tau}) + \lambda (\mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}), \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in S, \quad (3.7)$$

$$Re \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + 2(1 - \alpha) (\mathbf{d}(\mathbf{u}), \mathbf{d}(\mathbf{v})) + (\boldsymbol{\sigma}, \mathbf{d}(\mathbf{v})) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in Z, \quad (3.8)$$

$$\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad (3.9)$$

$$\boldsymbol{\sigma}(0, x) = \boldsymbol{\sigma}_0(x). \quad (3.10)$$

To describe the finite element framework let T_h be a triangulation of the discretized domain $\Omega \subset \mathbb{R}^{\acute{d}}$. Then

$$\bar{\Omega} = \cup K, \quad K \in T_h.$$

We assume that there exist constants c_1 and c_2 such that

$$c_1 h \leq h_K \leq c_2 \rho_K,$$

where h_k is the diameter of triangle K , ρ_K is the diameter of the greatest ball (sphere) included in K , and $h = \max_{K \in T_h} h_K$. Let $P_k(A)$ denote the space of polynomials on A of degree no greater

than k and $C(\bar{\Omega})^{\dot{d}}$ the space of vector valued functions with $\dot{d}$ components which are continuous on $\bar{\Omega}$. Then the associated finite element spaces are defined by:

$$\begin{aligned} X_h &:= \left\{ \mathbf{v} \in X \cap C(\bar{\Omega})^{\dot{d}} : \mathbf{v}|_K \in P_k(K) \forall K \in T_h \right\}, \\ S_h &:= \left\{ \boldsymbol{\tau} \in S \cap C(\bar{\Omega})^{\dot{d} \times \dot{d}} : \boldsymbol{\tau}|_K \in P_m(K) \forall K \in T_h \right\}, \\ Q_h &:= \left\{ q \in Q \cap C(\bar{\Omega}) : q|_K \in P_q(K) \forall K \in T_h \right\}, \\ Z_h &:= \left\{ \mathbf{v} \in X_h : (q, \nabla \cdot \mathbf{v}) = 0 \forall q \in Q_h \right\}. \end{aligned}$$

Analogically to the continuous spaces assume that X_h and Q_h satisfy the discrete *inf-sup* condition:

$$\inf_{q \in Q_h} \sup_{\mathbf{v} \in X_h} \frac{(q, \nabla \cdot \mathbf{v})}{\|q\| \|\mathbf{v}\|_1} \geq \beta > 0. \quad (3.11)$$

Let $\mathcal{U}$ and $\mathcal{S}$ denote the L^2 projections of $\mathbf{u}$ and $\boldsymbol{\sigma}$ onto Z_h and S_h , respectively, and define:

$$\begin{aligned} \boldsymbol{\Lambda}^{(n)} &= \mathbf{u}^{(n)} - \mathcal{U}^{(n)}, & \mathbf{E}^{(n)} &= \mathcal{U}^{(n)} - \mathbf{u}_h^{(n)}, \\ \boldsymbol{\Gamma}^{(n)} &= \boldsymbol{\sigma}^{(n)} - \mathcal{S}^{(n)}, & \mathbf{F}^{(n)} &= \mathcal{S}^{(n)} - \boldsymbol{\sigma}_h^{(n)}, \\ e_{\mathbf{u}}^{(n)} &= \mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}, & e_{\boldsymbol{\sigma}}^{(n)} &= \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}_h^{(n)}. \end{aligned}$$

We define the discrete temporal operator

$$d_t f^{(n+1)} := \frac{f(t_{n+1}) - f(t_n)}{\Delta t}.$$

When $v(\mathbf{x}, t)$ is defined on the entire time interval $(0, T)$,

$$\|v\|_{\infty, k} := \sup_{0 < t < T} \|v(\cdot, t)\|_k, \quad \|v\|_{0, k} := \left(\int_0^T \|v(\cdot, t)\|_k^2 dt \right)^{1/2}, \quad \|v\|(t) := \|v(\cdot, t)\|.$$

The following discrete norms are used:

$$|||v|||_{\infty, k} := \max_{1 \leq n \leq N} \|v^{(n)}\|_k, \quad |||v|||_{0, k} := \left(\sum_{n=1}^N \Delta t \|v^{(n)}\|_k^2 \right)^{\frac{1}{2}}.$$

4 Analysis

In this section we investigate the numerical approximation method corresponding to (3.7)-(3.10). First the discrete variational formulation of the θ -method is given. Then computability of the algorithm is shown and a priori error estimates given.

4.1 Discrete Variational Approximation

To stabilize the constitutive equation a streamline upwind Petrov-Galerkin (SUPG) discretization is used to control spurious oscillations in the approximation. This is implemented by testing all

terms in the constitutive equation (except the discretized temporal derivative) against modified test elements of the form $\boldsymbol{\tau}_{\delta(n)}$ where

$$\boldsymbol{\tau}_{\delta(n)} := \boldsymbol{\tau} + \delta \mathbf{u}_h^{(n)} \cdot \nabla \boldsymbol{\tau}. \quad (4.1)$$

Note that if δ is set to zero the standard Galerkin method is obtained. The variational formulations for the steps in the θ -method approximation are as follows.

Step 1a: Find $\boldsymbol{\sigma}_h^{(n+\theta)} \in S_h$ such that

$$\begin{aligned} \frac{\lambda}{\theta \Delta t} \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau} \right) + \omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta(n)} \right) &= \frac{\lambda}{\theta \Delta t} \left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau} \right) - (1-\omega) \left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta(n)} \right) - \lambda \left(\mathbf{u}_h^{(n)} \cdot \nabla \boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta(n)} \right) \\ &\quad - \lambda \left(g_a(\boldsymbol{\sigma}_h^{(n)}, \nabla \mathbf{u}_h^{(n)}), \boldsymbol{\tau}_{\delta(n)} \right) + 2\alpha \left(\mathbf{d}(\mathbf{u}_h^{(n)}), \boldsymbol{\tau}_{\delta(n)} \right), \quad \forall \boldsymbol{\tau} \in S_h. \end{aligned} \quad (4.2)$$

Step 1b: Find $\mathbf{u}_h^{(n+\theta)} \in Z_h$ such that

$$\begin{aligned} \frac{Re}{\theta \Delta t} \left(\mathbf{u}_h^{(n+\theta)}, \mathbf{v} \right) + 2(1-\alpha) \left(\mathbf{d}(\mathbf{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v}) \right) \\ = \frac{Re}{\theta \Delta t} \left(\mathbf{u}_h^{(n)}, \mathbf{v} \right) + \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) - \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (4.3)$$

Step 2a: Find $\mathbf{u}_h^{(n+\bar{\theta})} \in Z_h$ such that

$$\begin{aligned} \frac{Re}{(1-2\theta)\Delta t} \left(\mathbf{u}_h^{(n+\bar{\theta})}, \mathbf{v} \right) &= \frac{Re}{(1-2\theta)\Delta t} \left(\mathbf{u}_h^{(n+\theta)}, \mathbf{v} \right) \\ &\quad - 2(1-\alpha) \left(\mathbf{d}(\mathbf{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v}) \right) + \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) - \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (4.4)$$

Step 2b: Find $\boldsymbol{\sigma}_h^{(n+\bar{\theta})} \in S_h$ such that

$$\begin{aligned} \frac{\lambda}{(1-2\theta)\Delta t} \left(\boldsymbol{\sigma}_h^{(n+\bar{\theta})}, \boldsymbol{\tau} \right) + (1-\omega) \left(\boldsymbol{\sigma}_h^{(n+\bar{\theta})}, \boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right) &+ \lambda \left(\mathbf{u}_h^{(n+\bar{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\bar{\theta})}, \boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right) \\ &+ \lambda \left(g_a(\boldsymbol{\sigma}_h^{(n+\bar{\theta})}, \nabla \mathbf{u}_h^{(n+\bar{\theta})}), \boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right) - 2\alpha \left(\mathbf{d}(\mathbf{u}_h^{(n+\bar{\theta})}), \boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right) \\ &= \frac{\lambda}{(1-2\theta)\Delta t} \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau} \right) - \omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right), \quad \forall \boldsymbol{\tau} \in S_h. \end{aligned} \quad (4.5)$$

Step 3a: Find $\boldsymbol{\sigma}_h^{(n+1)} \in S_h$ such that

$$\begin{aligned} \frac{\lambda}{\theta \Delta t} \left(\boldsymbol{\sigma}_h^{(n+1)}, \boldsymbol{\tau} \right) + \omega \left(\boldsymbol{\sigma}_h^{(n+1)}, \boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right) &= \frac{\lambda}{\theta \Delta t} \left(\boldsymbol{\sigma}_h^{(n+\bar{\theta})}, \boldsymbol{\tau} \right) - (1-\omega) \left(\boldsymbol{\sigma}_h^{(n+\bar{\theta})}, \boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right) \\ &\quad - \lambda \left(\mathbf{u}_h^{(n+\bar{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\bar{\theta})}, -\boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right) + \lambda \left(g_a(\boldsymbol{\sigma}_h^{(n+\bar{\theta})}, \nabla -\mathbf{u}_h^{(n+\bar{\theta})}), \boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right) \\ &\quad + 2\alpha \left(-\mathbf{d}(\mathbf{u}_h^{(n+\bar{\theta})}), \boldsymbol{\tau}_{\delta(n+\bar{\theta})} \right), \quad \forall \boldsymbol{\tau} \in S_h. \end{aligned} \quad (4.6)$$

Step 3b: Find $\mathbf{u}_h^{(n+1)} \in Z_h$ such that

$$\begin{aligned} \frac{Re}{\theta \Delta t} \left(\mathbf{u}_h^{(n+1)}, \mathbf{v} \right) + 2(1 - \alpha) \left(\mathbf{d}(\mathbf{u}_h^{(n+1)}), \mathbf{d}(\mathbf{v}) \right) \\ = \frac{Re}{\theta \Delta t} \left(\mathbf{u}_h^{(n+\tilde{\theta})}, \mathbf{v} \right) + \left(\mathbf{f}^{(n+1)}, \mathbf{v} \right) - \left(\boldsymbol{\sigma}_h^{(n+1)}, \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (4.7)$$

4.2 Existence and Uniqueness

Before error estimates are presented the computability of the θ -method is shown. This is accomplished by proving that the associated coefficient matrices used in each step of the algorithm are invertible.

The following induction hypothesis is used: There exists a constant K such that for $n = 1, \dots, N$

$$\left\| \mathbf{u}_h^{(n)} \right\|_{\infty}, \left\| \mathbf{u}_h^{(n+\tilde{\theta})} \right\|_{\infty} \leq K. \quad (\text{IH1})$$

The justification of (IH1) is established below.

Lemma 1 (Step 1a) *Assume (IH1) is true. For $\delta \leq Ch$ and Δt sufficiently small there exists a unique solution $\boldsymbol{\sigma}_h^{(n+\theta)} \in S_h$ satisfying (4.2).*

Proof: Equation (4.2) can be equivalently written as

$$\begin{aligned} \mathcal{A}_1 \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau} \right) &= \frac{\lambda}{\theta \Delta t} \left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau} \right) - \left((1 - \omega) \boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta(n)} \right) \\ &\quad - \lambda \left(\left(\mathbf{u}_h^{(n)} \cdot \nabla \boldsymbol{\sigma}_h^{(n)} + g_a(\boldsymbol{\sigma}_h^{(n)}, \nabla \mathbf{u}_h^{(n)}) \right), \boldsymbol{\tau}_{\delta(n)} \right) + 2\alpha \left(\mathbf{d}(\mathbf{u}_h^{(n)}), \boldsymbol{\tau}_{\delta(n)} \right), \quad \forall \boldsymbol{\tau} \in S_h, \end{aligned} \quad (4.8)$$

where

$$\mathcal{A}_1 \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau} \right) := \frac{\lambda}{\theta \Delta t} \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau} \right) + \omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta(n)} \right).$$

Here (4.8) represents a square linear system of equations $A\mathbf{x} = \mathbf{b}$. With the choice $\boldsymbol{\tau} = \boldsymbol{\sigma}_h^{(n+\theta)}$, examining the individual terms in $\mathcal{A}_1$ we see that

$$\begin{aligned} \frac{\lambda}{\theta \Delta t} \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\sigma}_h^{(n+\theta)} \right) &= \frac{\lambda}{\theta \Delta t} \left\| \boldsymbol{\sigma}_h^{(n+\theta)} \right\|^2, \\ \omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\sigma}_h^{(n+\theta)} \right) &= \omega \left\| \boldsymbol{\sigma}_h^{(n+\theta)} \right\|^2, \end{aligned}$$

and

$$\begin{aligned} \omega \delta \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \mathbf{u}_h^{(n)} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\theta)} \right) &= \frac{\omega \delta}{2} \left(\mathbf{u}_h^{(n)}, \nabla \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\sigma}_h^{(n+\theta)} \right) \right) \\ &\leq \frac{\omega \delta}{2} \left\| \mathbf{u}_h^{(n)} \right\|_{\infty} Ch^{-1} \left\| \boldsymbol{\sigma}_h^{(n+\theta)} \right\|^2 \\ &\leq \frac{\omega \delta K Ch^{-1}}{2} \left\| \boldsymbol{\sigma}_h^{(n+\theta)} \right\|^2. \end{aligned}$$

Provided $\delta \leq Ch$, and $\Delta t \leq (2\lambda)/(\theta\omega KC)$ then $\mathcal{A}_1 \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\sigma}_h^{(n+\theta)} \right) > 0$, and thus the $\ker(\mathcal{A}_1) = \{\mathbf{0}\}$. It then follows that (4.2) has a unique solution. ■

Lemma 2 (Step 1b) *There exists a unique solution $\mathbf{u}_h^{(n+\theta)} \in Z_h$ satisfying (4.3).*

Proof: Equation (4.3) can be equivalently written as

$$\mathcal{A}_2 \left(\mathbf{u}_h^{(n+\theta)}, \mathbf{v} \right) = \frac{Re}{\theta\Delta t} \left(\mathbf{u}_h^{(n)}, \mathbf{v} \right) + \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) - \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h,$$

where

$$\mathcal{A}_2 \left(\mathbf{u}_h^{(n+\theta)}, \mathbf{v} \right) := \frac{Re}{\theta\Delta t} \left(\mathbf{u}_h^{(n+\theta)}, \mathbf{v} \right) + 2(1-\alpha) \left(\mathbf{d}(\mathbf{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v}) \right).$$

Note that choosing $\mathbf{v} = \mathbf{u}_h^{(n+\theta)}$

$$\mathcal{A}_2 \left(\mathbf{u}_h^{(n+\theta)}, \mathbf{u}_h^{(n+\theta)} \right) = \frac{Re}{\theta\Delta t} \left(\mathbf{u}_h^{(n+\theta)}, \mathbf{u}_h^{(n+\theta)} \right) + 2(1-\alpha) \left(\mathbf{d}(\mathbf{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{u}_h^{(n+\theta)}) \right) > 0.$$

and existence and uniqueness of a solution to (4.3) has been shown. ■

Lemma 3 (Step 2a) *There exists a unique solution $\mathbf{u}_h^{(n+\tilde{\theta})} \in Z_h$ satisfying (4.4).*

Proof: First write equation (4.4) as

$$\begin{aligned} \mathcal{A}_3 \left(\mathbf{u}_h^{(n+\tilde{\theta})}, \mathbf{v} \right) &= \frac{Re}{(1-2\theta)\Delta t} \left(\mathbf{u}_h^{(n+\theta)}, \mathbf{v} \right) - 2(1-\alpha) \left(\mathbf{d}(\mathbf{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v}) \right) \\ &\quad + \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) - \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \mathbf{d}(\mathbf{v}) \right), \end{aligned}$$

where

$$\mathcal{A}_3 \left(\mathbf{u}_h^{(n+\tilde{\theta})}, \mathbf{v} \right) := \frac{Re}{(1-2\theta)\Delta t} \left(\mathbf{u}_h^{(n+\tilde{\theta})}, \mathbf{v} \right).$$

For the choice of $\mathbf{v} = \mathbf{u}_h^{(n+\tilde{\theta})}$, the system $\mathcal{A}_3 \left(\mathbf{u}_h^{(n+\tilde{\theta})}, \mathbf{u}_h^{(n+\tilde{\theta})} \right) > 0$, and the proof is complete. ■

Lemma 4 (Step 2b) *Assume (IH1) is true. For $\delta \leq Ch$ and Δt sufficiently small there exists a unique solution $\boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \in S_h$ satisfying (4.5).*

Proof: Write (4.5) as

$$\mathcal{A}_4 \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau} \right) = \frac{\lambda}{(1-2\theta)\Delta t} \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau} \right) - \omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) + 2\alpha \left(\mathbf{d}(\mathbf{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right),$$

with

$$\begin{aligned} \mathcal{A}_4 \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau} \right) &:= \frac{\lambda}{(1-2\theta)\Delta t} \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau} \right) + (1-\omega) \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) \\ &\quad + \lambda \left(\mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) + \lambda \left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla \mathbf{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right). \end{aligned}$$

Estimating the terms in $\mathcal{A}_4\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right)$ yields

$$\begin{aligned}
\frac{\lambda}{(1-2\theta)\Delta t} \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right) &= \frac{\lambda}{(1-2\theta)\Delta t} \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2, \\
(1-\omega) \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right) &= (1-\omega) \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2, \\
(1-\omega) \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \delta \mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right) &\leq \epsilon_0 \delta \left\| \mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2 + \frac{(1-\omega)^2 \delta}{4\epsilon_0} \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2, \\
\lambda \left(\mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right) &\leq \frac{\lambda \left\| \mathbf{u}_h^{(n+\tilde{\theta})} \right\|_{\infty} C h^{-1}}{2} \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2 \\
&\leq \frac{\lambda K C h^{-1}}{2} \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2, \\
\lambda \left(\mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \delta \mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right) &= \lambda \delta \left\| \mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2, \\
\lambda \left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla \mathbf{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right) &\leq 4\lambda \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \nabla \mathbf{u}_h^{(n+\tilde{\theta})} \right\| \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\| \\
&\leq 4d^{1/2} \lambda \left\| \nabla \mathbf{u}_h^{(n+\tilde{\theta})} \right\|_{\infty} \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2 \\
&\leq 4d^{1/2} \lambda C h^{-1} \left\| \mathbf{u}_h^{(n+\tilde{\theta})} \right\|_{\infty} \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2 \\
&\leq 4d^{1/2} \lambda C h^{-1} K \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2, \\
\left(\lambda g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla \mathbf{u}_h^{(n+\tilde{\theta})}), \delta \mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right) &\leq 4\lambda \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \nabla \mathbf{u}_h^{(n+\tilde{\theta})} \right\| \left\| \delta \mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\| \\
&\leq 4d^{1/2} \lambda \left\| \nabla \mathbf{u}_h^{(n+\tilde{\theta})} \right\|_{\infty} \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\| \left\| \delta \mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\| \\
&\leq \frac{4d\lambda C^2 h^{-2} K^2 \delta}{\epsilon_1} \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2 + \lambda \epsilon_1 \delta \left\| \mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{A}_4(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}) &\geq \left(\frac{\lambda}{(1-2\theta)\Delta t} + (1-\omega) - \frac{(1-\omega)^2 \delta}{4\epsilon_0} \right. \\
&\quad \left. - \lambda K C h^{-1} \left(\frac{1}{2} + 4d^{1/2} + \frac{4dKCh^{-1}\delta}{\epsilon_1} \right) \right) \left\| \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2 \\
&\quad + \delta (\lambda - \epsilon_0 - \lambda \epsilon_1) \left\| \mathbf{u}_h^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right\|^2.
\end{aligned}$$

Choosing $\epsilon_0 = \frac{\lambda}{3}$, $\epsilon_1 = \frac{1}{3}$, $\delta \leq Ch^{-1}$, and $\Delta t \leq Ch^{-1}$ establishes $\mathcal{A}_4(\boldsymbol{\sigma}_h^{(n+\bar{\theta})}, \boldsymbol{\sigma}_h^{(n+\bar{\theta})}) > 0$. Hence, a unique solution exists for (4.2). ■

The unique solvability of (4.6) and (4.7) of the algorithm follows exactly as (4.2) and (4.3).

4.3 Error Estimates

In this section supporting analysis for the θ -method described in Section 2 is given. The θ -method decouples the stress updates from the velocity-pressure updates. Though the modeling system is nonlinear, the updates for the stress and velocity-pressure only require linear systems of equations to be solved. Motivated by this decoupling (as a first step toward an analysis of the fully discretized system) we investigate separately the stress and velocity-pressure approximations. First a θ -method for the constitutive equation given by (4.2), (4.5), and (4.6) is investigated assuming the velocity and pressure are known. Then a θ -method for the Stokes-like problem given by (4.3), (4.4), and (4.7) is analyzed assuming the stress is known. A priori error estimates for each scheme are given in Theorems 4.1 and 4.2, and a discussion of the proofs is given below. (Detailed proofs are given in section 5, and 6.)

It is convenient to define the following notation:

$$\begin{aligned}\tilde{\mathbf{u}}_h &:= \text{discrete approximation using true } \boldsymbol{\sigma}, \\ \tilde{\boldsymbol{\sigma}}_h &:= \text{discrete approximation using true } \mathbf{u}, \text{ and } p.\end{aligned}$$

Theorem 4.1 (Assuming u and p are known) *For sufficiently smooth solutions $\boldsymbol{\sigma}$, $\mathbf{u}$, p such that*

$$\|\boldsymbol{\sigma}\|_\infty, \|\boldsymbol{\sigma}_t\|_\infty, \|\mathbf{u}\|_\infty, \|\mathbf{u}_t\|_\infty, \|\mathbf{u}_{tt}\|_\infty, \|\nabla \mathbf{u}\|_\infty, \|(\nabla \mathbf{u})_t\|_\infty, \text{ and } \|(\nabla \mathbf{u})_{tt}\|_\infty \leq M, \quad \forall t \in [0, T],$$

$\Delta t \leq Ch^2$, the fractional step θ -method approximation, $\tilde{\boldsymbol{\sigma}}_h$ given by Step 1a, Step 2b, and Step 3a converges to $\boldsymbol{\sigma}$ on the interval $(0, T]$ as $\Delta t, h \rightarrow 0$, and satisfies the error estimates:

$$\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\|_{\infty,0} \leq F\boldsymbol{\sigma}(\Delta t, h, \delta), \quad (4.9)$$

$$\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\|_{0,0} \leq F\boldsymbol{\sigma}(\Delta t, h, \delta), \quad (4.10)$$

and

$$\|\mathbf{u} \cdot \nabla(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)\|_{\bar{\theta}} := \left(\sum_{n=1}^N \Delta t \left\| \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \left(\boldsymbol{\sigma}^{(n+\bar{\theta})} - \tilde{\boldsymbol{\sigma}}_h^{(n+\bar{\theta})} \right) \right\|^2 \right)^{\frac{1}{2}} \leq F\boldsymbol{\sigma}(\Delta t, h, \delta), \quad (4.11)$$

where

$$\begin{aligned}F\boldsymbol{\sigma}(\Delta t, h, \delta) &:= C(\Delta t)^2 \left(\|\boldsymbol{\sigma}_{ttt}\|_{0,0} + \|\boldsymbol{\sigma}_{tt}\|_{0,1} + \|\boldsymbol{\sigma}_t\|_{0,1} + \|\boldsymbol{\sigma}\|_{0,1} \right. \\ &\quad \left. + \|\boldsymbol{\sigma}_{tt}\|_{0,0} + \|\boldsymbol{\sigma}_t\|_{0,0} + \|\boldsymbol{\sigma}\|_{0,0} + C_T \right) \\ &\quad + C(\Delta t)\delta \left(\|\boldsymbol{\sigma}\|_{0,1} + \|\boldsymbol{\sigma}_t\|_{0,1} + \|\boldsymbol{\sigma}\|_{0,0} + \|\boldsymbol{\sigma}_t\|_{0,0} + C_T \right) \\ &\quad + C(h^{m+1} + h^m + \delta h^m) \|\boldsymbol{\sigma}\|_{0,m+1} \\ &\quad + Ch^{m+1} \|\boldsymbol{\sigma}_t\|_{0,m+1} + C\delta \|\boldsymbol{\sigma}_t\|_{0,0} + Ch^{m+1} \|\boldsymbol{\sigma}\|_{\infty,0}. \quad (4.12)\end{aligned}$$

Theorem 4.2 (Assuming σ is known) For a sufficiently smooth solutions $\mathbf{u}$, σ , p such that $\|\sigma\|_\infty \leq M$, $\forall t \in [0, T]$, and $\Delta t \leq Ch^2$, the fractional step θ -method approximation, $\tilde{\mathbf{u}}_h$ given by Step 1b, Step 2a, and Step 3b converges to $\mathbf{u}$ on the interval $(0, T]$ as $\Delta t, h \rightarrow 0$, and satisfies the error estimates:

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{\infty, 0} \leq F\mathbf{u}(\Delta t, h, \delta), \quad (4.13)$$

and

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{0, 1} \leq F\mathbf{u}(\Delta t, h, \delta), \quad (4.14)$$

where

$$\begin{aligned} F\mathbf{u}(\Delta t, h, \delta) &:= Ch^{k+1} \|\mathbf{u}_t\|_{0, k+1} + Ch^k \|\mathbf{u}\|_{0, k+1} + Ch^{q+1} \|p\|_{0, q+1} \\ &+ C(\Delta t)^2 \|\mathbf{u}_{ttt}\|_{0, 0} + C(\Delta t)^2 \|\mathbf{u}_{tt}\|_{0, 1} + C(\Delta t)^2 \|\mathbf{f}_{tt}\|_{0, 0} \\ &+ C(\Delta t)^2 C_T + Ch^k \|\mathbf{u}\|_{\infty, k+1}. \end{aligned} \quad (4.15)$$

Outline of the proof. In [6], and [7] the analysis for a fractional step θ -method for the convection diffusion equation was presented. The proof of Theorems 4.1 and 4.2 are done in an analogous manner to the proof presented in [7]. Here we present an outline of the proof for Theorem 4.1, and note that the proof of Theorem 4.2 is accomplished in a similar manner.

Step 1. When obtaining a priori error estimates for the time dependent approximation schemes it is useful to examine a *unit stride*, i.e. the terms analyzed are distance Δt apart. For the θ -method this is accomplished by considering linear combinations of the approximation methods steps. In order to obtain *unit strides* from $\tilde{\sigma}_h^{(n)}$ to $\tilde{\sigma}_h^{(n+1)}$, $\tilde{\sigma}_h^{(n-\theta)}$ to $\tilde{\sigma}_h^{(n+\tilde{\theta})}$, and $\tilde{\sigma}_h^{(n+\theta-1)}$ to $\tilde{\sigma}_h^{(n+\theta)}$ the following linear combinations are formed:

$$\theta \Delta t(4.2) + (1 - 2\theta) \Delta t(4.5) + \theta \Delta t(4.6), \quad (4.16)$$

$$\theta \Delta t(4.2) + (1 - 2\theta) \Delta t(4.5) + \theta \Delta t((4.6) \text{ with } n \rightarrow n - 1), \quad (4.17)$$

$$\theta \Delta t(4.2) + (1 - 2\theta) \Delta t((4.5) \text{ with } n \rightarrow n - 1) + \theta \Delta t((4.6) \text{ with } n \rightarrow n - 1). \quad (4.18)$$

Step 2. Evaluate (3.7) at the midpoint of each *unit stride* and subtract equations (4.16)-(4.18). Rearrange terms to obtain the equations:

$$\begin{aligned} &\left(\left(\sigma^{(n+1)} - \tilde{\sigma}_h^{(n+1)} \right) - \left(\sigma^{(n)} - \tilde{\sigma}_h^{(n)} \right), \tau \right) + \Delta t \mathcal{B}_{1_{pos}} \left(\left(\sigma^{(n+1)} - \tilde{\sigma}_h^{(n+1)} \right), \tau \right) \\ &= \Delta t \mathcal{B}_{1_{rem}} \left(\Delta t, \sigma, \mathbf{u}, \tilde{\sigma}_h^{(n+1)}, \tilde{\sigma}_h^{(n+\tilde{\theta})}, \tilde{\sigma}_h^{(n+\theta)}, \tilde{\sigma}_h^{(n)}, \tau \right), \end{aligned} \quad (4.19)$$

$$\begin{aligned} &\left(\left(\sigma^{(n+\tilde{\theta})} - \tilde{\sigma}_h^{(n+\tilde{\theta})} \right) - \left(\sigma^{(n-\theta)} - \tilde{\sigma}_h^{(n-\theta)} \right), \tau \right) + \Delta t \mathcal{B}_{2_{pos}} \left(\left(\sigma^{(n+\tilde{\theta})} - \tilde{\sigma}_h^{(n+\tilde{\theta})} \right), \tau \right) \\ &= \Delta t \mathcal{B}_{2_{rem}} \left(\Delta t, \sigma, \mathbf{u}, \tilde{\sigma}_h^{(n+\tilde{\theta})}, \tilde{\sigma}_h^{(n+\theta)}, \tilde{\sigma}_h^{(n-\theta)}, \tilde{\sigma}_h^{(n)}, \tau \right), \end{aligned} \quad (4.20)$$

$$\begin{aligned} & \left(\left(\boldsymbol{\sigma}^{(n+\theta)} - \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)} \right) - \left(\boldsymbol{\sigma}^{(n+\theta-1)} - \tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)} \right), \boldsymbol{\tau} \right) + \Delta t \mathcal{B}_{3_{pos}} \left(\left(\boldsymbol{\sigma}^{(n+\theta)} - \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)} \right), \boldsymbol{\tau} \right) \\ & = \Delta t \mathcal{B}_{3_{rem}} \left(\Delta t, \boldsymbol{\sigma}, \mathbf{u}, \tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)}, \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}, \tilde{\boldsymbol{\sigma}}_h^{(n-\theta)}, \tilde{\boldsymbol{\sigma}}_h^{(n)}, \boldsymbol{\tau} \right), \end{aligned} \quad (4.21)$$

where $\mathcal{B}_{1_{pos}}$, $\mathcal{B}_{2_{pos}}$, and $\mathcal{B}_{3_{pos}}$ denote the positive part of the operators.

Step 3. Use $\boldsymbol{\sigma}^{(n)} - \tilde{\boldsymbol{\sigma}}_h^{(n)} = \tilde{\boldsymbol{\Gamma}}^{(n)} + \tilde{\boldsymbol{F}}^{(n)}$, and choose $\boldsymbol{\tau} = \tilde{\boldsymbol{F}}^{(n)}$, in (4.19) to obtain an expression of the form

$$\left\| \mathbf{F}^{(n+1)} \right\|^2 - \left\| \mathbf{F}^{(n)} \right\|^2 + \Delta t \mathcal{B}_{1_{pos}} \left(\tilde{\mathbf{F}}^{(n+1)}, \tilde{\mathbf{F}}^{(n+1)} \right) \leq \Delta t \tilde{\mathcal{B}}_{1_{rem}} \left(\Delta t, \boldsymbol{\sigma}, \mathbf{u}, \tilde{\boldsymbol{\Gamma}}, \tilde{\mathbf{F}} \right). \quad (4.22)$$

Assuming that $\mathbf{F}^{(0)} = 0$, (4.22) is then summed from $n = 0$ to $n = l - 1$ so that the expression *telescopes* to

$$\left\| \mathbf{F}^l \right\|^2 + \Delta t \sum_{n=0}^{l-1} \mathcal{B}_{1_{pos}} \left(\tilde{\mathbf{F}}^{(n+1)}, \tilde{\mathbf{F}}^{(n+1)} \right) = \Delta t \mathcal{R}_1 \left(\Delta t, \boldsymbol{\sigma}, \mathbf{u}, \tilde{\boldsymbol{\Gamma}}, \tilde{\mathbf{F}} \right). \quad (4.23)$$

A similar approach is taken with (4.20) and (4.21) where $\boldsymbol{\tau}$ is chosen to be $\mathbf{F}^{(n+\bar{\theta})}$ and $\mathbf{F}^{(n+\theta)}$, respectively, giving equations for $\left\| \mathbf{F}^{(l-\theta)} \right\|^2$, and $\left\| \mathbf{F}^{(l-1+\theta)} \right\|^2$. These three equations are then added together to form a single equation.

Step 4. Suitable inequalities/estimates are then applied to the terms in the equation.

Step 5. Gronwall's lemma is applied to get an estimate for $\left\| \tilde{\mathbf{F}}^{l-1+\theta} \right\|^2 + \left\| \tilde{\mathbf{F}}^{l-\theta} \right\|^2 + \left\| \tilde{\mathbf{F}}^l \right\|^2$, and then using the triangle inequality we obtain the error estimate for $\left\| \boldsymbol{\sigma}^{(l)} - \tilde{\boldsymbol{\sigma}}_h^{(l)} \right\| + \left\| \boldsymbol{\sigma}^{(l-\theta)} - \tilde{\boldsymbol{\sigma}}_h^{(l-\theta)} \right\| + \left\| \boldsymbol{\sigma}^{(l-1+\theta)} - \tilde{\boldsymbol{\sigma}}_h^{(l-1+\theta)} \right\|$. ■

In Section 7 the numerical results presented use continuous, piecewise linear approximations for $\boldsymbol{\sigma}$, and p , and a continuous, piecewise quadratic approximation for $\mathbf{u}$. With these approximations we have the following estimates:

Corollary 1 For S_h the space of continuous, piecewise linear functions, $\Delta t \leq Ch^2$, and $\boldsymbol{\sigma}, \mathbf{u}, p$ sufficiently smooth, the approximation $\tilde{\boldsymbol{\sigma}}_h$ satisfies the error estimate:

$$\left\| \mathbf{u} \cdot \nabla (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h) \right\|_{\bar{\theta}} \leq C \left((\Delta t)^2 + \Delta t \delta + h \delta + h + \delta \right). \quad (4.24)$$

Corollary 2 For X_h the space of continuous, piecewise quadratic functions, and Q_h the space of continuous, piecewise linear functions, $\Delta t \leq Ch^2$, and $\boldsymbol{\sigma}, \mathbf{u}, p$ sufficiently smooth, the approximation $\tilde{\mathbf{u}}_h$ satisfies the error estimate:

$$\left\| \mathbf{u} - \tilde{\mathbf{u}}_h \right\|_{0,1} \leq C \left((\Delta t)^2 + h^2 \right). \quad (4.25)$$

5 Proof of Theorem 4.1

In this Section the proof of Theorem 4.1 is accomplished using the steps outlined in Section 4. We define the upwinded test element using true solution $\mathbf{u}^{(n)}$ as :

$$\boldsymbol{\tau}_{\delta^{(n)}} := \boldsymbol{\tau} + \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\tau}. \quad (5.1)$$

For notational simplicity the ‘tilde’ is dropped from $\tilde{\boldsymbol{\sigma}}_h$, the discrete approximation using the true $\mathbf{u}$, and p .

5.1 Set up linear combinations:

Three linear combinations are set up as stated in (4.16), (4.17), and (4.18).

Linear combination 1: The first linear combination is given by

$$\theta(4.2) + (1 - 2\theta)(4.5) + \theta(4.6), \quad (5.2)$$

and yields the expression:

$$\begin{aligned} \mathcal{B}_1^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) &:= \lambda \left(d_t \left(\boldsymbol{\sigma}_h^{(n+1)} \right), \boldsymbol{\tau} \right) + \theta \omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n)}} \right) \\ &+ (1 - 2\theta) \omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) + \theta \omega \left(\boldsymbol{\sigma}_h^{(n+1)}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\ &+ \tilde{\theta} \tilde{\omega} \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) + \theta \tilde{\omega} \left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n)}} \right) + \tilde{\theta} \lambda \left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\ &+ \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n)}} \right) + \tilde{\theta} \lambda \left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\ &+ \theta \lambda \left(g_a(\boldsymbol{\sigma}_h^n, \nabla \mathbf{u}^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}} \right) - (1 - \theta) 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\ &- \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}} \right) = 0, \quad \forall \boldsymbol{\tau} \in S_h. \end{aligned} \quad (5.3)$$

The true solution for the constitutive equation evaluated at $t_{n+\frac{1}{2}}$ satisfies:

$$\begin{aligned} 0 &= -\lambda \left(\frac{\partial \boldsymbol{\sigma}^{(n+\frac{1}{2})}}{\partial t}, \boldsymbol{\tau} \right) - \left(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau} \right) + 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}), \boldsymbol{\tau} \right) \\ &- \lambda \left(\mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})} + g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}), \boldsymbol{\tau} \right), \quad \forall \boldsymbol{\tau} \in S. \end{aligned} \quad (5.4)$$

An appropriately upwinded version of the true solution is obtained by considering

$$\theta * (5.4) \Big|_{\boldsymbol{\tau}_{\delta^{(n)}}} + \tilde{\theta} * (5.4) \Big|_{\boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}}. \quad (5.5)$$

Next we add and subtract terms to expression (5.5) yielding:

$$\mathcal{B}_1^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \mathcal{I}_1^n(\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in S, \quad (5.6)$$

where

$$\begin{aligned}
\mathcal{I}_1^n(\boldsymbol{\tau}) := & \lambda \left(d_t \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \boldsymbol{\tau} \right) + \lambda \theta \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\tau} \right) \\
& + \lambda \tilde{\theta} \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\tau} \right) + \omega \theta \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n)} \right) \\
& + \omega \theta \left(\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) + \omega (1-2\theta) \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) \\
& + \tilde{\omega} \theta \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n)} \right) + \tilde{\omega} \tilde{\theta} \left(\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) \\
& + \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n)} \right) \\
& + \tilde{\theta} \lambda \left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) \\
& + \theta \lambda \left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}), \boldsymbol{\tau}_{\delta(n)} \right) \\
& + \tilde{\theta} \lambda \left(g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}), \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) \\
& + \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}), \boldsymbol{\tau}_{\delta(n)} \right) + \tilde{\theta} 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right). \quad (5.7)
\end{aligned}$$

Subtraction of (5.3) from (5.6) gives:

$$\mathcal{B}_1^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_1^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) = \mathcal{I}_1^n(\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in S_h, \quad (5.8)$$

the first expression that will be bounded to complete the analysis.

Linear combination 2 : The second linear combination gives a full discrete time step of size Δt between $t_{n-\theta}$ and $t_{n+1-\theta}$ once we multiply the resulting expression by Δt in a later step. Using (4.17) divided by Δt yields,

$$\begin{aligned}
\mathcal{B}_2^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) := & \lambda \left(d_t \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right), \boldsymbol{\tau} \right) + \theta \omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta(n)} \right) + (1-2\theta) \omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) \\
& + \theta \omega \left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta(n-\theta)} \right) + (1-2\theta) \tilde{\omega} \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) + \theta \tilde{\omega} \left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta(n)} \right) \\
& + \theta \tilde{\omega} \left(\boldsymbol{\sigma}_h^{(n-\theta)}, \boldsymbol{\tau}_{\delta(n-\theta)} \right) + (1-2\theta) \lambda \left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) \\
& + \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta(n)} \right) + \theta \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}_h^{(n-\theta)}, \boldsymbol{\tau}_{\delta(n-\theta)} \right) \\
& + (1-2\theta) \lambda \left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) + \theta \lambda \left(g_a(\boldsymbol{\sigma}_h^{(n)}, \nabla \mathbf{u}^{(n)}), \boldsymbol{\tau}_{\delta(n)} \right) \\
& + \theta \lambda \left(g_a(\boldsymbol{\sigma}_h^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \boldsymbol{\tau}_{\delta(n-\theta)} \right) - (1-2\theta) 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta(n+\tilde{\theta})} \right) \\
& - \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n)}), \boldsymbol{\tau}_{\delta(n)} \right) - \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n-\theta)}), \boldsymbol{\tau}_{\delta(n-\theta)} \right) = 0, \quad \forall \boldsymbol{\tau} \in S_h. \quad (5.9)
\end{aligned}$$

The true solution to the constitutive equation evaluated at the midpoint between $t_{n-\theta}$ and $t_{n+1-\theta}$ is

$$\begin{aligned}
0 = & -\lambda \left(\frac{\partial \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}}{\partial t}, \boldsymbol{\tau} \right) - \left(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \boldsymbol{\tau} \right) + 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}), \boldsymbol{\tau} \right) \\
& - \lambda \left(\mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} + g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \boldsymbol{\tau} \right), \quad \forall \boldsymbol{\tau} \in S. \quad (5.10)
\end{aligned}$$

In a manner similar to (5.5) we form the linear combination

$$\theta(5.10) \Big|_{\tau_{\delta(n)}} + (1-2\theta)(5.10) \Big|_{\tau_{\delta(n+\bar{\theta})}} + \theta(5.10) \Big|_{\tau_{\delta(n-\theta)}}, \quad (5.11)$$

and write the expression as

$$\mathcal{B}_2^n(\sigma, \tau) = \mathcal{I}_2^n(\tau), \quad \forall \tau \in S, \quad (5.12)$$

where

$$\begin{aligned} \mathcal{I}_2^n(\tau) := & \lambda \left(d_t \sigma^{(n+\bar{\theta})} - \sigma_t^{(n+\frac{1}{2}-\theta)}, \tau \right) + \lambda \theta \left(-\sigma_t^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \tau \right) \\ & + \lambda (1-2\theta) \left(-\sigma_t^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \tau \right) + \lambda \theta \left(-\sigma_t^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \tau \right) \\ & + \theta \omega \left(\sigma^{(n+\theta)} - \sigma^{(n+\frac{1}{2}-\theta)}, \tau_{\delta(n)} \right) + (1-2\theta) \omega \left(\sigma^{(n+\theta)} - \sigma^{(n+\frac{1}{2}-\theta)}, \tau_{\delta(n+\bar{\theta})} \right) \\ & + \theta \omega \left(\sigma^{(n)} - \sigma^{(n+\frac{1}{2}-\theta)}, \tau_{\delta(n-\theta)} \right) + (1-2\theta) \tilde{\omega} \left(\sigma^{(n+\bar{\theta})} - \sigma^{(n+\frac{1}{2}-\theta)}, \tau_{\delta(n+\bar{\theta})} \right) \\ & + \theta \tilde{\omega} \left(\sigma^{(n)} - \sigma^{(n+\frac{1}{2}-\theta)}, \tau_{\delta(n)} \right) + \theta \tilde{\omega} \left(\sigma^{(n-\theta)} - \sigma^{(n+\frac{1}{2}-\theta)}, \tau_{\delta(n-\theta)} \right) \\ & + (1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \sigma^{(n+\bar{\theta})} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \sigma^{(n+\frac{1}{2}-\theta)}, \tau_{\delta(n+\bar{\theta})} \right) \\ & + \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \sigma^{(n)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \sigma^{(n+\frac{1}{2}-\theta)}, \tau_{\delta(n)} \right) \\ & + \theta \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \sigma^{(n-\theta)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \sigma^{(n+\frac{1}{2}-\theta)}, \tau_{\delta(n-\theta)} \right) \\ & + (1-2\theta) \lambda \left(g_a(\sigma^{(n+\bar{\theta})}, \nabla \mathbf{u}^{(n+\bar{\theta})}) - g_a(\sigma^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \tau_{\delta(n+\bar{\theta})} \right) \\ & + \theta \lambda \left(g_a(\sigma^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\sigma^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \tau_{\delta(n)} \right) \\ & + \theta \lambda \left(g_a(\sigma^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\sigma^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \tau_{\delta(n-\theta)} \right) \\ & + \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n)}), \tau_{\delta(n)} \right) + (1-2\theta) 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n+\bar{\theta})}), \tau_{\delta(n+\bar{\theta})} \right) \\ & + \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}), \tau_{\delta(n-\theta)} \right). \end{aligned} \quad (5.13)$$

An error expression is now formed by subtracting (5.9) from (5.12):

$$\mathcal{B}_2^n(\sigma, \tau) - \mathcal{B}_2^n(\sigma_h, \tau) = \mathcal{I}_2^n(\tau), \quad \forall \tau \in S_h. \quad (5.14)$$

Linear combination 3: We form the third linear combination (such that when multiplied by Δt a full discrete time step between $t_{n+\theta-1}$ and $t_{n+\theta}$ may be analyzed) by considering

$$\theta * (4.2) + (1-2\theta) * ((4.5) \text{ with } n \rightarrow n-1) + \theta * ((4.6) \text{ with } n \rightarrow n-1). \quad (5.15)$$

Using (5.15) we obtain

$$\begin{aligned} \mathcal{B}_3^n(\sigma_h, \tau) := & \lambda \left(d_t \left(\sigma_h^{(n+\theta)} \right), \tau \right) + \theta \omega \left(\sigma_h^{(n+\theta)}, \tau_{\delta(n)} \right) + \theta \omega \left(\sigma_h^{(n)}, \tau_{\delta(n-\theta)} \right) \\ & + (1-2\theta) \omega \left(\sigma_h^{(n+\theta-1)}, \tau_{\delta(n-\theta)} \right) + \theta \tilde{\omega} \left(\sigma_h^{(n)}, \tau_{\delta(n)} \right) + \tilde{\theta} \tilde{\omega} \left(\sigma_h^{(n-\theta)}, \tau_{\delta(n-\theta)} \right) \\ & + \tilde{\theta} \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \sigma_h^{(n-\theta)}, \tau_{\delta(n-\theta)} \right) + \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \sigma_h^{(n)}, \tau_{\delta(n)} \right) \\ & + \tilde{\theta} \lambda \left(g_a(\sigma_h^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \tau_{\delta(n-\theta)} \right) + \theta \lambda \left(g_a(\sigma_h^{(n)}, \nabla \mathbf{u}^{(n)}), \tau_{\delta(n)} \right) \\ & - \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n)}), \tau_{\delta(n)} \right) - \tilde{\theta} 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n-\theta)}), \tau_{\delta(n-\theta)} \right) = 0, \quad \forall \tau \in S_h. \end{aligned} \quad (5.16)$$

Evaluating the true solution at the midpoint between $t_{n+\theta-1}$ and $t_{n+\theta}$:

$$0 = -\lambda \left(\frac{\partial \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}}{\partial t}, \boldsymbol{\tau} \right) - \left(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \boldsymbol{\tau} \right) + 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}), \boldsymbol{\tau} \right) \\ - \lambda \left(\mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} + g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}), \boldsymbol{\tau} \right), \quad \forall \boldsymbol{\tau} \in S. \quad (5.17)$$

An appropriately upwinded true solution is found using the linear combination

$$\theta(5.17) \Big|_{\boldsymbol{\tau}_{\delta(n)}} + \tilde{\theta}(5.17) \Big|_{\boldsymbol{\tau}_{\delta(n-\theta)}}. \quad (5.18)$$

Adding $\mathcal{B}_3^n(\boldsymbol{\sigma}, \boldsymbol{\tau})$ to both sides of expression (5.18) gives

$$\mathcal{B}_3^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \mathcal{I}_3^n(\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in S, \quad (5.19)$$

where

$$\begin{aligned} \mathcal{I}_3^n(\boldsymbol{\tau}) := & \lambda \left(d_t \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})}, \boldsymbol{\tau} \right) + \lambda \theta \left(-\boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\tau} \right) \\ & + \lambda \tilde{\theta} \left(-\boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\tau} \right) + \theta \omega \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n)} \right) \\ & + \theta \omega \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n-\theta)} \right) + (1-2\theta) \omega \left(\boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n-\theta)} \right) \\ & + \theta \tilde{\omega} \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n)} \right) + \tilde{\theta} \tilde{\omega} \left(\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n-\theta)} \right) \\ & + \tilde{\theta} \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n-\theta)} \right) \\ & + \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \boldsymbol{\tau}_{\delta(n)} \right) \\ & + \tilde{\theta} \lambda \left(g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}), \boldsymbol{\tau}_{\delta(n-\theta)} \right) \\ & + \theta \lambda \left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}), \boldsymbol{\tau}_{\delta(n)} \right) \\ & + \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}), \boldsymbol{\tau}_{\delta(n)} \right) \\ & + \tilde{\theta} 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}), \boldsymbol{\tau}_{\delta(n-\theta)} \right). \end{aligned} \quad (5.20)$$

Subtraction of (5.16) from (5.19) yields the final error expression needed for analysis of the constitutive equation

$$\mathcal{B}_3^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_3^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) = \mathcal{I}_3^n(\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in S_h. \quad (5.21)$$

Bounds are now found for (5.8), (5.14), and (5.21).

5.1.1 Bounding $\mathcal{B}_1^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_1^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau})$

In this section bounds on the term $\mathcal{B}_1^n(\boldsymbol{\sigma}, \mathbf{F}^{(n+1)}) - \mathcal{B}_1^n(\boldsymbol{\sigma}_h, \mathbf{F}^{(n+1)})$ in expression (5.8) are examined. Recall for this proof we have dropped the ‘tilde’ from $\tilde{\boldsymbol{\sigma}}_h$. Thus,

$$\boldsymbol{\Gamma}^{(n)} = \boldsymbol{\sigma}^{(n)} - \mathcal{S}^{(n)}, \quad \text{and} \quad \mathbf{F}^{(n)} = \mathcal{S}^{(n)} - \tilde{\boldsymbol{\sigma}}_h^{(n)}.$$

It is also convenient to use the notation:

$$\mathbf{F}_{\delta^{(\mu)}}^{(\nu)} := \mathbf{F}^{(\nu)} + \delta \mathbf{u}^{(\mu)} \cdot \nabla \mathbf{F}^{(\nu)}. \quad (5.22)$$

Here,

$$\begin{aligned} \mathcal{B}_1^n(\boldsymbol{\sigma}, \mathbf{F}^{(n+1)}) - \mathcal{B}_1^n(\boldsymbol{\sigma}_h, \mathbf{F}^{(n+1)}) &= \lambda \left(d_t(e_{\boldsymbol{\sigma}}^{(n+1)}), \mathbf{F}^{(n+1)} \right) + \theta \omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta^{(n)}}^{(n+1)} \right) \\ &+ \theta \omega \left(e_{\boldsymbol{\sigma}}^{(n+1)}, \mathbf{F}_{\delta^{(n+\bar{\theta})}}^{(n+1)} \right) + (1 - 2\theta) \omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta^{(n+\bar{\theta})}}^{(n+1)} \right) \\ &+ \tilde{\theta} \tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n+\bar{\theta})}, \mathbf{F}_{\delta^{(n+\bar{\theta})}}^{(n+1)} \right) + \theta \tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta^{(n)}}^{(n+1)} \right) \\ &+ \tilde{\theta} \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n+\bar{\theta})}, \mathbf{F}_{\delta^{(n+\bar{\theta})}}^{(n+1)} \right) + \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta^{(n)}}^{(n+1)} \right) \\ &+ \tilde{\theta} \lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n+\bar{\theta})}, \nabla \mathbf{u}^{(n+\bar{\theta})}), \mathbf{F}_{\delta^{(n+\bar{\theta})}}^{(n+1)} \right) + \theta \lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}_{\delta^{(n)}}^{(n+1)} \right). \end{aligned} \quad (5.23)$$

Each of the terms in (5.23) is now bounded.

$$\lambda \left(d_t(e_{\boldsymbol{\sigma}}^{(n+1)}), \mathbf{F}^{(n+1)} \right) = \lambda \left(d_t(\mathbf{F}^{(n+1)}), \mathbf{F}^{(n+1)} \right) + \lambda \left(d_t(\boldsymbol{\Gamma}^{(n+1)}), \mathbf{F}^{(n+1)} \right), \quad (5.24)$$

where

$$\begin{aligned} \lambda \left(d_t(\mathbf{F}^{(n+1)}), \mathbf{F}^{(n+1)} \right) &= \frac{\lambda}{\Delta t} \left(\left(\mathbf{F}^{(n+1)}, \mathbf{F}^{(n+1)} \right) - \left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+1)} \right) \right) \\ &\geq \frac{\lambda}{\Delta t} \left(\left\| \mathbf{F}^{(n+1)} \right\|^2 - \left\| \mathbf{F}^{(n+1)} \right\| \left\| \mathbf{F}^{(n)} \right\| \right) \\ &\geq \frac{\lambda}{2\Delta t} \left(\left\| \mathbf{F}^{(n+1)} \right\|^2 - \left\| \mathbf{F}^{(n)} \right\|^2 \right), \end{aligned} \quad (5.25)$$

and,

$$\lambda \left(d_t(\boldsymbol{\Gamma}^{(n+1)}), \mathbf{F}^{(n+1)} \right) \leq \frac{\lambda}{4} \left\| d_t(\boldsymbol{\Gamma}^{(n+1)}) \right\|^2 + \lambda \left\| \mathbf{F}^{(n+1)} \right\|^2. \quad (5.26)$$

Each of the next eight terms in (5.23) is rewritten as the sum of four terms. Starting with

$$\begin{aligned} \theta \omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta^{(n)}}^{(n+1)} \right) &= \theta \omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+1)} \right) + \theta \omega \left(\mathbf{F}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\ &+ \theta \omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+1)} \right) + \theta \omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right). \end{aligned} \quad (5.27)$$

we have the bounds

$$\begin{aligned} \theta \omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+1)} \right) &\leq \theta \omega \left\| \mathbf{F}^{(n+\theta)} \right\| \left\| \mathbf{F}^{(n+1)} \right\| \\ &\leq \frac{\theta \omega}{2} \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \frac{\theta \omega}{2} \left\| \mathbf{F}^{(n+1)} \right\|^2, \end{aligned} \quad (5.28)$$

$$\begin{aligned} \theta \omega \left(\mathbf{F}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) &\leq \theta \omega \left\| \mathbf{F}^{(n+\theta)} \right\| \left\| \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right\| \\ &\leq \delta \theta \omega \left\| \mathbf{F}^{(n+\theta)} \right\| \left\| \mathbf{u}^{(n)} \right\|_{\infty} \hat{d}^{\frac{1}{2}} \left\| \nabla \mathbf{F}^{(n+1)} \right\| \\ &\leq \delta \theta \omega M \hat{d}^{\frac{1}{2}} \left\| \mathbf{F}^{(n+\theta)} \right\| \left\| \nabla \mathbf{F}^{(n+1)} \right\| \\ &\leq \frac{\theta \omega M^2 \hat{d}}{4} \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \delta^2 \theta \omega \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \end{aligned} \quad (5.29)$$

$$\theta\omega \left(\mathbf{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+1)} \right) \leq \frac{\theta\omega}{2} \left\| \mathbf{\Gamma}^{(n+\theta)} \right\|^2 + \frac{\theta\omega}{2} \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.30)$$

and

$$\theta\omega \left(\mathbf{\Gamma}^{(n+\theta)}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \leq \frac{\theta\omega M^2 \acute{d}}{4} \left\| \mathbf{\Gamma}^{(n+\theta)} \right\|^2 + \delta^2 \theta\omega \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2. \quad (5.31)$$

Splitting the next term into its pieces gives:

$$\begin{aligned} \theta\omega \left(e_{\boldsymbol{\sigma}}^{(n+1)}, \mathbf{F}_{\delta^{(n+\bar{\theta})}}^{(n+1)} \right) &= \theta\omega \left(\mathbf{F}^{(n+1)}, \mathbf{F}^{(n+1)} \right) + \theta\omega \left(\mathbf{F}^{(n+1)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\ &\quad + \theta\omega \left(\mathbf{\Gamma}^{(n+1)}, \mathbf{F}^{(n+1)} \right) + \theta\omega \left(\mathbf{\Gamma}^{(n+1)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right). \end{aligned} \quad (5.32)$$

Bounding each term we have

$$\theta\omega \left(\mathbf{F}^{(n+1)}, \mathbf{F}^{(n+1)} \right) = \theta\omega \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.33)$$

$$\theta\omega \left(\mathbf{F}^{(n+1)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \leq \frac{\theta\omega M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n+1)} \right\|^2 + \delta^2 \theta\omega \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.34)$$

$$\theta\omega \left(\mathbf{\Gamma}^{(n+1)}, \mathbf{F}^{(n+1)} \right) \leq \frac{\theta\omega}{2} \left\| \mathbf{\Gamma}^{(n+1)} \right\|^2 + \frac{\theta\omega}{2} \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.35)$$

and

$$\theta\omega \left(\mathbf{\Gamma}^{(n+1)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \leq \frac{\theta\omega M^2 \acute{d}}{4} \left\| \mathbf{\Gamma}^{(n+1)} \right\|^2 + \delta^2 \theta\omega \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2. \quad (5.36)$$

Next

$$\begin{aligned} (1-2\theta)\omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta^{(n+\bar{\theta})}}^{(n+1)} \right) &= (1-2\theta)\omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+1)} \right) \\ &\quad + (1-2\theta)\omega \left(\mathbf{F}^{(n+\theta)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\ &\quad + (1-2\theta)\omega \left(\mathbf{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+1)} \right) \\ &\quad + (1-2\theta)\omega \left(\mathbf{\Gamma}^{(n+\theta)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right). \end{aligned} \quad (5.37)$$

Bounding each term

$$(1-2\theta)\omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+1)} \right) \leq \frac{(1-2\theta)\omega}{2} \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \frac{(1-2\theta)\omega}{2} \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.38)$$

$$\begin{aligned} (1-2\theta)\omega \left(\mathbf{F}^{(n+\theta)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) &\leq \frac{(1-2\theta)\omega M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n+\theta)} \right\|^2 \\ &\quad + (1-2\theta)\omega \delta^2 \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2 \end{aligned} \quad (5.39)$$

$$(1-2\theta)\omega \left(\mathbf{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+1)} \right) \leq \frac{(1-2\theta)\omega}{2} \left\| \mathbf{\Gamma}^{(n+\theta)} \right\|^2 + \frac{(1-2\theta)\omega}{2} \left\| \mathbf{F}^{(n+1)} \right\|^2 \quad (5.40)$$

$$\begin{aligned} (1-2\theta)\omega \left(\mathbf{\Gamma}^{(n+\theta)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) &\leq \frac{(1-2\theta)\omega M^2 \acute{d}}{4} \left\| \mathbf{\Gamma}^{(n+\theta)} \right\|^2 \\ &\quad + \delta^2 (1-2\theta)\omega \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2 \end{aligned} \quad (5.41)$$

Splitting the following gives:

$$\begin{aligned}\tilde{\theta}\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \mathbf{F}_{\delta(n+\tilde{\theta})}^{(n+1)}\right) &= \tilde{\theta}\tilde{\omega}\left(\mathbf{F}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+1)}\right) + \tilde{\theta}\tilde{\omega}\left(\mathbf{F}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right) \\ &\quad + \tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+1)}\right) + \tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right),\end{aligned}$$

and these terms are bounded as

$$\tilde{\theta}\tilde{\omega}\left(\mathbf{F}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+\tilde{\theta})}\right\|^2 + \frac{\tilde{\theta}\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+1)}\right\|^2, \quad (5.42)$$

$$\tilde{\theta}\tilde{\omega}\left(\mathbf{F}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{4} \left\|\mathbf{F}^{(n+\tilde{\theta})}\right\|^2 + \delta^2\tilde{\theta}\tilde{\omega} \left\|\nabla \mathbf{F}^{(n+1)}\right\|^2, \quad (5.43)$$

$$\tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}}{2} \left\|\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2 + \frac{\tilde{\theta}\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+1)}\right\|^2, \quad (5.44)$$

and

$$\tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{4} \left\|\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2 + \delta^2\tilde{\theta}\tilde{\omega} \left\|\nabla \mathbf{F}^{(n+1)}\right\|^2. \quad (5.45)$$

Continuing

$$\begin{aligned}\theta\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n)}^{(n+1)}\right) &= \theta\tilde{\omega}\left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+1)}\right) + \theta\tilde{\omega}\left(\mathbf{F}^{(n)}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)}\right) \\ &\quad + \theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \mathbf{F}^{(n+1)}\right) + \theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)}\right).\end{aligned} \quad (5.46)$$

These terms are bounded as:

$$\theta\tilde{\omega}\left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+1)}\right) \leq \frac{\theta\tilde{\omega}}{2} \left\|\mathbf{F}^{(n)}\right\|^2 + \frac{\theta\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+1)}\right\|^2, \quad (5.47)$$

$$\theta\tilde{\omega}\left(\mathbf{F}^{(n)}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)}\right) \leq \frac{\theta\tilde{\omega}M^2\acute{d}}{4} \left\|\mathbf{F}^{(n)}\right\|^2 + \delta^2\theta\tilde{\omega} \left\|\nabla \mathbf{F}^{(n+1)}\right\|^2, \quad (5.48)$$

$$\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \mathbf{F}^{(n+1)}\right) \leq \frac{\theta\tilde{\omega}}{2} \left\|\boldsymbol{\Gamma}^{(n)}\right\|^2 + \frac{\theta\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+1)}\right\|^2, \quad (5.49)$$

and

$$\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)}\right) \leq \frac{\theta\tilde{\omega}M^2\acute{d}}{4} \left\|\boldsymbol{\Gamma}^{(n)}\right\|^2 + \delta^2\theta\tilde{\omega} \left\|\nabla \mathbf{F}^{(n+1)}\right\|^2. \quad (5.50)$$

Next the convective terms are handled. Thus,

$$\begin{aligned}\tilde{\theta}\lambda\left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \mathbf{F}_{\delta(n+\tilde{\theta})}^{(n+1)}\right) &= \tilde{\theta}\lambda\left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+1)}\right) \\ &\quad + \tilde{\theta}\lambda\left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right) + \tilde{\theta}\lambda\left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+1)}\right) \\ &\quad + \tilde{\theta}\lambda\left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right).\end{aligned} \quad (5.51)$$

We bound each part in the following manner:

$$\tilde{\theta}\lambda\left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\lambda\acute{d}M^2}{4} \left\|\mathbf{F}^{(n+1)}\right\|^2 + \tilde{\theta}\lambda \left\|\nabla \mathbf{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.52)$$

$$\begin{aligned}\tilde{\theta}\lambda\left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right) &\leq \frac{\delta \tilde{\theta} \lambda d M^2}{2} \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2 \\ &\quad + \frac{\delta \tilde{\theta} \lambda d M^2}{2} \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2,\end{aligned}\quad (5.53)$$

$$\tilde{\theta}\lambda\left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{\Gamma}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+1)}\right) \leq \frac{\tilde{\theta} \lambda d M^2}{4} \left\| \mathbf{F}^{(n+1)} \right\|^2 + \tilde{\theta} \lambda \left\| \nabla \mathbf{\Gamma}^{(n+\tilde{\theta})} \right\|^2, \quad (5.54)$$

and

$$\begin{aligned}\tilde{\theta}\lambda\left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{\Gamma}^{(n+\tilde{\theta})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right) &\leq \frac{\delta^2 \tilde{\theta} \lambda d M^2}{2} \left\| \nabla \mathbf{\Gamma}^{(n+\tilde{\theta})} \right\|^2 \\ &\quad + \frac{\tilde{\theta} \lambda d M^2}{2} \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2.\end{aligned}\quad (5.55)$$

Similarly the second convective term is split as:

$$\begin{aligned}\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n)}^{(n+1)}\right) &= \theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \mathbf{F}^{(n+1)}\right) + \theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)}\right) \\ &\quad + \theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+1)}\right) + \theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{\Gamma}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)}\right),\end{aligned}\quad (5.56)$$

and bounded term by term with

$$\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \mathbf{F}^{(n+1)}\right) \leq \frac{\theta \lambda d M^2}{4} \left\| \mathbf{F}^{(n+1)} \right\|^2 + \theta \lambda \left\| \nabla \mathbf{F}^{(n)} \right\|^2, \quad (5.57)$$

$$\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)}\right) \leq \frac{\delta \theta \lambda d M^2}{2} \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2 + \frac{\delta \theta \lambda d M^2}{2} \left\| \nabla \mathbf{F}^{(n)} \right\|^2, \quad (5.58)$$

$$\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+1)}\right) \leq \frac{\theta \lambda d M^2}{4} \left\| \mathbf{F}^{(n+1)} \right\|^2 + \theta \lambda \left\| \nabla \mathbf{\Gamma}^{(n)} \right\|^2, \quad (5.59)$$

and

$$\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{\Gamma}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)}\right) \leq \frac{\theta \lambda d M^2}{2} \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2 + \frac{\delta^2 \theta \lambda d M^2}{2} \left\| \nabla \mathbf{\Gamma}^{(n)} \right\|^2. \quad (5.60)$$

Lastly the $g_a(\cdot, \cdot)$ terms are considered. Splitting,

$$\begin{aligned}\tilde{\theta}\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}_{\delta(n+\tilde{\theta})}^{(n+1)}\right) &= \tilde{\theta}\lambda\left(g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+1)}\right) \\ &\quad + \tilde{\theta}\lambda\left(g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right) + \tilde{\theta}\lambda\left(g_a(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+1)}\right) \\ &\quad + \tilde{\theta}\lambda\left(g_a(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)}\right),\end{aligned}\quad (5.61)$$

and bounding each of these terms gives:

$$\begin{aligned}\tilde{\theta}\lambda\left(g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+1)}\right) &\leq \tilde{\theta}\lambda \left\| g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) \right\| \left\| \mathbf{F}^{(n+1)} \right\| \\ &\leq 4d\tilde{\theta}\lambda \left\| \nabla \mathbf{u}^{(n+\tilde{\theta})} \right\|_{\infty} \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\| \left\| \mathbf{F}^{(n+1)} \right\| \\ &\leq 4d\tilde{\theta}\lambda M \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\| \left\| \mathbf{F}^{(n+1)} \right\| \\ &\leq 4\tilde{\theta}\lambda d M \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \tilde{\theta}\lambda d M \left\| \mathbf{F}^{(n+1)} \right\|^2,\end{aligned}\quad (5.62)$$

$$\begin{aligned}
\tilde{\theta}\lambda \left(g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) &\leq \tilde{\theta}\lambda\delta \left\| g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) \right\| \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right\| \\
&\leq 4d^2\tilde{\theta}\lambda\delta \left\| \nabla \mathbf{u}^{(n+\tilde{\theta})} \right\|_\infty^2 \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\| \left\| \nabla \mathbf{F}^{(n+1)} \right\| \\
&\leq 4d^2\tilde{\theta}\lambda\delta M^2 \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\| \left\| \nabla \mathbf{F}^{(n+1)} \right\| \\
&\leq 4d^4\tilde{\theta}\lambda M^4 \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \delta^2\tilde{\theta}\lambda \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.63)
\end{aligned}$$

$$\tilde{\theta}\lambda \left(g_a(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+1)} \right) \leq 4\tilde{\theta}\lambda dM \left\| \mathbf{\Gamma}^{(n+\tilde{\theta})} \right\|^2 + \tilde{\theta}\lambda dM \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.64)$$

and

$$\tilde{\theta}\lambda \left(g_a(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \leq 4d^4\tilde{\theta}\lambda M^4 \left\| \mathbf{\Gamma}^{(n+\tilde{\theta})} \right\|^2 + \delta^2\tilde{\theta}\lambda \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2. \quad (5.65)$$

Similarly,

$$\begin{aligned}
\theta\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}_{\delta^{(n)}}^{(n+1)} \right) &= \theta\lambda \left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+1)} \right) \\
&\quad + \theta\lambda \left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) + \theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+1)} \right) \\
&\quad + \theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right), \quad (5.66)
\end{aligned}$$

and has the following bounds:

$$\theta\lambda \left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+1)} \right) \leq 4\theta\lambda dM \left\| \mathbf{F}^{(n)} \right\|^2 + \theta\lambda dM \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.67)$$

$$\theta\lambda \left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \leq 4d^4\theta\lambda M^4 \left\| \mathbf{F}^{(n)} \right\|^2 + \theta\lambda\delta^2 \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.68)$$

$$\theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+1)} \right) \leq 4\theta\lambda dM \left\| \mathbf{\Gamma}^{(n)} \right\|^2 + \theta\lambda dM \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.69)$$

and

$$\theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \leq 4\theta\lambda d^4 M^4 \left\| \mathbf{\Gamma}^{(n)} \right\|^2 + \delta^2\theta\lambda \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2. \quad (5.70)$$

5.1.2 Bounding $\mathcal{B}_2^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_2^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau})$

We continue in this section by determining bounds on the terms $\mathcal{B}_2^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_2^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau})$ in expression (5.14). Note that the value of $\boldsymbol{\tau}$ is chosen to be $\mathbf{F}^{(n+\tilde{\theta})}$. Thus,

$$\begin{aligned}
\mathcal{B}_2^n(\boldsymbol{\sigma}, \mathbf{F}^{(n+\tilde{\theta})}) - \mathcal{B}_2^n(\boldsymbol{\sigma}_h, \mathbf{F}^{(n+\tilde{\theta})}) &= \lambda \left(d_t e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+\tilde{\theta})} \right) + \theta\omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta^{(n)}}^{(n+\tilde{\theta})} \right) \\
&\quad + (1-2\theta)\omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta^{(n+\tilde{\theta})}}^{(n+\tilde{\theta})} \right) + \theta\omega \left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta^{(n-\theta)}}^{(n+\tilde{\theta})} \right) + (1-2\theta)\tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \mathbf{F}_{\delta^{(n+\tilde{\theta})}}^{(n+\tilde{\theta})} \right) \\
&\quad + \theta\tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta^{(n)}}^{(n+\tilde{\theta})} \right) + \theta\tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n-\theta)}, \mathbf{F}_{\delta^{(n-\theta)}}^{(n+\tilde{\theta})} \right) + (1-2\theta)\lambda \left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \mathbf{F}_{\delta^{(n+\tilde{\theta})}}^{(n+\tilde{\theta})} \right) \\
&\quad + \theta\lambda \left(\mathbf{u}^{(n)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta^{(n)}}^{(n+\tilde{\theta})} \right) + \theta\lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n-\theta)}, \mathbf{F}_{\delta^{(n-\theta)}}^{(n+\tilde{\theta})} \right) \\
&\quad + (1-2\theta)\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}_{\delta^{(n+\tilde{\theta})}}^{(n+\tilde{\theta})} \right) + \theta\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}_{\delta^{(n)}}^{(n+\tilde{\theta})} \right) \\
&\quad + \theta\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}_{\delta^{(n-\theta)}}^{(n+\tilde{\theta})} \right). \quad (5.71)
\end{aligned}$$

Each term in (5.71) is now bounded. Here

$$\lambda \left(d_t \left(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})} \right), \mathbf{F}^{(n+\tilde{\theta})} \right) = \lambda \left(d_t \left(\mathbf{F}^{(n+\tilde{\theta})} \right), \mathbf{F}^{(n+\tilde{\theta})} \right) + \lambda \left(d_t \left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})} \right), \mathbf{F}^{(n+\tilde{\theta})} \right), \quad (5.72)$$

with

$$\lambda \left(d_t \left(\mathbf{F}^{(n+\tilde{\theta})} \right), \mathbf{F}^{(n+\tilde{\theta})} \right) \geq \frac{\lambda}{2\Delta t} \left(\left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 - \left\| \mathbf{F}^{(n-\theta)} \right\|^2 \right), \quad (5.73)$$

and

$$\lambda \left(d_t \left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})} \right), \mathbf{F}^{(n+\tilde{\theta})} \right) \leq \lambda \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \frac{\lambda}{4} \left\| d_t \left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})} \right) \right\|^2. \quad (5.74)$$

Then

$$\begin{aligned} \theta \omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta(n)}^{(n+\tilde{\theta})} \right) &= \theta \omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) + \theta \omega \left(\mathbf{F}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &\quad + \theta \omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) + \theta \omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right), \end{aligned} \quad (5.75)$$

and the terms are bounded as:

$$\theta \omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \leq \frac{\theta \omega}{2} \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \frac{\theta \omega}{2} \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.76)$$

$$\theta \omega \left(\mathbf{F}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \leq \frac{\theta \omega M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \delta^2 \theta \omega \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.77)$$

$$\theta \omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \leq \frac{\theta \omega}{2} \left\| \boldsymbol{\Gamma}^{(n+\theta)} \right\|^2 + \frac{\theta \omega}{2} \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.78)$$

and

$$\theta \omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \leq \frac{\theta \omega M^2 \acute{d}}{4} \left\| \boldsymbol{\Gamma}^{(n+\theta)} \right\|^2 + \delta^2 \theta \omega \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2. \quad (5.79)$$

Splitting the next term:

$$\begin{aligned} (1-2\theta) \omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta(n+\tilde{\theta})}^{(n+\tilde{\theta})} \right) &= (1-2\theta) \omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &\quad + (1-2\theta) \omega \left(\mathbf{F}^{(n+\theta)}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &\quad + (1-2\theta) \omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) + (1-2\theta) \omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right). \end{aligned} \quad (5.80)$$

Bounding these terms:

$$(1-2\theta) \omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \leq \frac{(1-2\theta) \omega}{2} \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \frac{(1-2\theta) \omega}{2} \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.81)$$

$$\begin{aligned} (1-2\theta) \omega \left(\mathbf{F}^{(n+\theta)}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \\ \leq \frac{(1-2\theta) \omega}{4\epsilon_1} \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \delta^2 (1-2\theta) \omega \epsilon_1 \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \end{aligned} \quad (5.82)$$

$$(1-2\theta) \omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \leq \frac{(1-2\theta) \omega}{2} \left\| \boldsymbol{\Gamma}^{(n+\theta)} \right\|^2 + \frac{(1-2\theta) \omega}{2} \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.83)$$

and

$$\begin{aligned} (1-2\theta)\omega\left(\mathbf{\Gamma}^{(n+\theta)}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \\ \leq \frac{(1-2\theta)\omega}{4\epsilon_1} \left\|\mathbf{\Gamma}^{(n+\theta)}\right\|^2 + \delta^2(1-2\theta)\omega\epsilon_1 \left\|\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right\|^2. \end{aligned} \quad (5.84)$$

Next

$$\begin{aligned} \theta\omega\left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\tilde{\theta})}\right) &= \theta\omega\left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+\tilde{\theta})}\right) + \theta\omega\left(\mathbf{F}^{(n)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \\ &\quad + \theta\omega\left(\mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+\tilde{\theta})}\right) + \theta\omega\left(\mathbf{\Gamma}^{(n)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right), \end{aligned} \quad (5.85)$$

and each term is bounded as

$$\theta\omega\left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\omega}{2} \left\|\mathbf{F}^{(n)}\right\|^2 + \frac{\theta\omega}{2} \left\|\mathbf{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.86)$$

$$\theta\omega\left(\mathbf{F}^{(n)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\omega M^2 \acute{d}}{4} \left\|\mathbf{F}^{(n)}\right\|^2 + \delta^2\theta\omega \left\|\nabla \mathbf{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.87)$$

$$\theta\omega\left(\mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\omega}{2} \left\|\mathbf{\Gamma}^{(n)}\right\|^2 + \frac{\theta\omega}{2} \left\|\mathbf{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.88)$$

and

$$\theta\omega\left(\mathbf{\Gamma}^{(n)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\omega M^2 \acute{d}}{4} \left\|\mathbf{\Gamma}^{(n)}\right\|^2 + \delta^2\theta\omega \left\|\nabla \mathbf{F}^{(n+\tilde{\theta})}\right\|^2. \quad (5.89)$$

The term

$$\begin{aligned} (1-2\theta)\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \mathbf{F}_{\delta(n+\tilde{\theta})}^{(n+\tilde{\theta})}\right) &= (1-2\theta)\tilde{\omega}\left(\mathbf{F}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+\tilde{\theta})}\right) \\ &\quad + (1-2\theta)\tilde{\omega}\left(\mathbf{F}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) + (1-2\theta)\tilde{\omega}\left(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+\tilde{\theta})}\right) \\ &\quad + (1-2\theta)\tilde{\omega}\left(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right), \end{aligned} \quad (5.90)$$

is bounded as

$$(1-2\theta)\tilde{\omega}\left(\mathbf{F}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+\tilde{\theta})}\right) = (1-2\theta)\tilde{\omega} \left\|\mathbf{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.91)$$

$$\begin{aligned} (1-2\theta)\tilde{\omega}\left(\mathbf{F}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \\ \leq \frac{(1-2\theta)\tilde{\omega}}{4\epsilon_1} \left\|\mathbf{F}^{(n+\tilde{\theta})}\right\|^2 + \delta^2(1-2\theta)\tilde{\omega}\epsilon_1 \left\|\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right\|^2, \end{aligned} \quad (5.92)$$

$$(1-2\theta)\tilde{\omega}\left(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \mathbf{F}^{(n+\tilde{\theta})}\right) \leq \frac{(1-2\theta)\tilde{\omega}}{2} \left\|\mathbf{\Gamma}^{(n+\tilde{\theta})}\right\|^2 + \frac{(1-2\theta)\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.93)$$

and

$$\begin{aligned} (1-2\theta)\tilde{\omega}\left(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \delta\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \\ \leq \frac{(1-2\theta)\tilde{\omega}}{4\epsilon_1} \left\|\mathbf{\Gamma}^{(n+\tilde{\theta})}\right\|^2 + \delta^2(1-2\theta)\tilde{\omega}\epsilon_1 \left\|\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right\|^2. \end{aligned} \quad (5.94)$$

The term

$$\begin{aligned} \theta \tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta^{(n)}}^{(n+\bar{\theta})} \right) &= \theta \tilde{\omega} \left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+\bar{\theta})} \right) + \theta \tilde{\omega} \left(\mathbf{F}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ &\quad + \theta \tilde{\omega} \left(\boldsymbol{\Gamma}^{(n)}, \mathbf{F}^{(n+\bar{\theta})} \right) + \theta \tilde{\omega} \left(\boldsymbol{\Gamma}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right), \end{aligned} \quad (5.95)$$

with bounds

$$\theta \tilde{\omega} \left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \tilde{\omega}}{2} \left\| \mathbf{F}^{(n)} \right\|^2 + \frac{\theta \tilde{\omega}}{2} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.96)$$

$$\theta \tilde{\omega} \left(\mathbf{F}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \tilde{\omega} M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n)} \right\|^2 + \delta^2 \theta \tilde{\omega} \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.97)$$

$$\theta \tilde{\omega} \left(\boldsymbol{\Gamma}^{(n)}, \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \tilde{\omega}}{2} \left\| \boldsymbol{\Gamma}^{(n)} \right\|^2 + \frac{\theta \tilde{\omega}}{2} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.98)$$

and

$$\theta \tilde{\omega} \left(\boldsymbol{\Gamma}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \tilde{\omega} M^2 \acute{d}}{4} \left\| \boldsymbol{\Gamma}^{(n)} \right\|^2 + \delta^2 \theta \tilde{\omega} \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2. \quad (5.99)$$

Considering,

$$\begin{aligned} \theta \tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n-\theta)}, \mathbf{F}_{\delta^{(n-\theta)}}^{(n+\bar{\theta})} \right) &= \theta \tilde{\omega} \left(\mathbf{F}^{(n-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) + \theta \tilde{\omega} \left(\mathbf{F}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ &\quad + \theta \tilde{\omega} \left(\boldsymbol{\Gamma}^{(n-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) + \theta \tilde{\omega} \left(\boldsymbol{\Gamma}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right). \end{aligned} \quad (5.100)$$

We have the bounds

$$\theta \tilde{\omega} \left(\mathbf{F}^{(n-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \tilde{\omega}}{2} \left\| \mathbf{F}^{(n-\theta)} \right\|^2 + \frac{\theta \tilde{\omega}}{2} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.101)$$

$$\theta \tilde{\omega} \left(\mathbf{F}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \tilde{\omega} M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n-\theta)} \right\|^2 + \delta^2 \theta \tilde{\omega} \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.102)$$

$$\theta \tilde{\omega} \left(\boldsymbol{\Gamma}^{(n-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \tilde{\omega}}{2} \left\| \boldsymbol{\Gamma}^{(n-\theta)} \right\|^2 + \frac{\theta \tilde{\omega}}{2} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.103)$$

and

$$\theta \tilde{\omega} \left(\boldsymbol{\Gamma}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \tilde{\omega} M^2 \acute{d}}{4} \left\| \boldsymbol{\Gamma}^{(n-\theta)} \right\|^2 + \delta^2 \theta \tilde{\omega} \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2. \quad (5.104)$$

Examining the convective terms:

$$\begin{aligned} (1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n+\bar{\theta})}, \mathbf{F}_{\delta^{(n+\bar{\theta})}}^{(n+\bar{\theta})} \right) &= (1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})}, \mathbf{F}^{(n+\bar{\theta})} \right) \\ &\quad + (1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})}, \delta \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) + (1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \boldsymbol{\Gamma}^{(n+\bar{\theta})}, \mathbf{F}^{(n+\bar{\theta})} \right) \\ &\quad + (1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \boldsymbol{\Gamma}^{(n+\bar{\theta})}, \delta \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right). \end{aligned} \quad (5.105)$$

Bounding each piece, using $\mathbf{u}|_{\partial\Omega} = 0$, and $\nabla \cdot \mathbf{u}^{(n+\bar{\theta})} = 0$

$$\begin{aligned} (1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})}, \mathbf{F}^{(n+\bar{\theta})} \right) &= -(1-2\theta) \lambda \left(\nabla \cdot \mathbf{u}^{(n+\bar{\theta})} \mathbf{F}^{(n+\bar{\theta})}, \mathbf{F}^{(n+\bar{\theta})} \right) \\ &\quad - (1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})}, \mathbf{F}^{(n+\bar{\theta})} \right) \\ &= -(1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})}, \mathbf{F}^{(n+\bar{\theta})} \right). \end{aligned}$$

Thus,

$$(1 - 2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})}, \mathbf{F}^{(n+\bar{\theta})} \right) = 0, \quad (5.106)$$

$$(1 - 2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})}, \delta \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) = \delta (1 - 2\theta) \lambda \left\| \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.107)$$

$$\begin{aligned} (1 - 2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \Gamma^{(n+\bar{\theta})}, \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \frac{(1 - 2\theta) \lambda M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2 + (1 - 2\theta) \lambda \left\| \nabla \Gamma^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.108)$$

and

$$\begin{aligned} (1 - 2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \Gamma^{(n+\bar{\theta})}, \delta \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \delta^2 (1 - 2\theta) \lambda \epsilon_1 \left\| \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2 + \frac{(1 - 2\theta) \lambda M^2 \acute{d}}{4\epsilon_1} \left\| \nabla \Gamma^{(n+\bar{\theta})} \right\|^2. \end{aligned} \quad (5.109)$$

The next convective term is split as

$$\begin{aligned} \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n)}^{(n+\bar{\theta})} \right) &= \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \mathbf{F}^{(n+\bar{\theta})} \right) + \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ &+ \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \Gamma^{(n)}, \mathbf{F}^{(n+\bar{\theta})} \right) + \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \Gamma^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right), \end{aligned} \quad (5.110)$$

and bounded by

$$\theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \lambda M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2 + \theta \lambda \left\| \nabla \mathbf{F}^{(n)} \right\|^2, \quad (5.111)$$

$$\theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\delta \theta \lambda M^2 \acute{d}}{2} \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2 + \frac{\delta \theta \lambda M^2 \acute{d}}{2} \left\| \nabla \mathbf{F}^{(n)} \right\|^2, \quad (5.112)$$

$$\theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \Gamma^{(n)}, \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \lambda M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2 + \theta \lambda \left\| \nabla \Gamma^{(n)} \right\|^2, \quad (5.113)$$

and

$$\theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \Gamma^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \lambda M^2 \acute{d}}{2} \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2 + \frac{\delta^2 \theta \lambda M^2 \acute{d}}{2} \left\| \nabla \Gamma^{(n)} \right\|^2. \quad (5.114)$$

Next

$$\begin{aligned} \theta \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n-\theta)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\bar{\theta})} \right) &= \theta \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) \\ &+ \theta \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) + \theta \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \Gamma^{(n-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) \\ &+ \theta \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \Gamma^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right), \end{aligned} \quad (5.115)$$

with the bounds

$$\theta \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\theta \lambda M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2 + \theta \lambda \left\| \nabla \mathbf{F}^{(n-\theta)} \right\|^2, \quad (5.116)$$

$$\theta\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \leq \frac{\delta\theta\lambda M^2 \acute{d}}{2} \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \frac{\delta\theta\lambda M^2 \acute{d}}{2} \left\| \nabla \mathbf{F}^{(n-\theta)} \right\|^2, \quad (5.117)$$

$$\theta\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{\Gamma}^{(n-\theta)}, \mathbf{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\lambda M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \theta\lambda \left\| \nabla \mathbf{\Gamma}^{(n-\theta)} \right\|^2, \quad (5.118)$$

and

$$\theta\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{\Gamma}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\lambda M^2 \acute{d}}{2} \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \frac{\delta^2\theta\lambda M^2 \acute{d}}{2} \left\| \nabla \mathbf{\Gamma}^{(n-\theta)} \right\|^2. \quad (5.119)$$

For the first $g_a(\cdot, \cdot)$ term

$$\begin{aligned} (1-2\theta)\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}_{\delta(n+\tilde{\theta})}^{(n+\tilde{\theta})}\right) &= (1-2\theta)\lambda\left(g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+\tilde{\theta})}\right) \\ + (1-2\theta)\lambda\left(g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) &+ (1-2\theta)\lambda\left(g_a(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+\tilde{\theta})}\right) \\ &+ (1-2\theta)\lambda\left(g_a(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right), \end{aligned} \quad (5.120)$$

with each part bounded by

$$(1-2\theta)\lambda\left(g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+\tilde{\theta})}\right) \leq (1-2\theta)\lambda 4M\acute{d} \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.121)$$

$$\begin{aligned} (1-2\theta)\lambda\left(g_a(\mathbf{F}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \\ \leq 4(1-2\theta)\lambda \acute{d}^4 M^4 \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \delta^2(1-2\theta)\lambda \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \end{aligned} \quad (5.122)$$

$$\begin{aligned} (1-2\theta)\lambda\left(g_a(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+\tilde{\theta})}\right) \\ \leq 4(1-2\theta)\lambda M^2 \acute{d}^2 \left\| \mathbf{\Gamma}^{(n+\tilde{\theta})} \right\|^2 + (1-2\theta)\lambda \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \end{aligned} \quad (5.123)$$

and

$$\begin{aligned} (1-2\theta)\lambda\left(g_a(\mathbf{\Gamma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) \\ \leq 4(1-2\theta)\lambda \acute{d}^4 M^4 \left\| \mathbf{\Gamma}^{(n+\tilde{\theta})} \right\|^2 + \delta^2(1-2\theta)\lambda \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2. \end{aligned} \quad (5.124)$$

Next,

$$\begin{aligned} \theta\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}_{\delta(n)}^{(n+\tilde{\theta})}\right) &= \theta\lambda\left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+\tilde{\theta})}\right) \\ &+ \theta\lambda\left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right) + \theta\lambda\left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+\tilde{\theta})}\right) \\ &+ \theta\lambda\left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})}\right), \end{aligned} \quad (5.125)$$

and has the bounds

$$\theta\lambda\left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+\tilde{\theta})}\right) \leq 4\theta\lambda M^2 \acute{d}^2 \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \theta\lambda \left\| \mathbf{F}^{(n)} \right\|^2, \quad (5.126)$$

$$\theta\lambda \left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \leq 4\theta\lambda M^4 \dot{d}^4 \left\| \mathbf{F}^{(n)} \right\|^2 + \delta^2 \theta\lambda \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.127)$$

$$\theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+\tilde{\theta})} \right) \leq 4\theta\lambda M^2 \dot{d}^2 \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \theta\lambda \left\| \mathbf{\Gamma}^{(n)} \right\|^2, \quad (5.128)$$

and

$$\theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \leq 4\theta\lambda M^4 \dot{d}^4 \left\| \mathbf{\Gamma}^{(n)} \right\|^2 + \delta^2 \theta\lambda \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2. \quad (5.129)$$

The last term in this grouping is

$$\begin{aligned} \theta\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}_{\delta(n-\theta)}^{(n+\tilde{\theta})} \right) &= \theta\lambda \left(g_a(\mathbf{F}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &+ \theta\lambda \left(g_a(\mathbf{F}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) + \theta\lambda \left(g_a(\mathbf{\Gamma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &+ \theta\lambda \left(g_a(\mathbf{\Gamma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right), \end{aligned} \quad (5.130)$$

and we use the bounds

$$\theta\lambda \left(g_a(\mathbf{F}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\tilde{\theta})} \right) \leq 4\theta\lambda M^2 \dot{d}^2 \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \theta\lambda \left\| \mathbf{F}^{(n-\theta)} \right\|^2, \quad (5.131)$$

$$\theta\lambda \left(g_a(\mathbf{F}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \leq 4\theta\lambda M^4 \dot{d}^4 \left\| \mathbf{F}^{(n-\theta)} \right\|^2 + \delta^2 \theta\lambda \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.132)$$

$$\theta\lambda \left(g_a(\mathbf{\Gamma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\tilde{\theta})} \right) \leq 4\theta\lambda M^2 \dot{d}^2 \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \theta\lambda \left\| \mathbf{\Gamma}^{(n-\theta)} \right\|^2, \quad (5.133)$$

and

$$\theta\lambda \left(g_a(\mathbf{\Gamma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \leq 4\theta\lambda M^4 \dot{d}^4 \left\| \mathbf{\Gamma}^{(n-\theta)} \right\|^2 + \delta^2 \theta\lambda \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2. \quad (5.134)$$

5.1.3 Bounding $\mathcal{B}_3^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_3^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau})$

In this sub section we determine bounds on the terms in expression (5.21), $\mathcal{B}_3^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_3^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau})$, for $\boldsymbol{\tau} = \mathbf{F}^{(n+\theta)}$. Thus,

$$\begin{aligned} \mathcal{B}_3^n(\boldsymbol{\sigma}, \mathbf{F}^{(n+\theta)}) - \mathcal{B}_3^n(\boldsymbol{\sigma}_h, \mathbf{F}^{(n+\theta)}) &:= \lambda \left(d_t e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}^{(n+\theta)} \right) + \theta\omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta(n)}^{(n+\theta)} \right) + \theta\omega \left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)} \right) \\ &+ (1-2\theta)\omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta-1)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)} \right) + \theta\tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n)}^{(n+\theta)} \right) + \tilde{\theta}\tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n-\theta)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)} \right) \\ &+ \tilde{\theta}\lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n-\theta)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)} \right) + \theta\lambda \left(\mathbf{u}^{(n)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n)}^{(n+\theta)} \right) \\ &+ \tilde{\theta}\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)} \right) + \theta\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}_{\delta(n)}^{(n+\theta)} \right). \end{aligned} \quad (5.135)$$

Each of the terms in expression (5.135) are now bounded. Here

$$\lambda \left(d_t \left(e_{\boldsymbol{\sigma}}^{(n+\theta)} \right), \mathbf{F}^{(n+\theta)} \right) = \lambda \left(d_t \left(\mathbf{F}^{(n+\theta)} \right), \mathbf{F}^{(n+\theta)} \right) + \lambda \left(d_t \left(\mathbf{\Gamma}^{(n+\theta)} \right), \mathbf{F}^{(n+\theta)} \right), \quad (5.136)$$

with

$$\lambda \left(d_t \left(\mathbf{F}^{(n+\theta)} \right), \mathbf{F}^{(n+\theta)} \right) \geq \frac{\lambda}{2\Delta t} \left(\left\| \mathbf{F}^{(n+\theta)} \right\|^2 - \left\| \mathbf{F}^{(n+\theta)-1} \right\|^2 \right), \quad (5.137)$$

and

$$\lambda \left(d_t \left(\mathbf{\Gamma}^{(n+\theta)} \right), \mathbf{F}^{(n+\theta)} \right) \leq \frac{\lambda}{4} \left\| d_t \left(\mathbf{\Gamma}^{(n+\theta)} \right) \right\|^2 + \lambda \left\| \mathbf{F}^{(n+\theta)} \right\|^2. \quad (5.138)$$

Next,

$$\begin{aligned} \theta \omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \mathbf{F}_{\delta(n)}^{(n+\theta)} \right) &= \theta \omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+\theta)} \right) + \theta \omega \left(\mathbf{F}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\ &\quad + \theta \omega \left(\mathbf{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+\theta)} \right) + \theta \omega \left(\mathbf{\Gamma}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right). \end{aligned} \quad (5.139)$$

Bounding each term we have

$$\theta \omega \left(\mathbf{F}^{(n+\theta)}, \mathbf{F}^{(n+\theta)} \right) = \theta \omega \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.140)$$

$$\theta \omega \left(\mathbf{F}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq \frac{\theta \omega M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \delta^2 \theta \omega \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.141)$$

$$\theta \omega \left(\mathbf{\Gamma}^{(n+\theta)}, \mathbf{F}^{(n+\theta)} \right) \leq \frac{\theta \omega}{2} \left\| \mathbf{\Gamma}^{(n+\theta)} \right\|^2 + \frac{\theta \omega}{2} \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.142)$$

and

$$\theta \omega \left(\mathbf{\Gamma}^{(n+\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq \frac{\theta \omega M^2 \acute{d}}{4} \left\| \mathbf{\Gamma}^{(n+\theta)} \right\|^2 + \delta^2 \theta \omega \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2. \quad (5.143)$$

Again, expanding the term

$$\begin{aligned} \theta \omega \left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)} \right) &= \theta \omega \left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+\theta)} \right) + \theta \omega \left(\mathbf{F}^{(n)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\ &\quad + \theta \omega \left(\mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+\theta)} \right) + \theta \omega \left(\mathbf{\Gamma}^{(n)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right). \end{aligned} \quad (5.144)$$

Bounding each of these

$$\theta \omega \left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+\theta)} \right) \leq \frac{\theta \omega}{2} \left\| \mathbf{F}^{(n)} \right\|^2 + \frac{\theta \omega}{2} \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.145)$$

$$\theta \omega \left(\mathbf{F}^{(n)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq \frac{\theta \omega M^2 \acute{d}}{4} \left\| \mathbf{F}^{(n)} \right\|^2 + \delta^2 \theta \omega \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.146)$$

$$\theta \omega \left(\mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+\theta)} \right) \leq \frac{\theta \omega}{2} \left\| \mathbf{\Gamma}^{(n)} \right\|^2 + \frac{\theta \omega}{2} \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.147)$$

and

$$\theta \omega \left(\mathbf{\Gamma}^{(n)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq \frac{\theta \omega M^2 \acute{d}}{4} \left\| \mathbf{\Gamma}^{(n)} \right\|^2 + \delta^2 \theta \omega \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2. \quad (5.148)$$

Next

$$\begin{aligned} (1-2\theta) \omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta-1)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)} \right) &= (1-2\theta) \omega \left(\mathbf{F}^{(n+\theta-1)}, \mathbf{F}^{(n+\theta)} \right) \\ &\quad + (1-2\theta) \omega \left(\mathbf{F}^{(n+\theta-1)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) + (1-2\theta) \omega \left(\mathbf{\Gamma}^{(n+\theta-1)}, \mathbf{F}^{(n+\theta)} \right) \\ &\quad + (1-2\theta) \omega \left(\mathbf{\Gamma}^{(n+\theta-1)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right), \end{aligned} \quad (5.149)$$

and has bounds

$$(1-2\theta) \omega \left(\mathbf{F}^{(n+\theta-1)}, \mathbf{F}^{(n+\theta)} \right) \leq \frac{(1-2\theta) \omega}{2} \left\| \mathbf{F}^{(n+\theta-1)} \right\|^2 + \frac{(1-2\theta) \omega}{2} \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.150)$$

$$\begin{aligned}
(1-2\theta)\omega\left(\mathbf{F}^{(n+\theta-1)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right) \\
\leq \frac{(1-2\theta)\omega M^2 \acute{d}}{4} \left\|\mathbf{F}^{(n+\theta-1)}\right\|^2 + \delta^2(1-2\theta)\omega \left\|\nabla\mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.151)
\end{aligned}$$

$$(1-2\theta)\omega\left(\mathbf{\Gamma}^{(n+\theta-1)}, \mathbf{F}^{(n+\theta)}\right) \leq \frac{(1-2\theta)\omega}{2} \left\|\mathbf{\Gamma}^{(n+\theta-1)}\right\|^2 + \frac{(1-2\theta)\omega}{2} \left\|\mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.152)$$

and

$$\begin{aligned}
(1-2\theta)\omega\left(\mathbf{\Gamma}^{(n+\theta-1)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right) \\
\leq \frac{(1-2\theta)\omega M^2 \acute{d}}{4} \left\|\mathbf{\Gamma}^{(n+\theta-1)}\right\|^2 + \delta^2(1-2\theta)\omega \left\|\nabla\mathbf{F}^{(n+\theta)}\right\|^2. \quad (5.153)
\end{aligned}$$

Splitting

$$\begin{aligned}
\theta\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n)}^{(n+\theta)}\right) &= \theta\tilde{\omega}\left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+\theta)}\right) + \theta\tilde{\omega}\left(\mathbf{F}^{(n)}, \delta\mathbf{u}^{(n)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right) \\
&\quad + \theta\tilde{\omega}\left(\mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+\theta)}\right) + \theta\tilde{\omega}\left(\mathbf{\Gamma}^{(n)}, \delta\mathbf{u}^{(n)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right), \quad (5.154)
\end{aligned}$$

we have

$$\theta\tilde{\omega}\left(\mathbf{F}^{(n)}, \mathbf{F}^{(n+\theta)}\right) \leq \frac{\theta\tilde{\omega}}{2} \left\|\mathbf{F}^{(n)}\right\|^2 + \frac{\theta\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.155)$$

$$\theta\tilde{\omega}\left(\mathbf{F}^{(n)}, \delta\mathbf{u}^{(n)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right) \leq \frac{\theta\tilde{\omega}M^2 \acute{d}}{4} \left\|\mathbf{F}^{(n)}\right\|^2 + \delta^2\theta\tilde{\omega} \left\|\nabla\mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.156)$$

$$\theta\tilde{\omega}\left(\mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+\theta)}\right) \leq \frac{\theta\tilde{\omega}}{2} \left\|\mathbf{\Gamma}^{(n)}\right\|^2 + \frac{\theta\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.157)$$

and

$$\theta\tilde{\omega}\left(\mathbf{\Gamma}^{(n)}, \delta\mathbf{u}^{(n)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right) \leq \frac{\theta\tilde{\omega}M^2 \acute{d}}{4} \left\|\mathbf{\Gamma}^{(n)}\right\|^2 + \delta^2\theta\tilde{\omega} \left\|\nabla\mathbf{F}^{(n+\theta)}\right\|^2. \quad (5.158)$$

The term

$$\begin{aligned}
\tilde{\theta}\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n-\theta)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)}\right) &= \tilde{\theta}\tilde{\omega}\left(\mathbf{F}^{(n-\theta)}, \mathbf{F}^{(n+\theta)}\right) + \tilde{\theta}\tilde{\omega}\left(\mathbf{F}^{(n-\theta)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right) \\
&\quad + \tilde{\theta}\tilde{\omega}\left(\mathbf{\Gamma}^{(n-\theta)}, \mathbf{F}^{(n+\theta)}\right) + \tilde{\theta}\tilde{\omega}\left(\mathbf{\Gamma}^{(n-\theta)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right). \quad (5.159)
\end{aligned}$$

Here each term is treated in the following manner.

$$\tilde{\theta}\tilde{\omega}\left(\mathbf{F}^{(n-\theta)}, \mathbf{F}^{(n+\theta)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}}{2} \left\|\mathbf{F}^{(n-\theta)}\right\|^2 + \frac{\tilde{\theta}\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.160)$$

$$\tilde{\theta}\tilde{\omega}\left(\mathbf{F}^{(n-\theta)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}M^2 \acute{d}}{4} \left\|\mathbf{F}^{(n-\theta)}\right\|^2 + \delta^2\tilde{\theta}\tilde{\omega} \left\|\nabla\mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.161)$$

$$\tilde{\theta}\tilde{\omega}\left(\mathbf{\Gamma}^{(n-\theta)}, \mathbf{F}^{(n+\theta)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}}{2} \left\|\mathbf{\Gamma}^{(n-\theta)}\right\|^2 + \frac{\tilde{\theta}\tilde{\omega}}{2} \left\|\mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.162)$$

and

$$\tilde{\theta}\tilde{\omega}\left(\mathbf{\Gamma}^{(n-\theta)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla\mathbf{F}^{(n+\theta)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}M^2 \acute{d}}{4} \left\|\mathbf{\Gamma}^{(n-\theta)}\right\|^2 + \delta^2\tilde{\theta}\tilde{\omega} \left\|\nabla\mathbf{F}^{(n+\theta)}\right\|^2. \quad (5.163)$$

Examining the convective terms,

$$\begin{aligned}\tilde{\theta}\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n-\theta)}, \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)}\right) &= \tilde{\theta}\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n-\theta)}, \mathbf{F}^{(n+\theta)}\right) \\ &+ \tilde{\theta}\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right) + \tilde{\theta}\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{\Gamma}^{(n-\theta)}, \mathbf{F}^{(n+\theta)}\right) \\ &+ \tilde{\theta}\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{\Gamma}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right).\end{aligned}\quad (5.164)$$

Here each part is bounded

$$\tilde{\theta}\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n-\theta)}, \mathbf{F}^{(n+\theta)}\right) \leq \frac{\tilde{\theta}\lambda\dot{d}M^2}{4} \left\|\mathbf{F}^{(n+\theta)}\right\|^2 + \tilde{\theta}\lambda \left\|\nabla \mathbf{F}^{(n-\theta)}\right\|^2, \quad (5.165)$$

$$\tilde{\theta}\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right) \leq \frac{\delta\tilde{\theta}\lambda\dot{d}M^2}{2} \left\|\nabla \mathbf{F}^{(n-\theta)}\right\|^2 + \frac{\delta\tilde{\theta}\lambda\dot{d}M^2}{2} \left\|\nabla \mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.166)$$

$$\tilde{\theta}\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{\Gamma}^{(n-\theta)}, \mathbf{F}^{(n+\theta)}\right) \leq \frac{\tilde{\theta}\lambda\dot{d}M^2}{4} \left\|\mathbf{F}^{(n+\theta)}\right\|^2 + \tilde{\theta}\lambda \left\|\nabla \mathbf{\Gamma}^{(n-\theta)}\right\|^2, \quad (5.167)$$

and

$$\tilde{\theta}\lambda\left(\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{\Gamma}^{(n-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right) \leq \frac{\tilde{\theta}\lambda\dot{d}M^2}{2} \left\|\nabla \mathbf{F}^{(n+\theta)}\right\|^2 + \frac{\delta^2\tilde{\theta}\lambda\dot{d}M^2}{2} \left\|\nabla \mathbf{\Gamma}^{(n-\theta)}\right\|^2. \quad (5.168)$$

The second convective term is:

$$\begin{aligned}\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n)}, \mathbf{F}_{\delta(n)}^{(n+\theta)}\right) &= \theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \mathbf{F}^{(n+\theta)}\right) + \theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right) \\ &+ \theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+\theta)}\right) + \theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{\Gamma}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right),\end{aligned}\quad (5.169)$$

with bounds given by

$$\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \mathbf{F}^{(n+\theta)}\right) \leq \frac{\theta\lambda\dot{d}M^2}{4} \left\|\mathbf{F}^{(n+\theta)}\right\|^2 + \theta\lambda \left\|\nabla \mathbf{F}^{(n)}\right\|^2, \quad (5.170)$$

$$\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right) \leq \frac{\delta\theta\lambda\dot{d}M^2}{2} \left\|\nabla \mathbf{F}^{(n)}\right\|^2 + \frac{\delta\theta\lambda\dot{d}M^2}{2} \left\|\nabla \mathbf{F}^{(n+\theta)}\right\|^2, \quad (5.171)$$

$$\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{\Gamma}^{(n)}, \mathbf{F}^{(n+\theta)}\right) \leq \frac{\theta\lambda\dot{d}M^2}{4} \left\|\mathbf{F}^{(n+\theta)}\right\|^2 + \theta\lambda \left\|\nabla \mathbf{\Gamma}^{(n)}\right\|^2, \quad (5.172)$$

and

$$\theta\lambda\left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{\Gamma}^{(n)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right) \leq \frac{\theta\lambda\dot{d}M^2}{2} \left\|\nabla \mathbf{F}^{(n+\theta)}\right\|^2 + \frac{\delta^2\theta\lambda\dot{d}M^2}{2} \left\|\nabla \mathbf{\Gamma}^{(n)}\right\|^2. \quad (5.173)$$

Lastly the $g_a(\cdot, \cdot)$ terms are considered. Specifically

$$\begin{aligned}\tilde{\theta}\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}_{\delta(n-\theta)}^{(n+\theta)}\right) &= \tilde{\theta}\lambda\left(g_a(\mathbf{F}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\theta)}\right) \\ &+ \tilde{\theta}\lambda\left(g_a(\mathbf{F}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right) + \tilde{\theta}\lambda\left(g_a(\mathbf{\Gamma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\theta)}\right) \\ &+ \tilde{\theta}\lambda\left(g_a(\mathbf{\Gamma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)}\right).\end{aligned}\quad (5.174)$$

Handling each part

$$\tilde{\theta}\lambda \left(g_a(\mathbf{F}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\theta)} \right) \leq 4\tilde{\theta}\lambda M^2 \hat{d}^2 \left\| \mathbf{F}^{(n-\theta)} \right\|^2 + \tilde{\theta}\lambda \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.175)$$

$$\tilde{\theta}\lambda \left(g_a(\mathbf{F}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq 4\tilde{\theta}\lambda M^4 \hat{d}^4 \left\| \mathbf{F}^{(n-\theta)} \right\|^2 + \delta^2 \tilde{\theta}\lambda \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.176)$$

$$\tilde{\theta}\lambda \left(g_a(\mathbf{\Gamma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\theta)} \right) \leq 4\tilde{\theta}\lambda M^2 \hat{d}^2 \left\| \mathbf{\Gamma}^{(n-\theta)} \right\|^2 + \tilde{\theta}\lambda \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.177)$$

and

$$\tilde{\theta}\lambda \left(g_a(\mathbf{\Gamma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq 4\tilde{\theta}\lambda M^4 \hat{d}^4 \left\| \mathbf{\Gamma}^{(n-\theta)} \right\|^2 + \delta^2 \tilde{\theta}\lambda \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2. \quad (5.178)$$

For the last term:

$$\begin{aligned} \theta\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}_{\delta^{(n)}}^{(n+\theta)} \right) &= \theta\lambda \left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+\theta)} \right) \\ &\quad + \theta\lambda \left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\ &\quad + \theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+\theta)} \right) + \theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right), \end{aligned} \quad (5.179)$$

and each term is bounded by

$$\theta\lambda \left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+\theta)} \right) \leq 4\theta\lambda M^2 \hat{d}^2 \left\| \mathbf{F}^{(n)} \right\|^2 + \theta\lambda \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.180)$$

$$\theta\lambda \left(g_a(\mathbf{F}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq 4\theta\lambda M^4 \hat{d}^4 \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \delta^2 \theta\lambda \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.181)$$

$$\theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \mathbf{F}^{(n+\theta)} \right) \leq 4\theta\lambda M^2 \hat{d}^2 \left\| \mathbf{F}^{(n+\theta)} \right\|^2 + \theta\lambda \left\| \mathbf{\Gamma}^{(n)} \right\|^2, \quad (5.182)$$

and

$$\theta\lambda \left(g_a(\mathbf{\Gamma}^{(n)}, \nabla \mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq 4\theta\lambda M^4 \hat{d}^4 \left\| \mathbf{\Gamma}^{(n)} \right\|^2 + \delta^2 \theta\lambda \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2. \quad (5.183)$$

5.1.4 Bound the $\mathcal{I}_1^n(\mathbf{F}^{(n+1)})$ Terms in (5.7)

The bounds on the $\mathcal{I}_1^n(\boldsymbol{\tau})$ term are now found with $\boldsymbol{\tau} = \mathbf{F}^{(n+1)}$. This term is broken into two pieces in the following manner:

$$\mathcal{I}_1^n(\mathbf{F}^{(n+1)}) = \text{Int}_{\boldsymbol{\sigma}_{1g}}(\mathbf{F}^{(n+1)}) + \text{Int}_{\boldsymbol{\sigma}_{1up}}(\mathbf{F}^{(n+1)}),$$

with

$$\begin{aligned} \text{Int}_{\boldsymbol{\sigma}_{1g}}(\mathbf{F}^{(n+1)}) &= \lambda \left(d_t \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \mathbf{F}^{(n+1)} \right) + \omega \left(\tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \mathbf{F}^{(n+1)} \right) \\ &\quad + \tilde{\omega} \left(\theta \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \mathbf{F}^{(n+1)} \right) \\ &\quad + \lambda \left(\theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \mathbf{F}^{(n+1)} \right) \\ &\quad + \lambda \left(\theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) + \tilde{\theta} g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}), \mathbf{F}^{(n+1)} \right) \\ &\quad + 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+1)} \right), \end{aligned}$$

and

$$\begin{aligned}
Int_{\sigma_{1up}}(\mathbf{F}^{(n+1)}) &= \lambda \theta \left(-\sigma_t^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \lambda \tilde{\theta} \left(-\sigma_t^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \omega \theta \left(\sigma^{(n+\theta)} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \omega \theta \left(\sigma^{(n+1)} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \omega (1 - 2\theta) \left(\sigma^{(n+\theta)} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \tilde{\omega} \theta \left(\sigma^{(n)} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \tilde{\omega} \tilde{\theta} \left(\sigma^{(n+\tilde{\theta})} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \sigma^{(n)} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \tilde{\theta} \lambda \left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \sigma^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \theta \lambda \left(g_a(\sigma^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\sigma^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \tilde{\theta} \lambda \left(g_a(\sigma^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\sigma^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
&+ \tilde{\theta} 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right).
\end{aligned}$$

Bounding each of the $Int_{\sigma_{1g}}(\mathbf{F}^{(n+1)})$ terms:

$$\lambda \left(d_t \sigma^{(n+1)} - \sigma_t^{(n+\frac{1}{2})}, \mathbf{F}^{(n+1)} \right) \leq \frac{\lambda}{4} \left\| d_t \sigma^{(n+1)} - \sigma_t^{(n+\frac{1}{2})} \right\|^2 + \lambda \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.184)$$

$$\omega \left(\tilde{\theta} \sigma^{(n+\theta)} + \theta \sigma^{(n+1)} - \sigma^{(n+\frac{1}{2})}, \mathbf{F}^{(n+1)} \right) \leq \frac{\omega}{4} \left\| \tilde{\theta} \sigma^{(n+\theta)} + \theta \sigma^{(n+1)} - \sigma^{(n+\frac{1}{2})} \right\|^2 \quad (5.185)$$

$$+ \omega \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.186)$$

$$\tilde{\omega} \left(\theta \sigma^{(n)} + \tilde{\theta} \sigma^{(n+\tilde{\theta})} - \sigma^{(n+\frac{1}{2})}, \mathbf{F}^{(n+1)} \right) \leq \frac{\tilde{\omega}}{4} \left\| \theta \sigma^{(n)} + \tilde{\theta} \sigma^{(n+\tilde{\theta})} - \sigma^{(n+\frac{1}{2})} \right\|^2 + \tilde{\omega} \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.187)$$

$$\begin{aligned}
&\lambda \left(\theta \mathbf{u}^{(n)} \cdot \nabla \sigma^{(n)} + \tilde{\theta} \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \sigma^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \sigma^{(n+\frac{1}{2})}, \mathbf{F}^{(n+1)} \right) \\
&\leq \frac{\lambda}{4} \left\| \theta \mathbf{u}^{(n)} \cdot \nabla \sigma^{(n)} + \tilde{\theta} \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \sigma^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \sigma^{(n+\frac{1}{2})} \right\|^2 + \lambda \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.188)
\end{aligned}$$

$$\begin{aligned}
&\lambda \left(\theta g_a(\sigma^{(n)}, \nabla \mathbf{u}^{(n)}) + \tilde{\theta} g_a(\sigma^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\sigma^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}), \mathbf{F}^{(n+1)} \right) \\
&\leq \frac{\lambda}{4} \left\| \theta g_a(\sigma^{(n)}, \nabla \mathbf{u}^{(n)}) + \tilde{\theta} g_a(\sigma^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\sigma^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}) \right\|^2 \\
&\quad + \lambda \left\| \mathbf{F}^{(n+1)} \right\|^2, \quad (5.189)
\end{aligned}$$

and

$$\begin{aligned}
2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}), \mathbf{F}^{(n+1)} \right) \\
\leq \frac{\alpha}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}) \right\|^2 + 2\alpha \left\| \mathbf{F}^{(n+1)} \right\|^2. \quad (5.190)
\end{aligned}$$

Bounding each of the $Int_{\sigma_{1up}}(\mathbf{F}^{(n+1)})$ terms:

$$\lambda \theta \left(-\sigma_t^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \leq \frac{\lambda \theta \delta^2 M^2 \acute{d}}{4} \left\| -\sigma_t^{(n+\frac{1}{2})} \right\|^2 + \lambda \theta \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.191)$$

$$\lambda \tilde{\theta} \left(-\sigma_t^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \leq \frac{\lambda \tilde{\theta} \delta^2 M^2 \acute{d}}{4} \left\| -\sigma_t^{(n+\frac{1}{2})} \right\|^2 + \lambda \tilde{\theta} \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.192)$$

$$\begin{aligned}
\omega \theta \left(\sigma^{(n+\theta)} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) &\leq \frac{\omega \theta \delta^2 M^2 \acute{d}}{4} \left\| \sigma^{(n+\theta)} - \sigma^{(n+\frac{1}{2})} \right\|^2 \\
&+ \omega \theta \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.193)
\end{aligned}$$

$$\begin{aligned}
\omega \theta \left(\sigma^{(n+1)} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) &\leq \frac{\omega \theta \delta^2 M^2 \acute{d}}{4} \left\| \sigma^{(n+1)} - \sigma^{(n+\frac{1}{2})} \right\|^2 \\
&+ \omega \theta \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.194)
\end{aligned}$$

$$\begin{aligned}
\omega (1-2\theta) \left(\sigma^{(n+\theta)} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) &\leq \frac{\omega (1-2\theta) \delta^2 M^2 \acute{d}}{4} \left\| \sigma^{(n+\theta)} - \sigma^{(n+\frac{1}{2})} \right\|^2 \\
&+ \omega (1-2\theta) \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.195)
\end{aligned}$$

$$\tilde{\omega} \theta \left(\sigma^{(n)} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \leq \frac{\tilde{\omega} \theta \delta^2 M^2 \acute{d}}{4} \left\| \sigma^{(n)} - \sigma^{(n+\frac{1}{2})} \right\|^2 + \tilde{\omega} \theta \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.196)$$

$$\begin{aligned}
\tilde{\omega} \tilde{\theta} \left(\sigma^{(n+\tilde{\theta})} - \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) &\leq \frac{\tilde{\omega} \tilde{\theta} \delta^2 M^2 \acute{d}}{4} \left\| \sigma^{(n+\tilde{\theta})} - \sigma^{(n+\frac{1}{2})} \right\|^2 \\
&+ \tilde{\omega} \tilde{\theta} \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.197)
\end{aligned}$$

$$\begin{aligned}
\theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \sigma^{(n)} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
\leq \frac{\theta \lambda \delta^2 M^2 \acute{d}}{4} \left\| \mathbf{u}^{(n)} \cdot \nabla \sigma^{(n)} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \sigma^{(n+\frac{1}{2})} \right\|^2 + \theta \lambda \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.198)
\end{aligned}$$

$$\begin{aligned}
\tilde{\theta} \lambda \left(\mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \sigma^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \sigma^{(n+\frac{1}{2})}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\
\leq \frac{\tilde{\theta} \lambda \delta^2 M^2 \acute{d}}{4} \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \sigma^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \sigma^{(n+\frac{1}{2})} \right\|^2 + \tilde{\theta} \lambda \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.199)
\end{aligned}$$

$$\begin{aligned} & \theta \lambda \left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\ & \leq \frac{\theta \lambda \delta^2 M^2 \dot{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}) \right\|^2 + \theta \lambda \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.200) \end{aligned}$$

$$\begin{aligned} & \tilde{\theta} \lambda \left(g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\ & \leq \frac{\tilde{\theta} \lambda \delta^2 M^2 \dot{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}) \right\|^2 + \tilde{\theta} \lambda \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.201) \end{aligned}$$

$$\begin{aligned} & \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\ & \leq \frac{\theta \alpha \delta^2 M^2 \dot{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}) \right\|^2 + \theta 2\alpha \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2, \quad (5.202) \end{aligned}$$

$$\begin{aligned} & \tilde{\theta} 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}), \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+1)} \right) \\ & \leq \frac{\tilde{\theta} \alpha \delta^2 M^2 \dot{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}) \right\|^2 + \tilde{\theta} 2\alpha \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2. \quad (5.203) \end{aligned}$$

5.1.5 Bounding the $\mathcal{I}_2^n(\mathbf{F}^{(n+\tilde{\theta})})$ Terms in (5.13)

Here we bound the terms in $\mathcal{I}_2^n(\mathbf{F}^{(n+\tilde{\theta})})$ in expression (5.13) in two phases, first handling the standard Galerkin terms, and then the upwinded terms. Consider

$$\mathcal{I}_2^n(\mathbf{F}^{(n+\tilde{\theta})}) = \mathcal{I}_{2_g}^n(\mathbf{F}^{(n+\tilde{\theta})}) + \mathcal{I}_{2_{up}}^n(\mathbf{F}^{(n+\tilde{\theta})}), \quad (5.204)$$

where

$$\begin{aligned} \mathcal{I}_{2_g}^n(\mathbf{F}^{(n+\tilde{\theta})}) &= \lambda \left(d_t \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &+ \omega \left(\theta \boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &+ \tilde{\omega} \left((1-2\theta) \boldsymbol{\sigma}^{(n+\tilde{\theta})} + \theta \boldsymbol{\sigma}^{(n)} + \theta \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &+ \lambda \left((1-2\theta) \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} + \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \theta \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} \right. \\ &\quad \left. - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &+ \lambda \left((1-2\theta) g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) \right. \\ &\quad \left. + \theta g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \mathbf{F}^{(n+\tilde{\theta})} \right) \\ &+ 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - (1-2\theta) \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}) \right. \\ &\quad \left. - \theta \mathbf{d}(\mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\tilde{\theta})} \right), \end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{2\text{up}}^n(\mathbf{F}^{(n+\bar{\theta})}) &= \lambda\theta \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \lambda(1-2\theta) \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \lambda\theta \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) + \theta\omega \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ (1-2\theta)\omega \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \theta\omega \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ (1-2\theta)\tilde{\omega} \left(\boldsymbol{\sigma}^{(n+\bar{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \theta\tilde{\omega} \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \theta\tilde{\omega} \left(\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ (1-2\theta)\lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\bar{\theta})} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \theta\lambda \left(\mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \theta\lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ (1-2\theta)\lambda \left(g_a(\boldsymbol{\sigma}^{(n+\bar{\theta})}, \nabla \mathbf{u}^{(n+\bar{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \theta\lambda \left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \theta\lambda \left(g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n)}), \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ (1-2\theta) 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n+\bar{\theta})}), \delta\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\
&+ \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}), \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right).
\end{aligned}$$

Bounding the $\mathcal{I}_{2g}^n(\mathbf{F}^{(n+\bar{\theta})})$ terms:

$$\lambda \left(d_t \boldsymbol{\sigma}^{(n+\bar{\theta})} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) \leq \frac{\lambda}{4} \left\| d_t \boldsymbol{\sigma}^{(n+\bar{\theta})} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)} \right\|^2 + \lambda \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.205)$$

$$\begin{aligned}
&\omega \left(\theta \boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) \\
&\leq \frac{\omega}{4} \left\| \theta \boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \omega \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.206)
\end{aligned}$$

$$\begin{aligned}
&\tilde{\omega} \left((1-2\theta) \boldsymbol{\sigma}^{(n+\bar{\theta})} + \theta \boldsymbol{\sigma}^{(n)} + \theta \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{F}^{(n+\bar{\theta})} \right) \\
&\leq \frac{\tilde{\omega}}{4} \left\| (1-2\theta) \boldsymbol{\sigma}^{(n+\bar{\theta})} + \theta \boldsymbol{\sigma}^{(n)} + \theta \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \tilde{\omega} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \quad (5.207)
\end{aligned}$$

$$\begin{aligned}
& \lambda \left((1-2\theta) \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} + \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \theta \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{F}^{(n+\tilde{\theta})} \right) \\
& \leq \frac{\lambda}{4} \left\| (1-2\theta) \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} + \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \theta \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& \quad + \lambda \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.208)
\end{aligned}$$

$$\begin{aligned}
& \lambda \left((1-2\theta) g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) \right. \\
& \quad \left. + \theta g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \mathbf{F}^{(n+\tilde{\theta})} \right) \\
& \leq \frac{\lambda}{4} \left\| (1-2\theta) g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) \right. \\
& \quad \left. + \theta g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}) \right\|^2 + \lambda \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.209)
\end{aligned}$$

and

$$\begin{aligned}
& 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - (1-2\theta) \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}) - \theta \mathbf{d}(\mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\tilde{\theta})} \right) \\
& \leq \frac{\alpha}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - (1-2\theta) \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}) - \theta \mathbf{d}(\mathbf{u}^{(n-\theta)}) \right\|^2 + 2\alpha \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2. \quad (5.210)
\end{aligned}$$

Bounding the terms in $\mathcal{I}_{2_{up}}^n(\mathbf{F}^{(n+\tilde{\theta})})$

$$\lambda \theta \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \leq \frac{\lambda \theta \delta^2 M^2 \acute{d}}{4} \left\| -\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)} \right\|^2 + \lambda \theta \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.211)$$

$$\begin{aligned}
& \lambda (1-2\theta) \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \\
& \leq \frac{\lambda (1-2\theta) \delta^2 M^2 \acute{d}}{4} \left\| -\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)} \right\|^2 + \lambda (1-2\theta) \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.212)
\end{aligned}$$

$$\lambda \theta \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \leq \frac{\lambda \theta \delta^2 M^2 \acute{d}}{4} \left\| -\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)} \right\|^2 + \lambda \theta \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.213)$$

$$\begin{aligned}
& \theta \omega \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \\
& \leq \frac{\theta \omega \delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \theta \omega \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.214)
\end{aligned}$$

$$\begin{aligned}
& (1-2\theta) \omega \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right) \\
& \leq \frac{(1-2\theta) \omega \delta^2}{4\epsilon_1} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + (1-2\theta) \omega \epsilon_1 \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2, \quad (5.215)
\end{aligned}$$

$$\begin{aligned} \theta\omega \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \frac{\theta\omega\delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \theta\omega \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.216)$$

$$\begin{aligned} (1-2\theta) \tilde{\omega} \left(\boldsymbol{\sigma}^{(n+\bar{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \frac{(1-2\theta) \tilde{\omega}}{4\epsilon_1} \left\| \boldsymbol{\sigma}^{(n+\bar{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \delta^2 (1-2\theta) \tilde{\omega} \epsilon_1 \left\| \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.217)$$

$$\begin{aligned} \theta\tilde{\omega} \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \frac{\theta\tilde{\omega}\delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \theta\tilde{\omega} \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.218)$$

$$\begin{aligned} \theta\tilde{\omega} \left(\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \frac{\theta\tilde{\omega}\delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \theta\tilde{\omega} \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.219)$$

$$\begin{aligned} (1-2\theta) \lambda \left(\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\bar{\theta})} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \frac{(1-2\theta) \lambda}{4\epsilon_1} \left\| \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\bar{\theta})} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\ + \delta^2 (1-2\theta) \lambda \epsilon_1 \left\| \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.220)$$

$$\begin{aligned} \theta\lambda \left(\mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \frac{\theta\lambda\delta^2 M^2 \acute{d}}{4} \left\| \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \theta\lambda \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.221)$$

$$\begin{aligned} \theta\lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \frac{\theta\lambda\delta^2 M^2 \acute{d}}{4} \left\| \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \theta\lambda \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.222)$$

$$\begin{aligned} (1-2\theta) \lambda \left(g_a(\boldsymbol{\sigma}^{(n+\bar{\theta})}, \nabla \mathbf{u}^{(n+\bar{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \delta \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ \leq \frac{(1-2\theta) \lambda}{4\epsilon_1} \left\| g_a(\boldsymbol{\sigma}^{(n+\bar{\theta})}, \nabla \mathbf{u}^{(n+\bar{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}) \right\|^2 \\ + \delta^2 (1-2\theta) \lambda \epsilon_1 \left\| \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.223)$$

$$\begin{aligned} & \theta \lambda \left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ & \leq \frac{\theta \lambda \delta^2 M^2 \dot{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}) \right\|^2 + \theta \lambda \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.224)$$

$$\begin{aligned} & \theta \lambda \left(g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ & \leq \frac{\theta \lambda \delta^2 M^2 \dot{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}) \right\|^2 + \theta \lambda \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.225)$$

$$\begin{aligned} & \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ & \leq \frac{\theta \alpha \delta^2 M^2 \dot{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n)}) \right\|^2 + \theta 2\alpha \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.226)$$

$$\begin{aligned} & (1-2\theta) 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n+\bar{\theta})}), \delta \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ & \leq \frac{(1-2\theta) \alpha}{2\epsilon_1} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n+\bar{\theta})}) \right\|^2 + \delta^2 (1-2\theta) 2\alpha \epsilon_1 \left\| \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2, \end{aligned} \quad (5.227)$$

and

$$\begin{aligned} & \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right) \\ & \leq \frac{\theta 2\alpha \delta^2 M^2 \dot{d}}{4} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}) \right\|^2 + \theta 2\alpha \left\| \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2. \end{aligned} \quad (5.228)$$

5.1.6 Bounding the $\mathcal{I}_3^n(\mathbf{F}^{(n+\theta)})$ Terms in (5.20)

Bounding the $\mathcal{I}_3^n(\mathbf{F}^{(n+\theta)})$ terms in (5.20) is done in two phases, first the standard Galerkin terms and then the upwinded terms.

$$\mathcal{I}_3^n(\mathbf{F}^{(n+\theta)}) = \mathcal{I}_{3g}^n(\mathbf{F}^{(n+\theta)}) + \mathcal{I}_{3up}^n(\mathbf{F}^{(n+\theta)}), \quad (5.229)$$

where

$$\begin{aligned} \mathcal{I}_{3g}^n(\mathbf{F}^{(n+\theta)}) = & \lambda \left(d_t \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})}, \mathbf{F}^{(n+\theta)} \right) \\ & + \tilde{\omega} \left(\theta \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{F}^{(n+\theta)} \right) \\ & + \omega \left(\theta \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{F}^{(n+\theta)} \right) \\ & + \lambda \left(\tilde{\theta} \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} + \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{F}^{(n+\theta)} \right) \\ & + \lambda \left(\tilde{\theta} g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}), \mathbf{F}^{(n+\theta)} \right) \\ & + 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\theta)} \right), \end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{3\text{ up}}^n(\mathbf{F}^{(n+\theta)}) &= \lambda\theta \left(-\boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) + \lambda\tilde{\theta} \left(-\boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \theta\omega \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \theta\omega \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ (1-2\theta)\omega \left(\boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \theta\tilde{\omega} \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \tilde{\theta}\tilde{\omega} \left(\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \tilde{\theta}\lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \theta\lambda \left(\mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \tilde{\theta}\lambda \left(g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}), \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \theta\lambda \left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}), \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}), \delta\mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
&+ \tilde{\theta} 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}), \delta\mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right).
\end{aligned}$$

Bounding the terms in $\mathcal{I}_{3g}^n(\mathbf{F}^{(n+\theta)})$:

$$\lambda \left(d_t \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})}, \mathbf{F}^{(n+\theta)} \right) \leq \frac{\lambda}{4} \left\| d_t \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})} \right\|^2 + \lambda \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.230)$$

$$\tilde{\omega} \left(\theta \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{F}^{(n+\theta)} \right) \leq \frac{\tilde{\omega}}{4} \left\| \theta \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 \quad (5.231)$$

$$+ \tilde{\omega} \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.232)$$

$$\begin{aligned}
&\omega \left(\theta \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{F}^{(n+\theta)} \right) \\
&\leq \frac{\omega}{4} \left\| \theta \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \omega \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.233)
\end{aligned}$$

$$\begin{aligned}
&\lambda \left(\tilde{\theta} \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} + \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{F}^{(n+\theta)} \right) \\
&\leq \frac{\lambda}{4} \left\| \tilde{\theta} \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} + \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \lambda \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.234)
\end{aligned}$$

$$\begin{aligned}
&\lambda \left(\tilde{\theta} g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}), \mathbf{F}^{(n+\theta)} \right) \\
&\leq \frac{\lambda}{4} \left\| \tilde{\theta} g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}) \right\|^2 \\
&\quad + \lambda \left\| \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.235)
\end{aligned}$$

$$\begin{aligned}
& 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n-\theta)}), \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\alpha}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n-\theta)}) \right\|^2 + 2\alpha \left\| \mathbf{F}^{(n+\theta)} \right\|^2. \quad (5.236)
\end{aligned}$$

Bounding the $\mathcal{I}_{3up}^n(\mathbf{F}^{(n+\theta)})$ terms:

$$\lambda \theta \left(-\boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq \frac{\lambda \theta \delta^2 M^2 \acute{d}}{4} \left\| -\boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})} \right\|^2 + \lambda \theta \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.237)$$

$$\lambda \tilde{\theta} \left(-\boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \leq \frac{\lambda \tilde{\theta} \delta^2 M^2 \acute{d}}{4} \left\| -\boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})} \right\|^2 + \lambda \tilde{\theta} \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.238)$$

$$\begin{aligned}
& \theta \omega \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\theta \omega \delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \theta \omega \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.239)
\end{aligned}$$

$$\begin{aligned}
& \theta \omega \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\theta \omega \delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \theta \omega \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.240)
\end{aligned}$$

$$\begin{aligned}
& (1-2\theta) \omega \left(\boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{(1-2\theta) \omega \delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + (1-2\theta) \omega \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.241)
\end{aligned}$$

$$\begin{aligned}
& \theta \tilde{\omega} \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\theta \tilde{\omega} \delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \theta \tilde{\omega} \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.242)
\end{aligned}$$

$$\begin{aligned}
& \tilde{\theta} \tilde{\omega} \left(\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\tilde{\theta} \tilde{\omega} \delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \tilde{\theta} \tilde{\omega} \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.243)
\end{aligned}$$

$$\begin{aligned}
& \tilde{\theta} \lambda \left(\mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\tilde{\theta} \lambda \delta^2 M^2 \acute{d}}{4} \left\| \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \tilde{\theta} \lambda \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.244)
\end{aligned}$$

$$\begin{aligned}
& \theta \lambda \left(\mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\theta \lambda \delta^2 M^2 \acute{d}}{4} \left\| \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \theta \lambda \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.245)
\end{aligned}$$

$$\begin{aligned}
& \tilde{\theta}\lambda \left(g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\tilde{\theta}\lambda\delta^2 M^2 \acute{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}) \right\|^2 + \tilde{\theta}\lambda \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.246)
\end{aligned}$$

$$\begin{aligned}
& \theta\lambda \left(g_a(\boldsymbol{\sigma}^n, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\theta\lambda\delta^2 M^2 \acute{d}}{4} \left\| g_a(\boldsymbol{\sigma}^n, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}) \right\|^2 + \theta\lambda \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.247)
\end{aligned}$$

$$\begin{aligned}
& \theta 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}), \delta \mathbf{u}^{(n)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\theta\alpha\delta^2 M^2 \acute{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}) \right\|^2 + \theta 2\alpha \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2, \quad (5.248)
\end{aligned}$$

$$\begin{aligned}
& \tilde{\theta} 2\alpha \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}), \delta \mathbf{u}^{(n-\theta)} \cdot \nabla \mathbf{F}^{(n+\theta)} \right) \\
& \leq \frac{\tilde{\theta}\alpha\delta^2 M^2 \acute{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}) \right\|^2 + \tilde{\theta} 2\alpha \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2. \quad (5.249)
\end{aligned}$$

5.1.7 Bring all bounds together:

The goal is to now bring all the bounded expressions together into a single expression so that the discrete Gronwall lemma may be applied, and the error estimate established. Thus, bringing all the bounds on the three linear combinations (4.16), (4.17), and (4.18) together into a single expression we obtain

$$\begin{aligned}
& \frac{\lambda}{2\Delta t} \left(\left\| \mathbf{F}^{(n+1)} \right\|^2 - \left\| \mathbf{F}^{(n)} \right\|^2 + \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 - \left\| \mathbf{F}^{(n-\theta)} \right\|^2 + \left\| \mathbf{F}^{(n+\theta)} \right\|^2 - \left\| \mathbf{F}^{(n+\theta)-1} \right\|^2 \right) \\
& + \theta \omega \left\| \mathbf{F}^{(n+1)} \right\|^2 + (1-2\theta) \tilde{\omega} \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 + \theta \omega \left\| \mathbf{F}^{(n+\theta)} \right\|^2 \\
& + (1-2\theta) (\delta\lambda - \epsilon_1 \delta^2 (2+3\lambda+2\alpha)) \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 \\
\leq & \left\| \mathbf{F}^{(n)} \right\|^2 \left(\theta\lambda + \theta\omega + \frac{3\theta\tilde{\omega}}{2} + 4\theta\lambda M\acute{d} + 4\theta\lambda M^2 \acute{d}^2 + 8\theta\lambda M^4 \acute{d}^4 + \frac{\theta\omega M^2 \acute{d}}{2} + \frac{3\theta\tilde{\omega} M^2 \acute{d}}{4} \right) \\
& + \left\| \mathbf{F}^{(n+1)} \right\|^2 \left(1 + 4\lambda + 2\alpha + \theta + \tilde{\theta}\tilde{\omega} + (1-2\theta)\omega + \frac{\theta\omega}{2} + \frac{\theta\omega M^2 \acute{d}}{4} + \frac{\lambda\acute{d}M^2}{2} + 2\lambda\acute{d}M \right) \\
& + \left\| \mathbf{F}^{(n+\theta-1)} \right\|^2 \left(\frac{(1-2\theta)\omega}{2} \left(1 + \frac{M^2 \acute{d}}{2} \right) \right) \\
& + \left\| \mathbf{F}^{(n+\theta)} \right\|^2 \left(1 + 5\lambda + \theta\omega + \tilde{\omega} + \tilde{\theta}\lambda + 2\alpha + \frac{\theta\omega M^2 \acute{d}}{2} + (1-2\theta)\omega + \frac{\tilde{\theta}\omega}{2} + \frac{(1-2\theta)\omega M^2 \acute{d}}{4} \right. \\
& \quad \left. + \frac{\theta\omega M^2 \acute{d}}{4} + \frac{\omega}{2} + \frac{\lambda\acute{d}M^2}{2} + 4\theta\lambda M^4 \acute{d}^4 + 4\theta\lambda M^2 \acute{d}^2 + \frac{(1-2\theta)\omega}{4\epsilon_1} \right) \\
& + \left\| \mathbf{F}^{(n-\theta)} \right\|^2 \left(\frac{\tilde{\omega}}{2} + \theta\lambda + \frac{\tilde{\omega} M^2 \acute{d}}{4} + 4\tilde{\theta}\lambda M^2 \acute{d}^2 + 4\lambda M^4 \acute{d}^4 \right) \\
& + \left\| \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 \left(1 + 4\lambda + \omega + \theta + \theta(1-2\omega) + (1-2\theta)\lambda + 2\alpha + \frac{\tilde{\theta}\tilde{\omega}}{2} + \frac{(1-2\theta)\tilde{\omega}}{2} + \frac{\tilde{\theta}\tilde{\omega} M^2 \acute{d}}{4} \right. \\
& \quad \left. + \frac{(1-2\theta)\lambda M^2 \acute{d}}{4} + 4\tilde{\theta}\lambda\acute{d}M + \theta\lambda M^2 \acute{d} + 4(1-2\theta)\lambda M\acute{d} \right. \\
& \quad \left. + 4(1-2\theta)\lambda M^4 \acute{d}^4 + 4\tilde{\theta}\lambda M^4 \acute{d}^4 + 16\theta\lambda M^2 \acute{d}^2 + \frac{(1-2\theta)\tilde{\omega}}{4\epsilon_1} \right) \\
& + \left\| \nabla \mathbf{F}^{(n-\theta)} \right\|^2 \left(\lambda + \frac{\delta\lambda M^2 \acute{d}}{2} + \delta^2 \tilde{\theta}\lambda \right) + \left\| \nabla \mathbf{F}^{(n)} \right\|^2 \left(3\lambda\theta \left(1 + \frac{\delta\acute{d}M^2}{2} \right) \right) \\
& + \left\| \nabla \mathbf{F}^{(n+\tilde{\theta})} \right\|^2 \left(2\lambda + 2\theta + 3\theta\lambda + 4\alpha\theta + \frac{\lambda M^2 \acute{d}}{2} + \frac{\theta\lambda M^2 \acute{d}}{2} + \theta\delta\lambda M^2 \acute{d} + 2\delta^2(\lambda + 2\theta) \right) \\
& + \left\| \nabla \mathbf{F}^{(n+\theta)} \right\|^2 \left(1 + 3\lambda + 2\alpha + \frac{\lambda\acute{d}M^2}{2} + \frac{\delta\lambda\acute{d}M^2}{2} + \delta^2(\omega + 2\theta + 2\lambda + (1-2\theta)\omega + 2\tilde{\theta}\tilde{\omega}) \right) \\
& + \left\| \nabla \mathbf{F}^{(n+1)} \right\|^2 \left(1 + 3\lambda + 2\alpha + \frac{\delta\lambda\acute{d}M^2}{2} + \frac{\lambda\acute{d}M^2}{2} + \delta^2\omega(2-\theta) + 2\delta^2\tilde{\omega} + 2\delta^2\lambda \right) \\
& + H_{\mathbf{\Gamma}} + H_g + \delta^2 H_{upA} + \delta^2 H_{upB}. \tag{5.250}
\end{aligned}$$

Here the terms $H_{\mathbf{\Gamma}}$, H_g , H_{upA} , and H_{upB} in (5.250) are defined by

$$\begin{aligned}
H_g := & \frac{\lambda}{4} \left\| d_t \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2})} \right\|^2 + \frac{\lambda}{4} \left\| d_t \boldsymbol{\sigma}^{(n+\bar{\theta})} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{\lambda}{4} \left\| d_t \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})} \right\|^2 + \frac{\omega}{4} \left\| \tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& + \frac{\omega}{4} \left\| \theta \boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{\omega}{4} \left\| \theta \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 \\
& + \frac{\tilde{\omega}}{4} \left\| \theta \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \boldsymbol{\sigma}^{(n+\bar{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& + \frac{\tilde{\omega}}{4} \left\| (1-2\theta) \boldsymbol{\sigma}^{(n+\bar{\theta})} + \theta \boldsymbol{\sigma}^{(n)} + \theta \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{\tilde{\omega}}{4} \left\| \theta \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 \\
& + \frac{\lambda}{4} \left\| \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\bar{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& + \frac{\lambda}{4} \left\| (1-2\theta) \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\bar{\theta})} + \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} \right. \\
& \quad \left. + \theta \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{\lambda}{4} \left\| \tilde{\theta} \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} + \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 \\
& + \frac{\lambda}{4} \left\| \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) + \tilde{\theta} g_a(\boldsymbol{\sigma}^{(n+\bar{\theta})}, \nabla \mathbf{u}^{(n+\bar{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}) \right\|^2 \\
& + \frac{\lambda}{4} \left\| (1-2\theta) g_a(\boldsymbol{\sigma}^{(n+\bar{\theta})}, \nabla \mathbf{u}^{(n+\bar{\theta})}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) \right. \\
& \quad \left. + \theta g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}) \right\|^2 \\
& + \frac{\lambda}{4} \left\| \tilde{\theta} g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}) \right\|^2 \\
& + \frac{\alpha}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\bar{\theta})}) \right\|^2 \\
& + \frac{\alpha}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - (1-2\theta) \mathbf{d}(\mathbf{u}^{(n+\bar{\theta})}) - \theta \mathbf{d}(\mathbf{u}^{(n-\theta)}) \right\|^2 \\
& + \frac{\alpha}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n-\theta)}) \right\|^2, \tag{5.251}
\end{aligned}$$

$$\begin{aligned}
H_{upA} = & \frac{\lambda M^2 \dot{d}}{4} \left\| -\boldsymbol{\sigma}_t^{(n+\frac{1}{2})} \right\|^2 + \frac{\lambda M^2 \dot{d}}{4} \left\| -\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)} \right\|^2 + \frac{\lambda M^2 \dot{d}}{4} \left\| -\boldsymbol{\sigma}_t^{(n+\theta-\frac{1}{2})} \right\|^2 \\
& + \frac{\omega \theta M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 + \frac{\omega \theta M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& + \frac{\omega (1-2\theta) M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& + \frac{\theta \omega M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \frac{\tilde{\omega} \theta M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& + \frac{\tilde{\omega} \tilde{\theta} M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 + \frac{(1-2\theta) \omega}{4\epsilon_1} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{(1-2\theta) \tilde{\omega}}{4\epsilon_1} \left\| \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \frac{\theta \omega M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{\theta \tilde{\omega} M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \frac{\theta \tilde{\omega} M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{\theta \omega M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \frac{\theta \omega M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 \\
& + \frac{(1-2\theta) \omega M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \frac{\theta \tilde{\omega} M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 \\
& + \frac{\tilde{\theta} \tilde{\omega} M^2 \dot{d}}{4} \left\| \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 \\
& + \frac{\theta \lambda M^2 \dot{d}}{4} \left\| \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& + \frac{\tilde{\theta} \lambda M^2 \dot{d}}{4} \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& + \frac{(1-2\theta) \lambda}{4\epsilon_1} \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{\theta \lambda M^2 \dot{d}}{4} \left\| \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{\theta \lambda M^2 \dot{d}}{4} \left\| \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
& + \frac{\tilde{\theta} \lambda M^2 \dot{d}}{4} \left\| \mathbf{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 \\
& + \frac{\theta \lambda M^2 \dot{d}}{4} \left\| \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \mathbf{u}^{(n+\theta-\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2, \tag{5.252}
\end{aligned}$$

$$\begin{aligned}
H_{upB} = & \frac{\tilde{\theta}\lambda M^2 \dot{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}) \right\|^2 \\
& + \frac{(1-2\theta)\lambda}{4\epsilon_1} \left\| g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}) \right\|^2 \\
& + \frac{\theta\lambda M^2 \dot{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}) \right\|^2 \\
& + \frac{\theta\lambda M^2 \dot{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \mathbf{u}^{(n+\frac{1}{2}-\theta)}) \right\|^2 \\
& + \frac{\tilde{\theta}\lambda M^2 \dot{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \mathbf{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}) \right\|^2 \\
& + \frac{\theta\lambda M^2 \dot{d}}{4} \left\| g_a(\boldsymbol{\sigma}^n, \nabla \mathbf{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \mathbf{u}^{(n+\theta-\frac{1}{2})}) \right\|^2 \\
& + \frac{\theta\alpha M^2 \dot{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}) \right\|^2 + \frac{\tilde{\theta}\alpha M^2 \dot{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}) \right\|^2 \\
& + \frac{\theta\alpha M^2 \dot{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n)}) \right\|^2 + \frac{(1-2\theta)\alpha}{2\epsilon_1} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}) \right\|^2 \\
& + \frac{\theta 2\alpha M^2 \dot{d}}{4} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}) \right\|^2 + \frac{\theta\alpha M^2 \dot{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n)}) \right\|^2 \\
& + \frac{\tilde{\theta}\alpha M^2 \dot{d}}{2} \left\| \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}) \right\|^2, \tag{5.253}
\end{aligned}$$

and

$$\begin{aligned}
H_{\mathbf{\Gamma}} := & \frac{\lambda}{4} \left\| d_t(\mathbf{\Gamma}^{(n+\theta)}) \right\|^2 + \frac{\lambda}{4} \left\| d_t(\mathbf{\Gamma}^{(n+1)}) \right\|^2 + \frac{\lambda}{4} \left\| d_t(\mathbf{\Gamma}^{(n+\tilde{\theta})}) \right\|^2 \\
& + \left\| \mathbf{\Gamma}^{(n-\theta)} \right\|^2 \left(\theta\lambda + \frac{\tilde{\omega}}{2} + \frac{\tilde{\omega}M^2 \dot{d}}{4} + 4\tilde{\theta}\lambda M^2 \dot{d}^2 + 4\lambda M^4 \dot{d}^4 \right) \\
& + \left\| \mathbf{\Gamma}^{(n+\theta)} \right\|^2 \left(\frac{\omega}{2} + \frac{\theta\omega}{2} + 3\frac{\theta\omega M^2 \dot{d}}{4} + \frac{(1-2\theta)\omega}{4\epsilon_1} + \frac{(1-2\theta)\omega}{2} + \frac{(1-2\theta)\omega M^2 \dot{d}}{4} \right) \\
& + \left\| \mathbf{\Gamma}^{(n+1)} \right\|^2 \left(\frac{\theta\omega}{2} \right) + \left\| \mathbf{\Gamma}^{(n+\theta-1)} \right\|^2 \left(\frac{(1-2\theta)\omega}{2} + \frac{(1-2\theta)\omega M^2 \dot{d}}{4} \right) \\
& + \left\| \mathbf{\Gamma}^{(n+\tilde{\theta})} \right\|^2 \left(\frac{\tilde{\theta}\tilde{\omega}}{2} + \frac{(1-2\theta)\tilde{\omega}}{2} + \frac{\tilde{\theta}\tilde{\omega}M^2 \dot{d}}{4} + \frac{(1-2\theta)\tilde{\omega}}{4\epsilon_1} \right. \\
& \quad \left. + 4\tilde{\theta}\lambda M \dot{d} + 4(1-2\theta)\lambda M^2 \dot{d}^2 + 4\tilde{\theta}\lambda M^4 \dot{d}^4 + 4(1-2\theta)\lambda M^4 \dot{d}^4 \right) \\
& + \left\| \mathbf{\Gamma}^{(n)} \right\|^2 \left(2\theta\lambda + \theta\omega + 3\frac{\theta\tilde{\omega}}{2} + 4\theta\lambda M \dot{d} + 2\frac{\theta\omega M^2 \dot{d}}{4} + 3\frac{\theta\tilde{\omega}M^2 \dot{d}}{4} + 12\theta\lambda M^4 \dot{d}^4 \right) \\
& + \left\| \nabla \mathbf{\Gamma}^{(n-\theta)} \right\|^2 \left(\theta\lambda + \frac{\delta^2 \lambda M^2 \dot{d}}{2} \right) + \left\| \nabla \mathbf{\Gamma}^{(n)} \right\|^2 \left(3\theta\lambda + 3\frac{\delta^2 \theta \lambda \dot{d} M^2}{2} \right) \\
& + \left\| \nabla \mathbf{\Gamma}^{(n+\tilde{\theta})} \right\|^2 \left(\tilde{\theta}\lambda + (1-2\theta)\lambda + (1-3\theta)\frac{\delta^2 \lambda M^2 \dot{d}}{2} + \frac{(1-2\theta)\lambda M^2 \dot{d}}{4\epsilon_1} + \frac{2\theta\delta^2 \lambda M^2 \dot{d}}{2} \right).
\end{aligned}$$

Multiplying (5.250) by $2\Delta t/\lambda$ and summing from an initial time $t = 0$ to time $t = l$ a telescoping of terms is achieved. An inverse inequality is also applied to the $\|\nabla \mathbf{F}\|$ terms on the RHS of the

expression (5.250). Bounds on $\|\mathbf{F}^\theta\|^2$ and $\|\mathbf{F}^{1-\theta}\|^2$ (the terms that do not telescope) are stated and proved in the appendix. Thus,

$$\begin{aligned}
& \left\| \mathbf{F}^{l+1} \right\|^2 - \left\| \mathbf{F}^0 \right\|^2 + \left\| \mathbf{F}^{l+1-\theta} \right\|^2 - \left\| \mathbf{F}^{1-\theta} \right\|^2 + \left\| \mathbf{F}^{l+\theta} \right\|^2 - \left\| \mathbf{F}^\theta \right\|^2 \\
& + \Delta t \sum_{n=0}^l \frac{2}{\lambda} \theta \omega \left\| \mathbf{F}^{(n+1)} \right\|^2 + \Delta t \sum_{n=0}^l \frac{2}{\lambda} (1-2\theta) \tilde{\omega} \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2 + \Delta t \sum_{n=0}^l \frac{2}{\lambda} \theta \omega \left\| \mathbf{F}^{(n+\theta)} \right\|^2 \\
& + \Delta t \sum_{n=0}^l \frac{2}{\lambda} (1-2\theta) (\delta\lambda - \epsilon_1 \delta^2 (2+3\lambda+2\alpha)) \left\| \mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})} \right\|^2 \\
\leq & \Delta t \sum_{n=0}^l \left\| \mathbf{F}^{(n)} \right\|^2 \frac{1}{\lambda} \left(2\theta\lambda + 2\theta\omega + 3\theta\tilde{\omega} + 8\theta\lambda M\dot{d} + 8\theta\lambda M^2 \dot{d}^2 + 16\theta\lambda M^4 \dot{d}^4 + \theta\omega M^2 \dot{d} + \frac{3\theta\tilde{\omega} M^2 \dot{d}}{2} \right. \\
& \left. + Ch^{-2} \left(6\lambda\theta \left(2 + \delta\dot{d} M^2 \right) \right) \right) \\
& + \Delta t \sum_{n=0}^l \left\| \mathbf{F}^{(n+1)} \right\|^2 \frac{1}{\lambda} \left(2 + 8\lambda + 4\alpha + 2\theta + 2\tilde{\theta}\tilde{\omega} + 2(1-2\theta)\omega + \theta\omega + \frac{\theta\omega M^2 \dot{d}}{2} + \lambda\dot{d} M^2 + 4\lambda\dot{d} M \right. \\
& \left. + Ch^{-2} \left(2 + 6\lambda + 4\alpha + \delta\lambda\dot{d} M^2 + \lambda\dot{d} M^2 + 2\delta^2\omega(2-\theta) + 4\delta^2\tilde{\omega} + 4\delta^2\lambda \right) \right) \\
& + \Delta t \sum_{n=0}^l \left\| \mathbf{F}^{(n+\theta-1)} \right\|^2 \frac{1}{\lambda} \left((1-2\theta)\omega 2 \left(1 + \frac{M^2 \dot{d}}{2} \right) \right) \\
& + \Delta t \sum_{n=0}^l \left\| \mathbf{F}^{(n+\theta)} \right\|^2 \frac{1}{\lambda} \left(2 + 10\lambda + 2\theta\omega + 2\tilde{\omega} + 2\tilde{\theta}\lambda + 4\alpha + \theta\omega M^2 \dot{d} + 2(1-2\theta)\omega + \tilde{\theta}\omega \right. \\
& \left. + \frac{(1-2\theta)\omega M^2 \dot{d}}{2} + \frac{\theta\omega M^2 \dot{d}}{2} + \omega + \lambda\dot{d} M^2 + 8\theta\lambda M^4 \dot{d}^4 + 8\theta\lambda M^2 \dot{d}^2 + \frac{(1-2\theta)\omega}{2\epsilon_1} \right. \\
& \left. + Ch^{-2} \left(2 + 6\lambda + 4\alpha + \lambda\dot{d} M^2 + \delta\lambda\dot{d} M^2 + 2\delta^2 \left(\omega + 2\theta + 2\lambda + (1-2\theta)\omega + 2\tilde{\theta}\tilde{\omega} \right) \right) \right) \\
& + \Delta t \sum_{n=0}^l \left\| \mathbf{F}^{(n-\theta)} \right\|^2 \frac{1}{\lambda} \left(\tilde{\omega} + 2\theta\lambda + \frac{\tilde{\omega} M^2 \dot{d}}{2} + 8\tilde{\theta}\lambda M^2 \dot{d}^2 + 8\lambda M^4 \dot{d}^4 \right. \\
& \left. + Ch^{-2} \left(2\lambda + \delta\lambda M^2 \dot{d} + 2\delta^2 \tilde{\theta}\lambda \right) \right) \\
& + \Delta t \sum_{n=0}^l \left\| \mathbf{F}^{(n+\bar{\theta})} \right\|^2 \frac{1}{\lambda} \left(2 + 8\lambda + 2\omega + 2\theta + 2\theta(1-2\omega) + 2(1-2\theta)\lambda + 4\alpha + \tilde{\theta}\tilde{\omega} + (1-2\theta)\tilde{\omega} \right. \\
& \left. + \frac{\tilde{\theta}\tilde{\omega} M^2 \dot{d}}{2} + \frac{(1-2\theta)\lambda M^2 \dot{d}}{2} + 8\tilde{\theta}\lambda\dot{d} M + 2\theta\lambda M^2 \dot{d} + 8(1-2\theta)\lambda M\dot{d} \right. \\
& \left. + 8(1-2\theta)\lambda M^4 \dot{d}^4 + 8\tilde{\theta}\lambda M^4 \dot{d}^4 + 32\theta\lambda M^2 \dot{d}^2 + \frac{(1-2\theta)\tilde{\omega}}{2\epsilon_1} \right. \\
& \left. + Ch^{-2} \left(4\lambda + 4\theta + 6\theta\lambda + 8\alpha\theta + \lambda M^2 \dot{d} + \theta\lambda M^2 \dot{d} + 2\theta\delta\lambda M^2 \dot{d} + 4\delta^2(\lambda + 2\theta) \right) \right) \\
& + \Delta t \sum_{n=0}^l \frac{2}{\lambda} H_{\mathbf{\Gamma}} + \Delta t \sum_{n=0}^l \frac{2}{\lambda} H_g + \Delta t \sum_{n=0}^l \frac{2}{\lambda} \delta^2 H_{upA} + \Delta t \sum_{n=0}^l \frac{2}{\lambda} \delta^2 H_{upB}. \tag{5.254}
\end{aligned}$$

To allow for the application of the discrete Gronwall's Lemma the inequalities stated in (5.255),

(5.256), and (5.257) must be satisfied.

$$\begin{aligned}
1 \geq & \Delta t \frac{1}{\lambda} \left(2\theta\lambda + 2\theta\omega + 3\theta\tilde{\omega} + 8\theta\lambda M\dot{d} + 8\theta\lambda M^2\dot{d}^2 + 16\theta\lambda M^4\dot{d}^4 + \theta\omega M^2\dot{d} + \frac{3\theta\tilde{\omega}M^2\dot{d}}{2} \right. \\
& 2 + 8\lambda + 4\alpha + 2\theta + 2\tilde{\theta}\tilde{\omega} + 2(1-2\theta)\omega + \theta\omega + \frac{\theta\omega M^2\dot{d}}{2} + \lambda\dot{d}M^2 + 4\lambda\dot{d}M \\
& + Ch^{-2} \left(6\lambda\theta \left(2 + \delta\dot{d}M^2 \right) + 2 + 6\lambda + 4\alpha + \delta\lambda\dot{d}M^2 \right. \\
& \left. \left. + \lambda\dot{d}M^2 + 2\delta^2\omega(2-\theta) + 4\delta^2\tilde{\omega} + 4\delta^2\lambda \right) \right) \quad (5.255)
\end{aligned}$$

$$\begin{aligned}
1 \geq & \Delta t \frac{1}{\lambda} \left(2 + 2(1-2\theta)\omega \left(1 + \frac{M^2\dot{d}}{2} \right) + 10\lambda + 2\theta\omega + 2\tilde{\omega} + 2\tilde{\theta}\lambda + 4\alpha + \theta\omega M^2\dot{d} + 2(1-2\theta)\omega \right. \\
& + \tilde{\theta}\omega + \frac{(1-2\theta)\omega M^2\dot{d}}{2} + \frac{\theta\omega M^2\dot{d}}{2} + \omega + \lambda\dot{d}M^2 + 8\theta\lambda M^4\dot{d}^4 + 8\theta\lambda M^2\dot{d}^2 + \frac{(1-2\theta)\omega}{2\epsilon_1} \\
& \left. + Ch^{-2} \left(2 + 6\lambda + 4\alpha + \lambda\dot{d}M^2 + \delta\lambda\dot{d}M^2 + 2\delta^2 \left(\omega + 2\theta + 2\lambda + (1-2\theta)\omega + 2\tilde{\theta}\tilde{\omega} \right) \right) \right) \quad (5.256)
\end{aligned}$$

$$\begin{aligned}
1 \geq & \Delta t \frac{1}{\lambda} \left(2 + \tilde{\omega} + 2\theta\lambda + \frac{\tilde{\omega}M^2\dot{d}}{2} + 8\tilde{\theta}\lambda M^2\dot{d}^2 + 8\lambda M^4\dot{d}^4 \right. \\
& + 8\lambda + 2\omega + 2\theta + 2\theta(1-2\omega) + 2(1-2\theta)\lambda + 4\alpha + \tilde{\theta}\tilde{\omega} + (1-2\theta)\tilde{\omega} \\
& + \frac{\tilde{\theta}\tilde{\omega}M^2\dot{d}}{2} + \frac{(1-2\theta)\lambda M^2\dot{d}}{2} + 8\tilde{\theta}\lambda\dot{d}M + 2\theta\lambda M^2\dot{d} + 8(1-2\theta)\lambda M\dot{d} \\
& + 8(1-2\theta)\lambda M^4\dot{d}^4 + 8\tilde{\theta}\lambda M^4\dot{d}^4 + 32\theta\lambda M^2\dot{d}^2 + \frac{(1-2\theta)\tilde{\omega}}{2\epsilon_1} \\
& \left. + Ch^{-2} \left(2\lambda + \delta\lambda M^2\dot{d} + 2\delta^2\tilde{\theta}\lambda + 4\lambda + 4\theta + 6\theta\lambda + 8\alpha\theta \right. \right. \\
& \left. \left. + \lambda M^2\dot{d} + \theta\lambda M^2\dot{d} + 2\theta\delta\lambda M^2\dot{d} + 4\delta^2(\lambda + 2\theta) \right) \right) \quad (5.257)
\end{aligned}$$

Summarizing expressions (5.255), (5.256), and (5.257) gives a stability result needed for the discrete Gronwall's Lemma of the form:

$$\Delta t (C + Ch^{-2} + \delta C + \delta Ch^{-2} + \delta^2 Ch^{-2}) \leq 1. \quad (5.258)$$

This leads to the computationally restrictive stability relationship

$$\Delta t \leq Ch^2. \quad (5.259)$$

At this junction in order to complete the proof of the a priori error estimate we need to consider the bounds on each of the $H_{\mathbf{\Gamma}}, H_g, H_{upA}$, and H_{upB} terms in expression (5.254) using interpolation properties, and some of the lemmas from the appendix.

5.1.8 Bounding $H_{\mathbf{\Gamma}}$

Using interpolation properties of the approximation space, and elements of order m for the stress we approximate the terms in $\Delta t \sum_{n=0}^l \frac{2}{\lambda} H_{\mathbf{\Gamma}}$. With C and C_i denoting constants independent of the discretization and upwinding parameters $h, \Delta t$, and δ , exemplary terms are handled as,

$$\sum_{n=0}^l \frac{\Delta t}{2} \left\| d_t \left(\mathbf{\Gamma}^{(n+1)} \right) \right\|^2 = \sum_{n=0}^l \frac{\Delta t}{2} \left\| \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \frac{\partial \mathbf{\Gamma}}{\partial t} dt \right\|^2 \quad (5.260)$$

$$\begin{aligned} &\leq \sum_{n=0}^l \frac{1}{2\Delta t} \int_{\Omega} \int_{t_n}^{t_{n+1}} 1^2 dt \int_{t_n}^{t_{n+1}} \left(\frac{\partial \mathbf{\Gamma}}{\partial t} \right)^2 dt dA \\ &= \sum_{n=0}^l \frac{1}{2} \int_{\Omega} \int_{t_n}^{t_{n+1}} \left(\frac{\partial \mathbf{\Gamma}}{\partial t} \right)^2 dt dA \\ &\leq Ch^{2m+2} \|\boldsymbol{\sigma}_t\|_{0,m+1}^2, \end{aligned} \quad (5.261)$$

$$\begin{aligned} \sum_{n=0}^l \Delta t (C_1) \left\| \mathbf{\Gamma}^{(n+\theta)} \right\|^2 &\leq (Ch^{2m+2}) \left(\sum_{n=0}^l \Delta t \left\| \boldsymbol{\sigma}^{(n+\theta)} \right\|_{m+1}^2 \right) \\ &\leq Ch^{2m+2} \|\boldsymbol{\sigma}\|_{0,m+1}^2, \end{aligned} \quad (5.262)$$

$$\begin{aligned} \sum_{n=0}^l \Delta t (C_2 + \delta^2 C_3) \left\| \nabla \mathbf{\Gamma}^{(n)} \right\|^2 &\leq (Ch^{2m} + \delta Ch^{2m}) \left(\sum_{n=0}^l \Delta t \left\| \boldsymbol{\sigma}^{(n)} \right\|_{m+1}^2 \right) \\ &\leq Ch^{2m} \|\boldsymbol{\sigma}\|_{0,m+1}^2 + \delta^2 Ch^{2m} \|\boldsymbol{\sigma}\|_{0,m+1}^2, \end{aligned} \quad (5.263)$$

Thus,

$$\begin{aligned} \Delta t \sum_{n=0}^l \frac{2}{\lambda} H_{\mathbf{\Gamma}} &\leq C (h^{2m+2} + h^{2m} + \delta^2 h^{2m}) \|\boldsymbol{\sigma}\|_{0,m+1}^2 \\ &\quad + Ch^{2m+2} \|\boldsymbol{\sigma}_t\|_{0,m+1}^2. \end{aligned} \quad (5.264)$$

5.2 Bounding H_g

For each of the terms in H_g a typical term is bounded here. To complete the bounds we use Taylor series expansions. The first bound is shown in detail here, and then the reader is referred to a series

of useful lemmas given in the appendix that aid in bounding the remaining terms. Examine

$$\begin{aligned}
& \sum_{n=0}^l \frac{\Delta t}{2} \left\| d_t \sigma^{(n+1)} - \sigma_t^{(n+\frac{1}{2})} \right\|^2 = \sum_{n=0}^l \frac{\Delta t}{2} \left\| \frac{1}{\Delta t} \sigma^{(n+1)} - \frac{1}{\Delta t} \sigma^{(n)} - \sigma_t^{(n+\frac{1}{2})} \right\|^2 \\
&= \sum_{n=0}^l \frac{\Delta t}{2} \int_{\Omega} \left(\frac{1}{\Delta t} \sigma^{(n+1)} - \frac{1}{\Delta t} \sigma^{(n)} - \sigma_t^{(n+\frac{1}{2})} \right)^2 dA \\
&= \sum_{n=0}^l \frac{\Delta t}{2} \int_{\Omega} \left(\frac{1}{\Delta t} \left(\sigma^{(n+\frac{1}{2})} + \frac{\Delta t}{2} \sigma_t^{(n+\frac{1}{2})} + \frac{(\Delta t)^2}{8} \sigma_{tt}^{(n+\frac{1}{2})} + \frac{1}{2} \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \sigma_{ttt}(\cdot, t) (t_{n+1} - t)^2 dt \right) \right. \\
&\quad \left. - \frac{1}{\Delta t} \left(\sigma^{(n+\frac{1}{2})} - \frac{\Delta t}{2} \sigma_t^{(n+\frac{1}{2})} + \frac{(\Delta t)^2}{8} \sigma_{tt}^{(n+\frac{1}{2})} - \frac{1}{2} \int_{t_n}^{t_{n+\frac{1}{2}}} \sigma_{ttt}(\cdot, t) (t_{n+1} - t)^2 dt \right) - \sigma_t^{(n+\frac{1}{2})} \right)^2 dA \\
&= \sum_{n=0}^l \frac{\Delta t}{2} \int_{\Omega} \left(\frac{1}{2\Delta t} \right)^2 \left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \sigma_{ttt}(\cdot, t) (t_{n+1} - t)^2 dt + \int_{t_n}^{t_{n+\frac{1}{2}}} \sigma_{ttt}(\cdot, t) (t_{n+1} - t)^2 dt \right)^2 dA \\
&\leq \sum_{n=0}^l \frac{\Delta t}{2} \int_{\Omega} 2 \left(\frac{1}{2\Delta t} \right)^2 \left(\left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \sigma_{ttt}(\cdot, t) (t_{n+1} - t)^2 dt \right)^2 \right. \\
&\quad \left. + \left(\int_{t_n}^{t_{n+\frac{1}{2}}} \sigma_{ttt}(\cdot, t) (t_{n+1} - t)^2 dt \right)^2 \right) dA \\
&\leq \sum_{n=0}^l \frac{\Delta t}{2} \int_{\Omega} 2 \left(\frac{1}{2\Delta t} \right)^2 \left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \sigma_{ttt}(\cdot, t)^2 dt \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} (t_{n+1} - t)^4 dt \right. \\
&\quad \left. + \int_{t_n}^{t_{n+\frac{1}{2}}} \sigma_{ttt}(\cdot, t)^2 dt \int_{t_n}^{t_{n+\frac{1}{2}}} (t_{n+1} - t)^4 dt \right) dA \\
&= \sum_{n=0}^l \frac{\Delta t}{2} \int_{\Omega} \frac{\Delta t^3}{5(2^6)} \left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \sigma_{ttt}(\cdot, t)^2 dt + \int_{t_n}^{t_{n+\frac{1}{2}}} \sigma_{ttt}(\cdot, t)^2 dt \right) dA \\
&= \sum_{n=0}^l \frac{\Delta t^4}{5(2^7)} \int_{t_n}^{t_{n+1}} \|\sigma_{ttt}(\cdot, t)\|^2 dt \\
&\leq C(\Delta t)^4 \|\sigma_{ttt}\|_{0,0}^2. \tag{5.265}
\end{aligned}$$

Similarly,

$$\sum_{n=0}^l \frac{\omega \Delta t}{2\lambda} \left\| \theta \sigma^{(n+\theta)} + \tilde{\theta} \sigma^{(n+1)} - \sigma^{(n+\frac{1}{2})} \right\|^2 \leq \sum_{n=0}^l C(\Delta t)^4 \int_{t_{n+\theta}}^{t_{n+1}} \|\sigma_{tt}(\cdot, t)\|^2 dt \tag{5.266}$$

$$\leq C(\Delta t)^4 \|\sigma_{tt}\|_{0,0}^2. \tag{5.267}$$

Next consider the convective term, and use lemma C.2

$$\begin{aligned}
& \sum_{n=0}^l \frac{\Delta t}{2} \left\| \theta \mathbf{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& \leq \sum_{n=0}^l \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1}-\theta} \|(\mathbf{u} \cdot \nabla \boldsymbol{\sigma})_{tt}\|^2 dt \\
& = \sum_{n=0}^l \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1}-\theta} \|\mathbf{u} \cdot \nabla \boldsymbol{\sigma}_{tt} + 2\mathbf{u}_t \cdot \nabla \boldsymbol{\sigma}_t + \mathbf{u}_{tt} \cdot \nabla \boldsymbol{\sigma}\|^2 dt \\
& \leq \sum_{n=0}^l \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1}-\theta} \left(4\|\mathbf{u} \cdot \nabla \boldsymbol{\sigma}_{tt}\|^2 + 4\|2\mathbf{u}_t \cdot \nabla \boldsymbol{\sigma}_t\|^2 + 4\|\mathbf{u}_{tt} \cdot \nabla \boldsymbol{\sigma}\|^2 \right) dt \\
& \leq \sum_{n=0}^l \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1}-\theta} \left(4\|\mathbf{u}\|_{\infty}^2 \dot{d} \|\nabla \boldsymbol{\sigma}_{tt}\|^2 + 16\|\mathbf{u}\|_{\infty}^2 \dot{d} \|\nabla \boldsymbol{\sigma}_t\|^2 + 4\|\mathbf{u}\|_{\infty}^2 \dot{d} \|\nabla \boldsymbol{\sigma}\|^2 \right) dt \\
& \leq \sum_{n=0}^l \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1}-\theta} \left(4M^2 \dot{d} \|\nabla \boldsymbol{\sigma}_{tt}\|^2 + 16M^2 \dot{d} \|\nabla \boldsymbol{\sigma}_t\|^2 + 4M^2 \dot{d} \|\nabla \boldsymbol{\sigma}\|^2 \right) dt \\
& \leq (\Delta t)^4 C \left(\|\boldsymbol{\sigma}_{tt}\|_{0,1}^2 + \|\boldsymbol{\sigma}_t\|_{0,1}^2 + \|\boldsymbol{\sigma}\|_{0,1}^2 \right). \tag{5.268}
\end{aligned}$$

Again using lemma (C.2) the $g_a(\cdot, \cdot)$ terms are handled as:

$$\begin{aligned}
& \sum_{n=0}^l \frac{\Delta t}{2} \left\| \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \mathbf{u}^{(n)}) + \tilde{\theta} g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}) \right\|^2 \\
& \leq \sum_{n=0}^l \frac{(\Delta t)^4}{48} \int_{t_{n+\theta}}^{t_{n+1}} \|(g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}))_{tt}\|^2 dt \\
& \leq \sum_{n=0}^l (\Delta t)^4 C_1 \int_{t_{n+\theta}}^{t_{n+1}} \|(\boldsymbol{\sigma} \nabla \mathbf{u})_{tt}\|^2 dt \\
& = \sum_{n=0}^l (\Delta t)^4 C_1 \int_{t_{n+\theta}}^{t_{n+1}} \|\boldsymbol{\sigma}(\nabla \mathbf{u})_{tt} + 2\boldsymbol{\sigma}_t(\nabla \mathbf{u})_t + \boldsymbol{\sigma}_{tt} \nabla \mathbf{u}\|^2 dt \\
& \leq \sum_{n=0}^l (\Delta t)^4 C_2 \int_{t_{n+\theta}}^{t_{n+1}} \left(\|\boldsymbol{\sigma}(\nabla \mathbf{u})_{tt}\|^2 + \|\boldsymbol{\sigma}_t(\nabla \mathbf{u})_t\|^2 + \|\boldsymbol{\sigma}_{tt} \nabla \mathbf{u}\|^2 \right) dt \\
& \leq \sum_{n=0}^l (\Delta t)^4 C_2 \int_{t_{n+\theta}}^{t_{n+1}} \left(\|(\nabla \mathbf{u})_{tt}\|_{\infty}^2 \dot{d}^2 \|\boldsymbol{\sigma}\|^2 + \|(\nabla \mathbf{u})_t\|_{\infty}^2 \dot{d}^2 \|\boldsymbol{\sigma}_t\|^2 + \|\nabla \mathbf{u}\|_{\infty}^2 \dot{d}^2 \|\boldsymbol{\sigma}_{tt}\|^2 \right) dt \\
& \leq \sum_{n=0}^l (\Delta t)^4 C_2 M^2 \dot{d}^2 \int_{t_{n+\theta}}^{t_{n+1}} \left(\|\boldsymbol{\sigma}\|^2 + \|\boldsymbol{\sigma}_t\|^2 + \|\boldsymbol{\sigma}_{tt}\|^2 \right) dt \\
& \leq (\Delta t)^4 C \left(\|\boldsymbol{\sigma}_{tt}\|_{0,0}^2 + \|\boldsymbol{\sigma}_t\|_{0,0}^2 + \|\boldsymbol{\sigma}\|_{0,0}^2 \right). \tag{5.269}
\end{aligned}$$

Lastly for the H_g terms we consider

$$\begin{aligned}
& \sum_{n=0}^l \frac{\alpha \Delta t}{\lambda} \left\| \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) - \theta \mathbf{d}(\mathbf{u}^{(n)}) - \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\tilde{\theta})}) \right\|^2 \\
& \leq \sum_{n=0}^l \frac{\alpha (\Delta t)^4}{24\lambda} \int_{t_n}^{t_{n+1}-\theta} \|\mathbf{d}(\mathbf{u})_{tt}\|^2 dt \\
& \leq \sum_{n=0}^l \frac{\alpha (\Delta t)^4}{24\lambda} \int_{t_n}^{t_{n+1}-\theta} \|(\nabla \mathbf{u})_{tt}\|^2 dt \\
& \leq \sum_{n=0}^l \frac{\alpha (\Delta t)^4}{24\lambda} \int_{t_n}^{t_{n+1}-\theta} \|(\nabla \mathbf{u})_{tt}\|_{\infty}^2 \hat{d}^2 \|1\|^2 dt \\
& \leq (\Delta t)^4 C C_T,
\end{aligned} \tag{5.270}$$

where C_T is a constant dependent on the final time T .

In summary we have

$$\begin{aligned}
\Delta t \sum_{n=0}^l \frac{2}{\lambda} H_g & \leq C(\Delta t)^4 \left(\|\sigma_{ttt}\|_{0,0}^2 + \|\sigma_{tt}\|_{0,1}^2 + \|\sigma_t\|_{0,1}^2 + \|\sigma\|_{0,1}^2 \right. \\
& \quad \left. + \|\sigma_{tt}\|_{0,0}^2 + \|\sigma_t\|_{0,0}^2 + \|\sigma\|_{0,0}^2 + C_T \right).
\end{aligned} \tag{5.271}$$

5.3 Bounding H_{upA} and H_{upB}

For each of the H_{upA} terms a characteristic term is treated below. All parts of the H_{Int} terms will match one of the following cases, and is bounded similarly. Thus,

$$\frac{\delta^2 M^2 \dot{d}}{2} \sum_{n=0}^l \Delta t \left\| \sigma_t^{(n+\frac{1}{2})} \right\|^2 \leq C \delta^2 \|\sigma_t\|_{0,0}^2. \tag{5.272}$$

Using lemma C.7 we bound

$$\begin{aligned}
\sum_{n=0}^l \frac{\delta^2 \omega \theta M^2 \dot{d} \Delta t}{2\lambda} \left\| \sigma^{(n+\theta)} - \sigma^{(n+\frac{1}{2})} \right\|^2 & \leq \sum_{n=0}^l \frac{\delta^2 \omega \theta M^2 \dot{d} (\Delta t)^2}{2\lambda} \int_{t_{n+\theta}}^{t_{n+\frac{1}{2}}} \|\sigma_t\|^2 dt \\
& \leq C(\Delta t)^2 \delta^2 \|\sigma_t\|_{0,0}^2.
\end{aligned} \tag{5.273}$$

Using lemma C.8

$$\begin{aligned}
& \sum_{n=0}^l \frac{\delta^2 \tilde{\theta} \lambda M^2 \dot{d} \Delta t}{2\lambda} \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \mathbf{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
& \leq \sum_{n=0}^l \frac{\delta^2 \tilde{\theta} \lambda M^2 \dot{d}(\Delta t)^2}{2\lambda} \int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}} \|(\mathbf{u} \cdot \nabla \boldsymbol{\sigma})_t\|^2 dt \\
& = \sum_{n=0}^l \frac{\delta^2 \tilde{\theta} \lambda M^2 \dot{d}(\Delta t)^2}{2\lambda} \int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}} \|\mathbf{u}_t \cdot \nabla \boldsymbol{\sigma} + \mathbf{u} \cdot \nabla \boldsymbol{\sigma}_t\|^2 dt \\
& \leq \sum_{n=0}^l \frac{\delta^2 \tilde{\theta} \lambda M^2 \dot{d}(\Delta t)^2}{\lambda} \int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}} \|\mathbf{u}_t \cdot \nabla \boldsymbol{\sigma}\|^2 + \|\mathbf{u} \cdot \nabla \boldsymbol{\sigma}_t\|^2 dt \\
& \leq C \delta^2 (\Delta t)^2 \left(\|\boldsymbol{\sigma}\|_{0,1}^2 + \|\boldsymbol{\sigma}_t\|_{0,1}^2 \right), \tag{5.274}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{n=0}^l \frac{\delta^2 \tilde{\theta} M^2 \dot{d}(\Delta t)}{2} \left\| g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \mathbf{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \mathbf{u}^{(n+\frac{1}{2})}) \right\|^2 \\
& \leq \sum_{n=0}^l \frac{\delta^2 \tilde{\theta} M^2 \dot{d}(\Delta t)^2}{2} \int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}} \|(g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}))_t\|^2 dt \\
& \leq \sum_{n=0}^l C_1 \delta^2 (\Delta t)^2 \int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}} \|(\boldsymbol{\sigma} \nabla \mathbf{u})_t\|^2 dt \\
& = \sum_{n=0}^l C_1 \delta^2 (\Delta t)^2 \int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}} \|\boldsymbol{\sigma}_t \nabla \mathbf{u} + \boldsymbol{\sigma} \nabla \mathbf{u}_t\|^2 dt \\
& \leq \sum_{n=0}^l C_2 \delta^2 (\Delta t)^2 \int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}} \|\boldsymbol{\sigma}_t \nabla \mathbf{u}\|^2 + \|\boldsymbol{\sigma} \nabla \mathbf{u}_t\|^2 dt \\
& \leq C \delta^2 (\Delta t)^2 \left(\|\boldsymbol{\sigma}_t\|_{0,0}^2 + \|\boldsymbol{\sigma}\|_{0,0}^2 \right). \tag{5.275}
\end{aligned}$$

Lastly using lemma C.14

$$\begin{aligned}
& \sum_{n=0}^l \frac{\delta^2 \tilde{\theta} \alpha M^2 \dot{d}(\Delta t)}{\lambda} \left\| \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\mathbf{u}^{(n-\theta)}) \right\|^2 \\
& \leq \sum_{n=0}^l \frac{\delta^2 \tilde{\theta} \alpha M^2 \dot{d}(\Delta t)^2}{\lambda} \int_{t_{n-\theta}}^{t_{n+\theta-\frac{1}{2}}} \|\mathbf{d}(\mathbf{u})_t\|^2 dt \\
& \leq \sum_{n=0}^l \frac{\delta^2 \tilde{\theta} \alpha M^2 \dot{d}(\Delta t)^2}{\lambda} \int_{t_{n-\theta}}^{t_{n+\theta-\frac{1}{2}}} \|(\nabla \mathbf{u})_t\|^2 dt \\
& \leq \delta^2 (\Delta t)^2 C C_T. \tag{5.276}
\end{aligned}$$

This gives

$$\begin{aligned} \sum_{n=0}^l \frac{2}{\lambda} \delta^2 \Delta t (H_{upA} + H_{upB}) &\leq C \delta^2 (\Delta t)^2 \left(\|\boldsymbol{\sigma}\|_{0,1}^2 + \|\boldsymbol{\sigma}_t\|_{0,1}^2 + \|\boldsymbol{\sigma}\|_{0,0}^2 + \|\boldsymbol{\sigma}_t\|_{0,0}^2 \right) \\ &\quad + C \delta^2 \|\boldsymbol{\sigma}_t\|_{0,0}^2 + \delta^2 (\Delta t)^2 C C_T. \end{aligned} \quad (5.277)$$

5.4 Apply the discrete Gronwall's lemma

Using the defined bounds on the H terms, the lemma A.1 and lemma A.2 for bounds on $\|\mathbf{F}^{(\theta)}\|^2$ and $\|\mathbf{F}^{(\bar{\theta})}\|^2$, and assuming Δt and h satisfy the relationship given in (5.259) the discrete Gronwall's lemma is applied to (5.254) giving

$$\begin{aligned} &\|\mathbf{F}^{l+1}\|^2 + \|\mathbf{F}^{l+1-\theta}\|^2 + \|\mathbf{F}^{l+\theta}\|^2 \\ &\quad + \Delta t \sum_{n=0}^l \frac{2}{\lambda} (1-2\theta) (\delta\lambda - \epsilon_1 \delta^2 (2+3\lambda+2\alpha)) \|\mathbf{u}^{(n+\bar{\theta})} \cdot \nabla \mathbf{F}^{(n+\bar{\theta})}\|^2 \\ &\leq G_{\boldsymbol{\sigma}}(\Delta t, h, \delta), \end{aligned} \quad (5.278)$$

where

$$\begin{aligned} G_{\boldsymbol{\sigma}}(\Delta t, h, \delta) &:= C(\Delta t)^4 \left(\|\boldsymbol{\sigma}_{ttt}\|_{0,0}^2 + \|\boldsymbol{\sigma}_{tt}\|_{0,1}^2 + \|\boldsymbol{\sigma}_t\|_{0,1}^2 + \|\boldsymbol{\sigma}\|_{0,1}^2 \right. \\ &\quad \left. + \|\boldsymbol{\sigma}_{tt}\|_{0,0}^2 + \|\boldsymbol{\sigma}_t\|_{0,0}^2 + \|\boldsymbol{\sigma}\|_{0,0}^2 + C_T \right) \\ &\quad + C(\Delta t)^2 \delta^2 \left(\|\boldsymbol{\sigma}\|_{0,1}^2 + \|\boldsymbol{\sigma}_t\|_{0,1}^2 + \|\boldsymbol{\sigma}\|_{0,0}^2 + \|\boldsymbol{\sigma}_t\|_{0,0}^2 + C_T \right) \\ &\quad + C(h^{2m+2} + h^{2m} + \delta^2 h^{2m}) \|\boldsymbol{\sigma}\|_{0,m+1}^2 \\ &\quad + C h^{2m+2} \|\boldsymbol{\sigma}_t\|_{0,m+1}^2 + C \delta^2 \|\boldsymbol{\sigma}_t\|_{0,0}^2. \end{aligned}$$

Using (5.278) it can be seen that

$$\|\mathbf{F}\|_{\infty,0}^2 \leq G_{\boldsymbol{\sigma}}(\Delta t, h, \delta), \quad (5.279)$$

$$\|\mathbf{F}\|_{0,0}^2 \leq T G_{\boldsymbol{\sigma}}(\Delta t, h, \delta), \quad (5.280)$$

and

$$\|\mathbf{u} \cdot \nabla \mathbf{F}\|_{\bar{\theta}}^2 \leq \mathcal{C} G_{\boldsymbol{\sigma}}(\Delta t, h, \delta). \quad (5.281)$$

Then

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\infty,0}^2 &\leq \|\mathbf{F}\|_{\infty,0}^2 + \|\boldsymbol{\Gamma}\|_{\infty,0}^2 \\ &\leq G_{\boldsymbol{\sigma}}(\Delta t, h, \delta) + C h^{2m+2} \|\boldsymbol{\sigma}\|_{\infty,0}^2, \end{aligned} \quad (5.282)$$

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,0}^2 &\leq \|\mathbf{F}\|_{0,0}^2 + \|\boldsymbol{\Gamma}\|_{0,0}^2 \\ &\leq T G_{\boldsymbol{\sigma}}(\Delta t, h, \delta) + C h^{2m+2} \|\boldsymbol{\sigma}\|_{0,m+1}^2, \end{aligned} \quad (5.283)$$

and (see (4.11))

$$\begin{aligned}
|||\mathbf{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)|||_{\tilde{\theta}}^2 &= |||\mathbf{u} \cdot \nabla(\mathbf{F} + \boldsymbol{\Gamma})|||_{\tilde{\theta}}^2 \\
&= |||\mathbf{u} \cdot \nabla \mathbf{F} + \mathbf{u} \cdot \nabla \boldsymbol{\Gamma}|||_{\tilde{\theta}}^2 \\
&\leq |||\mathbf{u} \cdot \nabla \mathbf{F}|||_{\tilde{\theta}}^2 + |||\mathbf{u} \cdot \nabla \boldsymbol{\Gamma}|||_{\tilde{\theta}}^2 \\
&\leq \mathcal{C}G\boldsymbol{\sigma}(\Delta t, h, \delta) + Ch^m |||\boldsymbol{\sigma}|||_{0,m+1}^2,
\end{aligned} \tag{5.284}$$

establishing the estimates in Theorem 4.1. ■

6 Proof of Theorem 4.2

In this section the proof of Theorem 4.2 is established. The proof is done in a manner similar to that used for Theorem 4.1. For notational simplicity the ‘tilde’ is dropped from $\tilde{\mathbf{u}}_h$ and $\tilde{p}$, the discrete approximations of velocity and pressure using the true stress. We start by examining appropriate linear combinations.

Linear combination 1: Consider the linear combination of

$$\theta \times (4.3) \Big|_{t=t_n} + (1 - 2\theta) \times (4.4) \Big|_{t=t_n} + \theta \times (4.7) \Big|_{t=t_n}. \tag{6.1}$$

When we multiply the expression resulting from (6.1) by Δt (done in a later step) we have a unit stride of size Δt between t_n and t_{n+1} with

$$\begin{aligned}
Re \left(\frac{\mathbf{u}_h^{(n+1)} - \mathbf{u}_h^{(n)}}{\Delta t}, \mathbf{v} \right) + \tilde{\theta} \left(\boldsymbol{\sigma}^{(n+\theta)}, \mathbf{d}(\mathbf{v}) \right) + \theta \left(\boldsymbol{\sigma}^{(n+1)}, \mathbf{d}(\mathbf{v}) \right) + 2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{u}_h^{(n+1)}), \mathbf{d}(\mathbf{v}) \right) \\
+ 2\tilde{\alpha}\tilde{\theta} \left(\mathbf{d}(\mathbf{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v}) \right) = \tilde{\theta} \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) + \theta \left(\mathbf{f}^{(n+1)}, \mathbf{v} \right), \quad \forall \mathbf{v} \in Z_h.
\end{aligned} \tag{6.2}$$

$$\implies \mathcal{B}_4^n(\mathbf{u}_h, \mathbf{v}) = \tilde{\theta} \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) + \theta \left(\mathbf{f}^{(n+1)}, \mathbf{v} \right) \tag{6.3}$$

Terms in (6.3) are defined by those in (6.2). The true solution of the momentum equation evaluated at the midpoint between t_n and t_{n+1} satisfies:

$$\begin{aligned}
0 = \left(\mathbf{f}^{(n+\frac{1}{2})}, \mathbf{v} \right) + \left(p^{(n+\frac{1}{2})}, \nabla \cdot \mathbf{v} \right) - Re \left(\mathbf{u}_t^{(n+\frac{1}{2})}, \mathbf{v} \right) \\
- 2\tilde{\alpha} \left(\mathbf{d} \left(\mathbf{u}^{(n+\frac{1}{2})} \right), \mathbf{d}(\mathbf{v}) \right) - \left(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h.
\end{aligned} \tag{6.4}$$

Adding $\mathcal{B}_4^n(\mathbf{u}, \mathbf{v})$ to both sides (6.4) yields

$$\mathcal{B}_4^n(\mathbf{u}, \mathbf{v}) = \left(\mathbf{f}^{(n+\frac{1}{2})}, \mathbf{v} \right) + \left(p^{(n+\frac{1}{2})}, \nabla \cdot \mathbf{v} \right) + \mathcal{I}_4^n(\mathbf{v}), \quad \forall \mathbf{v} \in Z_h, \tag{6.5}$$

where the term

$$\begin{aligned}
\mathcal{I}_4^n(\mathbf{v}) := Re \left(d_t \mathbf{u}^{(n+1)} - \mathbf{u}_t^{(n+\frac{1}{2})}, \mathbf{v} \right) + \left(\tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \mathbf{d}(\mathbf{v}) \right) \\
+ 2\tilde{\alpha} \left(\theta \mathbf{d}(\mathbf{u}^{(n+1)}) + \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\theta)}) - \mathbf{d} \left(\mathbf{u}^{(n+\frac{1}{2})} \right), \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h.
\end{aligned} \tag{6.6}$$

Subtraction of (6.2) from (6.5) gives the error equation:

$$\mathcal{B}_4^n(e\mathbf{u}, \mathbf{v}) = \left(\mathbf{f}^{(n+\frac{1}{2})} - \tilde{\theta}\mathbf{f}^{(n+\theta)} - \theta\mathbf{f}^{(n+1)}, \mathbf{v} \right) + \left(p^{(n+\frac{1}{2})}, \nabla \cdot \mathbf{v} \right) + \mathcal{I}_4^n(\mathbf{v}), \quad \forall \mathbf{v} \in Z_h. \quad (6.7)$$

This procedure is now repeated, setting up two more error expressions.

Linear combination 2: Using the linear combination of

$$\theta \times (4.3) \Big|_{t=t_n} + (1-2\theta) \times (4.4) \Big|_{t=t_n} + \theta \times (4.7) \Big|_{t=t_{n-1}}, \quad (6.8)$$

will provide a unit stride between $t_{n-\theta}$ and $t_{n+1-\theta}$ when we multiply the resulting expression by Δt in a later step.

$$\begin{aligned} Re \left(\frac{\mathbf{u}_h^{(n+\tilde{\theta})} - \mathbf{u}_h^{(n-\theta)}}{\Delta t}, \mathbf{v} \right) + \tilde{\theta} \left(\boldsymbol{\sigma}^{(n+\theta)}, \mathbf{d}(\mathbf{v}) \right) + \theta \left(\boldsymbol{\sigma}^{(n)}, \mathbf{d}(\mathbf{v}) \right) + \tilde{\theta} 2\tilde{\alpha} \left(\mathbf{d}(\mathbf{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v}) \right) \\ + \theta 2\tilde{\alpha} \left(\mathbf{d}(\mathbf{u}_h^{(n)}), \mathbf{d}(\mathbf{v}) \right) = \tilde{\theta} \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) + \theta \left(\mathbf{f}^{(n)}, \mathbf{v} \right), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (6.9)$$

$$\implies \mathcal{B}_5^n(\mathbf{u}_h, \mathbf{v}) = \tilde{\theta} \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) + \theta \left(\mathbf{f}^{(n)}, \mathbf{v} \right). \quad (6.10)$$

Again the terms in (6.10) are defined by those in (6.9). The true solution of the momentum equation evaluated at the midpoint between $t_{n-\theta}$ and $t_{n+1-\theta}$ satisfies:

$$\begin{aligned} 0 = \left(\mathbf{f}^{(n+\frac{1}{2}-\theta)}, \mathbf{v} \right) + \left(p^{(n+\frac{1}{2}-\theta)}, \nabla \cdot \mathbf{v} \right) - Re \left(\mathbf{u}_t^{(n+\frac{1}{2}-\theta)}, \mathbf{v} \right) \\ - 2\tilde{\alpha} \left(\mathbf{d} \left(\mathbf{u}^{(n+\frac{1}{2}-\theta)} \right), \mathbf{d}(\mathbf{v}) \right) - \left(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (6.11)$$

$\mathcal{B}_5^n(\mathbf{u}, \mathbf{v})$ is now added to both sides (6.11) giving

$$\mathcal{B}_5^n(\mathbf{u}, \mathbf{v}) = \left(\mathbf{f}^{(n+\frac{1}{2}-\theta)}, \mathbf{v} \right) + \left(p^{(n+\frac{1}{2}-\theta)}, \nabla \cdot \mathbf{v} \right) + \mathcal{I}_5^n(\mathbf{v}). \quad (6.12)$$

Here we define

$$\begin{aligned} \mathcal{I}_5^n(\mathbf{v}) := Re \left(d_t \mathbf{u}^{(n+\tilde{\theta})} - \mathbf{u}_t^{(n+\frac{1}{2}-\theta)}, \mathbf{v} \right) + \left(\tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{d}(\mathbf{v}) \right) \\ + 2\tilde{\alpha} \left(\tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta \mathbf{d}(\mathbf{u}^{(n)}) - \mathbf{d} \left(\mathbf{u}^{(n+\frac{1}{2}-\theta)} \right), \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (6.13)$$

Lastly the error expression is obtained by subtracting (6.9) from (6.12) giving

$$\mathcal{B}_5^n(e\mathbf{u}, \mathbf{v}) = \left(\mathbf{f}^{(n+\frac{1}{2}-\theta)} - \tilde{\theta}\mathbf{f}^{(n+\theta)} - \theta\mathbf{f}^{(n)}, \mathbf{v} \right) + \left(p^{(n+\frac{1}{2}-\theta)}, \nabla \cdot \mathbf{v} \right) + \mathcal{I}_5^n(\mathbf{v}), \quad \forall \mathbf{v} \in Z_h. \quad (6.14)$$

Linear combination 3: Multiplying the linear combination of

$$\theta \times (4.3) \Big|_{t=t_n} + (1-2\theta) \times (4.4) \Big|_{t=t_{n-1}} + \theta \times (4.7) \Big|_{t=t_{n-1}}, \quad (6.15)$$

in a later step by Δt provides a unit stride between $t_{n+\theta-1}$ and $t_{n+\theta}$.

$$\begin{aligned} & Re \left(\frac{\mathbf{u}_h^{(n+\theta)} - \mathbf{u}_h^{(n+\theta-1)}}{\Delta t}, \mathbf{v} \right) + 2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v}) \right) + 2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{u}_h^{(n)}), \mathbf{d}(\mathbf{v}) \right) \\ & + 2\tilde{\alpha}(1-2\theta) \left(\mathbf{d}(\mathbf{u}_h^{(n+\theta-1)}), \mathbf{d}(\mathbf{v}) \right) + \theta \left(\boldsymbol{\sigma}^{(n+\theta)}, \mathbf{d}(\mathbf{v}) \right) + (1-2\theta) \left(\boldsymbol{\sigma}^{(n+\theta-1)}, \mathbf{d}(\mathbf{v}) \right) \\ & + \theta \left(\boldsymbol{\sigma}^{(n)}, \mathbf{d}(\mathbf{v}) \right) = \theta \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) + (1-2\theta) \left(\mathbf{f}^{(n+\theta-1)}, \mathbf{v} \right) + \theta \left(\mathbf{f}^{(n)}, \mathbf{v} \right), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (6.16)$$

$$\implies \mathcal{B}_6^n(\mathbf{u}_h, \mathbf{v}) = \theta \left(\mathbf{f}^{(n+\theta)}, \mathbf{v} \right) + (1-2\theta) \left(\mathbf{f}^{(n+\theta-1)}, \mathbf{v} \right) + \theta \left(\mathbf{f}^{(n)}, \mathbf{v} \right). \quad (6.17)$$

The terms in (6.17) are defined by those in (6.16). The true solution of the momentum equation evaluated at the midpoint between $t_{n+\theta-1}$ and $t_{n+\theta}$ satisfies:

$$\begin{aligned} 0 = & \left(\mathbf{f}^{(n+\theta-\frac{1}{2})}, \mathbf{v} \right) + \left(p^{(n+\theta-\frac{1}{2})}, \nabla \cdot \mathbf{v} \right) - Re \left(\mathbf{u}_t^{(n+\theta-\frac{1}{2})}, \mathbf{v} \right) \\ & - 2\tilde{\alpha} \left(\mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}), \mathbf{d}(\mathbf{v}) \right) - \left(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (6.18)$$

Adding $\mathcal{B}_6^n(\mathbf{u}, \mathbf{v})$ to both sides of (6.18) we obtain:

$$\mathcal{B}_6^n(\mathbf{u}, \mathbf{v}) = \left(\mathbf{f}^{(n+\theta-\frac{1}{2})}, \mathbf{v} \right) + \left(p^{(n+\theta-\frac{1}{2})}, \nabla \cdot \mathbf{v} \right) + \mathcal{I}_6^n(\mathbf{v}), \quad \forall \mathbf{v} \in Z_h, \quad (6.19)$$

with

$$\begin{aligned} \mathcal{I}_6^n(\mathbf{v}) := & Re \left(d_t \mathbf{u}^{(n+\theta)} - \mathbf{u}_t^{(n+\theta-\frac{1}{2})}, \mathbf{v} \right) + \left(\theta \boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta-1)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{d}(\mathbf{v}) \right) \\ & + 2\tilde{\alpha} \left(\theta \mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta \mathbf{d}(\mathbf{u}^{(n)}) + (1-2\theta) \mathbf{d}(\mathbf{u}^{(n+\theta-1)}) - \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}), \mathbf{d}(\mathbf{v}) \right), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (6.20)$$

Subtracting (6.16) from (6.19) gives the final error expression for the conservation equations.

$$\begin{aligned} \mathcal{B}_6^n(e\mathbf{u}, \mathbf{v}) = & \left(\mathbf{f}^{(n+\theta-\frac{1}{2})} - \theta \mathbf{f}^{(n+\theta)} - (1-2\theta) \mathbf{f}^{(n+\theta-1)} - \theta \mathbf{f}^{(n)}, \mathbf{v} \right) \\ & + \left(p^{(n+\frac{1}{2}-\theta)}, \nabla \cdot \mathbf{v} \right) + \mathcal{I}_6^n(\mathbf{v}), \quad \forall \mathbf{v} \in Z_h. \end{aligned} \quad (6.21)$$

In order to complete the analysis bounds are now found on (6.7), (6.14), and (6.21), with the value of the test function $\mathbf{v}$ set as $\mathbf{E}^{(n+1)}$, $\mathbf{E}^{(n+\tilde{\theta})}$, and $\mathbf{E}^{(n+\theta)}$ respectively.

6.1 Bounding $\mathcal{B}_4^n(e\mathbf{u}, \mathbf{E}^{(n+1)})$, $\mathcal{B}_5^n(e\mathbf{u}, \mathbf{E}^{(n+\tilde{\theta})})$, and $\mathcal{B}_6^n(e\mathbf{u}, \mathbf{E}^{(n+\theta)})$

In order to complete the analysis for the conservation equation we need to consider bounds on (6.7), (6.14), and (6.21). In order to consider the bounds on the terms in $\mathcal{B}_4^n(e\mathbf{u}, \mathbf{E}^{(n+1)})$, $\mathcal{B}_5^n(e\mathbf{u}, \mathbf{E}^{(n+\tilde{\theta})})$, and $\mathcal{B}_6^n(e\mathbf{u}, \mathbf{E}^{(n+\theta)})$ we use the following estimates:

$$Re \left(d_t e\mathbf{u}^{(n+1)}, \mathbf{E}^{(n+1)} \right) = Re \left(d_t \mathbf{E}^{(n+1)}, \mathbf{E}^{(n+1)} \right) + Re \left(d_t \boldsymbol{\Lambda}^{(n+1)}, \mathbf{E}^{(n+1)} \right), \quad (6.22)$$

with bounds

$$Re \left(d_t \mathbf{E}^{(n+1)}, \mathbf{E}^{(n+1)} \right) \geq \frac{Re}{2\Delta t} \left(\left\| \mathbf{E}^{(n+1)} \right\|^2 - \left\| \mathbf{E}^{(n)} \right\|^2 \right), \quad (6.23)$$

and

$$Re \left(d_t \mathbf{\Lambda}^{(n+1)}, \mathbf{E}^{(n+1)} \right) \leq \frac{Re}{4} \left\| d_t \mathbf{\Lambda}^{(n+1)} \right\|^2 + Re \left\| \mathbf{E}^{(n+1)} \right\|^2. \quad (6.24)$$

Similarly,

$$Re \left(d_t e_{\mathbf{u}}^{(n+\tilde{\theta})}, \mathbf{E}^{(n+\tilde{\theta})} \right) = Re \left(d_t \mathbf{E}^{(n+\tilde{\theta})}, \mathbf{E}^{(n+\tilde{\theta})} \right) + Re \left(d_t \mathbf{\Lambda}^{(n+\tilde{\theta})}, \mathbf{E}^{(n+\tilde{\theta})} \right), \quad (6.25)$$

and is bounded as

$$Re \left(d_t \mathbf{E}^{(n+\tilde{\theta})}, \mathbf{E}^{(n+\tilde{\theta})} \right) \geq \frac{Re}{2\Delta t} \left(\left\| \mathbf{E}^{(n+\tilde{\theta})} \right\|^2 - \left\| \mathbf{E}^{(n-\theta)} \right\|^2 \right), \quad (6.26)$$

and

$$Re \left(d_t \mathbf{\Lambda}^{(n+\tilde{\theta})}, \mathbf{E}^{(n+\tilde{\theta})} \right) \leq \frac{Re}{4} \left\| d_t \mathbf{\Lambda}^{(n+\tilde{\theta})} \right\|^2 + Re \left\| \mathbf{E}^{(n+\tilde{\theta})} \right\|^2. \quad (6.27)$$

We bound

$$Re \left(d_t e_{\mathbf{u}}^{(n+\theta)}, \mathbf{E}^{(n+\theta)} \right) = Re \left(d_t \mathbf{E}^{(n+\theta)}, \mathbf{E}^{(n+\theta)} \right) + Re \left(d_t \mathbf{\Lambda}^{(n+\theta)}, \mathbf{E}^{(n+\theta)} \right), \quad (6.28)$$

as

$$Re \left(d_t \mathbf{E}^{(n+\theta)}, \mathbf{E}^{(n+\theta)} \right) \geq \frac{Re}{2\Delta t} \left(\left\| \mathbf{E}^{(n+\theta)} \right\|^2 - \left\| \mathbf{E}^{(n+\theta-1)} \right\|^2 \right), \quad (6.29)$$

and

$$Re \left(d_t \mathbf{\Lambda}^{(n+\theta)}, \mathbf{E}^{(n+\theta)} \right) \leq \frac{Re}{4} \left\| d_t \mathbf{\Lambda}^{(n+\theta)} \right\|^2 + Re \left\| \mathbf{E}^{(n+\theta)} \right\|^2. \quad (6.30)$$

For the deformation tensor terms we have,

$$2\tilde{\alpha}\theta \left(\mathbf{d}(e_{\mathbf{u}}^{(n+1)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right) = 2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{E}^{(n+1)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right) + 2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{\Lambda}^{(n+1)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right), \quad (6.31)$$

with bounds

$$2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{E}^{(n+1)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right) = 2\tilde{\alpha}\theta \left\| \mathbf{d}(\mathbf{E}^{(n+1)}) \right\|^2, \quad (6.32)$$

and

$$2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{\Lambda}^{(n+1)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right) \leq \frac{2\tilde{\alpha}\theta}{4\epsilon_2} \left\| \nabla \mathbf{\Lambda}^{(n+1)} \right\|^2 + 2\tilde{\alpha}\theta\epsilon_2 \left\| \mathbf{d}(\mathbf{E}^{(n+1)}) \right\|^2. \quad (6.33)$$

The term

$$2\tilde{\alpha}\tilde{\theta} \left(\mathbf{d}(e_{\mathbf{u}}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right) = 2\tilde{\alpha}\tilde{\theta} \left(\mathbf{d}(\mathbf{E}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right) + 2\tilde{\alpha}\tilde{\theta} \left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right), \quad (6.34)$$

and is bounded such that

$$2\tilde{\alpha}\tilde{\theta} \left(\mathbf{d}(\mathbf{E}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right) \leq \frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2} \left\| \nabla \mathbf{E}^{(n+\theta)} \right\|^2 + 2\tilde{\alpha}\tilde{\theta}\epsilon_2 \left\| \mathbf{d}(\mathbf{E}^{(n+1)}) \right\|^2, \quad (6.35)$$

and

$$2\tilde{\alpha}\tilde{\theta} \left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right) \leq \frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2} \left\| \nabla \mathbf{\Lambda}^{(n+\theta)} \right\|^2 + 2\tilde{\alpha}\tilde{\theta}\epsilon_2 \left\| \mathbf{d}(\mathbf{E}^{(n+1)}) \right\|^2. \quad (6.36)$$

We have

$$\begin{aligned} (1-\theta)2\tilde{\alpha} \left(\mathbf{d}(e_{\mathbf{u}}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) \\ = (1-\theta)2\tilde{\alpha} \left(\mathbf{d}(\mathbf{E}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) + (1-\theta)2\tilde{\alpha} \left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right), \end{aligned} \quad (6.37)$$

with bounds

$$(1 - \theta)2\tilde{\alpha} \left(\mathbf{d}(\mathbf{E}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) \leq \frac{(1 - \theta)2\tilde{\alpha}}{4\epsilon_3} \left\| \nabla \mathbf{E}^{(n+\tilde{\theta})} \right\|^2 + (1 - \theta)2\tilde{\alpha}\epsilon_3 \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2, \quad (6.38)$$

and

$$(1 - \theta)2\tilde{\alpha} \left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) \leq \frac{(1 - \theta)2\tilde{\alpha}}{2} \left\| \nabla \mathbf{\Lambda}^{(n+\theta)} \right\|^2 + \frac{(1 - \theta)2\tilde{\alpha}}{2} \left\| \nabla \mathbf{E}^{(n+\tilde{\theta})} \right\|^2. \quad (6.39)$$

Next

$$\theta 2\tilde{\alpha} \left(\mathbf{d}(e_{\mathbf{u}}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) = \theta 2\tilde{\alpha} \left(\mathbf{d}(\mathbf{E}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) + \theta 2\tilde{\alpha} \left(\mathbf{d}(\mathbf{\Lambda}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right), \quad (6.40)$$

with

$$\theta 2\tilde{\alpha} \left(\mathbf{d}(\mathbf{E}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) \leq \theta \tilde{\alpha} \left\| \nabla \mathbf{E}^{(n)} \right\|^2 + \theta \tilde{\alpha} \left\| \nabla \mathbf{E}^{(n+\tilde{\theta})} \right\|^2, \quad (6.41)$$

and

$$\theta 2\tilde{\alpha} \left(\mathbf{d}(\mathbf{\Lambda}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) \leq \theta \tilde{\alpha} \left\| \nabla \mathbf{\Lambda}^{(n)} \right\|^2 + \theta \tilde{\alpha} \left\| \nabla \mathbf{E}^{(n+\tilde{\theta})} \right\|^2. \quad (6.42)$$

Splitting

$$2\tilde{\alpha}\theta \left(\mathbf{d}(e_{\mathbf{u}}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) = 2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{E}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) + 2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right), \quad (6.43)$$

with bounds

$$2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{E}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) = 2\tilde{\alpha}\theta \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2, \quad (6.44)$$

and

$$2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) \leq \frac{\tilde{\alpha}\theta}{2\epsilon_3} \left\| \nabla \mathbf{\Lambda}^{(n+\theta)} \right\|^2 + 2\tilde{\alpha}\theta\epsilon_3 \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2. \quad (6.45)$$

Then

$$2\tilde{\alpha}\theta \left(\mathbf{d}(e_{\mathbf{u}}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) = 2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{E}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) + 2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{\Lambda}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right), \quad (6.46)$$

with each terms handled as:

$$2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{E}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) \leq \frac{\tilde{\alpha}\theta}{2\epsilon_3} \left\| \nabla \mathbf{E}^{(n)} \right\|^2 + 2\tilde{\alpha}\theta\epsilon_3 \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2, \quad (6.47)$$

$$2\tilde{\alpha}\theta \left(\mathbf{d}(\mathbf{\Lambda}^{(n)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) \leq \frac{2\tilde{\alpha}\theta}{4\epsilon_3} \left\| \nabla \mathbf{\Lambda}^{(n)} \right\|^2 + 2\tilde{\alpha}\theta\epsilon_3 \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2. \quad (6.48)$$

Lastly

$$\begin{aligned} & 2\tilde{\alpha} (1 - 2\theta) \left(\mathbf{d}(e_{\mathbf{u}}^{(n+\theta-1)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) \\ &= 2\tilde{\alpha} (1 - 2\theta) \left(\mathbf{d}(\mathbf{E}^{(n+\theta-1)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) + 2\tilde{\alpha} (1 - 2\theta) \left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta-1)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right), \end{aligned} \quad (6.49)$$

where

$$\begin{aligned} & 2\tilde{\alpha} (1 - 2\theta) \left(\mathbf{d}(\mathbf{E}^{(n+\theta-1)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) \\ & \leq \frac{\tilde{\alpha} (1 - 2\theta)}{2\epsilon_3} \left\| \nabla \mathbf{E}^{(n+\theta-1)} \right\|^2 + 2\tilde{\alpha} (1 - 2\theta) \epsilon_3 \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2, \end{aligned} \quad (6.50)$$

and

$$\begin{aligned} & 2\tilde{\alpha} (1 - 2\theta) \left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta-1)}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) \\ & \leq \frac{\tilde{\alpha} (1 - 2\theta)}{2\epsilon_3} \left\| \nabla \mathbf{\Lambda}^{(n+\theta-1)} \right\|^2 + 2\tilde{\alpha} (1 - 2\theta) \epsilon_3 \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2. \end{aligned} \quad (6.51)$$

6.2 Bounding $\mathcal{I}_4^n(\mathbf{E}^{(n+1)})$

Bounds on the interpolation error are now considered for $\mathcal{I}_4^n(\mathbf{E}^{(n+1)})$ (see (6.6)). Thus,

$$\begin{aligned} \mathcal{I}_4^n(\mathbf{E}^{(n+1)}) := & Re \left(d_t \mathbf{u}^{(n+1)} - \mathbf{u}_t^{(n+\frac{1}{2})}, \mathbf{E}^{(n+1)} \right) + \left(\tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \mathbf{d}(\mathbf{E}^{(n+1)}) \right) \\ & + 2\tilde{\alpha} \left(\theta \mathbf{d}(\mathbf{u}^{(n+1)}) + \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\theta)}) - \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right). \end{aligned} \quad (6.52)$$

We bound each of these terms as

$$Re \left(d_t \mathbf{u}^{(n+1)} - \mathbf{u}_t^{(n+\frac{1}{2})}, \mathbf{E}^{(n+1)} \right) \leq \frac{Re}{4} \left\| d_t \mathbf{u}^{(n+1)} - \mathbf{u}_t^{(n+\frac{1}{2})} \right\|^2 + Re \left\| \mathbf{E}^{(n+1)} \right\|^2, \quad (6.53)$$

$$\begin{aligned} & \left(\tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \mathbf{d}(\mathbf{E}^{(n+1)}) \right) \\ & \leq \frac{1}{4\epsilon_2} \left\| \tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 + \epsilon_2 \left\| \mathbf{d}(\mathbf{E}^{(n+1)}) \right\|^2, \end{aligned} \quad (6.54)$$

and

$$\begin{aligned} & 2\tilde{\alpha} \left(\theta \mathbf{d}(\mathbf{u}^{(n+1)}) + \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\theta)}) - \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}), \mathbf{d}(\mathbf{E}^{(n+1)}) \right) \\ & \leq \frac{\tilde{\alpha}}{2\epsilon_2} \left\| \theta \mathbf{d}(\mathbf{u}^{(n+1)}) + \tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\theta)}) - \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})}) \right\|^2 + 2\tilde{\alpha}\epsilon_2 \left\| \mathbf{d}(\mathbf{E}^{(n+1)}) \right\|^2. \end{aligned} \quad (6.55)$$

6.3 Bounding $\mathcal{I}_5^n(\mathbf{E}^{(n+\tilde{\theta})})$

Similar to the previous section we now bound the $\mathcal{I}_5^n$ term (see (6.13)). Here

$$\begin{aligned} \mathcal{I}_5^n(\mathbf{E}^{(n+\tilde{\theta})}) := & Re \left(d_t \mathbf{u}^{(n+\tilde{\theta})} - \mathbf{u}_t^{(n+\frac{1}{2}-\theta)}, \mathbf{E}^{(n+\tilde{\theta})} \right) + \left(\tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) \\ & + 2\tilde{\alpha} \left(\tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta \mathbf{d}(\mathbf{u}^{(n)}) - \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right). \end{aligned} \quad (6.56)$$

Then

$$Re \left(d_t \mathbf{u}^{(n+\tilde{\theta})} - \mathbf{u}_t^{(n+\frac{1}{2}-\theta)}, \mathbf{E}^{(n+\tilde{\theta})} \right) \leq \frac{Re}{4} \left\| d_t \mathbf{u}^{(n+\tilde{\theta})} - \mathbf{u}_t^{(n+\frac{1}{2}-\theta)} \right\|^2 + Re \left\| \mathbf{E}^{(n+\tilde{\theta})} \right\|^2, \quad (6.57)$$

$$\begin{aligned} & \left(\tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) \\ & \leq \frac{1}{4} \left\| \tilde{\theta} \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \left\| \nabla \mathbf{E}^{(n+\tilde{\theta})} \right\|^2, \end{aligned} \quad (6.58)$$

and

$$\begin{aligned} & 2\tilde{\alpha} \left(\tilde{\theta} \mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta \mathbf{d}(\mathbf{u}^{(n)}) - \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}), \mathbf{d}(\mathbf{E}^{(n+\tilde{\theta})}) \right) \\ & \leq \frac{2\tilde{\alpha}}{4} \left\| (1-\theta) \mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta \mathbf{d}(\mathbf{u}^{(n)}) - \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)}) \right\|^2 + 2\tilde{\alpha} \left\| \nabla \mathbf{E}^{(n+\tilde{\theta})} \right\|^2. \end{aligned} \quad (6.59)$$

6.4 Bounding $\mathcal{I}_6^n(\mathbf{E}^{(n+\theta)})$

Here we examine the bounds given for the $\mathcal{I}_6^n$ term (see (6.20)) with the choice of $\mathbf{v} = \mathbf{E}^{(n+\theta)}$.

$$\begin{aligned} \mathcal{I}_6^n(\mathbf{E}^{(n+\theta)}) &:= \operatorname{Re} \left(d_t \mathbf{u}^{(n+\theta)} - \mathbf{u}_t^{(n+\theta-\frac{1}{2})}, \mathbf{E}^{(n+\theta)} \right) \\ &\quad + \left(\theta \boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta-1)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) \\ &\quad + 2\tilde{\alpha} \left(\theta \mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta \mathbf{d}(\mathbf{u}^{(n)}) + (1-2\theta) \mathbf{d}(\mathbf{u}^{(n+\theta-1)}) - \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right). \end{aligned} \quad (6.60)$$

Each piece is bounded as

$$\operatorname{Re} \left(d_t \mathbf{u}^{(n+\theta)} - \mathbf{u}_t^{(n+\theta-\frac{1}{2})}, \mathbf{E}^{(n+\theta)} \right) \leq \frac{Re}{4} \left\| d_t \mathbf{u}^{(n+\theta)} - \mathbf{u}_t^{(n+\theta-\frac{1}{2})} \right\|^2 + Re \left\| \mathbf{E}^{(n+\theta)} \right\|^2, \quad (6.61)$$

$$\begin{aligned} &\left(\theta \boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta-1)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) \\ &\leq \frac{1}{4\epsilon_3} \left\| \theta \boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta) \boldsymbol{\sigma}^{(n+\theta-1)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})} \right\|^2 + \epsilon_3 \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2, \end{aligned} \quad (6.62)$$

and

$$\begin{aligned} &2\tilde{\alpha} \left(\theta \mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta \mathbf{d}(\mathbf{u}^{(n)}) + (1-2\theta) \mathbf{d}(\mathbf{u}^{(n+\theta-1)}) - \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}), \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right) \\ &\leq \frac{\tilde{\alpha}}{2\epsilon_3} \left\| \theta \mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta \mathbf{d}(\mathbf{u}^{(n)}) + (1-2\theta) \mathbf{d}(\mathbf{u}^{(n+\theta-1)}) - \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})}) \right\|^2 + 2\tilde{\alpha}\epsilon_3 \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2. \end{aligned} \quad (6.63)$$

6.5 Bounding the $\mathbf{f}$ and p terms

Bounds for the terms containing the forcing function $\mathbf{f}$ are

$$\left(\mathbf{f}^{(n+\frac{1}{2})} - \tilde{\theta} \mathbf{f}^{(n+\theta)} - \theta \mathbf{f}^{(n+1)}, \mathbf{E}^{(n+1)} \right) \leq \frac{1}{4} \left\| \mathbf{f}^{(n+\frac{1}{2})} - \tilde{\theta} \mathbf{f}^{(n+\theta)} - \theta \mathbf{f}^{(n+1)} \right\|^2 + \left\| \mathbf{E}^{(n+1)} \right\|^2, \quad (6.64)$$

$$\left(\mathbf{f}^{(n+\frac{1}{2}-\theta)} - \tilde{\theta} \mathbf{f}^{(n+\theta)} - \theta \mathbf{f}^{(n)}, \mathbf{E}^{(n+\tilde{\theta})} \right) \leq \frac{1}{4} \left\| \mathbf{f}^{(n+\frac{1}{2}-\theta)} - \tilde{\theta} \mathbf{f}^{(n+\theta)} - \theta \mathbf{f}^{(n)} \right\|^2 + \left\| \mathbf{E}^{(n+\tilde{\theta})} \right\|^2, \quad (6.65)$$

and

$$\begin{aligned} &\left(\mathbf{f}^{(n+\theta-\frac{1}{2})} - \theta \mathbf{f}^{(n+\theta)} - (1-2\theta) \mathbf{f}^{(n+\theta-1)} - \theta \mathbf{f}^{(n)}, \mathbf{E}^{(n+\theta)} \right) \\ &\leq \frac{1}{4} \left\| \mathbf{f}^{(n+\theta-\frac{1}{2})} - \theta \mathbf{f}^{(n+\theta)} - (1-2\theta) \mathbf{f}^{(n+\theta-1)} - \theta \mathbf{f}^{(n)} \right\|^2 + \left\| \mathbf{E}^{(n+\theta)} \right\|^2. \end{aligned} \quad (6.66)$$

The pressure terms are bounded using $p^{(\cdot)} \in Z_h$

$$\begin{aligned} \left(p^{(n+\frac{1}{2})}, \nabla \cdot \mathbf{E}^{(n+1)} \right) &= \left(p^{(n+\frac{1}{2})} - \mathcal{P}^{(n+\frac{1}{2})}, \nabla \cdot \mathbf{E}^{(n+1)} \right) \\ &\leq \left\| p^{(n+\frac{1}{2})} - \mathcal{P}^{(n+\frac{1}{2})} \right\| \dot{d}^{\frac{1}{2}} \left\| \nabla \mathbf{E}^{(n+1)} \right\|^2 \\ &\leq \frac{\dot{d}^{\frac{1}{2}}}{4} \left\| p^{(n+\frac{1}{2})} - \mathcal{P}^{(n+\frac{1}{2})} \right\|^2 + \left\| \nabla \mathbf{E}^{(n+1)} \right\|^2, \end{aligned} \quad (6.67)$$

$$\left(p^{(n+\frac{1}{2}-\theta)}, \nabla \cdot \mathbf{E}^{(n+\tilde{\theta})}\right) \leq \frac{\dot{d}^{\frac{1}{2}}}{4} \left\|p^{(n+\frac{1}{2}-\theta)} - \mathcal{P}^{(n+\frac{1}{2}-\theta)}\right\|^2 + \left\|\nabla \mathbf{E}^{(n+\tilde{\theta})}\right\|^2, \quad (6.68)$$

and

$$\left(p^{(n+\theta-\frac{1}{2})}, \nabla \cdot \mathbf{E}^{(n+\theta)}\right) \leq \frac{\dot{d}^{\frac{1}{2}}}{4} \left\|p^{(n+\theta-\frac{1}{2})} - \mathcal{P}^{(n+\theta-\frac{1}{2})}\right\|^2 + \left\|\nabla \mathbf{E}^{(n+\theta)}\right\|^2. \quad (6.69)$$

6.6 Bring all the parts together for the conservation equations

Using the estimates given in (6.22) - (6.69) a single error expression is given by

$$\begin{aligned} & \frac{Re}{2\Delta t} \left(\left\|\mathbf{E}^{(n+1)}\right\|^2 - \left\|\mathbf{E}^{(n)}\right\|^2 + \left\|\mathbf{E}^{(n+\tilde{\theta})}\right\|^2 - \left\|\mathbf{E}^{(n-\theta)}\right\|^2 + \left\|\mathbf{E}^{(n+\theta)}\right\|^2 - \left\|\mathbf{E}^{(n+\theta-1)}\right\|^2 \right) \\ & + \left(2\tilde{\alpha}\theta - \epsilon_2 \left(\tilde{\alpha}\theta + 2\tilde{\alpha} + 4\tilde{\alpha}\tilde{\theta} + 1\right)\right) \left\|\mathbf{d}(\mathbf{E}^{(n+1)})\right\|^2 \\ & + (2\tilde{\alpha}\theta - \epsilon_3 (2\tilde{\alpha}(1-\theta) + 6\tilde{\alpha}\theta + 4\tilde{\alpha}(1-2\theta) + 2\tilde{\alpha} + 1)) \left\|\mathbf{d}(\mathbf{E}^{(n+\theta)})\right\|^2 \\ \leq & (2Re + 1) \left\|\mathbf{E}^{(n+1)}\right\|^2 + (2Re + 1) \left\|\mathbf{E}^{(n+\tilde{\theta})}\right\|^2 + (2Re + 1) \left\|\mathbf{E}^{(n+\theta)}\right\|^2 \\ & + \left(\theta\tilde{\alpha} + \frac{\tilde{\alpha}\theta}{2\epsilon_3}\right) \left\|\nabla \mathbf{E}^{(n)}\right\|^2 + \left(1 + \frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2}\right) \left\|\nabla \mathbf{E}^{(n+\theta)}\right\|^2 + \frac{\tilde{\alpha}(1-2\theta)}{2\epsilon_3} \left\|\nabla \mathbf{E}^{(n+\theta-1)}\right\|^2 \\ & + \left(2 + (1+\theta)2\tilde{\alpha} + \frac{(1-\theta)2\tilde{\alpha}}{4\epsilon_3} + \frac{(1-\theta)2\tilde{\alpha}}{2}\right) \left\|\nabla \mathbf{E}^{(n+\tilde{\theta})}\right\|^2 + \left\|\nabla \mathbf{E}^{(n+1)}\right\|^2 \\ & + H_{\mathbf{\Lambda}} + H_{\mathcal{I}^n} + H_{\mathbf{f}} + H_p, \end{aligned} \quad (6.70)$$

where

$$\begin{aligned} H_{\mathbf{\Lambda}} &:= \frac{Re}{4} \left(\left\|d_t \mathbf{\Lambda}^{(n+1)}\right\|^2 + \left\|d_t \mathbf{\Lambda}^{(n+\tilde{\theta})}\right\|^2 + \left\|d_t \mathbf{\Lambda}^{(n+\theta)}\right\|^2 \right) \\ &+ \frac{2\tilde{\alpha}\theta}{4\epsilon_2} \left\|\nabla \mathbf{\Lambda}^{(n+1)}\right\|^2 + \left(\theta\tilde{\alpha} + \frac{2\tilde{\alpha}\theta}{4\epsilon_3}\right) \left\|\nabla \mathbf{\Lambda}^{(n)}\right\|^2 + \frac{\tilde{\alpha}(1-2\theta)}{2\epsilon_3} \left\|\nabla \mathbf{\Lambda}^{(n+\theta-1)}\right\|^2 \\ &+ \left(\frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2} + \frac{(1-\theta)2\tilde{\alpha}}{2} + \frac{\tilde{\alpha}\theta}{2\epsilon_3}\right) \left\|\nabla \mathbf{\Lambda}^{(n+\theta)}\right\|^2, \end{aligned} \quad (6.71)$$

$$\begin{aligned} H_{\mathcal{I}^n} &:= \frac{Re}{4} \left(\left\|d_t \mathbf{u}^{(n+1)} - \mathbf{u}_t^{(n+\frac{1}{2})}\right\|^2 + \left\|d_t \mathbf{u}^{(n+\tilde{\theta})} - \mathbf{u}_t^{(n+\frac{1}{2}-\theta)}\right\|^2 + \left\|d_t \mathbf{u}^{(n+\theta)} - \mathbf{u}_t^{(n+\theta-\frac{1}{2})}\right\|^2 \right) \\ &+ \frac{1}{4\epsilon_2} \left\|\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2 + \frac{1}{4} \left\|\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}\right\|^2 \\ &+ \frac{1}{4\epsilon_3} \left\|\theta\boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta)\boldsymbol{\sigma}^{(n+\theta-1)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}\right\|^2 \\ &+ \frac{\tilde{\alpha}}{2\epsilon_2} \left\|\theta\mathbf{d}(\mathbf{u}^{(n+1)}) + \tilde{\theta}\mathbf{d}(\mathbf{u}^{(n+\theta)}) - \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2})})\right\|^2 \\ &+ \frac{\tilde{\alpha}}{2} \left\|(1-\theta)\mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta\mathbf{d}(\mathbf{u}^{(n)}) - \mathbf{d}(\mathbf{u}^{(n+\frac{1}{2}-\theta)})\right\|^2 \\ &+ \frac{\tilde{\alpha}}{2\epsilon_3} \left\|\theta\mathbf{d}(\mathbf{u}^{(n+\theta)}) + \theta\mathbf{d}(\mathbf{u}^{(n)}) + (1-2\theta)\mathbf{d}(\mathbf{u}^{(n+\theta-1)}) - \mathbf{d}(\mathbf{u}^{(n+\theta-\frac{1}{2})})\right\|^2, \end{aligned} \quad (6.72)$$

$$\begin{aligned}
H_{\mathbf{f}} := & \frac{1}{4} \left\| \mathbf{f}^{(n+\frac{1}{2})} - \tilde{\theta} \mathbf{f}^{(n+\theta)} - \theta \mathbf{f}^{(n+1)} \right\|^2 + \frac{1}{4} \left\| \mathbf{f}^{(n+\frac{1}{2}-\theta)} - \tilde{\theta} \mathbf{f}^{(n+\theta)} - \theta \mathbf{f}^{(n)} \right\|^2 \\
& + \frac{1}{4} \left\| \mathbf{f}^{(n+\theta-\frac{1}{2})} - \theta \mathbf{f}^{(n+\theta)} - (1-2\theta) \mathbf{f}^{(n+\theta-1)} - \theta \mathbf{f}^{(n)} \right\|^2,
\end{aligned} \tag{6.73}$$

and

$$H_p := \frac{d^{\frac{1}{2}}}{4} \left(\left\| p^{(n+\frac{1}{2})} - \mathcal{P}^{(n+\frac{1}{2})} \right\|^2 + \left\| p^{(n+\frac{1}{2}-\theta)} - \mathcal{P}^{(n+\frac{1}{2}-\theta)} \right\|^2 + \left\| p^{(n+\theta-\frac{1}{2})} - \mathcal{P}^{(n+\theta-\frac{1}{2})} \right\|^2 \right). \tag{6.74}$$

Multiplying expression (6.70) by $\frac{2\Delta t}{Re}$ and summing from time $n = 0$ to l gives

$$\begin{aligned}
& \left\| \mathbf{E}^{(l+1)} \right\|^2 - \left\| \mathbf{E}^{(0)} \right\|^2 + \left\| \mathbf{E}^{(l+\tilde{\theta})} \right\|^2 - \left\| \mathbf{E}^{(\tilde{\theta})} \right\|^2 + \left\| \mathbf{E}^{(l+\theta)} \right\|^2 - \left\| \mathbf{E}^{(\theta)} \right\|^2 \\
& + \left(4\tilde{\alpha}\theta - \epsilon_2 \left(2\tilde{\alpha}\theta + 4\tilde{\alpha} + 8\tilde{\alpha}\tilde{\theta} + 2 \right) \right) \frac{\Delta t}{Re} \sum_{n=0}^l \left\| \mathbf{d}(\mathbf{E}^{(n+1)}) \right\|^2 \\
& + \left(4\tilde{\alpha}\theta - \epsilon_3 \left(4\tilde{\alpha}(1-\theta) + 12\tilde{\alpha}\theta + 8\tilde{\alpha}(1-2\theta) + 4\tilde{\alpha} + 2 \right) \right) \frac{\Delta t}{Re} \sum_{n=0}^l \left\| \mathbf{d}(\mathbf{E}^{(n+\theta)}) \right\|^2 \\
\leq & \Delta t \sum_{n=0}^l \left\| \mathbf{E}^{(n+1)} \right\|^2 \frac{2}{Re} \left((2Re+1) + Ch^{-2} \right) + \Delta t \sum_{n=0}^l \left\| \mathbf{E}^{(n)} \right\|^2 \frac{2Ch^{-2}}{Re} \left(\theta\tilde{\alpha} + \frac{\tilde{\alpha}\theta}{2\epsilon_3} \right) \\
& + \Delta t \sum_{n=0}^l \left\| \mathbf{E}^{(n+\tilde{\theta})} \right\|^2 \frac{2}{Re} \left((2Re+1) + Ch^{-2} \left(2 + \frac{6\tilde{\alpha}}{2} + \frac{(\theta)2\tilde{\alpha}}{2} + \frac{(1-\theta)2\tilde{\alpha}}{4\epsilon_3} \right) \right) \\
& + \Delta t \sum_{n=0}^l \left\| \mathbf{E}^{(n+\theta)} \right\|^2 \frac{2}{Re} \left((2Re+1) + Ch^{-2} \left(1 + \frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2} \right) \right) \\
& + \Delta t \sum_{n=0}^l \left\| \mathbf{E}^{(n+\theta-1)} \right\|^2 \frac{2Ch^{-2}\tilde{\alpha}(1-2\theta)}{2Re\epsilon_3} \\
& + \Delta t \sum_{n=0}^l \frac{2}{Re} H_{\mathbf{\Lambda}} + \Delta t \sum_{n=0}^l \frac{2}{Re} H_{\mathcal{I}^n} + \Delta t \sum_{n=0}^l \frac{2}{Re} H_{\mathbf{f}} + \Delta t \sum_{n=0}^l \frac{2}{Re} H_p.
\end{aligned} \tag{6.75}$$

In order to apply the discrete Gronwall's lemma we need the inequalities stated in (6.76), (6.77), and (6.78) to hold.

$$1 \geq \frac{2\Delta t}{Re} \left(((2Re+1) + Ch^{-2}) + Ch^{-2} \left(\theta\tilde{\alpha} + \frac{\tilde{\alpha}\theta}{2\epsilon_3} \right) \right), \tag{6.76}$$

$$1 \geq \frac{2\Delta t}{Re} \left((2Re+1) + Ch^{-2} \left(2 + \frac{6\tilde{\alpha}}{2} + \frac{(\theta)2\tilde{\alpha}}{2} + \frac{(1-\theta)2\tilde{\alpha}}{4\epsilon_3} \right) \right), \tag{6.77}$$

$$1 \geq \frac{2\Delta t}{Re} \left((2Re+1) + Ch^{-2} \left(1 + \frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2} \right) + \frac{Ch^{-2}\tilde{\alpha}(1-2\theta)}{2\epsilon_3} \right). \tag{6.78}$$

Summarizing (6.76), (6.77), and (6.78) yields that

$$\Delta t (C + Ch^{-2}) \leq 1.$$

This gives the stability condition

$$\Delta t \leq Ch^2. \tag{6.79}$$

6.7 The H terms in (6.75)

Using interpolation properties, and elements of order k , and q for velocity and pressure respectively, bounds are found for the $H_{\mathbf{\Lambda}}, H_{\mathcal{I}^n}, H_{\mathbf{f}}$, and H_p expressions in (6.75). Here

$$\sum_{n=0}^l \frac{2\Delta t}{Re} H_{\mathbf{\Lambda}} \leq Ch^{2k+2} \|\mathbf{u}_t\|_{0,k+1}^2 + Ch^{2k} \|\mathbf{u}\|_{0,k+1}. \quad (6.80)$$

Bounding,

$$\begin{aligned} \sum_{n=0}^l \frac{2\Delta t}{Re} H_{\mathcal{I}^n} &\leq C(\Delta t)^4 \|\mathbf{u}_{ttt}\|_{0,0}^2 + C(\Delta t)^4 \|\boldsymbol{\sigma}_{tt}\|_{0,0}^2 + C(\Delta t)^4 \|(\nabla \mathbf{u})_{tt}\|_{0,0}^2 \\ &\leq C(\Delta t)^4 \|\mathbf{u}_{ttt}\|_{0,0}^2 + C(\Delta t)^4 \|(\nabla \mathbf{u})_{tt}\|_{0,0}^2 + C(\Delta t)^4 C_T \\ &\leq C(\Delta t)^4 \|\mathbf{u}_{ttt}\|_{0,0}^2 + C(\Delta t)^4 \|\mathbf{u}_{tt}\|_{0,1}^2 + C(\Delta t)^4 C_T. \end{aligned} \quad (6.81)$$

For the forcing function and the pressure we have:

$$\sum_{n=0}^l \frac{2\Delta t}{Re} H_{\mathbf{f}} \leq C(\Delta t)^4 \|\mathbf{f}_{tt}\|_{0,0}^2, \quad (6.82)$$

and

$$\sum_{n=0}^l \frac{2\Delta t}{Re} H_p \leq Ch^{2q+2} \|p\|_{0,q+1}^2. \quad (6.83)$$

Applying the discrete Gronwall's lemma to (6.75) we obtain

$$\begin{aligned} &\|\mathbf{E}^{l+1}\|^2 - \|\mathbf{E}^{(0)}\|^2 + \|\mathbf{E}^{(l+\tilde{\theta})}\|^2 - \|\mathbf{E}^{(\tilde{\theta})}\|^2 + \|\mathbf{E}^{(l+\theta)}\|^2 - \|\mathbf{E}^{(\theta)}\|^2 \\ &+ (4\tilde{\alpha}\theta - 2\epsilon_2(6\tilde{\alpha} - 3\tilde{\alpha}\theta - 1)) \frac{\Delta t}{Re} \sum_{n=0}^l \|\mathbf{d}(\mathbf{E}^{(n+1)})\|^2 \\ &+ (4\tilde{\alpha}\theta - 2\epsilon_3(8\tilde{\alpha} - 4\tilde{\alpha}\theta + 1)) \frac{\Delta t}{Re} \sum_{n=0}^l \|\mathbf{d}(\mathbf{E}^{(n+\theta)})\|^2 \\ &\leq G_{\mathbf{u}}(\Delta t, h, \delta), \end{aligned} \quad (6.84)$$

where

$$\begin{aligned} G_{\mathbf{u}}(\Delta t, h, \delta) &:= Ch^{2k+2} \|\mathbf{u}_t\|_{0,k+1}^2 + Ch^{2k} \|\mathbf{u}\|_{0,k+1} + Ch^{2q+2} \|p\|_{0,q+1}^2 \\ &+ C(\Delta t)^4 \|\mathbf{u}_{ttt}\|_{0,0}^2 + C(\Delta t)^4 \|\mathbf{u}_{tt}\|_{0,1}^2 + C(\Delta t)^4 \|\mathbf{f}_{tt}\|_{0,0}^2 \\ &+ C(\Delta t)^4 C_T. \end{aligned} \quad (6.85)$$

Applying lemmas 5 and A.3 for bounds on $\|\mathbf{E}^{(\theta)}\|^2$ and $\|\mathbf{E}^{(\tilde{\theta})}\|^2$ respectively, and assuming $\|\mathbf{E}^{(0)}\|^2 = 0$, then from (6.84) we have (6.86). Lemma 5 and lemma A.3 are stated and proved in Appendix A.

$$\|\mathbf{E}\|^2 + C \|\mathbf{d}(\mathbf{E})\|_{0,0}^2 \leq G_{\mathbf{u}}(\Delta t, h, \delta), \quad (6.86)$$

which gives

$$\|\mathbf{E}\|_{\infty,0}^2 \leq G_{\mathbf{u}}(\Delta t, h, \delta), \quad (6.87)$$

and

$$|||\mathbf{E}|||_{0,0}^2 \leq TG\mathbf{u}(\Delta t, h, \delta). \quad (6.88)$$

Using Korn's lemma with (6.86) gives

$$|||\nabla \mathbf{E}|||_{0,0}^2 \leq CG\mathbf{u}(\Delta t, h, \delta). \quad (6.89)$$

Hence

$$|||\mathbf{E}|||_{0,1}^2 \leq CG\mathbf{u}(\Delta t, h, \delta). \quad (6.90)$$

To establish the error estimate in Theorem 4.2 note that

$$\begin{aligned} |||\mathbf{u} - \mathbf{u}_h|||_{\infty,0}^2 &\leq |||\mathbf{E}|||_{\infty,0}^2 + |||\Lambda|||_{\infty,0}^2 \\ &\leq G\mathbf{u}(\Delta t, h, \delta) + Ch^{2k+2} |||\mathbf{u}|||_{\infty,k+1}^2, \end{aligned} \quad (6.91)$$

and

$$\begin{aligned} |||\mathbf{u} - \mathbf{u}_h|||_{0,1}^2 &\leq |||\mathbf{E}|||_{0,1}^2 + |||\Lambda|||_{0,1}^2 \\ &\leq CG\mathbf{u}(\Delta t, h, \delta) + Ch^{2k} |||\mathbf{u}|||_{0,k+1}^2. \end{aligned} \quad (6.92)$$

Justification of (IH1) for $\tilde{\mathbf{u}}_h$.

Assume that (IH1) holds for $n = 1, 2, \dots, l-1$. Then using inverse estimates, interpolation properties, and (4.15)

$$\begin{aligned} \|\mathbf{u}_h^l\|_{\infty} &\leq \|\mathbf{u}_h^l - \mathbf{u}^l\|_{\infty} + \|\mathbf{u}^l\|_{\infty} \\ &\leq \|\mathbf{E}^l\|_{\infty} + \|\Lambda^l\|_{\infty} + M \\ &\leq Ch^{\frac{-d}{2}} \|\mathbf{E}^l\| + Ch^{\frac{-d}{2}} \|\Lambda^l\| + M \\ &\leq C \left(h^{k-\frac{d}{2}} + h^{q-\frac{d}{2}+1} + (\Delta t)^2 h^{\frac{-d}{2}} + h^{k-\frac{d}{2}+1} \right) + M. \end{aligned} \quad (6.93)$$

Setting $k \geq \frac{d}{2}$, and $q \geq \frac{d}{2} - 1$ and choosing h , and Δt such that

$$h^{k-\frac{d}{2}}, h^{q-\frac{d}{2}+1} \leq \frac{1}{C}, \quad \text{and} \quad \Delta t^2 \leq \frac{h^{\frac{d}{2}}}{C}, \quad (6.94)$$

we have

$$\|\mathbf{u}_h^l\|_{\infty} \leq M + 4.$$

Similarly it follows that $\|\mathbf{u}_h^{(n+\bar{\theta})}\|_{\infty} \leq M + 4$. ■

7 Numerical Results

In this section we present numerical results for the θ -method on two test problems. The first example uses a known analytical solution to verify numerical convergence rates (Cvge. Rate) for

the θ -method. In the second example, we considered a prototypal problem of viscoelastic flow, flow through a 4:1 planar contraction. In both examples finite element computations were carried out using the FreeFem++ integrated development environment [13]. Continuous piecewise quadratic elements were used for modeling the velocity, and continuous piecewise linear elements were used for the pressure and stress. The constitutive equation was stabilized using a SUPG discretization with a parameter δ .

For the (optimal) value of $\theta = 1 - \sqrt{2}/2 \approx 0.29289$ the local temporal discretization errors are $O((\Delta t)^2)$. The influence of the value for θ on the numerical approximations is illustrated in Figure 7.1. for computations performed on Example 1 (described below). All the other computations reported in this paper were obtained using $\theta = 1 - \sqrt{2}/2$. The constitutive equation splitting parameter ω was set to $1/2$ in all computations.

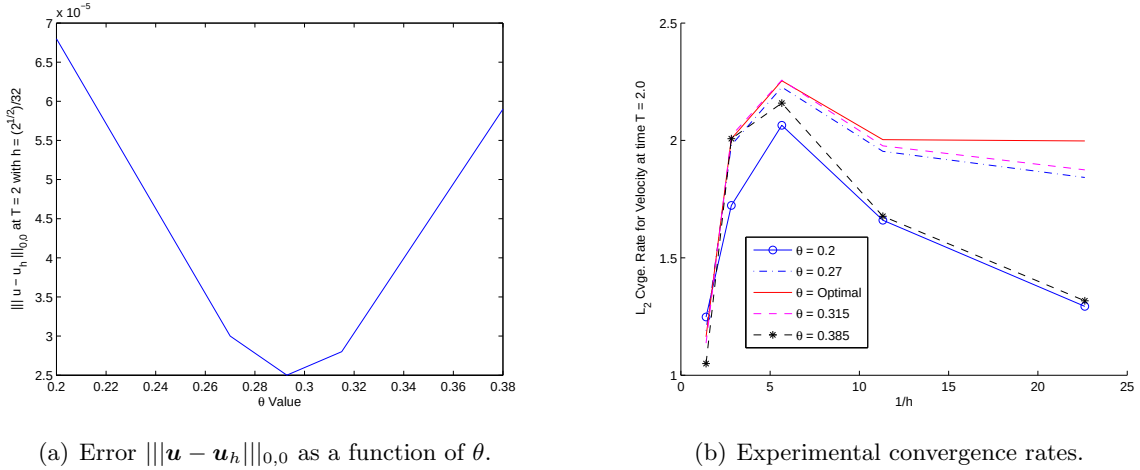


Figure 7.1: Optimal θ value

7.1 Example 1

In order to investigate the predicted convergence rates we consider fluid flow across a unit square with a known solution.

Let $\Omega = (0, 1) \times (0, 1)$, $Re = 1$, $\alpha = 1/2$, $\lambda = 2$, and $a = 1$. For the true solution we use

$$\mathbf{u} = \begin{pmatrix} e^{(x+y-\frac{1}{2}t)}(x^2 - x)(y^2 - y) \\ -e^{(x+y-t)}(x^2 - x)(y^2 - y) \end{pmatrix}, \quad (7.1)$$

$$p = \cos(2\pi x)(y^2 - y), \quad (7.2)$$

$$\boldsymbol{\sigma} = 2\alpha \mathbf{d}(\mathbf{u}). \quad (7.3)$$

Remark: A right-hand-side function is added to (2.1) and $\mathbf{f}$ in (2.2) is calculated using (7.1)-(7.3).

For this example three sequences of computations were performed:

- (i) approximation of $\tilde{\boldsymbol{\sigma}}_h$, assuming $\mathbf{u}$ and p (Theorem 4.1),
- (ii) approximation of $\tilde{\mathbf{u}}_h$ and $\tilde{p}_h$, assuming $\boldsymbol{\sigma}$ (Theorem 4.2),
- (iii) approximation of $\mathbf{u}_h, p_h$ and $\boldsymbol{\sigma}_h$.

7.1.1 Approximating $\tilde{\sigma}_h$ with u and p known

We first consider the approximation of the stress $\tilde{\sigma}_h$ assuming the velocity and pressure functions are known. This is analogous to implementing Step 1a, Step 2b, and Step 3a of the θ -method. From Corollary 1 we have predicted asymptotic convergence rate

$$\|u \cdot \nabla (\sigma - \tilde{\sigma}_h)\|_{\tilde{\theta}} \leq C ((\Delta t)^2 + \Delta t \delta + h \delta + h + \delta),$$

which is consistent with the numerical convergence rates presented in Table 7.1. Note Table 7.1 shows the effect of the upwinding parameter δ on $\|\sigma - \tilde{\sigma}_h\|_{0,0}$.

For the proof of Theorem 4.1 we required the restriction $\Delta t \leq Ch^2$. The numerical computations were performed with $\Delta t \sim h$. It is an open question if the restriction $\Delta t \leq Ch^2$ is a necessary condition for (4.12).

Table 7.1: Approximation errors and experimental convergence rates at $T = 2$

$\delta \downarrow$	$(\Delta t, h) \rightarrow$	$\left(1, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$
0	$\ \sigma - \tilde{\sigma}_h\ _{0,0}$	2.1235e-1	6.6773e-2	1.9191e-2	5.0437e-3	1.2830e-3
	Cvge. Rate	-	1.7	1.8	1.9	2.0
	$\ u \cdot \nabla (\sigma - \tilde{\sigma}_h)\ _{\tilde{\theta}}$	1.1544e-1	6.1759e-2	2.9979e-2	1.5185e-2	7.6944e-3
	Cvge. Rate	-	0.9	1.0	1.0	1.0
$\frac{h}{\sqrt{2}}$	$\ \sigma - \tilde{\sigma}_h\ _{0,0}$	2.0070e-1	8.4563e-2	3.7449e-2	1.6980e-2	8.0428e-3
	Cvge. Rate	-	1.2	1.2	1.1	1.1
	$\ u \cdot \nabla (\sigma - \tilde{\sigma}_h)\ _{\tilde{\theta}}$	1.0263e-1	6.4336e-2	3.2570e-2	1.6325e-2	8.2071e-3
	Cvge. Rate	-	0.7	1.0	1.0	1.0
$\left(\frac{h}{\sqrt{2}}\right)^{\frac{3}{2}}$	$\ \sigma - \tilde{\sigma}_h\ _{0,0}$	2.0174e-1	7.3678e-2	2.3575e-2	6.9629e-3	2.0645e-3
	Cvge. Rate	-	1.5	1.6	1.8	1.8
	$\ u \cdot \nabla (\sigma - \tilde{\sigma}_h)\ _{\tilde{\theta}}$	1.0447e-1	6.2239e-2	3.0497e-2	1.5304e-2	7.7199e-3
	Cvge. Rate	-	0.7	1.0	1.0	1.0
$\left(\frac{h}{\sqrt{2}}\right)^2$	$\ \sigma - \tilde{\sigma}_h\ _{0,0}^2$	2.0346e-1	6.9501e-2	2.0245e-2	5.3281e-3	1.3546e-3
	Cvge. Rate	-	1.5	1.8	1.9	2.0
	$\ u \cdot \nabla (\sigma - \tilde{\sigma}_h)\ _{\tilde{\theta}}$	1.0664e-1	6.1737e-2	3.0104e-2	1.5203e-2	7.6968e-3
	Cvge. Rate	-	0.8	1.0	1.0	1.0

7.1.2 Approximating $\tilde{u}_h$ and $\tilde{p}_h$ with known σ

Numerical results for the approximation of velocity, $\tilde{u}_h$, and pressure, $\tilde{p}_h$ for a known stress, σ , are presented in Table 7.2. These results correspond to the analysis of step 1b, step 2a, and step 3b as stated in Theorem 4.2.

The numerical convergence rates observed are consistent with those predicted in Corollary 2 where

$$\|u - \tilde{u}_h\|_{0,1} \leq O((\Delta t)^2 + h^2).$$

Similar to the case in the previous subsection we make note that the proof of Theorem 4.2 required the restriction $\Delta t \leq Ch^2$. Here the numerical computations were performed with $\Delta t \sim h$.

Table 7.2: Approximation errors and experimental convergence rates at $T = 2$

$(\Delta t, h) \rightarrow$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$	$\left(\frac{1}{32}, \frac{\sqrt{2}}{64}\right)$	$\left(\frac{1}{64}, \frac{\sqrt{2}}{128}\right)$
$ \mathbf{u} - \tilde{\mathbf{u}}_h _{0,1}$	4.4196e-2	1.1485e-2	2.9707e-3	7.5759e-4	1.9142e-4	4.8129e-5
Cvge. Rate	-	1.9	2.0	2.0	2.0	2.0
$ \mathbf{u} - \tilde{\mathbf{u}}_h _{\infty,0}$	1.4734e-3	1.7996e-4	2.5014e-5	4.1759e-6	8.5692e-7	2.1636e-7
Cvge. Rate	-	3.0	2.8	2.6	2.3	2.0
$ p - \tilde{p}_h _{0,0}$	1.0859e-1	6.6842e-3	1.5033e-3	3.9097e-4	1.2086e-4	4.7884e-5
Cvge. Rate	-	4.0	2.2	1.9	1.7	1.3
$ p - \tilde{p}_h _{\infty,0}$	8.4003e-2	4.9703e-3	1.1343e-3	3.2878e-4	1.2797e-4	6.0659e-5
Cvge. Rate	-	4.1	2.1	1.8	1.4	1.1

7.1.3 Full θ -method approximation for viscoelasticity

Table 7.3 contains the results for the approximation of $\mathbf{u}$ and $\boldsymbol{\sigma}$ using the θ -method described in Step 1a - Step 3b. The numerical convergence rates are consistent with our expectations based on Theorems 4.1 and 4.2, i.e.

$$|||\mathbf{u} - \mathbf{u}_h|||_{0,1} + |||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|||_{0,0} + |||\mathbf{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)|||_{\tilde{\theta}} \sim O\left((\Delta t)^2 + (\Delta t)\delta + \delta + h\right).$$

7.2 Example 2

For a second example the numerical approximation of viscoelastic flow through a planar 4:1 contraction channel is presented. This has been a long standing benchmark problem for viscoelastic flow [16, 19, 20]. A diagram of the flow geometry is given in Figure 7.2. It is assumed that the channel lengths are sufficiently long for fully developed Poiseuille flow at both the inflow and outflow boundaries. In the computations the value of L in Figure 7.2 is set at $1/4$.

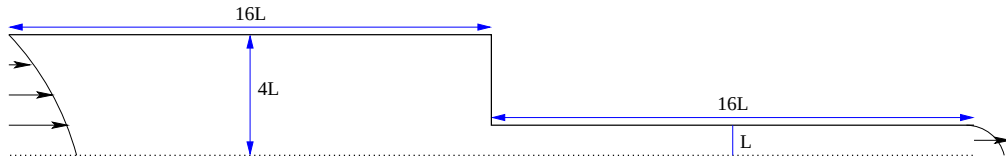


Figure 7.2: Plot of 4:1 contraction domain geometry.

The flow at $t = 0$ is assumed to be stationary and then slowly increased for $t > 0$ using $A(t) = 1 - e^{-t}$. The boundary conditions at the inflow of the channel are defined by

$$\mathbf{u} = A(t) \begin{pmatrix} \frac{1}{32} (1 - y^2) \\ 0 \end{pmatrix}, \quad (7.4)$$

Table 7.3: Approximation errors and experimental convergence rates at $T = 2$

$\delta \downarrow$	$(\Delta t, h) \rightarrow$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$	$\left(\frac{1}{32}, \frac{\sqrt{2}}{64}\right)$
0	$ \mathbf{u} - \mathbf{u}_h _{0,1}$	4.7608e-2	1.2323e-2	3.2034e-3	8.3187e-4	2.1793e-4
	Cvge. Rate	-	1.9	1.9	1.9	1.9
	$ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h _{0,0}$	6.5569e-2	1.9248e-2	5.0845e-3	1.3000e-3	3.2981e-4
$\frac{h}{\sqrt{2}}$	Cvge. Rate	-	1.8	1.9	2.0	2.0
	$ \mathbf{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) _{\tilde{\theta}}$	6.1217e-2	2.9982e-2	1.5191e-2	7.6984e-3	3.8800e-3
	Cvge. Rate	-	1.0	1.0	1.0	1.0
$\left(\frac{h}{\sqrt{2}}\right)^{\frac{3}{2}}$	$ \mathbf{u} - \mathbf{u}_h _{0,1}$	5.0150e-2	1.6193e-2	6.4636e-3	2.9147e-3	1.3989e-3
	Cvge. Rate	-	1.6	1.3	1.1	1.1
	$ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h _{0,0}$	8.1922e-2	3.5905e-2	1.5791e-2	7.3743e-3	3.5789e-3
$\left(\frac{h}{\sqrt{2}}\right)^2$	Cvge. Rate	-	1.2	1.2	1.1	1.0
	$ \mathbf{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) _{\tilde{\theta}}$	6.4087e-2	3.2589e-2	1.6309e-2	8.2007e-3	4.1180e-4
	Cvge. Rate	-	1.0	1.0	1.0	1.0
$\left(\frac{h}{\sqrt{2}}\right)^{\frac{3}{2}}$	$ \mathbf{u} - \mathbf{u}_h _{0,1}$	4.8620e-2	1.3135e-2	3.6334e-3	1.0192e-3	2.9513e-4
	Cvge. Rate	-	1.9	1.9	1.8	1.8
	$ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h _{0,0}$	7.1941e-2	2.3392e-2	6.8309e-3	1.9976e-3	6.0598e-4
$\left(\frac{h}{\sqrt{2}}\right)^2$	Cvge. Rate	-	1.6	1.8	1.8	1.7
	$ \mathbf{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) _{\tilde{\theta}}$	6.1860e-2	3.0510e-2	1.5305e-2	7.7221e-3	3.8851e-3
	Cvge. Rate	-	1.0	1.0	1.0	1.0
$\left(\frac{h}{\sqrt{2}}\right)^2$	$ \mathbf{u} - \mathbf{u}_h _{0,1}$	4.8022e-2	1.2507e-2	3.2644e-3	8.4818e-4	2.2193e-4
	Cvge. Rate	-	1.9	1.9	1.9	1.9
	$ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h _{0,0}$	6.8104e-2	2.0308e-2	5.3614e-3	1.3680e-3	3.4645e-4
$\left(\frac{h}{\sqrt{2}}\right)^2$	Cvge. Rate	-	1.7	1.9	2.0	2.0
	$ \mathbf{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) _{\tilde{\theta}}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3
	Cvge. Rate	-	1.0	1.0	1.0	1.0

and

$$\sigma_{11} = \frac{\lambda A(t)^2 y^2 \alpha (1+a)}{D(t)}, \quad \sigma_{12} = \frac{-16\alpha A(t)y}{D(t)}, \quad \text{and} \quad \sigma_{22} = \frac{\lambda A(t)^2 y^2 \alpha (a-1)}{D(t)}, \quad (7.5)$$

where

$$D(t) = 256 + A(t)^2 y^2 \lambda^2 (1+a)(1-a).$$

The outflow boundary condition is

$$\mathbf{u} = A(t) \begin{pmatrix} 2 \left(\frac{1}{16} - y^2 \right) \\ 0 \end{pmatrix}. \quad (7.6)$$

No slip boundary conditions were imposed for the velocity on the solid walls of the contraction, and a symmetry condition was imposed along the bottom of the computational domain. The computations were performed on a uniformly refined version of the mesh shown in Figure 7.3 with $\Delta x_{\min} = 0.0625$ and $\Delta y_{\min} = 0.015625$.

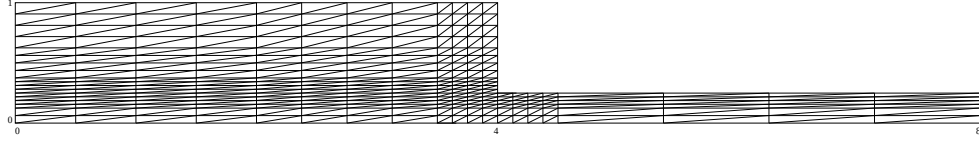


Figure 7.3: Sample Contraction Mesh

The computations were done using the full θ -method approximation given by (4.2) - (4.7) for an Oldroyd B fluid ($a = 1$), with $\lambda = 2$ and $Re = 1$. The value of α was set to $8/9$, which is commonly used in the literature [19]. The discrete time step and upwinding parameter were set to $\Delta t = 1/32$ and $\delta = (2/\Delta y_{\min})^2$. Figures 7.4 and 7.5 show streamlines for the fluid at times $t = 1/8, 1/2, 1$, and 4 superimposed on a contour plot showing the magnitude of velocity. Note that, consistent with expectations, as the velocity is increased, a vortex appears in the upper corner of the domain and grows with the magnitude of the velocity.

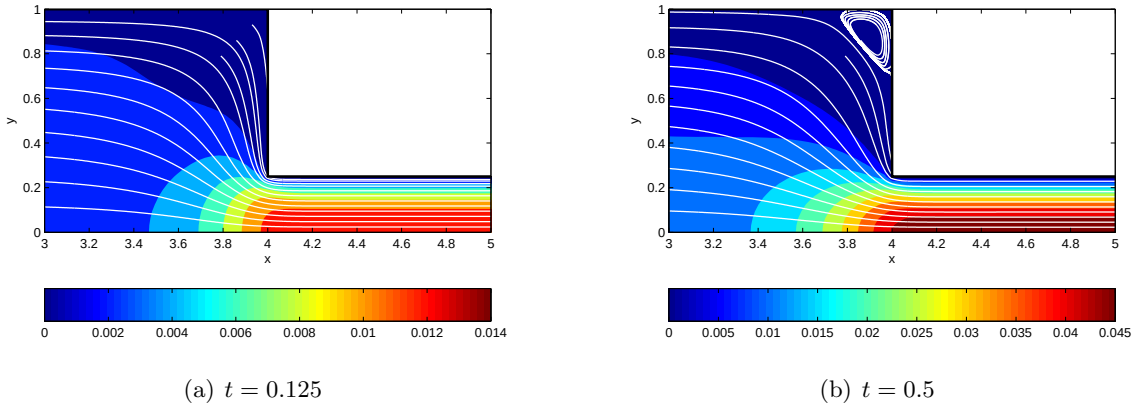
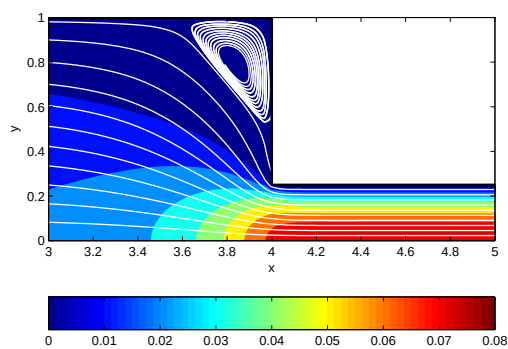
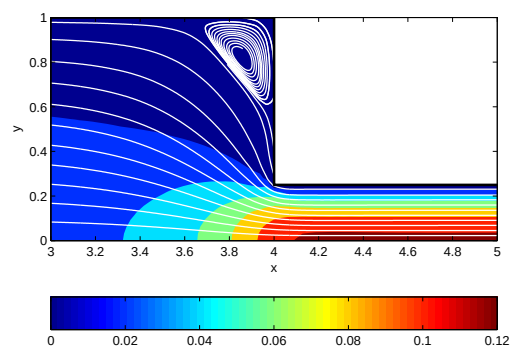


Figure 7.4: Streamlines and magnitude contours for $\mathbf{u}$



(a) $t = 1$



(b) $t = 4$

Figure 7.5: Streamline and magnitude contours for u

A Initial Bounds

In this section we establish error estimates for $\|\mathbf{F}^{(\theta)}\|^2$, $\|\mathbf{F}^{(\tilde{\theta})}\|^2$, $\|\mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})}\|^2$, $\|\mathbf{E}^{(\theta)}\|^2$, and $\|\mathbf{E}^{(\tilde{\theta})}\|^2$ used in the analysis of the θ -method.

$$\text{A.1} \quad \|\mathbf{F}^{(\theta)}\|^2, \|\mathbf{F}^{(\tilde{\theta})}\|^2, \text{ and } \|\mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})}\|^2$$

Lemma A.1

$$\begin{aligned} \|\mathbf{F}^{(\theta)}\|^2 &\leq Ch^{2m+2} + C\Delta t^2 h^{2m+2} + C\delta^2 \Delta t^2 h^{2m} \\ &\quad + C\Delta t^2 h^{2m} + C\delta^2 \Delta t^2 h^{2m-2} + C\Delta t^4 + C\delta^2 \Delta t^2 h^{-2} + C\Delta t^4 \delta^2 h^{-2}. \end{aligned} \quad (\text{A.1})$$

Proof: Subtracting ((4.2) with known $\mathbf{u}$ and $n = 0$) from ((3.2) at $t = 0$):

$$\begin{aligned} \frac{\lambda}{\theta \Delta t} (\mathbf{e}_{\boldsymbol{\sigma}}^{(\theta)}, \boldsymbol{\tau}) + \omega (\mathbf{e}_{\boldsymbol{\sigma}}^{(\theta)}, \boldsymbol{\tau}_{\delta(0)}) &= \frac{\lambda}{\theta \Delta t} (\mathbf{e}_{\boldsymbol{\sigma}}^{(0)}, \boldsymbol{\tau}) - (1 - \omega) (\mathbf{e}_{\boldsymbol{\sigma}}^{(0)}, \boldsymbol{\tau}_{\delta(0)}) - \lambda (\mathbf{u}^{(n)} \cdot \nabla \mathbf{e}_{\boldsymbol{\sigma}}^{(0)}, \boldsymbol{\tau}_{\delta(0)}) \\ &\quad - \lambda (g_a(\mathbf{e}_{\boldsymbol{\sigma}}^{(0)}, \nabla \mathbf{u}^{(0)}), \boldsymbol{\tau}_{\delta(0)}) + \lambda \left(\frac{\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}}{\theta \Delta t} - \boldsymbol{\sigma}_t^{(0)}, \boldsymbol{\tau} \right) \\ &\quad + \omega (\boldsymbol{\sigma}^{(0)} - \boldsymbol{\sigma}^{(\theta)}, \boldsymbol{\tau}_{\delta(0)}) + \lambda (-\boldsymbol{\sigma}_t^{(0)}, \delta \mathbf{u}^{(0)} \cdot \nabla \boldsymbol{\tau}). \end{aligned} \quad (\text{A.2})$$

Recall that $\mathbf{e}_{\boldsymbol{\sigma}}^{(\mu)} = \boldsymbol{\Gamma}^{(\mu)} + \mathbf{F}^{(\mu)}$. Let $\boldsymbol{\tau} = \mathbf{F}^{(\theta)}$ and (A.2) yields

$$\begin{aligned} &\|\mathbf{F}^{(\theta)}\|^2 + (\boldsymbol{\Gamma}^{(\theta)}, \mathbf{F}^{(\theta)}) + \frac{\omega \theta \Delta t}{\lambda} \|\mathbf{F}^{(\theta)}\|^2 + \frac{\omega \theta \Delta t}{\lambda} (\boldsymbol{\Gamma}^{(\theta)}, \mathbf{F}^{(\theta)}) \\ &\quad + \frac{\omega \theta \Delta t}{\lambda} (\mathbf{F}^{(\theta)}, \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}) + \frac{\omega \theta \Delta t}{\lambda} (\boldsymbol{\Gamma}^{(\theta)}, \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}) \\ &= (\mathbf{F}^{(0)}, \mathbf{F}^{(\theta)}) + (\boldsymbol{\Gamma}^{(0)}, \mathbf{F}^{(\theta)}) \\ &\quad - \frac{\tilde{\omega} \theta \Delta t}{\lambda} (\mathbf{F}^{(0)}, \mathbf{F}^{(\theta)}) - \frac{\tilde{\omega} \theta \Delta t}{\lambda} (\mathbf{F}^{(0)}, \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}) \\ &\quad - \frac{\tilde{\omega} \theta \Delta t}{\lambda} (\boldsymbol{\Gamma}^{(0)}, \mathbf{F}^{(\theta)}) - \frac{\tilde{\omega} \theta \Delta t}{\lambda} (\boldsymbol{\Gamma}^{(0)}, \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}) \\ &\quad - \theta \Delta t (\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(0)}, \mathbf{F}^{(\theta)}) - \theta \Delta t (\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(0)}, \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}) \\ &\quad - \theta \Delta t (\mathbf{u}^{(0)} \cdot \nabla \boldsymbol{\Gamma}^{(0)}, \mathbf{F}^{(\theta)}) - \theta \Delta t (\mathbf{u}^{(0)} \cdot \nabla \boldsymbol{\Gamma}^{(0)}, \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}) \\ &\quad - \theta \Delta t (g_a(\mathbf{F}^{(0)}, \nabla \mathbf{u}^{(0)}), \mathbf{F}^{(\theta)}) - \theta \Delta t (g_a(\mathbf{F}^{(0)}, \nabla \mathbf{u}^{(0)}), \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}) \\ &\quad - \theta \Delta t (g_a(\boldsymbol{\Gamma}^{(0)}, \nabla \mathbf{u}^{(0)}), \mathbf{F}^{(\theta)}) - \theta \Delta t (g_a(\boldsymbol{\Gamma}^{(0)}, \nabla \mathbf{u}^{(0)}), \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}) \\ &\quad - \theta \Delta t \omega (\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}, \mathbf{F}^{(\theta)}) - \theta \Delta t \omega (\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}, \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}) \\ &\quad + \theta \Delta t \left(\frac{\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}}{\theta \Delta t} - \boldsymbol{\sigma}_t^{(0)}, \mathbf{F}^{(\theta)} \right) + \theta \Delta t (-\boldsymbol{\sigma}_t^{(0)}, \delta \mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}). \end{aligned} \quad (\text{A.3})$$

Appropriate bounds can now be placed on each of the terms in (A.3). Thus,

$$\left(\mathbf{\Gamma}^{(\theta)}, \mathbf{F}^{(\theta)}\right) \leq \frac{1}{4\epsilon_1} \left\|\mathbf{\Gamma}^{(\theta)}\right\|^2 + \epsilon_1 \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.4})$$

$$\frac{\omega\theta\Delta t}{\lambda} \left(\mathbf{\Gamma}^{(\theta)}, \mathbf{F}^{(\theta)}\right) \leq \frac{\omega^2\theta^2(\Delta t)^2}{4\epsilon_2\lambda^2} \left\|\mathbf{\Gamma}^{(\theta)}\right\|^2 + \epsilon_2 \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.5})$$

$$\frac{\omega\theta\Delta t}{\lambda} \left(\mathbf{F}^{(\theta)}, \delta\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}\right) = 0, \quad (\text{A.6})$$

$$\frac{\omega\theta\Delta t}{\lambda} \left(\mathbf{\Gamma}^{(\theta)}, \delta\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}\right) \leq \frac{\delta^2 M^2 \dot{C} h^{-2} \omega^2 \theta^2 (\Delta t)^2}{4\epsilon_3 \lambda^2} \left\|\mathbf{\Gamma}^{(\theta)}\right\|^2 + \epsilon_3 \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.7})$$

$$\left(\mathbf{F}^{(0)}, \mathbf{F}^{(\theta)}\right) = 0, \quad (\text{A.8})$$

$$\left(\mathbf{\Gamma}^{(0)}, \mathbf{F}^{(\theta)}\right) \leq \frac{1}{4\epsilon_4} \left\|\mathbf{\Gamma}^{(0)}\right\|^2 + \epsilon_4 \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.9})$$

$$\frac{\tilde{\omega}\theta\Delta t}{\lambda} \left(\mathbf{F}^{(0)}, \mathbf{F}^{(\theta)}\right) = 0, \quad (\text{A.10})$$

$$\frac{\tilde{\omega}\theta\Delta t}{\lambda} \left(\mathbf{F}^{(0)}, \delta\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}\right) = 0, \quad (\text{A.11})$$

$$\frac{\tilde{\omega}\theta\Delta t}{\lambda} \left(\mathbf{\Gamma}^{(0)}, \mathbf{F}^{(\theta)}\right) \leq \frac{\tilde{\omega}^2\theta^2(\Delta t)^2}{4\lambda^2\epsilon_5} \left\|\mathbf{\Gamma}^{(0)}\right\|^2 + \epsilon_5 \left\|\mathbf{F}^{(\theta)}\right\|^2 \quad (\text{A.12})$$

$$\frac{\tilde{\omega}\theta\Delta t}{\lambda} \left(\mathbf{\Gamma}^{(0)}, \delta\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}\right) \leq \frac{\delta^2 M^2 \dot{C} \tilde{\omega}^2 \theta^2 (\Delta t)^2 C h^{-2}}{4\lambda^2\epsilon_6} \left\|\mathbf{\Gamma}^{(0)}\right\|^2 + \epsilon_6 \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.13})$$

$$\theta\Delta t \left(\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(0)}, \mathbf{F}^{(\theta)}\right) = 0, \quad (\text{A.14})$$

$$\theta\Delta t \left(\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(0)}, \delta\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}\right) = 0, \quad (\text{A.15})$$

$$\theta\Delta t \left(\mathbf{u}^{(0)} \cdot \nabla \mathbf{\Gamma}^{(0)}, \mathbf{F}^{(\theta)}\right) \leq \frac{M^2 \dot{d} \theta^2 (\Delta t)^2}{4\epsilon_7} \left\|\nabla \mathbf{\Gamma}^{(0)}\right\|^2 + \epsilon_7 \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.16})$$

$$\theta\Delta t \left(\mathbf{u}^{(0)} \cdot \nabla \mathbf{\Gamma}^{(0)}, \delta\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}\right) \leq \frac{\delta^2 \dot{d}^2 M^4 \theta^2 (\Delta t)^2 C h^{-2}}{4\epsilon_8} \left\|\nabla \mathbf{\Gamma}^{(0)}\right\|^2 + \epsilon_8 \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.17})$$

$$\theta\Delta t \left(g_a(\mathbf{F}^{(0)}, \nabla \mathbf{u}^{(0)}), \mathbf{F}^{(\theta)}\right) = 0, \quad (\text{A.18})$$

$$\theta\Delta t \left(g_a(\mathbf{F}^{(0)}, \nabla \mathbf{u}^{(0)}), \delta\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}\right) = 0, \quad (\text{A.19})$$

$$\theta\Delta t \left(g_a(\mathbf{\Gamma}^{(0)}, \nabla \mathbf{u}^{(0)}), \mathbf{F}^{(\theta)}\right) \leq \frac{4M^2 \dot{d}^2 \theta^2 (\Delta t)^2}{\epsilon_9} \left\|\mathbf{\Gamma}^{(0)}\right\|^2 + \epsilon_9 \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.20})$$

$$\theta\Delta t \left(g_a(\mathbf{\Gamma}^{(0)}, \nabla \mathbf{u}^{(0)}), \delta\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}\right) \leq \frac{\delta^2 4M^2 \dot{d}^3 \theta^2 (\Delta t)^2 C h^{-2}}{\epsilon_{10}} \left\|\mathbf{\Gamma}^{(0)}\right\|^2 + \epsilon_{10} \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.21})$$

$$\theta\Delta t \left(\frac{\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}}{\theta\Delta t} - \boldsymbol{\sigma}_t^{(0)}, \mathbf{F}^{(\theta)}\right) \leq \frac{\theta^2 (\Delta t)^2}{4\epsilon_{11}} \left\|\frac{\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}}{\theta\Delta t} - \boldsymbol{\sigma}_t^{(0)}\right\|^2 + \epsilon_{11} \left\|\mathbf{F}^{(\theta)}\right\|^2, \quad (\text{A.22})$$

$$\theta\Delta t \left(-\boldsymbol{\sigma}_t^{(0)}, \delta\mathbf{u}^{(0)} \cdot \nabla \mathbf{F}^{(\theta)}\right) \leq \frac{\delta^2 M^2 \dot{d} \theta^2 (\Delta t)^2 C h^{-2}}{4\epsilon_{12}} \left\|\boldsymbol{\sigma}_t^{(0)}\right\|^2 + \epsilon_{12} \left\|\mathbf{F}^{(\theta)}\right\|^2. \quad (\text{A.23})$$

$$\theta \Delta t \omega \left(\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}, \mathbf{F}^{(\theta)} \right) \leq \frac{\theta^2 (\Delta t)^2 \omega^2}{4\epsilon_{13}} \left\| \mathbf{u}^{(\theta)} - \mathbf{u}^{(0)} \right\|^2 + \epsilon_{13} \left\| \mathbf{F}^{(\theta)} \right\|^2 \quad (\text{A.24})$$

$$\theta \Delta t \omega \left(\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}, \delta \mathbf{u}^{(0)} \nabla \mathbf{F}^{(\theta)} \right) \leq \frac{\delta^2 \theta^2 (\Delta t)^2 \omega^2 d M^2 C h^{-2}}{4\epsilon_{14}} \left\| \boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)} \right\|^2 + \epsilon_{14} \left\| \mathbf{F}^{(\theta)} \right\|^2 \quad (\text{A.25})$$

Using (A.4) - (A.25) in (A.3) we obtain:

$$\begin{aligned} & \left\| \mathbf{F}^{(\theta)} \right\|^2 \left(1 + \frac{\omega \theta \Delta t}{\lambda} - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 - \epsilon_8 - \epsilon_9 - \epsilon_{10} - \epsilon_{11} - \epsilon_{12} - \epsilon_{13} - \epsilon_{14} \right) \\ & \leq \left\| \boldsymbol{\Gamma}^{(\theta)} \right\|^2 \left(\frac{1}{4\epsilon_1} + \frac{\omega^2 \theta^2 \Delta t^2}{4\lambda^2 \epsilon_2} + \frac{\delta^2 M^2 d C h^{-2} \omega^2 \theta^2 \Delta t^2}{4\epsilon_3 \lambda^2} \right) \\ & \quad + \left\| \boldsymbol{\Gamma}^{(0)} \right\|^2 \left(\frac{1}{4\epsilon_4} + \frac{\tilde{\omega}^2 \theta^2 \Delta t^2}{4\epsilon_5 \lambda^2} + \frac{\delta^2 M^2 d C h^{-2} \tilde{\omega}^2 \theta^2 \Delta t^2}{4\epsilon_6 \lambda^2} + \frac{4M^2 d^2 \theta^2 \Delta t^2}{\epsilon_9} + \frac{4M^2 d^3 C h^{-2} \delta^2 \theta^2 \Delta t^2}{\epsilon_{10}} \right) \\ & \quad + \left\| \nabla \boldsymbol{\Gamma}^{(0)} \right\|^2 \left(\frac{M^2 d \theta^2 \Delta t^2}{4\epsilon_7} + \frac{d^2 M^4 \delta^2 C h^{-2} \theta^2 \Delta t^2}{4\epsilon_8} \right) \\ & \quad + \left\| \boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)} \right\|^2 \left(\frac{\theta^2 (\Delta t)^2 \omega^2}{4\epsilon_{13}} + \frac{\delta^2 \theta^2 \Delta t^2 \omega^2 d M^2 C h^{-2}}{4\epsilon_{14}} \right) \\ & \quad + \frac{\theta^2 \Delta t^2}{4\epsilon_{11}} \left\| \frac{\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}}{\theta \Delta t} - \boldsymbol{\sigma}_t^{(0)} \right\|^2 + \frac{\delta^2 M^2 d \theta^2 \Delta t^2 C h^{-2}}{4\epsilon_{12}} \left\| \boldsymbol{\sigma}_t^{(0)} \right\|^2. \end{aligned} \quad (\text{A.26})$$

Choose $\epsilon_i = 1/24$, for $i \in \{1, \dots, 12\}$. Then using interpolation properties and lemmas C.17 and C.20 with (A.26) gives:

$$\begin{aligned} \left\| \mathbf{F}^{(\theta)} \right\|^2 & \leq C h^{2m+2} \left\| \boldsymbol{\sigma}^{(\theta)} \right\|_{m+1}^2 + C \Delta t^2 h^{2m+2} \left\| \boldsymbol{\sigma}^{(\theta)} \right\|_{m+1}^2 + C \delta^2 \Delta t^2 h^{2m} \left\| \boldsymbol{\sigma}^{(\theta)} \right\|_{m+1}^2 \\ & \quad + C h^{2m+2} \left\| \boldsymbol{\sigma}^{(0)} \right\|_{m+1}^2 + C \Delta t^2 h^{2m+2} \left\| \boldsymbol{\sigma}^{(0)} \right\|_{m+1}^2 + C \delta^2 \Delta t^2 h^{2m} \left\| \boldsymbol{\sigma}^{(0)} \right\|_{m+1}^2 \\ & \quad + C \Delta t^2 h^{2m} \left\| \boldsymbol{\sigma}^{(\theta)} \right\|_{m+1}^2 + C \Delta t^2 \delta^2 h^{2m-2} \left\| \boldsymbol{\sigma}^{(\theta)} \right\|_{m+1}^2 + C \Delta t^4 + C \delta^2 \Delta t^2 h^{-2} \left\| \boldsymbol{\sigma}_t^{(0)} \right\|^2 \\ & \quad + C \Delta t^4 \delta^2 h^{-2}, \end{aligned} \quad (\text{A.27})$$

$$\begin{aligned} \implies \left\| \mathbf{F}^{(\theta)} \right\|^2 & \leq C h^{2m+2} + C \Delta t^2 h^{2m+2} + C \delta^2 \Delta t^2 h^{2m} \\ & \quad + C \Delta t^2 h^{2m} + C \delta^2 \Delta t^2 h^{2m-2} + C \Delta t^4 \\ & \quad + C \delta^2 \Delta t^2 h^{-2} + C \Delta t^4 \delta^2 h^{-2}. \end{aligned} \quad (\text{A.28})$$

■

Lemma A.2 For sufficiently smooth solutions $\mathbf{u}$, $\boldsymbol{\sigma}$, and p ,

$$\left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2 \leq S(h, \Delta t, \delta), \quad \text{and} \quad \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2 \leq S(h, \Delta t, \delta), \quad (\text{A.29})$$

where

$$\begin{aligned}
S(h, \Delta t, \delta) &= C[h^{2m+2} + \Delta t^2 h^{2m+2} + \delta^2 \Delta t^2 h^{2m} + \Delta t^2 h^{2m} \\
&\quad + \delta^2 \Delta t^2 h^{2m-2} + \Delta t^4 + \delta^2 \Delta t^2 h^{-2} + C \Delta t^4 \delta^2 h^{-2}] (1 + \Delta t^2 + \Delta t \delta) \\
&\quad + C(h^{2m+2} + \Delta t^2 h^{2m+2} + \Delta t \delta h^{2m+2}) \\
&\quad + C(\Delta t^2 h^{2m} + \Delta t \delta h^{2m}) + C(\Delta t)^4 + C\delta^2 \Delta t^2.
\end{aligned}$$

Proof: Subtracting ((4.5) with known $\mathbf{u}$ and $n = 0$) from ((3.2) at $t = \tilde{\theta}$):

$$\begin{aligned}
&\frac{\lambda}{(1-2\theta)\Delta t} \left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\tilde{\theta})}, \boldsymbol{\tau} \right) + \tilde{\omega} \left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\tilde{\theta})}, \boldsymbol{\tau}_{\delta(\tilde{\theta})} \right) + \lambda \left(\mathbf{u}^{\tilde{\theta}} \cdot \nabla \mathbf{e}_{\boldsymbol{\sigma}}^{(\tilde{\theta})}, \boldsymbol{\tau}_{\delta(\tilde{\theta})} \right) + \lambda \left(g_a(\mathbf{e}_{\boldsymbol{\sigma}}^{(\tilde{\theta})}, \nabla \mathbf{u}^{(\tilde{\theta})}), \boldsymbol{\tau}_{\delta(\tilde{\theta})} \right) \\
&= \frac{\lambda}{(1-2\theta)\Delta t} \left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\theta)}, \boldsymbol{\tau} \right) - \omega \left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\theta)}, \boldsymbol{\tau}_{\delta(\tilde{\theta})} \right) + \lambda \left(\frac{\boldsymbol{\sigma}^{(\tilde{\theta})} - \boldsymbol{\sigma}^{(\theta)}}{(1-2\theta)\Delta t} - \boldsymbol{\sigma}_t^{(\tilde{\theta})}, \boldsymbol{\tau} \right) \\
&\quad + \lambda \left(-\boldsymbol{\sigma}_t^{(\tilde{\theta})}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \boldsymbol{\tau} \right) - \omega \left(\boldsymbol{\sigma}^{\tilde{\theta}} - \boldsymbol{\sigma}^{\theta}, \boldsymbol{\tau}_{\delta(\tilde{\theta})} \right). \quad (\text{A.30})
\end{aligned}$$

Observing that $\mathbf{e}_{\boldsymbol{\sigma}}^{(\mu)} = \mathbf{F}^{(\mu)} + \boldsymbol{\Gamma}^{(\mu)}$ and choose $\boldsymbol{\tau} = \mathbf{F}^{(\tilde{\theta})}$. Then from (A.30) we have

$$\begin{aligned}
&\left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2 + \left(\boldsymbol{\Gamma}^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) \\
&\quad + \frac{\tilde{\omega}(1-2\theta)\Delta t}{\lambda} \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2 + \frac{\tilde{\omega}(1-2\theta)\Delta t}{\lambda} \left(\mathbf{F}^{(\tilde{\theta})}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) \\
&\quad + \frac{\tilde{\omega}(1-2\theta)\Delta t}{\lambda} \left(\boldsymbol{\Gamma}^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) + \frac{\tilde{\omega}(1-2\theta)\Delta t}{\lambda} \left(\boldsymbol{\Gamma}^{(\tilde{\theta})}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) \\
&\quad + (1-2\theta)\Delta t \left(\mathbf{u}^{\tilde{\theta}} \cdot \nabla \mathbf{F}^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) + \delta(1-2\theta)\Delta t \left\| \mathbf{u}^{\tilde{\theta}} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2 \\
&\quad + (1-2\theta)\Delta t \left(\mathbf{u}^{\tilde{\theta}} \cdot \nabla \boldsymbol{\Gamma}^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) + (1-2\theta)\Delta t \left(\mathbf{u}^{\tilde{\theta}} \cdot \nabla \boldsymbol{\Gamma}^{(\tilde{\theta})}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) \\
&\quad + (1-2\theta)\Delta t \left(g_a(\mathbf{F}^{(\tilde{\theta})}, \nabla \mathbf{u}^{(\tilde{\theta})}), \mathbf{F}^{(\tilde{\theta})} \right) + (1-2\theta)\Delta t \left(g_a(\mathbf{F}^{(\tilde{\theta})}, \nabla \mathbf{u}^{(\tilde{\theta})}), \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) \\
&\quad + (1-2\theta)\Delta t \left(g_a(\boldsymbol{\Gamma}^{(\tilde{\theta})}, \nabla \mathbf{u}^{(\tilde{\theta})}), \mathbf{F}^{(\tilde{\theta})} \right) + (1-2\theta)\Delta t \left(g_a(\boldsymbol{\Gamma}^{(\tilde{\theta})}, \nabla \mathbf{u}^{(\tilde{\theta})}), \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) \\
&= \left(\mathbf{F}^{(\theta)}, \mathbf{F}^{(\tilde{\theta})} \right) + \left(\boldsymbol{\Gamma}^{(\theta)}, \mathbf{F}^{(\tilde{\theta})} \right) \\
&\quad - \frac{\omega(1-2\theta)\Delta t}{\lambda} \left(\mathbf{F}^{(\theta)}, \mathbf{F}^{(\tilde{\theta})} \right) - \frac{\omega(1-2\theta)\Delta t}{\lambda} \left(\mathbf{F}^{(\theta)}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) \\
&\quad - \frac{\omega(1-2\theta)\Delta t}{\lambda} \left(\boldsymbol{\Gamma}^{(\theta)}, \mathbf{F}^{(\tilde{\theta})} \right) - \frac{\omega(1-2\theta)\Delta t}{\lambda} \left(\boldsymbol{\Gamma}^{(\theta)}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) \\
&\quad + (1-2\theta)\Delta t \left(\frac{\boldsymbol{\sigma}^{(\tilde{\theta})} - \boldsymbol{\sigma}^{(\theta)}}{(1-2\theta)\Delta t} - \boldsymbol{\sigma}_t^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) + (1-2\theta)\Delta t \left(-\boldsymbol{\sigma}_t^{(\tilde{\theta})}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) \\
&\quad - \omega \left(\boldsymbol{\sigma}^{\tilde{\theta}} - \boldsymbol{\sigma}^{\theta}, \mathbf{F}^{(\tilde{\theta})} + \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right). \quad (\text{A.31})
\end{aligned}$$

Bounding all the terms in (A.31),

$$\left(\boldsymbol{\Gamma}^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) \leq \frac{1}{4\epsilon_1} \left\| \boldsymbol{\Gamma}^{(\tilde{\theta})} \right\|^2 + \epsilon_1 \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.32})$$

$$\frac{\tilde{\omega}(1-2\theta)\Delta t}{\lambda} \left(\mathbf{F}^{(\tilde{\theta})}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) = 0, \quad (\text{A.33})$$

$$\frac{\tilde{\omega}(1-2\theta)\Delta t}{\lambda} \left(\mathbf{\Gamma}^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) \leq \frac{\tilde{\omega}^2(1-2\theta)^2\Delta t^2}{\lambda^2 4\epsilon_2} \left\| \mathbf{\Gamma}^{(\tilde{\theta})} \right\|^2 + \epsilon_2 \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.34})$$

$$\begin{aligned} \frac{\tilde{\omega}(1-2\theta)\Delta t}{\lambda} \left(\mathbf{\Gamma}^{(\tilde{\theta})}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) &\leq \frac{\delta \tilde{\omega}^2(1-2\theta)\Delta t}{\lambda^2 4\epsilon_3} \left\| \mathbf{\Gamma}^{(\tilde{\theta})} \right\|^2 \\ &\quad + \epsilon_3 \delta(1-2\theta)\Delta t \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2, \end{aligned} \quad (\text{A.35})$$

$$+ (1-2\theta)\Delta t \left(\mathbf{u}^{\tilde{\theta}} \cdot \nabla \mathbf{F}^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) = 0, \quad (\text{A.36})$$

$$(1-2\theta)\Delta t \left(\mathbf{u}^{\tilde{\theta}} \cdot \nabla \mathbf{\Gamma}^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) \leq \frac{(1-2\theta)^2\Delta t^2 M^2 \acute{d}}{4\epsilon_4} \left\| \nabla \mathbf{\Gamma}^{(\tilde{\theta})} \right\|^2 + \epsilon_4 \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.37})$$

$$\begin{aligned} (1-2\theta)\Delta t \left(\mathbf{u}^{\tilde{\theta}} \cdot \nabla \mathbf{\Gamma}^{(\tilde{\theta})}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) &\leq \frac{\delta(1-2\theta)\Delta t M^2 \acute{d}}{4\epsilon_5} \left\| \nabla \mathbf{\Gamma}^{(\tilde{\theta})} \right\|^2 \\ &\quad + \epsilon_5 \delta(1-2\theta)\Delta t \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2, \end{aligned} \quad (\text{A.38})$$

$$(1-2\theta)\Delta t \left(g_a(\mathbf{F}^{(\tilde{\theta})}, \nabla \mathbf{u}^{(\tilde{\theta})}), \mathbf{F}^{(\tilde{\theta})} \right) \leq 4(1-2\theta)\Delta t M \acute{d} \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.39})$$

$$\begin{aligned} (1-2\theta)\Delta t \left(g_a(\mathbf{F}^{(\tilde{\theta})}, \nabla \mathbf{u}^{(\tilde{\theta})}), \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) \\ \leq \epsilon_6 \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2 + \frac{4\delta^2(1-2\theta)^2\Delta t^2 M^2 \acute{d}^2}{\epsilon_6} \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2, \end{aligned} \quad (\text{A.40})$$

$$(1-2\theta)\Delta t \left(g_a(\mathbf{\Gamma}^{(\tilde{\theta})}, \nabla \mathbf{u}^{(\tilde{\theta})}), \mathbf{F}^{(\tilde{\theta})} \right) \leq \frac{4(1-2\theta)^2\Delta t^2 M^2 \acute{d}^2}{\epsilon_7} \left\| \mathbf{\Gamma}^{(\tilde{\theta})} \right\|^2 + \epsilon_7 \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.41})$$

$$\begin{aligned} (1-2\theta)\Delta t \left(g_a(\mathbf{\Gamma}^{(\tilde{\theta})}, \nabla \mathbf{u}^{(\tilde{\theta})}), \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) &\leq \frac{4\delta(1-2\theta)\Delta t M^2 \acute{d}^2}{\epsilon_8} \left\| \mathbf{\Gamma}^{(\tilde{\theta})} \right\|^2 \\ &\quad + \delta(1-2\theta)\Delta t \epsilon_8 \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2, \end{aligned} \quad (\text{A.42})$$

$$\left(\mathbf{F}^{(\theta)}, \mathbf{F}^{(\tilde{\theta})} \right) \leq \frac{1}{4\epsilon_9} \left\| \mathbf{F}^{(\theta)} \right\|^2 + \epsilon_9 \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.43})$$

$$\left(\mathbf{\Gamma}^{(\theta)}, \mathbf{F}^{(\tilde{\theta})} \right) \leq \frac{1}{4\epsilon_{10}} \left\| \mathbf{\Gamma}^{(\theta)} \right\|^2 + \epsilon_{10} \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.44})$$

$$\frac{\omega(1-2\theta)\Delta t}{\lambda} \left(\mathbf{F}^{(\theta)}, \mathbf{F}^{(\tilde{\theta})} \right) \leq \frac{\omega^2(1-2\theta)^2\Delta t^2}{4\lambda^2\epsilon_{11}} \left\| \mathbf{F}^{(\theta)} \right\|^2 + \epsilon_{11} \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.45})$$

$$\begin{aligned} \frac{\omega(1-2\theta)\Delta t}{\lambda} \left(\mathbf{F}^{(\theta)}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) &\leq \frac{\delta \omega^2(1-2\theta)\Delta t}{\lambda^2 4\epsilon_{12}} \left\| \mathbf{F}^{(\theta)} \right\|^2 \\ &\quad + \delta(1-2\theta)\Delta t \epsilon_{12} \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2, \end{aligned} \quad (\text{A.46})$$

$$\frac{\omega(1-2\theta)\Delta t}{\lambda} \left(\mathbf{\Gamma}^{(\theta)}, \mathbf{F}^{(\tilde{\theta})} \right) \leq \frac{\omega^2(1-2\theta)^2\Delta t^2}{\lambda^2 4\epsilon_{13}} \left\| \mathbf{\Gamma}^{(\theta)} \right\|^2 + \epsilon_{13} \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.47})$$

$$\begin{aligned} \frac{\omega(1-2\theta)\Delta t}{\lambda} \left(\mathbf{\Gamma}^{(\theta)}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) &\leq \frac{\delta \omega^2(1-2\theta)\Delta t}{\lambda^2 4\epsilon_{14}} \left\| \mathbf{\Gamma}^{(\theta)} \right\|^2 \\ &\quad + \epsilon_{14} \delta(1-2\theta)\Delta t \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2, \end{aligned} \quad (\text{A.48})$$

$$\begin{aligned} (1-2\theta)\Delta t \left(\frac{\boldsymbol{\sigma}^{(\tilde{\theta})} - \boldsymbol{\sigma}^{(\theta)}}{(1-2\theta)\Delta t} - \boldsymbol{\sigma}_t^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) &\leq \frac{(1-2\theta)^2\Delta t^2}{4\epsilon_{15}} \left\| \frac{\boldsymbol{\sigma}^{(\tilde{\theta})} - \boldsymbol{\sigma}^{(\theta)}}{(1-2\theta)\Delta t} - \boldsymbol{\sigma}_t^{(\tilde{\theta})} \right\|^2 \\ &\quad + \epsilon_{15} \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2. \end{aligned} \quad (\text{A.49})$$

Using $\mathbf{u}|_{\partial\Omega} = 0$, and $\nabla \cdot \mathbf{u} = 0$,

$$\begin{aligned} (1-2\theta)\Delta t \left(\boldsymbol{\sigma}_t^{(\tilde{\theta})}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) &= -(1-2\theta)\delta\Delta t \left(\nabla \cdot \mathbf{u}^{(\tilde{\theta})}, \boldsymbol{\sigma}_t^{(\tilde{\theta})} : \mathbf{F}^{(\tilde{\theta})} \right) \\ &\quad - (1-2\theta)\delta\Delta t \left(\mathbf{u}^{(\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_t^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) \\ &= -(1-2\theta)\delta\Delta t \left(\mathbf{u}^{(\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_t^{(\tilde{\theta})}, \mathbf{F}^{(\tilde{\theta})} \right) \\ &\leq \frac{(1-2\theta)^2\delta^2\Delta t^2}{4\epsilon_{16}} \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_t^{(\tilde{\theta})} \right\|^2 + \epsilon_{16} \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2. \end{aligned} \quad (\text{A.50})$$

Lastly

$$\omega(1-2\theta)\Delta t \left(\boldsymbol{\sigma}^{\tilde{\theta}} - \boldsymbol{\sigma}^{\theta}, \mathbf{F}^{(\tilde{\theta})} \right) \leq \frac{\omega^2(1-2\theta)^2\Delta t^2}{4\epsilon_{17}} \left\| \boldsymbol{\sigma}^{\tilde{\theta}} - \boldsymbol{\sigma}^{\theta} \right\|^2 + \epsilon_{17} \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2, \quad (\text{A.51})$$

and

$$\begin{aligned} \omega(1-2\theta)\Delta t \left(\boldsymbol{\sigma}^{\tilde{\theta}} - \boldsymbol{\sigma}^{\theta}, \delta \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right) &\leq \frac{\delta^2\omega^2(1-2\theta)^2\Delta t^2 dM^2 C h^{-2}}{4\epsilon_{18}} \left\| \boldsymbol{\sigma}^{\tilde{\theta}} - \boldsymbol{\sigma}^{\theta} \right\|^2 \\ &\quad + \epsilon_{18} \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2. \end{aligned} \quad (\text{A.52})$$

Using the bounds stated in (A.32) - (A.52) with (A.31) yields

$$\begin{aligned}
& \left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2 \left(1 + \frac{\tilde{\omega} (1 - 2\theta) \Delta t}{\lambda} - \epsilon_1 - \epsilon_2 - \epsilon_4 - \epsilon_6 - \epsilon_7 - \epsilon_9 \right. \\
& \quad \left. - \epsilon_{10} - \epsilon_{11} - \epsilon_{13} - \epsilon_{15} - \epsilon_{16} - \epsilon_{ps17} - \epsilon_{ps18} - (1 - 2\theta) \Delta t 4M \dot{d} \right) \\
& + \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2 \delta (1 - 2\theta) \Delta t \left(1 - \epsilon_3 - \epsilon_5 - \epsilon_8 - \epsilon_{12} - \epsilon_{14} \right. \\
& \quad \left. - \frac{4\delta (1 - 2\theta) \Delta t M^2 \dot{d}^2}{\epsilon_6} \right) \\
\leq & \left\| \mathbf{F}^{(\theta)} \right\|^2 \left(\frac{1}{4\epsilon_9} + \frac{\omega^2 (1 - 2\theta)^2 \Delta t^2}{4\lambda^2 \epsilon_{11}} + \frac{\delta \omega^2 (1 - 2\theta) \Delta t}{\lambda^2 4\epsilon_{12}} \right) \\
& + \left\| \mathbf{\Gamma}^{(\theta)} \right\|^2 \left(\frac{1}{4\epsilon_{10}} + \frac{\omega^2 (1 - 2\theta)^2 \Delta t^2}{\lambda^2 4\epsilon_{13}} + \frac{\delta \omega^2 (1 - 2\theta) \Delta t}{\lambda^2 4\epsilon_{14}} \right) \\
& + \left\| \mathbf{\Gamma}^{(\tilde{\theta})} \right\|^2 \left(\frac{1}{4\epsilon_1} + \frac{\tilde{\omega}^2 (1 - 2\theta)^2 \Delta t^2}{\lambda^2 4\epsilon_2} + \frac{\delta \tilde{\omega}^2 (1 - 2\theta) \Delta t}{\lambda^2 4\epsilon_3} \right. \\
& \quad \left. + \frac{4 (1 - 2\theta)^2 \Delta t^2 M^2 \dot{d}^2}{\epsilon_7} + \frac{4\delta (1 - 2\theta) \Delta t M^2 \dot{d}^2}{\epsilon_8} \right) \\
& + \left\| \nabla \mathbf{\Gamma}^{(\tilde{\theta})} \right\|^2 \left(\frac{(1 - 2\theta)^2 \Delta t^2 M^2 \dot{d}}{4\epsilon_4} + \frac{\delta (1 - 2\theta) \Delta t M^2 \dot{d}}{4\epsilon_5} \right) \\
& + \left\| \boldsymbol{\sigma}^{\tilde{\theta}} - \boldsymbol{\sigma}^{\theta} \right\|^2 \left(\frac{\omega^2 (1 - 2\theta)^2 \Delta t^2}{4\epsilon_{17}} + \frac{\delta^2 \omega^2 (1 - 2\theta)^2 \Delta t^2 \dot{d} M^2 C h^{-2}}{4\epsilon_{18}} \right) \\
& + \frac{(1 - 2\theta)^2 \Delta t^2}{4\epsilon_{15}} \left\| \frac{\boldsymbol{\sigma}^{(\tilde{\theta})} - \boldsymbol{\sigma}^{(\theta)}}{(1 - 2\theta) \Delta t} - \boldsymbol{\sigma}_t^{(\tilde{\theta})} \right\|^2 + \frac{(1 - 2\theta)^2 \delta^2 \Delta t^2}{4\epsilon_{16}} \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_t^{(\tilde{\theta})} \right\|^2. \quad (\text{A.53})
\end{aligned}$$

Note that choosing $\epsilon_i (i \in \{1, \dots, 18\})$ and Δt sufficiently small the L.H.S. of (A.53) will remain positive. Thus, using interpolation properties, and lemmas C.19, A.1, and C.20 we obtain

$$\left\| \mathbf{F}^{(\tilde{\theta})} \right\|^2 \leq S(h, \Delta t, \delta) \quad \text{and} \quad \left\| \mathbf{u}^{(\tilde{\theta})} \cdot \nabla \mathbf{F}^{(\tilde{\theta})} \right\|^2 \leq S(h, \Delta t, \delta),$$

where

$$\begin{aligned}
S(h, \Delta t, \delta) = & C [h^{2m+2} + \Delta t^2 h^{2m+2} + \delta^2 \Delta t^2 h^{2m} + \Delta t^2 h^{2m} \\
& + \delta^2 \Delta t^2 h^{2m-2} + \Delta t^4 + \delta^2 \Delta t^2 h^{-2} + \Delta t^4 \delta^2 h^{-2}] (1 + \Delta t^2 + \Delta t \delta) \\
& + C (h^{2m+2} + \Delta t^2 h^{2m+2} + \Delta t \delta h^{2m+2}) \\
& + C (\Delta t^2 h^{2m} + \Delta t \delta h^{2m}) + C (\Delta t)^4 + C \delta^2 \Delta t^2 + C \delta^2 \Delta t^4 h^{-2}. \quad (\text{A.54})
\end{aligned}$$

A.2 $\left\| \mathbf{E}^{(\theta)} \right\|^2$ and $\left\| \mathbf{E}^{(\tilde{\theta})} \right\|^2$

Lemma 5

$$\left\| \mathbf{E}^{(\theta)} \right\|^2 + \frac{\theta 2(1-\alpha)\Delta t}{Re} \left\| \mathbf{d}(\mathbf{E}^{(\theta)}) \right\|^2 \leq Ch^{2k+2} + C(\Delta t)^4 + C\Delta th^{2k}. \quad (\text{A.55})$$

Proof: Subtracting ((4.3) with known $\boldsymbol{\sigma}$ and $n = 0$) from ((3.3) at $t = \theta\Delta t$) yields:

$$\frac{Re}{\theta\Delta t} \left(\mathbf{e}_{\mathbf{u}}^{(\theta)}, \mathbf{v} \right) + 2(1-\alpha) \left(\mathbf{d}(\mathbf{e}_{\mathbf{u}}^{(\theta)}), \mathbf{d}(\mathbf{v}) \right) = \frac{Re}{\theta\Delta t} \left(\mathbf{e}_{\mathbf{u}}^{(0)}, \mathbf{v} \right) + Re \left(\frac{\mathbf{u}^{(\theta)} - \mathbf{u}^{(0)}}{\theta\Delta t} - \mathbf{u}_t^{(\theta)}, \mathbf{v} \right), \quad \forall \mathbf{v} \in Z_h. \quad (\text{A.56})$$

Recall that $\mathbf{e}_{\mathbf{u}}^{(\theta)} = \boldsymbol{\Lambda}^{(\theta)} + \mathbf{E}^{(\theta)}$. Let $\mathbf{v} = \mathbf{E}^{(\theta)}$ and from (A.56) we obtain

$$\begin{aligned} & \left\| \mathbf{E}^{(\theta)} \right\|^2 + \frac{2(1-\alpha)\theta\Delta t}{Re} \left\| \mathbf{d}(\mathbf{E}^{(\theta)}) \right\|^2 \\ &= \left(\mathbf{u}^{(0)} - \mathbf{u}_h^{(0)}, \mathbf{E}^{(\theta)} \right) + \theta\Delta t \left(\frac{\mathbf{u}^{(\theta)} - \mathbf{u}^{(0)}}{\theta\Delta t} - \mathbf{u}_t^{(\theta)}, \mathbf{E}^{(\theta)} \right) \\ & \quad - \left(\boldsymbol{\Lambda}^{(\theta)}, \mathbf{E}^{(\theta)} \right) - \frac{2(1-\alpha)\theta\Delta t}{Re} \left(\mathbf{d}(\boldsymbol{\Lambda}^{(\theta)}), \mathbf{d}(\mathbf{E}^{(\theta)}) \right) \\ &\leq \frac{3}{2} \left\| \mathbf{u}^{(0)} - \mathbf{u}_h^{(0)} \right\|^2 + \frac{1}{6} \left\| \mathbf{E}^{(\theta)} \right\|^2 + \frac{3\theta^2\Delta t^2}{2} \left\| \frac{\mathbf{u}^{(\theta)} - \mathbf{u}^{(0)}}{\theta\Delta t} - \mathbf{u}_t^{(\theta)} \right\|^2 + \frac{1}{6} \left\| \mathbf{E}^{(\theta)} \right\|^2 \\ & \quad + \frac{3}{2} \left\| \boldsymbol{\Lambda}^{(\theta)} \right\|^2 + \frac{1}{6} \left\| \mathbf{E}^{(\theta)} \right\|^2 + \frac{(1-\alpha)\theta\Delta t}{Re} \left\| \mathbf{d}(\boldsymbol{\Lambda}^{(\theta)}) \right\|^2 + \frac{(1-\alpha)\theta\Delta t}{Re} \left\| \mathbf{d}(\mathbf{E}^{(\theta)}) \right\|^2. \quad (\text{A.57}) \end{aligned}$$

Simplifying expression (A.57) we obtain

$$\begin{aligned} & \left\| \mathbf{E}^{(\theta)} \right\|^2 + \frac{2(1-\alpha)\theta\Delta t}{Re} \left\| \mathbf{d}(\mathbf{E}^{(\theta)}) \right\|^2 \\ &\leq 3 \left\| \mathbf{u}^{(0)} - \mathbf{u}_h^{(0)} \right\|^2 + 3\theta^2\Delta t^2 \left\| \frac{\mathbf{u}^{(\theta)} - \mathbf{u}^{(0)}}{\theta\Delta t} - \mathbf{u}_t^{(\theta)} \right\|^2 \\ & \quad + 3 \left\| \boldsymbol{\Lambda}^{(\theta)} \right\|^2 + \frac{2(1-\alpha)\theta\Delta t}{Re} \left\| \mathbf{d}(\boldsymbol{\Lambda}^{(\theta)}) \right\|^2. \quad (\text{A.58}) \end{aligned}$$

Make note that

$$3 \left\| \mathbf{u}^{(0)} - \mathbf{u}_h^{(0)} \right\|^2 = \left\| \boldsymbol{\Lambda}^{(0)} \right\|^2 \leq Ch^{2k+2} \left\| \mathbf{u}(\cdot, 0) \right\|_{k+1} \leq Ch^{2k+2}. \quad (\text{A.59})$$

Applying lemma C.18 and interpolation properties completes the proof. ■

Lemma A.3

$$\left\| \mathbf{E}^{(\tilde{\theta})} \right\|^2 \leq Ch^{2k+2} + (1 + h^{-2}\Delta t) \left[Ch^{2k+2} + C(\Delta t)^4 + C\Delta th^{2k} \right] + C(\Delta t)^2 h^{2k-2}. \quad (\text{A.60})$$

Proof: Subtracting ((4.4) with $n = 0$) from ((3.3) at $t = \theta\Delta t$):

$$\begin{aligned} \frac{Re}{(1-2\theta)\Delta t} \left(\mathbf{e}_{\mathbf{u}}^{(\tilde{\theta})}, \mathbf{v} \right) &= \frac{Re}{(1-2\theta)\Delta t} \left(\mathbf{e}_{\mathbf{u}}^{(\theta)}, \mathbf{v} \right) \\ &\quad - 2(1-\alpha) \left(\mathbf{d}(\mathbf{e}_{\mathbf{u}}^{(\theta)}), \mathbf{d}(\mathbf{v}) \right) + Re \left(\frac{\mathbf{u}^{(\tilde{\theta})} - \mathbf{u}^{(\theta)}}{(1-2\theta)\Delta t} - \mathbf{u}_t^{(\theta)}, \mathbf{v} \right) \forall \mathbf{v} \in Z_h. \end{aligned} \quad (\text{A.61})$$

Let $\mathbf{v} = \mathbf{E}^{(\tilde{\theta})}$ then (A.61) gives

$$\begin{aligned} \left\| \mathbf{E}^{(\tilde{\theta})} \right\|^2 &= \left(\mathbf{E}^{(\theta)}, \mathbf{E}^{(\tilde{\theta})} \right) + \left(\mathbf{\Lambda}^{(\theta)}, \mathbf{E}^{(\tilde{\theta})} \right) - \left(\mathbf{\Lambda}^{(\tilde{\theta})}, \mathbf{E}^{(\tilde{\theta})} \right) + (1-2\theta)\Delta t \left(\frac{\mathbf{u}^{(\tilde{\theta})} - \mathbf{u}^{(\theta)}}{(1-2\theta)\Delta t} - \mathbf{u}_t^{(\theta)}, \mathbf{E}^{(\tilde{\theta})} \right) \\ &\quad - \frac{2(1-\alpha)(1-2\theta)\Delta t}{Re} \left(\mathbf{d}(\mathbf{E}^{(\theta)}), \mathbf{d}(\mathbf{E}^{(\tilde{\theta})}) \right) - \frac{2(1-\alpha)(1-2\theta)\Delta t}{Re} \left(\mathbf{d}(\mathbf{\Lambda}^{(\theta)}), \mathbf{d}(\mathbf{E}^{(\tilde{\theta})}) \right) \\ &\leq \frac{4}{12} \left\| \mathbf{E}^{(\tilde{\theta})} \right\|^2 + 3 \left\| \mathbf{E}^{(\theta)} \right\|^2 + 3 \left\| \mathbf{\Lambda}^{(\theta)} \right\|^2 + 3 \left\| \mathbf{\Lambda}^{(\tilde{\theta})} \right\|^2 + 3(1-2\theta)\Delta t \left\| \frac{\mathbf{u}^{(\tilde{\theta})} - \mathbf{u}^{(\theta)}}{(1-2\theta)\Delta t} - \mathbf{u}_t^{(\theta)} \right\|^2 \\ &\quad + \frac{2(1-\alpha)(1-2\theta)\Delta t}{Re} \left\| \mathbf{d}(\mathbf{E}^{(\theta)}) \right\| \left\| \mathbf{d}(\mathbf{E}^{(\tilde{\theta})}) \right\| \\ &\quad + \frac{2(1-\alpha)(1-2\theta)\Delta t}{Re} \left\| \mathbf{d}(\mathbf{\Lambda}^{(\theta)}) \right\| \left\| \mathbf{d}(\mathbf{E}^{(\tilde{\theta})}) \right\|. \end{aligned} \quad (\text{A.62})$$

Note that

$$\left\| \mathbf{d}(\mathbf{E}^{(\tilde{\theta})}) \right\| \leq \left\| \nabla \mathbf{E}^{(\tilde{\theta})} \right\| \leq Ch^{-1} \left\| \mathbf{E}^{(\tilde{\theta})} \right\|.$$

Considering the terms on the R.H.S. of expression (A.62) we have

$$\begin{aligned} \frac{2(1-\alpha)(1-2\theta)\Delta t}{Re} \left\| \mathbf{d}(\mathbf{E}^{(\theta)}) \right\| \left\| \mathbf{d}(\mathbf{E}^{(\tilde{\theta})}) \right\| &\leq \frac{2(1-\alpha)(1-2\theta)\Delta t}{Re} \left\| \mathbf{d}(\mathbf{E}^{(\theta)}) \right\| Ch^{-1} \left\| \mathbf{E}^{(\tilde{\theta})} \right\| \\ &\leq 3C^2h^{-2} \left(\frac{2(1-\alpha)(1-2\theta)^2\Delta t}{\theta Re} \right) \left(\frac{2(1-\alpha)\theta\Delta t}{Re} \right) \left\| \mathbf{d}(\mathbf{E}^{(\theta)}) \right\|^2 + \frac{1}{12} \left\| \mathbf{E}^{(\tilde{\theta})} \right\|^2 \\ &\leq 3C^2h^{-2} \left(\frac{2(1-\alpha)(1-2\theta)^2\Delta t}{\theta Re} \right) \left[Ch^{2k+2} + C(\Delta t)^4 + C\Delta th^{2k} \right] + \frac{1}{12} \left\| \mathbf{E}^{(\tilde{\theta})} \right\|^2, \end{aligned} \quad (\text{A.63})$$

and

$$\begin{aligned} \frac{2(1-\alpha)(1-2\theta)\Delta t}{Re} \left\| \mathbf{d}(\mathbf{\Lambda}^{(\theta)}) \right\| \left\| \mathbf{d}(\mathbf{E}^{(\tilde{\theta})}) \right\| \\ \leq 3C^2h^{-2} \left(\frac{4(1-\alpha)^2(1-2\theta)^2(\Delta t)^2}{Re^2} \right) Ch^{2k} + \frac{1}{12} \left\| \mathbf{E}^{(\tilde{\theta})} \right\|^2. \end{aligned} \quad (\text{A.64})$$

Applying lemma C.19, with the bounds established in (A.63), and (A.64) to (A.62) and we obtain:

$$\left\| \mathbf{E}^{(\tilde{\theta})} \right\|^2 \leq Ch^{2k+2} + (1+h^{-2}\Delta t) \left[Ch^{2k+2} + C(\Delta t)^4 + C\Delta th^{2k} \right] + C(\Delta t)^2 h^{2k-2}. \quad (\text{A.65})$$

■

B Taylor Expansions

The following are expressions for u at different times. Specifically the expressions listed are computations of Taylor series about the expansion points $u^{(n+\frac{1}{2})}$, $u^{(n+\frac{1}{2}-\theta)}$, and $u^{(n+\theta-\frac{1}{2})}$ that give an exact value for the truncation of the series. Integration by parts proves each of these statements.

Expanding about $u^{(n+\frac{1}{2})}$:

$$\begin{aligned}
u^{(n+1)} &= u^{(n+\frac{1}{2})} + \int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} u_t(\cdot, t) dt \\
&= u^{(n+\frac{1}{2})} + \frac{1}{2} \Delta t u_t^{(n+\frac{1}{2})} + \int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} u_{tt}(\cdot, t) (t_{(n+1)} - t) dt \\
&= u^{(n+\frac{1}{2})} + \frac{1}{2} \Delta t u_t^{(n+\frac{1}{2})} + \frac{1}{2} \frac{(\Delta t)^2}{4} u_{tt}^{(n+\frac{1}{2})} + \frac{1}{2} \int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} u_{ttt}(\cdot, t) (t_{(n+1)} - t)^2 dt \quad (\text{B.66})
\end{aligned}$$

$$\begin{aligned}
u^{(n+\theta)} &= u^{(n+\frac{1}{2})} - \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_t(\cdot, t) dt \\
&= u^{(n+\frac{1}{2})} - \left(\frac{1}{2} - \theta \right) \Delta t u_t^{(n+\frac{1}{2})} + \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta)}) dt \quad (\text{B.67})
\end{aligned}$$

$$\begin{aligned}
u^{(n+\tilde{\theta})} &= u^{(n+\frac{1}{2})} + \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\tilde{\theta})}} u_t(\cdot, t) dt \\
&= u^{(n+\frac{1}{2})} + \left(\frac{1}{2} - \theta \right) \Delta t u_t^{(n+\frac{1}{2})} + \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot, t) (t_{(n+\tilde{\theta})} - t) dt \quad (\text{B.68})
\end{aligned}$$

$$\begin{aligned}
u^{(n)} &= u^{(n+\frac{1}{2})} - \int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} u_t(\cdot, t) dt \\
&= u^{(n+\frac{1}{2})} - \left(\frac{1}{2} \right) \Delta t u_t^{(n+\frac{1}{2})} + \int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \\
&= u^{(n+\frac{1}{2})} - \left(\frac{1}{2} \right) \Delta t u_t^{(n+\frac{1}{2})} + \frac{1}{2} \frac{(\Delta t)^2}{4} u_{tt}^{(n+\frac{1}{2})} \\
&\quad - \frac{1}{2} \int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} u_{ttt}(\cdot, t) (t - t_{(n)})^2 dt \quad (\text{B.69})
\end{aligned}$$

Expanding about $u^{(n+\frac{1}{2}-\theta)}$:

$$\begin{aligned}
u^{(n)} &= u^{(n+\frac{1}{2}-\theta)} - \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_t(\cdot, t) dt \\
&= u^{(n+\frac{1}{2}-\theta)} - \left(\frac{1}{2} - \theta \right) \Delta t u_t^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \quad (\text{B.70})
\end{aligned}$$

$$\begin{aligned}
u^{(n+\theta)} &= u^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_t(\cdot, t) dt \\
&= u^{(n+\frac{1}{2}-\theta)} + \left(2\theta - \frac{1}{2}\right) \Delta t u_t^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \quad (\text{B.71})
\end{aligned}$$

$$\begin{aligned}
u^{(n-\theta)} &= u^{(n+\frac{1}{2}-\theta)} - \int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} u_t(\cdot, t) dt \\
&= u^{(n+\frac{1}{2}-\theta)} - \frac{\Delta t}{2} u_t^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \quad (\text{B.72})
\end{aligned}$$

$$\begin{aligned}
u^{(n+\tilde{\theta})} &= u^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\tilde{\theta})}} u_t(\cdot, t) dt \\
&= u^{(n+\frac{1}{2}-\theta)} + \frac{\Delta t}{2} u_t^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot, t) (t_{(n+\tilde{\theta})} - t) dt \quad (\text{B.73})
\end{aligned}$$

Expanding about $u^{(n+\theta-\frac{1}{2})}$:

$$\begin{aligned}
u^{(n)} &= u^{(n+\theta-\frac{1}{2})} + \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_t(\cdot, t) dt \\
&= u^{(n+\theta-\frac{1}{2})} + \left(\frac{1}{2} - \theta\right) \Delta t u_t^{(n+\theta-\frac{1}{2})} + \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_{tt}(\cdot, t) (t_{(n)} - t) dt \quad (\text{B.74})
\end{aligned}$$

$$\begin{aligned}
u^{(n-\theta)} &= u^{(n+\theta-\frac{1}{2})} - \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt \\
&= u^{(n+\theta-\frac{1}{2})} - \left(2\theta - \frac{1}{2}\right) \Delta t u_t^{(n+\theta-\frac{1}{2})} + \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \quad (\text{B.75})
\end{aligned}$$

$$\begin{aligned}
u^{(n+\theta-1)} &= u^{(n+\theta-\frac{1}{2})} - \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt \\
&= u^{(n+\theta-\frac{1}{2})} - \left(\frac{\Delta t}{2}\right) u_t^{(n+\theta-\frac{1}{2})} + \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta-1)}) dt \quad (\text{B.76})
\end{aligned}$$

$$\begin{aligned}
u^{(n+\theta)} &= u^{(n+\theta-\frac{1}{2})} + \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n+\theta)}} u_t(\cdot, t) dt \\
&= u^{(n+\theta-\frac{1}{2})} + \left(\frac{\Delta t}{2}\right) u_t^{(n+\theta-\frac{1}{2})} + \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \quad (\text{B.77})
\end{aligned}$$

Expanding about $u^{(n-\frac{1}{2})}$:

$$u^{(n)} = u^{(n-\frac{1}{2})} + \frac{\Delta t}{2} u_t^{(n-\frac{1}{2})} + \frac{1}{2} \frac{(\Delta t)^2}{4} u_{tt}^{(n-\frac{1}{2})} + \frac{1}{2} \int_{t_{n-\frac{1}{2}}}^{t_n} u_{ttt}(\cdot, t) (t_n - t)^2 dt \quad (\text{B.78})$$

$$u^{(n-1)} = u^{(n-\frac{1}{2})} - \frac{\Delta t}{2} u_t^{(n-\frac{1}{2})} + \frac{1}{2} \frac{(\Delta t)^2}{4} u_{tt}^{(n-\frac{1}{2})} - \frac{1}{2} \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} u_{ttt}(\cdot, t) (t - t_{n-1})^2 dt \quad (\text{B.79})$$

C Lemmas

Lemma C.1

$$\left\| \theta u^{(n+1)} + (1-\theta) u^{(n+\theta)} - u^{(n+\frac{1}{2})} \right\|^2 \leq \frac{(\Delta t)^3}{24} \int_{t_{(n+\theta)}}^{t_{(n+1)}} \|u_{tt}(\cdot, t)\|^2 dt \quad (\text{C.80})$$

Proof of Lemma C.1 using (B.66), (B.67) and choosing the value of $\theta = 1 - \frac{\sqrt{2}}{2}$:

$$\begin{aligned} & \left\| \theta u^{(n+1)} + (1-\theta) u^{(n+\theta)} - u^{(n+\frac{1}{2})} \right\|^2 = \int_{\Omega} \left(\theta u^{(n+1)} + (1-\theta) u^{(n+\theta)} - u^{(n+\frac{1}{2})} \right)^2 dA \\ &= \int_{\Omega} \left(\theta \left(u^{(n+\frac{1}{2})} + \frac{1}{2} \Delta t u_t^{(n+\frac{1}{2})} + \int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} u_{tt}(\cdot, t) (t_{(n+1)} - t) dt \right) \right. \\ & \quad \left. + (1-\theta) \left(u^{(n+\frac{1}{2})} - \left(\frac{1}{2} - \theta \right) \Delta t u_t^{(n+\frac{1}{2})} + \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta)}) dt \right) - u^{(n+\frac{1}{2})} \right)^2 dA \\ &= \int_{\Omega} \left(\left(-\theta^2 + 2\theta - \frac{1}{2} \right) \Delta t u_t^{(n+\frac{1}{2})} + \theta \int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} u_{tt}(\cdot, t) (t_{(n+1)} - t) dt \right. \\ & \quad \left. + (1-\theta) \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta)}) dt \right)^2 dA \\ &= \int_{\Omega} \left(\theta \int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} u_{tt}(\cdot, t) (t_{(n+1)} - t) dt + (1-\theta) \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta)}) dt \right)^2 dA \\ &\leq 2 \int_{\Omega} \left(\theta^2 \left(\int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} u_{tt}(\cdot, t) (t_{(n+1)} - t) dt \right)^2 + (1-\theta)^2 \left(\int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta)}) dt \right)^2 \right) dA \\ &\leq 2 \int_{\Omega} \left(\theta^2 \left(\int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} (t_{(n+1)} - t)^2 dt \right) \right. \\ & \quad \left. + (1-\theta)^2 \left(\int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} (t - t_{(n+\theta)})^2 dt \right) \right) dA \\ &= 2 \int_{\Omega} \left(\theta^2 \frac{1}{3} \left(\frac{\Delta t}{2} \right)^3 \left(\int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} (u_{tt}(\cdot, t))^2 dt \right) \right. \\ & \quad \left. + (1-\theta)^2 \frac{1}{3} \left(\left(\frac{1}{2} - \theta \right) \Delta t \right)^3 \left(\int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} (u_{tt}(\cdot, t))^2 dt \right) \right) dA \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3} \left(\frac{\Delta t}{2} \right)^3 \int_{\Omega} \left(\theta^2 \left(\int_{t_{(n+\frac{1}{2})}}^{t_{(n+1)}} (u_{tt}(\cdot, t))^2 dt \right) + (1-\theta)^2 (1-2\theta)^3 \left(\int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} (u_{tt}(\cdot, t))^2 dt \right) \right) dA \\
&\leq \frac{2}{3} \left(\frac{\Delta t}{2} \right)^3 \max \{ \theta^2, (1-\theta)^2, (1-2\theta)^3 \} \int_{\Omega} \left(\int_{t_{(n+\theta)}}^{t_{(n+1)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&= \frac{(\Delta t)^3}{12} (1-\theta)^2 \int_{t_{(n+\theta)}}^{t_{(n+1)}} \|u_{tt}(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.2

$$\left\| \theta u^{(n)} + (1-\theta) u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2})} \right\|^2 \leq \frac{(\Delta t)^3}{24} \int_{t_{(n)}}^{t_{(n+\bar{\theta})}} \|u_{tt}(\cdot, t)\|^2 dt \quad (\text{C.81})$$

Proof of Lemma C.2 using (B.68), (B.69), and the value of $\theta = 1 - \frac{\sqrt{2}}{2}$:

$$\begin{aligned}
&\left\| \theta u^{(n)} + (1-\theta) u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2})} \right\|^2 = \int_{\Omega} \left(\theta u^{(n)} + (1-\theta) u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left(\theta \left(u^{(n+\frac{1}{2})} - \left(\frac{1}{2} \right) \Delta t u_t^{(n+\frac{1}{2})} + \int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \right) \right. \\
&\quad \left. + (1-\theta) \left(u^{(n+\frac{1}{2})} + \left(\frac{1}{2} - \theta \right) \Delta t u_t^{(n+\frac{1}{2})} + \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} u_{tt}(\cdot, t) (t_{(n+\bar{\theta})} - t) dt \right) - u^{(n+\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left(\left(\theta^2 - 2\theta + \frac{1}{2} \right) \Delta t u_t^{(n+\frac{1}{2})} + \theta \int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \right. \\
&\quad \left. + (1-\theta) \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} u_{tt}(\cdot, t) (t_{(n+\bar{\theta})} - t) dt \right)^2 dA \\
&= \int_{\Omega} \left(\theta \int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n)}) dt + (1-\theta) \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} u_{tt}(\cdot, t) (t_{(n+\bar{\theta})} - t) dt \right)^2 dA \\
&\leq 2 \int_{\Omega} \left(\theta^2 \left(\int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \right)^2 + (1-\theta)^2 \left(\int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} u_{tt}(\cdot, t) (t_{(n+\bar{\theta})} - t) dt \right)^2 \right) dA \\
&\leq 2 \int_{\Omega} \left(\theta^2 \int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} (t - t_{(n)})^2 dt \right. \\
&\quad \left. + (1-\theta)^2 \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} (t_{(n+\bar{\theta})} - t)^2 dt \right) dA \\
&= \frac{2}{3} \left(\frac{\Delta t}{2} \right)^3 \int_{\Omega} \left(\theta^2 \int_{t_{(n)}}^{t_{(n+\frac{1}{2})}} (u_{tt}(\cdot, t))^2 dt + (1-\theta)^2 (1-2\theta)^3 \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} (u_{tt}(\cdot, t))^2 dt \right) dA
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{3} \left(\frac{\Delta t}{2} \right)^3 \max \left\{ \theta^2, (1-\theta)^2, (1-2\theta)^3 \right\} \int_{\Omega} \left(\int_{t_{(n)}}^{t_{(n+\bar{\theta})}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&= \frac{(\Delta t)^3}{12} (1-\theta)^2 \int_{t_{(n)}}^{t_{(n+\bar{\theta})}} \|u_{tt}(\cdot, t)\|^2 dt
\end{aligned}$$

■

Lemma C.3

$$\left\| \theta u^{(n)} + (1-\theta) u^{(n+\theta)} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 \leq \frac{(\Delta t)^3}{24} \int_{t_{(n)}}^{t_{(n+\theta)}} \|u_{tt}(\cdot, t)\|^2 dt \quad (\text{C.82})$$

Proof of Lemma C.3 using (B.70),(B.71) and choosing $\theta = \left(1 - \frac{\sqrt{2}}{2}\right)$:

$$\begin{aligned}
&\left\| \theta u^{(n)} + (1-\theta) u^{(n+\theta)} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 = \int_{\Omega} \left(\theta u^{(n)} + (1-\theta) u^{(n+\theta)} - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA \\
&= \int_{\Omega} \left(\theta \left(u^{(n+\frac{1}{2}-\theta)} - \left(\frac{1}{2} - \theta \right) \Delta t u_t^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \right) \right. \\
&\quad \left. + (1-\theta) \left(u^{(n+\frac{1}{2}-\theta)} + \left(2\theta - \frac{1}{2} \right) \Delta t u_t^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \right) - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA \\
&= \int_{\Omega} \left(\left(-\theta^2 + 2\theta - \frac{1}{2} \right) \Delta t u_t^{(n+\frac{1}{2}-\theta)} + \theta \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \right. \\
&\quad \left. + (1-\theta) \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \right)^2 dA \\
&= \int_{\Omega} \left(\theta \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n)}) dt + (1-\theta) \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \right)^2 dA \\
&\leq 2 \int_{\Omega} \left(\theta^2 \left(\int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \right)^2 + (1-\theta)^2 \left(\int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \right)^2 \right) dA \\
&\leq 2 \int_{\Omega} \left(\theta^2 \left(\int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} (t - t_{(n)})^2 dt \right) \right. \\
&\quad \left. + (1-\theta)^2 \left(\int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} (t_{(n+\theta)} - t)^2 dt \right) \right) dA
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3} \int_{\Omega} \left(\theta^2 \left(\left(\frac{1}{2} - \theta \right) \Delta t \right)^3 \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} (u_{tt}(\cdot, t))^2 dt \right. \\
&\quad \left. + (1-\theta)^2 \left(\left(2\theta - \frac{1}{2} \right) \Delta t \right)^3 \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&= \frac{2}{3} \left(\frac{\Delta t}{2} \right)^3 \int_{\Omega} \left(\theta^2 (1-2\theta)^3 \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} (u_{tt}(\cdot, t))^2 dt + (1-\theta)^2 (4\theta-1)^3 \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&\leq \frac{(\Delta t)^3}{12} \max \left\{ \theta^2, (1-2\theta)^3, (1-\theta)^2, (4\theta-1)^3 \right\} \int_{\Omega} \left(\int_{t_{(n)}}^{t_{(n+\theta)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&= \frac{(\Delta t)^3}{12} (1-\theta)^2 \int_{\Omega} \left(\int_{t_{(n)}}^{t_{(n+\theta)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&= \frac{(\Delta t)^3}{12} (1-\theta)^2 \int_{t_{(n)}}^{t_{(n+\theta)}} \|u_{tt}(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.4

$$\left\| (1-2\theta) u^{(n+\bar{\theta})} + \theta u^{(n)} + \theta u^{(n-\theta)} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 \leq \frac{(\Delta t)^3}{3} (3-2\sqrt{2}) \int_{t_{(n-\theta)}}^{t_{(n+\bar{\theta})}} \|u_{tt}(\cdot, t)\|^2 dt \quad (\text{C.83})$$

Proof of Lemma C.4 using (B.70), (B.72), (B.73) and choosing $\theta = \left(1 - \frac{\sqrt{2}}{2}\right)$:

$$\begin{aligned}
&\left\| (1-2\theta) u^{(n+\bar{\theta})} + \theta u^{(n)} + \theta u^{(n-\theta)} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 \\
&= \int_{\Omega} \left((1-2\theta) u^{(n+\bar{\theta})} + \theta u^{(n)} + \theta u^{(n-\theta)} - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA \\
&= \int_{\Omega} \left((1-2\theta) \left(u^{(n+\frac{1}{2}-\theta)} + \frac{\Delta t}{2} u_t^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} u_{tt}(\cdot, t) (t_{(n+\bar{\theta})} - t) dt \right) \right. \\
&\quad \left. + \theta \left(u^{(n+\frac{1}{2}-\theta)} - \left(\frac{1}{2} - \theta \right) \Delta t u_t^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \right) \right. \\
&\quad \left. + \theta \left(u^{(n+\frac{1}{2}-\theta)} - \frac{\Delta t}{2} u_t^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \right) - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA \\
&= \int_{\Omega} \left(((1-2\theta) - \theta(1-2\theta) - \theta) \frac{\Delta t}{2} u_t^{(n+\frac{1}{2}-\theta)} + (1-2\theta) \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} u_{tt}(\cdot, t) (t_{(n+\bar{\theta})} - t) dt \right. \\
&\quad \left. + \theta \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n)}) dt + \theta \int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \right)^2 dA \\
&= \int_{\Omega} \left((1-2\theta) \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} u_{tt}(\cdot, t) (t_{(n+\bar{\theta})} - t) dt + \theta \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \right. \\
&\quad \left. + \theta \int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \right)^2 dA
\end{aligned}$$

$$\begin{aligned}
&\leq 4 \int_{\Omega} \left((1-2\theta)^2 \left(\int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} u_{tt}(\cdot, t) (t_{(n+\bar{\theta})} - t) dt \right)^2 + \theta^2 \left(\int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n)}) dt \right)^2 \right. \\
&\quad \left. + \theta^2 \left(\int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \right)^2 \right) dA \\
&\leq 4 \int_{\Omega} \left((1-2\theta)^2 \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} (t_{(n+\bar{\theta})} - t)^2 dt \right. \\
&\quad + \theta^2 \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} (t - t_{(n)})^2 dt \\
&\quad \left. + \theta^2 \int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} (t - t_{(n-\theta)})^2 dt \right) dA \\
&= \frac{4}{3} \int_{\Omega} \left((1-2\theta)^2 \left(\frac{\Delta t}{2} \right)^3 \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} (u_{tt}(\cdot, t))^2 dt + \theta^2 \left((1-2\theta) \frac{\Delta t}{2} \right)^3 \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} (u_{tt}(\cdot, t))^2 dt \right. \\
&\quad \left. + \theta^2 \left(\frac{\Delta t}{2} \right)^3 \int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&= \frac{4}{3} \left(\frac{\Delta t}{2} \right)^3 \int_{\Omega} \left((1-2\theta)^2 \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} (u_{tt}(\cdot, t))^2 dt + \theta^2 (1-2\theta)^3 \int_{t_{(n)}}^{t_{(n+\frac{1}{2}-\theta)}} (u_{tt}(\cdot, t))^2 dt \right. \\
&\quad \left. + \theta^2 \int_{t_{(n-\theta)}}^{t_{(n+\frac{1}{2}-\theta)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&\leq \frac{4}{3} \left(\frac{\Delta t}{2} \right)^3 2 \max \left\{ (1-2\theta)^2, \theta^2, (1-2\theta)^3 \right\} \int_{\Omega} \left(\int_{t_{(n-\theta)}}^{t_{(n+\bar{\theta})}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&= \frac{(\Delta t)^3}{3} (1-2\theta)^2 \int_{t_{(n-\theta)}}^{t_{(n+\bar{\theta})}} \|u_{tt}(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.5

$$\left\| \theta u^{(n)} + (1-\theta) u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 \leq \frac{(\Delta t)^3}{24} \int_{t_{(n-\theta)}}^{t_{(n)}} \|u_{tt}(\cdot, t)\|^2 dt$$

Proof of Lemma C.5 using (B.74), (B.75) and the value of $\theta = \left(1 - \frac{\sqrt{2}}{2}\right)$.

$$\left\| \theta u^{(n)} + (1-\theta) u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 = \int_{\Omega} \left(\theta u^{(n)} + (1-\theta) u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right)^2 dA$$

$$\begin{aligned}
&= \int_{\Omega} \left(\theta u^{(n)} + (1-\theta) u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left(\theta \left(u^{(n+\theta-\frac{1}{2})} + \left(\frac{1}{2} - \theta \right) \Delta t u_t^{(n+\theta-\frac{1}{2})} + \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_{tt}(\cdot, t) (t_{(n)} - t) dt \right) \right. \\
&\quad \left. + (1-\theta) \left(u^{(n+\theta-\frac{1}{2})} - \left(2\theta - \frac{1}{2} \right) \Delta t u_t^{(n+\theta-\frac{1}{2})} + \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \right) - u^{(n+\theta-\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left(\left(\theta^2 - 2\theta + \frac{1}{2} \right) \Delta t u_t^{(n+\theta-\frac{1}{2})} + \theta \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_{tt}(\cdot, t) (t_{(n)} - t) dt \right. \\
&\quad \left. + (1-\theta) \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \right)^2 dA \\
&= \int_{\Omega} \left(\theta \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_{tt}(\cdot, t) (t_{(n)} - t) dt + (1-\theta) \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \right)^2 dA \\
&\leq 2 \int_{\Omega} \left(\theta^2 \left(\int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_{tt}(\cdot, t) (t_{(n)} - t) dt \right)^2 + (1-\theta)^2 \left(\int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n-\theta)}) dt \right)^2 \right) dA \\
&\leq 2 \int_{\Omega} \left(\theta^2 \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} (t_{(n)} - t)^2 dt \right. \\
&\quad \left. + (1-\theta)^2 \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} (t - t_{(n-\theta)})^2 dt \right) dA \\
&= \frac{2}{3} \left(\frac{\Delta t}{2} \right)^3 \int_{\Omega} \left(\theta^2 (1-2\theta)^3 \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} (u_{tt}(\cdot, t))^2 dt + (1-\theta)^2 (4\theta-1)^3 \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&\leq \frac{2}{3} \left(\frac{\Delta t}{2} \right)^3 \max \left\{ \theta^2, (1-2\theta)^3, (1-\theta)^2, (4\theta-1)^3 \right\} \int_{\Omega} \left(\int_{t_{(n-\theta)}}^{t_{(n)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&= \frac{(\Delta t)^3}{12} (1-\theta)^2 \int_{t_{(n-\theta)}}^{t_{(n)}} \|u_{tt}(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.6

$$\left\| (1-2\theta) u^{(n+\theta-1)} + \theta u^{(n)} + \theta u^{(n+\theta)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 \leq \frac{(\Delta t)^3}{3} (3-2\sqrt{2}) \int_{t_{(n+\theta-1)}}^{t_{(n+\theta)}} \|u_{tt}(\cdot, t)\|^2 dt$$

Proof of Lemma C.6 using (B.74),(B.76),(B.77) and the value of $\theta = \left(1 - \frac{\sqrt{2}}{2}\right)$:

$$\left\| (1-2\theta) u^{(n+\theta-1)} + \theta u^{(n)} + \theta u^{(n+\theta)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 = \int_{\Omega} \left((1-2\theta) u^{(n+\theta-1)} + \theta u^{(n)} + \theta u^{(n+\theta)} - u^{(n+\theta-\frac{1}{2})} \right)^2 dA$$

$$\begin{aligned}
&= \int_{\Omega} \left((1-2\theta) \left(u^{(n+\theta-\frac{1}{2})} - \left(\frac{\Delta t}{2} \right) u_t^{(n+\theta-\frac{1}{2})} + \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta-1)}) dt \right) \right. \\
&\quad \left. + \theta \left(u^{(n+\theta-\frac{1}{2})} + \left(\frac{1}{2} - \theta \right) \Delta t u_t^{(n+\theta-\frac{1}{2})} + \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_{tt}(\cdot, t) (t_{(n)} - t) dt \right) \right. \\
&\quad \left. + \theta \left(u^{(n+\theta-\frac{1}{2})} + \left(\frac{\Delta t}{2} \right) u_t^{(n+\theta-\frac{1}{2})} + \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \right) - u^{(n+\theta-\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left((-2\theta^2 + 4\theta - 1) \left(\frac{\Delta t}{2} \right) u_t^{(n+\theta-\frac{1}{2})} + (1-2\theta) \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta-1)}) dt \right. \\
&\quad \left. + \theta \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_{tt}(\cdot, t) (t_{(n)} - t) dt + \theta \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \right)^2 dA \\
&= \int_{\Omega} \left((1-2\theta) \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta-1)}) dt + \theta \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_{tt}(\cdot, t) (t_{(n)} - t) dt \right. \\
&\quad \left. + \theta \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \right)^2 dA \\
&\leq 4 \int_{\Omega} \left((1-2\theta)^2 \left(\int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_{tt}(\cdot, t) (t - t_{(n+\theta-1)}) dt \right)^2 + \theta^2 \left(\int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} u_{tt}(\cdot, t) (t_{(n)} - t) dt \right)^2 \right. \\
&\quad \left. + \theta^2 \left(\int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n+\theta)}} u_{tt}(\cdot, t) (t_{(n+\theta)} - t) dt \right)^2 \right) dA \\
&\leq 4 \int_{\Omega} \left((1-2\theta)^2 \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} (t - t_{(n+\theta-1)})^2 dt \right. \\
&\quad \left. + \theta^2 \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} (t_{(n)} - t)^2 dt \right. \\
&\quad \left. + \theta^2 \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n+\theta)}} (u_{tt}(\cdot, t))^2 dt \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n+\theta)}} (t_{(n+\theta)} - t)^2 dt \right) dA \\
&\leq \frac{4}{3} \left(\frac{\Delta t}{2} \right)^3 \int_{\Omega} \left((1-2\theta)^2 \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} (u_{tt}(\cdot, t))^2 dt + \theta^2 (1-2\theta)^3 \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n)}} (u_{tt}(\cdot, t))^2 dt \right. \\
&\quad \left. + \theta^2 \int_{t_{(n+\theta-\frac{1}{2})}}^{t_{(n+\theta)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&\leq \frac{8}{3} \left(\frac{\Delta t}{2} \right)^3 \max \{ (1-2\theta)^2, \theta^2, (1-2\theta)^3 \} \int_{\Omega} \left(\int_{t_{(n+\theta-1)}}^{t_{(n+\theta)}} (u_{tt}(\cdot, t))^2 dt \right) dA \\
&= \frac{(\Delta t)^3}{3} (1-2\theta)^2 \int_{t_{(n+\theta-1)}}^{t_{(n+\theta)}} \|u_{tt}(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.7

$$\left\| u^{(n+\theta)} - u^{(n+\frac{1}{2})} \right\|^2 \leq \Delta t \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} \|u_t(\cdot, t)\|^2 dt$$

Proof of Lemma C.7 using (B.67):

$$\begin{aligned} \left\| u^{(n+\theta)} - u^{(n+\frac{1}{2})} \right\|^2 &= \int_{\Omega} \left(u^{(n+\theta)} - u^{(n+\frac{1}{2})} \right)^2 dA \\ &= \int_{\Omega} \left(u^{(n+\frac{1}{2})} - \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_t(\cdot, t) dt - u^{(n+\frac{1}{2})} \right)^2 dA \\ &= \int_{\Omega} \left(- \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_t(\cdot, t) dt \right)^2 dA \\ &= \int_{\Omega} \left(\int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} 1 u_t(\cdot, t) dt \right)^2 dA \\ &\leq \int_{\Omega} \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} 1^2 dt \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} (u_t(\cdot, t))^2 dt dA \\ &\leq \int_{\Omega} \int_{t_{(n)}}^{t_{(n+1)}} 1^2 dt \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} (u_t(\cdot, t))^2 dt dA \\ &\leq \Delta t \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} \|u_t(\cdot, t)\|^2 dt \quad \blacksquare \end{aligned}$$

Lemma C.8

$$\left\| u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2})} \right\|^2 \leq \Delta t \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} \|u_t(\cdot, t)\|^2 dt$$

Proof of Lemma C.8 using (B.68):

$$\begin{aligned} \left\| u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2})} \right\|^2 &= \int_{\Omega} \left(u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2})} \right)^2 dA \\ &= \int_{\Omega} \left(u^{(n+\frac{1}{2})} + \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} u_t(\cdot, t) dt - u^{(n+\frac{1}{2})} \right)^2 dA \\ &= \int_{\Omega} \left(\int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} 1 u_t(\cdot, t) dt \right)^2 dA \\ &\leq \int_{\Omega} \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} 1^2 dt \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} (u_t(\cdot, t))^2 dt dA \\ &\leq \int_{\Omega} \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\frac{1}{2}+1)}} 1^2 dt \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} (u_t(\cdot, t))^2 dt dA \\ &\leq \Delta t \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} \|u_t(\cdot, t)\|^2 dt \quad \blacksquare \end{aligned}$$

Lemma C.9

$$\left\| \frac{1}{2}u^{(n+\theta)} + \frac{1}{2}u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2})} \right\|^2 \leq \frac{\Delta t}{2} \int_{t_{(n+\theta)}}^{t_{(n+\bar{\theta})}} \|u_t(\cdot, t)\|^2 dt$$

Proof of Lemma C.9 using (B.67) and (B.68):

$$\begin{aligned} \left\| \frac{1}{2}u^{(n+\theta)} + \frac{1}{2}u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2})} \right\|^2 &= \int_{\Omega} \left(\frac{1}{2}u^{(n+\theta)} + \frac{1}{2}u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2})} \right)^2 dA \\ &= \int_{\Omega} \left(\frac{1}{2} \left(u^{(n+\frac{1}{2})} - \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_t(\cdot, t) dt \right) \right. \\ &\quad \left. + \frac{1}{2} \left(u^{(n+\frac{1}{2})} + \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} u_t(\cdot, t) dt \right) - u^{(n+\frac{1}{2})} \right)^2 dA \\ &= \frac{1}{4} \int_{\Omega} \left(\int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} u_t(\cdot, t) dt - \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_t(\cdot, t) dt \right)^2 dA \\ &\leq \frac{1}{2} \int_{\Omega} \left(\int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} u_t(\cdot, t) dt \right)^2 + \left(\int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} u_t(\cdot, t) dt \right)^2 dA \\ &\leq \frac{1}{2} \int_{\Omega} \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} 1 dt \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} (u_t(\cdot, t))^2 dt \\ &\quad + \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} 1 dt \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} (u_t(\cdot, t))^2 dt dA \\ &\leq \frac{\Delta t}{2} \int_{\Omega} \int_{t_{(n+\frac{1}{2})}}^{t_{(n+\bar{\theta})}} (u_t(\cdot, t))^2 dt + \int_{t_{(n+\theta)}}^{t_{(n+\frac{1}{2})}} (u_t(\cdot, t))^2 dt dA \\ &\leq \frac{\Delta t}{2} \int_{t_{(n+\theta)}}^{t_{(n+\bar{\theta})}} \|u_t(\cdot, t)\|^2 dt \quad \blacksquare \end{aligned}$$

Lemma C.10

$$\left\| u^{(n+\theta)} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 \leq \Delta t \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} \|u_t(\cdot, t)\|^2 dt$$

Proof of Lemma C.10 using (B.71):

$$\left\| u^{(n+\theta)} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 = \int_{\Omega} \left(u^{(n+\theta)} - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA$$

$$\begin{aligned}
&= \int_{\Omega} \left(u^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_t(\cdot, t) dt - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA \\
&= \int_{\Omega} \left(\int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} 1 u_t(\cdot, t) dt \right)^2 dA \\
&\leq \int_{\Omega} \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} 1^2 dt \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} (u_t(\cdot, t))^2 dt dA \\
&\leq \Delta t \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} \|u_t(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.11

$$\left\| u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 \leq \Delta t \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} \|u_t(\cdot, t)\|^2 dt$$

Proof of Lemma C.11 using (B.73):

$$\begin{aligned}
\left\| u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 &= \int_{\Omega} \left(u^{(n+\bar{\theta})} - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA \\
&= \int_{\Omega} \left(u^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} u_t(\cdot, t) dt - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA \\
&= \int_{\Omega} \left(\int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} 1 u_t(\cdot, t) dt \right)^2 dA \\
&\leq \int_{\Omega} \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} 1^2 dt \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} (u_t(\cdot, t))^2 dt dA \\
&\leq \Delta t \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} \|u_t(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.12

$$\left\| \frac{1}{2} u^{(n+\bar{\theta})} + \frac{1}{2} u^{(n+\theta)} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 \leq \Delta t \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} \|u_t(\cdot, t)\|^2 dt$$

Proof of Lemma C.12 using (B.71), and (B.73):

$$\begin{aligned}
\left\| \frac{1}{2} u^{(n+\bar{\theta})} + \frac{1}{2} u^{(n+\theta)} - u^{(n+\frac{1}{2}-\theta)} \right\|^2 &= \int_{\Omega} \left(\frac{1}{2} u^{(n+\bar{\theta})} + \frac{1}{2} u^{(n+\theta)} - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA \\
&= \int_{\Omega} \left(\frac{1}{2} \left(u^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\bar{\theta})}} u_t(\cdot, t) dt \right) \right. \\
&\quad \left. + \frac{1}{2} \left(u^{(n+\frac{1}{2}-\theta)} + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_t(\cdot, t) dt \right) - u^{(n+\frac{1}{2}-\theta)} \right)^2 dA
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int_{\Omega} \left(\int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_t(\cdot, t) dt + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_t(\cdot, t) dt \right)^2 dA \\
&\leq \frac{1}{2} \int_{\Omega} \left(\left(\int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_t(\cdot, t) dt \right)^2 + \left(\int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} u_t(\cdot, t) dt \right)^2 \right) dA \\
&\leq \frac{1}{2} \int_{\Omega} \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} 1^2 dt \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} (u_t(\cdot, t))^2 dt + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} 1^2 dt \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} (u_t(\cdot, t))^2 dt dA \\
&\leq \frac{\Delta t}{2} \int_{\Omega} \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} (u_t(\cdot, t))^2 dt + \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} (u_t(\cdot, t))^2 dt dA \\
&\leq \Delta t \int_{t_{(n+\frac{1}{2}-\theta)}}^{t_{(n+\theta)}} \|u_t(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.13

$$\left\| u^{(n+\theta-1)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 \leq \Delta t \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} \|u_t(\cdot, t)\|^2 dt$$

Proof of Lemma C.13 using (B.76):

$$\begin{aligned}
\left\| u^{(n+\theta-1)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 &= \int_{\Omega} \left(u^{(n+\theta-1)} - u^{(n+\theta-\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left(u^{(n+\theta-\frac{1}{2})} - \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt - u^{(n+\theta-\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left(\int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt \right)^2 dA \\
&\leq \int_{\Omega} \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} 1^2 dt \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} (u_t(\cdot, t))^2 dt dA \\
&\leq \Delta t \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} \|u_t(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.14

$$\left\| u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 \leq \Delta t \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} \|u_t(\cdot, t)\|^2 dt$$

Proof of Lemma C.14 using (B.75):

$$\begin{aligned}
\left\| u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 &= \int_{\Omega} \left(u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left(u^{(n+\theta-\frac{1}{2})} - \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt - u^{(n+\theta-\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left(\int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt \right)^2 dA
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\Omega} \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} 1^2 dt \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} (u_t(\cdot, t))^2 dt dA \\
&\leq \Delta t \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} \|u_t(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.15

$$\left\| \frac{1}{2}u^{(n+\theta-1)} + \frac{1}{2}u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 \leq \Delta t \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} \|u_t(\cdot, t)\|^2 dt$$

Proof of Lemma C.15 using (B.75), and (B.76):

$$\begin{aligned}
\left\| \frac{1}{2}u^{(n+\theta-1)} + \frac{1}{2}u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right\|^2 &= \int_{\Omega} \left(\frac{1}{2}u^{(n+\theta-1)} + \frac{1}{2}u^{(n-\theta)} - u^{(n+\theta-\frac{1}{2})} \right)^2 dA \\
&= \int_{\Omega} \left(\frac{1}{2} \left(u^{(n+\theta-\frac{1}{2})} - \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt \right) \right. \\
&\quad \left. + \frac{1}{2} \left(u^{(n+\theta-\frac{1}{2})} - \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt \right) - u^{(n+\theta-\frac{1}{2})} \right)^2 dA \\
&= \frac{1}{4} \int_{\Omega} \left(\int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt + \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt \right)^2 dA \\
&\leq \frac{1}{2} \int_{\Omega} \left(\int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt \right)^2 + \left(\int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} u_t(\cdot, t) dt \right)^2 dA \\
&\leq \frac{1}{2} \int_{\Omega} \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} 1^2 dt \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} (u_t(\cdot, t))^2 dt \\
&\quad + \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} 1^2 dt \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} (u_t(\cdot, t))^2 dt dA \\
&\leq \frac{\Delta t}{2} \int_{\Omega} \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} (u_t(\cdot, t))^2 dt + \int_{t_{(n-\theta)}}^{t_{(n+\theta-\frac{1}{2})}} (u_t(\cdot, t))^2 dt dA \\
&\leq \Delta t \int_{t_{(n+\theta-1)}}^{t_{(n+\theta-\frac{1}{2})}} \|u_t(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.16

$$\left\| \frac{u^{(n)} - u^{(n-1)}}{\Delta t} - u_t^{(n-\frac{1}{2})} \right\|^2 \leq \frac{\Delta t^3}{5(2^6)} \int_{t_{n-1}}^{t_n} \|u_{ttt}(\cdot, t)\|^2 dt$$

Proof of Lemma C.16 using (B.78) and (B.79):

$$\begin{aligned}
&\left\| \frac{u^{(n)} - u^{(n-1)}}{\Delta t} - u_t^{(n-\frac{1}{2})} \right\|^2 = \left\| \frac{1}{\Delta t} \left(u^{(n)} - u^{(n-1)} \right) - u_t^{(n+\frac{1}{2})} \right\|^2 \\
&= \int_{\Omega} \left(\frac{1}{\Delta t} u^{(n-\frac{1}{2})} + \frac{1}{2} u_t^{(n-\frac{1}{2})} + \frac{1}{2} \frac{(\Delta t)}{4} u_{tt}^{(n-\frac{1}{2})} + \frac{1}{2\Delta t} \int_{t_{n-\frac{1}{2}}}^{t_n} u_{ttt}(\cdot, t) (t_n - t)^2 dt \right. \\
&\quad \left. - \frac{1}{\Delta t} u^{(n-\frac{1}{2})} + \frac{1}{2} u_t^{(n-\frac{1}{2})} - \frac{1}{2} \frac{(\Delta t)}{4} u_{tt}^{(n-\frac{1}{2})} + \frac{1}{2\Delta t} \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} u_{ttt}(\cdot, t) (t - t_{n-1})^2 dt - u_t^{(n-\frac{1}{2})} \right)^2 dA
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} \left(\frac{1}{2\Delta t} \int_{t_{n-\frac{1}{2}}}^{t_n} u_{ttt}(\cdot, t)(t_n - t)^2 dt + \frac{1}{2\Delta t} \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} u_{ttt}(\cdot, t)(t - t_{n-1})^2 dt \right)^2 dA \\
&\leq \int_{\Omega} \left(\frac{1}{2\Delta t} \right)^2 \left(2 \left(\int_{t_{n-\frac{1}{2}}}^{t_n} u_{ttt}(\cdot, t)(t_n - t)^2 dt \right)^2 + 2 \left(\int_{t_{n-1}}^{t_{n-\frac{1}{2}}} u_{ttt}(\cdot, t)(t - t_{n-1})^2 dt \right)^2 \right) dA \\
&\leq 2 \left(\frac{1}{2\Delta t} \right)^2 \int_{\Omega} \left(\int_{t_{n-\frac{1}{2}}}^{t_n} (u_{ttt}(\cdot, t))^2 dt \int_{t_{n-\frac{1}{2}}}^{t_n} (t_n - t)^4 dt \right. \\
&\quad \left. + \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (u_{ttt}(\cdot, t))^2 dt \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (t - t_{n-1})^4 dt \right) dA \\
&= 2 \left(\frac{1}{2\Delta t} \right)^2 \int_{\Omega} \left(\frac{1}{5} \left(\frac{\Delta t}{2} \right)^5 \left(\int_{t_{n-\frac{1}{2}}}^{t_n} (u_{ttt}(\cdot, t))^2 dt + \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (u_{ttt}(\cdot, t))^2 dt \right) \right) dA \\
&= \frac{\Delta t^3}{5(2^6)} \int_{\Omega} \int_{t_{n-1}}^{t_n} (u_{ttt}(\cdot, t))^2 dt dA \\
&= \frac{\Delta t^3}{5(2^6)} \int_{t_{n-1}}^{t_n} \|u_{ttt}(\cdot, t)\|^2 dt \quad \blacksquare
\end{aligned}$$

Lemma C.17 Assuming $\|u_{tt}(\cdot, t)\| < M$ then

$$\left\| \frac{u^{(\theta)} - u^{(0)}}{\theta \Delta t} - u_t^{(0)} \right\|^2 \leq C(\Delta t)^2.$$

Proof of Lemma C.17: Make note that

$$\begin{aligned}
u^{(\theta)} - u^{(0)} &= \int_0^{\theta \Delta t} u_t(\cdot, t) dt = (\theta \Delta t) u_t(\cdot, 0) - \int_0^{\theta \Delta t} (t - \theta \Delta t) u_{tt}(\cdot, t) dt. \\
\Rightarrow \left| \frac{u^{(\theta)} - u^{(0)}}{\theta \Delta t} - u_t^{(0)} \right| &= \frac{1}{\theta \Delta t} \left| \int_0^{\theta \Delta t} (\theta \Delta t - t) u_{tt}(\cdot, t) dt \right| \\
&\leq \frac{1}{\theta \Delta t} \left(\int_0^{\theta \Delta t} (\theta \Delta t - t)^2 dt \right)^{\frac{1}{2}} \left(\int_0^{\theta \Delta t} |u_{tt}(\cdot, t)|^2 dt \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{3}} (\theta \Delta t)^{\frac{1}{2}} \left(\int_0^{\theta \Delta t} |u_{tt}(\cdot, t)|^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\| \frac{u^{(\theta)} - u^{(0)}}{\theta \Delta t} - u_t^{(0)} \right\|^2 &\leq \frac{1}{3} \theta \Delta t \int_{\Omega} \int_0^{\theta \Delta t} |u_{tt}(\cdot, t)|^2 dt dA \\
&= \frac{1}{3} \theta \Delta t \int_0^{\theta \Delta t} \int_{\Omega} |u_{tt}(\cdot, t)|^2 dA dt \\
&\leq \frac{1}{3} (\theta \Delta t)^2 M^2 \\
&\leq C(\Delta t)^2. \quad \blacksquare
\end{aligned}$$

Lemma C.18 *Assuming $\|u_{tt}(\cdot, t)\| < M$ then*

$$\left\| \frac{u^{(\theta)} - u^{(0)}}{\theta \Delta t} - u_t^{(\theta)} \right\|^2 \leq C(\Delta t)^2.$$

Proof of Lemma C.18: Make note that

$$\begin{aligned} u^{(\theta)} - u^{(0)} &= \int_{t_0}^{t_\theta} u_t(\cdot, t) dt = (\theta \Delta t) u_t(\cdot, \theta \Delta t) - \int_{t_0}^{t_\theta} t u_{tt}(\cdot, t) dt. \\ \Rightarrow \left| \frac{u^{(\theta)} - u^{(0)}}{\theta \Delta t} - u_t^{(\theta)} \right| &= \frac{1}{\theta \Delta t} \left| \int_0^{\theta \Delta t} t u_{tt}(\cdot, t) dt \right| \\ &\leq \frac{1}{\theta \Delta t} \left(\int_{t_0}^{t_\theta} t^2 dt \right)^{\frac{1}{2}} \left(\int_{t_0}^{t_\theta} |u_{tt}(\cdot, t)|^2 dt \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{3}} (\theta \Delta t)^{\frac{1}{2}} \left(\int_{t_0}^{t_\theta} |u_{tt}(\cdot, t)|^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$\begin{aligned} \left\| \frac{u^{(\theta)} - u^{(0)}}{\theta \Delta t} - u_t^{(\theta)} \right\|^2 &\leq \frac{1}{3} \theta \Delta t \int_{\Omega} \int_0^{\theta \Delta t} |u_{tt}(\cdot, t)|^2 dt dA \\ &= \frac{1}{3} \theta \Delta t \int_{t_0}^{t_\theta} \int_{\Omega} |u_{tt}(\cdot, t)|^2 dA dt \\ &\leq \frac{1}{3} (\theta \Delta t)^2 M^2 \\ &\leq C(\Delta t)^2. \end{aligned} \quad \blacksquare$$

Lemma C.19 *Assuming $\|u_{tt}(\cdot, t)\| < M$ then*

$$\left\| \frac{u^{(\tilde{\theta})} - u^{(\theta)}}{(1 - 2\theta) \Delta t} - u_t^{(\theta)} \right\|^2 \leq C(\Delta t)^2.$$

Proof of Lemma C.19: Make note that

$$\begin{aligned} u^{(\tilde{\theta})} - u^{(\theta)} &= \int_{t_\theta}^{t_{\tilde{\theta}}} u_t(\cdot, t) dt = ((1 - 2\theta) \Delta t) u_t(\cdot, \theta \Delta t) - \int_{t_\theta}^{t_{\tilde{\theta}}} (t - \tilde{\theta} \Delta t) u_{tt}(\cdot, t) dt. \\ \Rightarrow \left| \frac{u^{(\tilde{\theta})} - u^{(\theta)}}{(1 - 2\theta) \Delta t} - u_t^{(\theta)} \right| &= \frac{1}{(1 - 2\theta) \Delta t} \left| \int_{t_\theta}^{t_{\tilde{\theta}}} (\tilde{\theta} \Delta t - t) u_{tt}(\cdot, t) dt \right| \\ &\leq \frac{1}{(1 - 2\theta) \Delta t} \left(\int_{t_\theta}^{t_{\tilde{\theta}}} (\tilde{\theta} \Delta t - t)^2 dt \right)^{\frac{1}{2}} \left(\int_{t_\theta}^{t_{\tilde{\theta}}} |u_{tt}(\cdot, t)|^2 dt \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{3}} ((1 - 2\theta) \Delta t)^{\frac{1}{2}} \left(\int_{t_\theta}^{t_{\tilde{\theta}}} |u_{tt}(\cdot, t)|^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$\begin{aligned}
\left\| \frac{u^{(\tilde{\theta})} - \mathbf{u}^{(\theta)}}{(1-2\theta)\Delta t} - u_t^{(\theta)} \right\|^2 &\leq \frac{1}{3} (1-2\theta) \Delta t \int_{\Omega} \int_{t_{\theta}}^{t_{\tilde{\theta}}} |u_{tt}(\cdot, t)|^2 dt dA \\
&= \frac{1}{3} (1-2\theta) \Delta t \int_{t_{\theta}}^{t_{\tilde{\theta}}} \int_{\Omega} |u_{tt}(\cdot, t)|^2 dA dt \\
&\leq \frac{1}{3} ((1-2\theta) \Delta t)^2 M^2 \\
&\leq C(\Delta t)^2. \quad \blacksquare
\end{aligned}$$

Lemma C.20 *Assuming $\|u_t\|_{\infty} < M$ then*

$$\left\| u^{(\theta)} - u^{(0)} \right\|^2 \leq C(\Delta t)^2.$$

Proof of Lemma C.20: Make note that

$$\begin{aligned}
u^{(\theta)} - u^{(0)} &= \int_{t_{\theta}}^{t_0} 1 u_t dt \\
&\leq \left(\int_{t_0}^{t_{\theta}} 1 dt \right)^{\frac{1}{2}} \left(\int_{t_0}^{t_{\theta}} u_t^2 dt \right)^{\frac{1}{2}} \\
&= \theta^{\frac{1}{2}} \Delta t^{\frac{1}{2}} \left(\int_{t_0}^{t_{\theta}} u_t^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\| u^{(\theta)} - u^{(0)} \right\|^2 &= \int_{\Omega} \left| u^{(\theta)} - u^{(0)} \right|^2 dA \\
&\leq \theta \Delta t \int_{\Omega} \int_{t_0}^{t_{\theta}} u_t^2 dt dA \\
&= \theta \Delta t \int_{t_0}^{t_{\theta}} \int_{\Omega} u_t^2 dA dt \\
&\leq \theta \Delta t M^2 \int_{t_0}^{t_{\theta}} dt \\
&= \theta^2 M^2 \Delta t^2. \quad \blacksquare
\end{aligned}$$

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