THE QUASIMONOTONICITY OF LINEAR DIFFERENTIAL SYSTEMS—THE COMPLEX SPECTRUM

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Abstract The method of vector Lyapunov functions to determine stability in dynamical systems requires that the comparison system be quasimonotone nondecreasing with respect to a cone contained in the nonnegative orthant. For linear comparison systems in $\mathbf{R}^n$ with real spectra, Heikkilä solved the problem for $n = 2$ and gave necessary conditions for $n > 2$. We previously showed a sufficient condition for $n > 2$, and here, for systems with complex eigenvalues, we give conditions for which the problem reduces to the nonnegative inverse eigenvalue problem.

KEY WORDS: Vector Lyapunov functions, cones, quasimonotonicity, dynamical systems.

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1. Introduction

The technique of vector Lyapunov functions has proven very useful in analyzing the stability of differential systems (see, e.g., [9]). In this method, however, the comparison system requires stability properties as well as quasimonotonicity relative to a cone (see [8]). For linear comparison systems, it is sufficient to consider square systems $A(t) \in \mathbf{R}^{n\times n}$, $t \geq 0$, and to find a cone contained in $\mathbf{R}^n_+$ with respect to which the system is quasimonotone (see [1]).

Heikkilä [6] showed that to solve the problem for linear comparison systems $A \in R^{n\times n}$, it is necessary and sufficient to find a nonnegative, nonsingular matrix $B \in \mathbf{R}^{n\times n}$ such that if $C = B^{-1}AB$, then $c_{ij} \geq 0$ for all $i \neq j$. In this case we say that A is *cross-positive* on the proper, simplicial cone determined by the columns of B (see [10]) and that C is *essentially nonnegative* (see [11]).

In [6], Heikkilä used Perron-Frobenius theory to show a necessary condition is for the matrix A to have a real eigenvalue with greatest real part and an associated nonnegative eigenvector. For $n = 2$, Heikkilä showed that it is necessary and sufficient that A have two nonnegative eigenvectors. For $n > 2$ he further showed that it is sufficient for a matrix A with a real spectrum to have all of its eigenvectors nonnegative, and in [7], Köksal and Fausett extended this result to include generalized eigenvectors. In [1], we showed that with $n \geq 2$, for matrices with real spectra it is sufficient that this first eigenvector be positive, and we discussed cases when it is nonnegative in terms of the reducibility of the matrix A.

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This problem has further applications to the positive orthant stabilizability problem (see [3]) and the "hit and hold" problem (see [2]) from control theory. In this paper we address the case of the general spectrum (when the eigenvalues of A are not strictly real) and we present conditions under which the problem reduces to the nonnegative inverse eigenvalue problem.

2. The General Spectrum

In the case of the real spectrum, all of our solutions used the fact that the diagonal (or Jordan canonical) form of the matrix A is essentially nonnegative, since all of its off-diagonal entries are zero (or 1). When a matrix C with real spectrum is essentially nonnegative, then there always exists an r such that $C + rI$ is nonnegative. However, when A has complex eigenvalues, it is possible that it is never similar to an essentially nonnegative matrix. For example, if the spectrum of A, $\sigma(A) = \{1, i, -i\}$, then no matrix B exists such that $C = B^{-1}AB$ is essentially nonnegative. This is because there is no real r for which the shifted spectrum $\sigma(C + rI) = \{1 + r, i + r, -i + r\}$ is the spectrum of a nonnegative matrix. (The spectrum $\sigma = \{1 + \epsilon, i, -i\}$ is in fact the spectrum of a nonnegative matrix for any $\epsilon > 0$.)

It is clear that in order for a matrix A to be quasimonotone nondecreasing with respect to a nonnegative cone, it is necessary that, for some real r, the shifted spectrum $\sigma(A + rI)$ be the spectrum of some nonnegative matrix, i.e., the shifted spectrum of A must solve the *nonnegative inverse eigenvalue problem*. Inverse eigenvalue problems are well-studied problems in linear algebra (see [5]) and perhaps the most important unsolved problem of this type is the nonnegative inverse eigenvalue problem, which asks when a set of numbers is the spectrum of a nonnegative matrix. Although the problem has been studied extensively, in general it remains unsolved (see, for example, [4]).

We offer a solution to our problem when the shifted spectrum of A (for some real r) solves the irreducible nonnegative inverse eigenvalue problem, in that $\sigma(A + rI)$ is the spectrum of an irreducible nonnegative matrix A_n. (The case where the shifted spectrum of A can be decomposed into subsets, each of which solves the irreducible nonnegative inverse eigenvalue problem individually, requires that we extend the theory of this paper using techniques similar to those shown in [1] for reducible matrices. However, unfortunately no general theory exists based strictly on the spectrum of A for such matrices.)

If there exists a $B \geq 0$ such that $A_e = B^{-1}AB$ is essentially nonnegative and irreducible, then Perron-Frobenius theory tells us that A_e has a real eigenvalue λ_1 with $\lambda_1 > \mathrm{Re}(\lambda_i)$ for $i = 2, ..., n$ and an associated positive eigenvector $\mathbf{y}_1$ (since $A_n = A_e + rI$ is nonnegative and irreducible). Therefore, it is necessary for A to have an eigenvalue λ_1 with $\lambda_1 > \mathrm{Re}(\lambda_i)$ for $i = 2, ..., n$ and an associated positive eigenvector $\mathbf{x}_1$ since $\mathbf{x}_1 = B\mathbf{y}_1$ with $\mathbf{y}_1 > 0$, $B \geq 0$, and B nonsingular. If A and A_e are similar, we show that it is sufficient that A have a positive first eigenvector in order for the change-of-basis matrix B to be nonnegative, so that A is quasimonotone with respect to a nonnegative cone. We note the similarity between this result and the result in [1] for matrices with real spectra, where it was also sufficient that A have a first eigenvector $\mathbf{x}_1 > 0$.

3. A Theorem For The General Spectrum

Theorem. *Let $A \in \mathbf{R}^{nxn}$ for $n > 2$. In order for A to be quasimonotone nondecreasing with respect to a nonnegative cone, it is necessary that for some real r the shifted spectrum $\sigma(A + rI)$ solve the nonnegative inverse eigenvalue problem, and that A have a first eigenvalue λ_1 where $\lambda_1 > \mathrm{Re}(\lambda_i)$ for $i = 2, ..., n$ with an associated nonnegative first eigenvector $\mathbf{x}_1 \geq 0$. Furthermore, it is sufficient that A be similar to an irreducible essentially nonnegative matrix A_e and that A have a positive first eigenvector $\mathbf{x}_1 > 0$.*

Proof. The necessary conditions follow from the previous discussion, and we note that for matrices with real spectra, the requirement on the shifted spectrum is always trivially satisfied. To show the sufficient conditions, let A_e be similar to A where A_e is irreducible and essentially nonnegative, so that $\sigma(A+rI)$ solves the irreducible nonnegative inverse eigenvalue problem. Furthermore, let A have a positive first eigenvector $\mathbf{x_1} > 0$ associated with the real eigenvalue of greatest real part. Let D be the real, block-diagonal canonical form of A and A_e so that $D = X^{-1}AX = Y^{-1}A_eY$ where the real and imaginary components of the (generalized) eigenvectors of A and A_e are the columns of X and Y respectively (further, let the first eigenvector $\mathbf{x}_1 > 0$ be the first column of X). Then $A_e = YX^{-1}AXY^{-1}$, but it is not necessarily true that $XY^{-1} \geq 0$, as required.

Let $A_n = A_e + rI$ with r chosen large enough that all diagonal elements of A_n are positive and A_n is nonsingular. Then A_n is nonnegative and primitive (irreducible, with only one eigenvalue of maximum modulus; see, e.g., [11]). Then a further change of basis using $(1/\mu_1\, A_n)^k$, where $\mu_1 = \lambda_1 + r$ is the first eigenvalue of A_n, yields $A_e = (1/\mu_1\, A_n)^{-k}YX^{-1}AXY^{-1}(1/\mu_1\, A_n)^k$. Since A_n is primitive and nonnegative, then as $k \to \infty$, $(1/\mu_1\, A_n)^k \to \mathbf{y}_1\mathbf{z}_1^T$, where $\mathbf{y}_1$ is the first eigenvector of A_n (and of A_e), $\mathbf{z}_1$ is the corresponding first left eigenvector, $\mathbf{y}_1 > 0$, $\mathbf{z}_1 > 0$, and $\mathbf{y}_1^T\mathbf{z}_1 = 1$. Hence, $XY^{-1}(1/\mu_1\, A_n)^k \to \mathbf{x}_1\mathbf{z}_1^T > 0$. Therefore, for some finite k this change of basis B will be nonnegative, $A_e = B^{-1}AB$ with $B \geq 0$, and A is quasimonotone nondecreasing with respect to a nonnegative cone. □

We note that the above theorem has, as a special case, our previous result for matrices with real spectra, since such shifted spectra always solve the nonnegative inverse eigenvalue problem, and we showed (using a different technique) that a positive first eigenvector was a sufficient condition for quasimonotonicity with respect to a nonnegative cone. Furthermore, our previous result held only for $n \geq 3$, as does this one as well, since a necessary condition for quasimonotonicity (a real eigenvalue) and the general spectrum (implying a pair of complex eigenvalues) require at least three eigenvalues.

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