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EFFICIENT EMPLOYMENT OF ADAPTIVE SENSOR

by

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We consider a sensor that is subject to false-positive and false-negative errors. The sensor searches for stationary threat objects such as ballistic missile launchers or improvised explosive devices. The objects are located in a certain area of interest, which is divided into area-cells. The area-cells are defined such that each one of them may contain, at most, one threat object. The task of the sensor is to determine if an area-cell contains a threat object, and the objective of the searcher is to maximize the number of correctly determined area-cells.

Since definitive identification of a threat object, and subsequent handling of that threat, are done by a limited number of available ground combat units, the correct determination of an area-cell is crucial for better allocating and directing these scarce resources. We develop an algorithm, rooted in large deviations theory and stochastic approximation theory, that leads to the optimal search effort. The computed allocation maximizes the expected number correctly determined area-cells as the number of available looks for searching becomes large.

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EFFICIENT EMPLOYMENT OF ADAPTIVE SENSOR

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ABSTRACT

We consider a sensor that is subject to false positive and false negative errors. The sensor searches for stationary threat objects such as ballistic missile launchers or improvised explosive devices. The objects are located in a certain area of interest, which is divided into area-cells. The area-cells are defined such that each of them may contain, at most, one threat object. The task of the sensor is to determine if an area-cell contains a threat object, and the objective of the searcher is to maximize the number of correctly determined area-cells.

Since definitive identification of a threat object, and subsequent handling of that threat, are done by a limited number of available ground combat units, the correct determination of an area-cell is crucial for better allocating and directing these scarce resources. We develop an algorithm, rooted in large deviations theory and stochastic approximation theory, that leads to the optimal search effort. The computed allocation maximizes the expected number correctly determined area-cells as the number of available looks for searching becomes large.

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EXECUTIVE SUMMARY

Advances in sensing, unmanned aerial vehicles (UAVs), and satellite technologies have vastly increased the military use of aerial or space sensors for detecting threat objects such as improvised explosive devices or missile launchers. These advanced technologies may generate powerful and effective sensors, which necessitate operational concepts in order to facilitate their efficient utilization. A typical scenario where such operational concepts are needed is related to maritime interdiction missions where a naval task force is patrolling a certain area of interest (e.g., Horn of Africa) attempting to detect and intercept threat objects (boats) such as terrorists, pirates and weapon smugglers. Such threat objects are relatively rare and therefore it is safe to assume that if the area of interest is divided into a grid of area-cells of size, say, five square miles, then each area-cell may contain at most one stationary threat object. The task force employs unmanned aerial vehicles, equipped with electro-optical or other sensor for surveillance and search. Once a threat object is detected, the task force dispatches an armed inspection unit to investigate the threat. Obviously, a false positive detection results in unnecessary deployment of the intercepting unit, while a false negative detection may result in severe operational consequences such as terrorist or pirate attacks. Arguably, the cost of a false positive detection is much smaller than the cost of false negative detection. The objective is to minimize the expected total cost.

In this thesis, we address operational concepts associated with employing sensors in persistent search missions over an extended search area. Specifically, we consider the problem of efficiently allocating *adaptive* sensors across a search area of interest. The sensors are adaptive in the sense that the search plan is not set in advance, but rather is updated in real time during the search process as new information is generated by the sensor.

Assume that one sensor is assigned an area of interest to search, which is partitioned into a grid of m area-cells. The sensor operates in glimpses or *looks*. A look may be viewed as a nominal period of time for inspecting a certain area-cell. The sensor may spend several looks (i.e., extended inspection time) in a certain area-cell. Each look

generates a cue or signal: *detection* or *no-detection*. A cue may be correct or erroneous (false positive or false negative). Suppose that the sensor has ℓ looks that it can apply to the search, and these looks are allocated to the various area-cells dynamically as the search mission evolves.

When the number of looks available ℓ is small, this can be done by solving a dynamic program. However, due to the curse of dimensionality [11], the computational cost of solving a dynamic program grows exponentially in ℓm, which precludes its use when the number of looks available ℓ is moderately large. This thesis is aimed at the situation when ℓ is large. The optimal search effort allocation is determined in two stages: First, presuming knowledge of the presence/absence of a threat object in each area-cell, use large deviations theory to characterize the optimal effort allocation; and second, use adaptive ideas to generate a search sequence that provably converges to the optimal effort allocation determined in the first stage.

Large deviations theory [4] suggests that the probability of incorrectly determining an area-cell decays roughly exponentially fast with the number of looks. Proposition 1 asserts that the expected number of incorrectly determined area-cells decays exponentially fast, with a rate that is the smallest decay rate amongst all area-cells. This in turn can be used to obtain the optimal fractional allocations; the exact expression is shown in Proposition 2. Unfortunately, the optimal fractional allocations require knowledge of the presence or absence of the threat objects, which is what we are trying to determine. However, one can take an adaptive approach, motivated by the large deviations results, which guarantees optimal allocation of the search resources when the number of looks available is large.

To the best of our knowledge, this is the first thesis that describes the optimal effort allocation in the context of target searching along multiple area-cells where the sensor has false positive and false negative errors, and that provides an implemental adaptive algorithm. We do not consider travel time among the area-cells because, for a large number of looks, its affect on the optimal effort allocations is not significant. This happens because the sensor could stay in the same area-cell for a large number of looks, then move to another area-cell and remain there for a long time, and so on, all the while

satisfying the optimal search effort allocation. When the number of looks is relatively small one should consider resource constraints; see, for example, [13].

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I. INTRODUCTION

A. OPERATIONAL MOTIVATION

Advances in sensing, unmanned aerial vehicles (UAVs), and satellite technologies have vastly increased the military use of aerial or space sensors for detecting threat objects such as improvised explosive devices or missile launchers. These advanced technologies may generate powerful and effective sensors, which necessitate operational concepts in order to facilitate their efficient utilization. A typical scenario where such operational concepts are needed is related to maritime interdiction missions where a naval task force is patrolling a certain area of interest (e.g., Horn of Africa) attempting to detect and intercept threats objects (boats) such as terrorists, pirates and weapon smugglers. Such threat objects are relatively rare and therefore it is safe to assume that if the area of interest is divided into a grid of area-cells of size, say five square miles, then each area-cell may contains at most one stationary threat object. The task force employs unmanned aerial vehicles, equipped with electro-optical or other sensor for surveillance and search. Once a threat object is detected, the task force dispatches an armed inspection unit to investigate the threat. Obviously, a false positive detection results in unnecessary deployment of the intercepting unit, while a false negative detection may result in serve operational consequences such as terrorist or pirate attacks. Arguably, the cost of a false positive detection is much smaller than the cost of false negative detection. The objective is to minimize the expected total cost.

In this thesis, we address operational concepts associated with employing sensors in persistent search missions over an extended search area. Specifically, we consider the problem of efficiently allocating *adaptive* sensors across a search area of interest. The sensors are adaptive in the sense that the search plan is not set in advance, but rather is updated in real time during the search process as new information is generated by the sensor.

B. BACKGROUND

The theory of optimal search has a history of principal importance in military operations. The theory has fundamental applications to anti-submarine warfare, counter-mine warfare, and search and rescue operations. The books [6] and [15] are classical references in this area; with [17] a valuable recent reference. Search problems with discrete time and space of the type addressed in this thesis are not new. Optimal whereabouts search, where we seek to maximize the probability of determining which box contains a certain object, is studied in [1] and [5]. Chew [3] considers an optimal search with a stopping rule where all search outcomes are independent, conditional on the location of the searched object, and the search policy. Wegener [18] investigates a search process where the search time of a cell depends on the number of searches so far. A minimum cost search problem is discussed in [12], where only one search mode is considered and the sensor has perfect specificity. The paper [14] deals with discrete search with multiple sensors in order to maximize the probability of successful search of a single threat during a specified time period. Other discrete search problems are studied in [2, 10, and 19]. However, all of the aforementioned references assume that the sensor has perfect specificity, that is, there are no false positive detections. Our models, which are related to [7], relax this assumption. The results we obtain in this thesis rely on the large deviations theory [4], and on stochastic approximation theory [8].

C. INTRODUCTION

Assume that one sensor is assigned an area of interest to search, which is partitioned into a grid of m area-cells. We assume that the area of interest can be partitioned in such a way that each-cell i, for $i = 1, 2, \ldots, m$, contains, at most, one threat object. The sensor operates in glimpses or *looks*. A look may be viewed as a normal period of time for inspecting a certain area-cell. The sensor may spend several looks (i.e., extended inspection time) in a certain area-cell. Each look generates a cue or signal: *detection* or *no detection*. A cues maybe correct or erroneous (false positive or false negative). Suppose that the sensor has ℓ looks that it can apply to the search, and these looks are allocated to the various area-cells dynamically as the mission evolves.

Let $\zeta_i = 1$ if area-cell i contains a threat object, and $\xi_i = 0$ otherwise, leading to the hypothesis $H_{i,0} : \zeta_i = 1$, and $H_{i,1} : \zeta_i = 0$. We suppose that there is some initial intelligence about the presence of threat objects, which is manifested by a prior probability,

$$\pi_i = P(\zeta_i = 1),$$

for $i = 1, \ldots, m$. This intelligence comes from exogenous sources such as satellite imaging, communication interception and human intelligence. The sensor is characterized by its sensitivity and specificity. For each area-cell i we have

$$a_i = P_1(\text{sensor indicates detection in area-cell } i \,|\, \zeta_i = 1),$$

which is called the sensitivity of the sensor; the specificity of the sensor is 1 - b_i, where

$$b_i = P_0(\text{sensor indicates detection in area-cell } i \,|\, \zeta_i = 0).$$

Although the a_i's and b_i's may depend on the area-cell, we assume that they do not depend on the number of looks. Without loss of generality, we take $a_i > b_i$, because we can reverse the cue if $a_i < b_i$. We explicitly assume that $a_i \neq b_i$, for otherwise, the sensor would not provide any valuable information. We assume that the collection of search results from the various looks is independent for a given area-cell (meaning that there is no systematic bias in the sensor), and that the results for different area-cells are independent.

D. RESEARCH MOTIVATION AND GOAL

The motivation of this thesis is to characterize effort allocation schemes that employ the sensor efficiently, where the measure of effectiveness is *expected number of correctly determined area-cells*. When the number of looks available, ℓ, is small, this can be done by solving a dynamic program. However, due to the curse of dimensionality [11], the computational cost of solving a dynamic program grows exponentially in ℓm, which precludes its use when the number of looks available ℓ is moderately large.

This thesis is aimed at the situation where ℓ is large. The optimal search effort allocation is determined in two stages: First, presuming knowledge of the true status of state of nature (i.e., ζ_i equal to 0 or 1, for $i = 1, \ldots, m$), use large deviations theory to

characterize the optimal effort allocation; and second, use adaptive ideas to generate a search sequence that provably converges to the optimal effort allocation determined in the first stage. To the best of our knowledge, this is the first thesis that describes the optimal effort allocation in the context of threat searching along multiple area-cells where the sensor has false positive and false negative errors, and provides an implementable adaptive algorithm. We do not consider travel time among the area-cells because, for a large number of looks, it affects on the optimal effort allocations is not significant. This happens because the sensor could stay in the same area-cell for a large number of looks, then move to another area-cell and remain there for a long time, and so on, all the while satisfying the optimal search effort allocation. When the number of looks is relatively small, one should consider resource constraints; see, for example, [13].

The remainder of this thesis is organized as follows. In Chapter II, we discuss what we mean by determining an area-cell, using results from the hypotheses testing literature. In Chapter III, we determine the optimal search effort allocations, and present an adaptive algorithm that results in sampling allocations, which converge almost surely to the optimal allocations. In Chapter IV, we present a few examples that illustrate the main results. The concluding remarks appear in the last chapter.

II. PROBLEM FRAMEWORK

A GOAL

A key issue is the determination of whether an area-cell contains a stationary threat object or not. Intuitively, the average number of detections in area-cell i should approach a_i if $\zeta_i = 1$ and b_i if $\zeta_i = 0$. Since $a_i > b_i$, a good decision rule should conclude that a threat is present if the average number of detection cues in that area is close to or above a_i, and that a threat is not present if the average is close to or below b_i. Hence, as each area-cell is looked at more, it will become less likely to have a wrong determination.

B. SET-UP

This problem has been well studied in the statistical literature; see, for example, [9]. In this section, we make the relevant ideas precise and set the stage for the main results.

Let the random variable $X_{i,j} = 1$ if the j'th look into area-cell i is a *detection* signal, and $X_{i,j} = 0$ otherwise, for $j = 1, \ldots, \ell_i$, where ℓ_i is the number of looks into area-cell i. For each area-cell i, $(X_{i,j} : j = 1,..., \ell_i)$ is a collection of independent and identically distributed random variables with Bernoulli (a_i) distribution if there is a threat object, or Bernoulli (b_i) otherwise.

A decision test S is a sequence of measurable maps with respect to $X_{i,1}, \ldots, X_{i,\ell_i}$ from $\{0,1\}^{\ell_i}$ into $\{0,1\}$ such that if $S_{\ell_i}(x_{i,1},...,x_{i,\ell_i}) = 1$ then $H_{i,0}$ is accepted, and if $S_{\ell_i}(x_{i,1},...,x_{i,\ell_i}) = 0$ then $H_{i,0}$ is rejected. The error probabilities produced by the decision test S are

$$\alpha_{\ell_i} = P_1(S_{\ell_i} \text{ rejects } H_{i,0}), \qquad \text{(Type I error probability)}$$

and

$$\beta_{\ell_i} = P_1(S_{\ell_i} \text{ accepts } H_{i,0}), \qquad \text{(Type II error probability)}$$

We define the sample average of the sensor signals, by

$$\bar{X}_i(\ell_i) = \frac{1}{\ell_i}\sum_{j=1}^{\ell_i} X_{i,j},$$

and let $S^*_{\ell_i}(x_{i,1},...,x_{i,\ell_i}) = 1$ if

$$\left(\frac{a_i}{b_i}\right)^{\ell_i \bar{x}_i(\ell_i)} \left(\frac{1-a_i}{1-b_i}\right)^{\ell_i[1-\bar{x}_i(\ell_i)]} > 1, \tag{1}$$

and $S^*_{\ell_i}(x_{i,1},...,x_{i,\ell_i}) = 0$ otherwise. In other words, $S^*_{\ell_i}(x_{i,1},...,x_{i,\ell_i}) = 1$ if and only if the likelihood of having a threat object is greater than the likelihood of not having a threat object.

The Bayes probability of error is given by $\alpha_{\ell_i}\pi_i + \beta_{\ell_i}(1-\pi_i)$. Chernoff's bound ([4], pp.93) asserts that if $0 < \pi_i < 1$ then

$$\inf_S \lim \inf_{\ell_i \to \infty} \frac{1}{\ell_i} \log[\alpha_{\ell_i}\pi_i + \beta_{\ell_i}(1-\pi_i)] = -\left[\gamma_i \log\left(\frac{\gamma_i}{a_i}\right) + (1-\gamma_i)\log\left(\frac{1-\gamma_i}{1-a_i}\right)\right],$$

where the infimum is taken over all the decision tests, and

$$\gamma_i = \frac{\log\left(\frac{1-b_i}{1-a_i}\right)}{\log\left(\frac{a_i}{1-a_i}\frac{1-b_i}{b_i}\right)}.$$

Several comments are in order. The infimum above is achieved by the decision test $S^*_{\ell_i}$ so that $S^*_{\ell_i}$ is optimal in the following sense: among all the decision tests, it minimizes the Bayes probability of error in log scale (i.e., it is the decision test with the largest Bayes probability of error exponential decay rate), as the number of looks $\ell_i \to \infty$. Hence, for the rest of this thesis, we deal with the decisions test $S^*_{\ell_i}$ for each area-cell i = 1, …, m.

By a straightforward algebraic manipulation, it can be seen that (1) holds if and only if $\bar{X}_i(\ell_i) > \gamma_i$. Thus the parameters γ_i are determination thresholds, meaning that if $\bar{X}_i(\ell_i) \le \gamma_i$, then the sensor operator declares that the area-cell does not contain a threat object, and if $\bar{X}_i(\ell_i) > \gamma_i$ then the area-cell is declared to contain a threat object.

III. IMPLEMENTATION OF ALGORITHM

A. ALGORITHM APPROACH

Let θ_i be the fraction of the search budget ℓ that corresponds to area-cell i, so that the number of looks into area-cell i is $\ell_i = \theta_i \ell$. To make the mathematical proceedings less cumbersome, we work with $\bar{X}_i(\theta_i \ell)$ (the average of the observations taken over $\theta_i \ell$ looks), rather than with $\bar{X}_i(\lfloor \theta_i \ell \rfloor)$, where $\lfloor \cdot \rfloor$ is the floor operator. Since our results hold for ℓ large, they continue to be true when the integrality condition is enforced, by working with a sequence that goes to infinity.

Assume, without loss of generality, that area-cells 1,… , r contain a threat object, and that area cells $r + 1$, … , m do not contain a threat object. Recall that the goal is to allocate the sensor in order to minimize the expected number of incorrectly determined area-cell. More precisely, the goal is to

$$\min\ g_\ell(\theta_1,...,\theta_m),$$

subject to

$$\sum_{i=1}^{m} \theta_i \le 1,$$
$$\theta_i \ge 0 \quad \text{for } i = 1, \ldots, m,$$

where

$$g_\ell(\theta_1,...,\theta_m) = \sum_{i=1}^{r} P[\bar{X}_i(\theta_i \ell) \le \gamma_i] + \sum_{i=r+1}^{m} P[\bar{X}_i(\theta_i \ell) > \gamma_i]. \tag{2}$$

Our contribution is two-fold: (i) we characterize fractional allocations $\theta_1^*,...,\theta_m^*$ that are optimal (in log scale) as $\ell \to \infty$; and (ii), we provide an easily implementable algorithm, rooted in stochastic approximation theory that results in sampling allocations that provably achieve the same performance (in log scale) as the optimal allocations in the limit as $\ell \to \infty$.

Large deviations theory [4] suggests that each of the summands in (2) decays roughly decays exponentially fast with the number of looks. The decay rate depends on

the large deviations rate function $I_i(\gamma_i) = \sup_\eta \{\eta\gamma_i - \log E\exp(\eta X_{i,1})\}$, which for a non-degenerate Bernoulli random variable $X_{i,1}$ with mean μ_i is given by

$$I_i(\gamma_i) = \gamma_i \log\left(\frac{\gamma_i}{\mu_i}\right) + (1-\gamma_i)\log\left(\frac{1-\gamma_i}{1-\mu_i}\right), \tag{3}$$

for $0 < \gamma_i < 1$; see Exercise 2.2.23 of [4]. The next result, whose proof can be found in [16], characterizes the decay rate of the expected number of wrongly determined area-cells.

1. Proposition 1 – Decay Rate

Suppose $a_i > b_i$ for each area-cell. Then

$$\lim_{\ell\to\infty}\frac{1}{\ell}\log g_\ell(\theta_1,...,\theta_m) = -\min_i \theta_i I_i(\gamma_i). \tag{4}$$

Proposition 1 asserts that the expected number of incorrectly determined area cells decay exponentially fast, with a rate that is the smallest decay rate amongst all area-cells. This suggests that a good allocation should maximize the slowest decay rate, i.e., the minimum of $\theta_i I_i(\gamma_i)$. The next result, from [16], shows that this approach is optimal among all feasible allocations.

2. Proposition 2 – Optimal Allocation

Suppose $a_i > b_i$ for each area-cell. Then

$$\lim_{\ell\to\infty}\frac{1}{\ell}\log g_\ell(\theta_1^*,...,\theta_m^*) = -\frac{1}{\sum_{k=1}^m I_k^{-1}(\gamma_k)}$$

where

$$\theta_i^* = \frac{I_i^{-1}(\gamma_i)}{\sum_{k=1}^m I_k^{-1}(\gamma_k)}. \tag{5}$$

For any other feasible $(\theta_1,...,\theta_m)$ we have

$$\liminf_{\ell\to\infty}\frac{1}{\ell}\log g_\ell(\theta_1,...,\theta_m) \geq -\frac{1}{\sum_{k=1}^m I_k^{-1}(\gamma_k)}.$$

The elements θ_i^* as the optimal allocation scheme, meaning that no other allocation achieves a higher exponential decay rate for $g_\ell(\cdot)$ as the number of looks goes to infinity. Proposition 2 results in

$$\theta_i^* = \frac{\left[\gamma_i \log\left(\frac{\gamma_i}{\mu_i}\right) + (1-\gamma_i)\log\left(\frac{1-\gamma_i}{1-\mu_i}\right)\right]^{-1}}{\sum_{k=1}^{r}\left[\gamma_k \log\left(\frac{\gamma_k}{\mu_k}\right) + (1-\gamma_k)\log\left(\frac{1-\gamma_k}{1-\mu_k}\right)\right]^{-1}}, \tag{6}$$

where $\mu_i = a_i$ for $i = 1, \ldots, r$, and $\mu_i = b_i$ for $i = r + 1, \ldots, m$. It can be seen from (6) that the optimal fractional allocations tend to be large when μ_i (i.e., a_i if a threat object is present, or b_i otherwise) is close to the determination threshold γ_i. This happens because, in that case, the probability of having $\overline{X}_{i,\ell_i}$ on "the wrong side" is relatively large, and more looks are needed to compensate for the bigger error probability.

While Proposition 2 characterizes the optimal search allocation, the fractions θ_i^* depend on knowledge about the presence / absence of the threat object, which is precisely what we are trying to determine. In the next section, we present a stochastic approximation algorithm that overcomes this issue and leads to fractional allocations that almost surely converge to the optimal θ_i^* allocations.

B. STOCHASTIC APPROXIMATION ALGORITHM

First, we present the algorithm, with the intuition behind it following. Initially we set

$$\tilde{X}_{i,0} = \tilde{x}_i,$$

for $0 < \tilde{x}_i < 1$, and

$$I_{i,0}(\gamma_i) = \gamma_i \log\left(\frac{\gamma_i}{\tilde{X}_{i,0}}\right) + (1-\gamma_i)\log\left(\frac{1-\gamma_i}{1-\tilde{X}_{i,0}}\right).$$

Based on prior intelligence, the value $\tilde{x}_i$ is our best guess of μ_i at stage 0; for instance, $\tilde{x}_i = a_i$ if $\pi_i > 0.5$ and $\tilde{x}_i = b_i$ if $\pi_i \leq 0.5$. The rate functions that lead to (6) are

estimated by substituting μ_i by $\tilde{X}_{i,0}$ in (3). The initial stage is $\ell = 0$ and the initial sample sizes are $\lambda_{i,0} = 0$.

1. Algorithm

1. Generate a replicate ξ from the probability mass function $I_{i,\ell}^{-1} / \sum_{k=1}^{m} I_{k,\ell}^{-1}$,

for $i = 1, \ldots, m$.

2. Update sample sizes: $\lambda_{\xi,\ell+1} = \lambda_{\xi,\ell} + 1$, and $\lambda_{i,\ell+1} = \lambda_{i,\ell}$ for $i \neq \xi$.

3. Generate a sample from area-cell ξ, (say) X_{ξ,λ_ξ}.

4. Update $\tilde{X}_{\xi,\ell+1}$ and $I_{\xi,\ell+1}$:

$$\tilde{X}_{\xi,\ell+1} = \tilde{X}_{\xi,\ell} + \frac{1}{\lambda_{\xi,\ell+1}}(X_{\xi,\lambda_\xi} - \tilde{X}_{\xi,\ell})$$

and

$$I_{\xi,\ell+1} = \gamma_\xi \log\left(\frac{\gamma_\xi}{\tilde{X}_{\xi,\ell+1}}\right) + (1-\gamma_\xi)\log\left(\frac{1-\gamma_\xi}{1-\tilde{X}_{\xi,\ell+1}}\right).$$

For $i \neq \xi$, set $\tilde{X}_{i,\ell+1} = \tilde{X}_{i,\ell}$, and $I_{i,\ell+1} = I_{i,\ell}$.

5. Increase $\ell \rightarrow \ell + 1$ and go to step 1.

To ensure that each area-cell is searched infinitely often, let $(\nu_\ell)_{\ell=0}^{\infty}$ be an increasing sequence such that $\nu_\ell \rightarrow \infty$ and $\ell^{-1}\sum_{k=1}^{\ell} J(\nu_k \leq \ell) \rightarrow 0$, where $J(\cdot)$ is the indicator function. We search all m area-cells at iteration $\nu_1, \nu_2, \ldots$, and update the parameters according to step 2 and 4 of the algorithm.

In step 1, we decide where to look next. This is accomplished by sampling from the probability mass function $I_{i,\ell}^{-1} / \sum_{k=1}^{\mathrm{m}} I_{k,\ell}^{-1}$, which is the best guess of the optimal allocation at stage ℓ. In step 3, the searcher generates an observation by sampling from a Bernoulli distribution with parameter a_ξ if area-cell ξ contains a threat object (i.e., ξ

$\in \{1, \ldots, r\}$), or from a Bernoulli distribution with parameter b_ξ otherwise. In step 4, we update the sample average and sample large deviations rate function of area-cell ξ.

The see why our algorithm leads to the optimal allocations, let $\theta_{i,\ell} = \lambda_{i,\ell} / \ell$ be the fractional allocation in stage ℓ of the algorithm. Hence, step 2 of the algorithm can be expressed as $\theta_{i,\ell+1} = \theta_{i,\ell} + (J(\xi_\ell = i) - \theta_{i,\ell}) / (\ell + 1)$, where ξ_ℓ is the ℓ'th replicate of ξ generated in step 1 of the algorithm. The recursion for $\theta_{i,\ell+1}$ can be re-written as

$$\theta_{i,\ell+1} = \theta_{i,\ell} + \frac{1}{\ell+1}(\theta_i^* - \theta_{i,\ell}) + \varepsilon_\ell,$$

where

$$\varepsilon_\ell = \frac{1}{\ell+1}[J(\xi_\ell = i) - q_{i,\ell}] + \frac{1}{\ell+1}(q_{i,\ell} - \theta_i^*),$$

and $q_{i,\ell} = I_{i,\ell}^{-1} / \sum_{j=1}^{m} I_{j,\ell}^{-1}$. If the error ε_ℓ becomes small relative to $(\theta_i^* - \theta_{i,\ell}) / (\ell + 1)$ term, then $\theta_{i,\ell}$ follows, as $\ell \to \infty$, the path of the solution of the ordinary differential equations

$$\theta_i^{'} = \theta_i^* - \theta_i, \quad i = 1, \ldots, m,$$

which has θ_i^* as the unique globally asymptotically stable point. This suggests that if the variability introduced by the error is sufficiently small, our algorithm provides fractional allocations that converge almost surely to the optimal allocations. The preceding argument leads to:

2. Conjecture 1 – Convergence to Optimal Allocation

The stochastic approximation algorithm search allocations converge with probability one to the optimal allocations determined by (5);

$$\frac{\lambda_{i,\ell}}{\ell} \to \theta_i^*,$$

almost surely as $\ell \to \infty$.

The proof of this conjecture will appear elsewhere.

Remark. $\theta_i^* I_i(\gamma_i) = \theta_k^* I_k(\gamma_k)$ suggests that step 2 of the algorithm could be replaced by $\xi = \arg\min_i \{\lambda_{i,\ell} I_{i,\ell}\}$.

The preceding can leads to the following conjecture: the expected number of incorrectly determined systems area-cells produced by the stochastic approximation algorithm approaches 0 at the best possible rate.

3. Conjecture 2 – Expected Number of Incorrectly Determined Area-cells Approaches Zero

The expected number of incorrect determinations satisfy

$$\frac{1}{\ell}\log\left[\sum_{i=1}^{r} P(\tilde{X}_{i,\ell} \le \gamma_i) + \sum_{i=r+1}^{m} P(\tilde{X}_{i,\ell} > \gamma_i)\right] \to -\frac{1}{\sum_{k=1}^{m} I_k^{-1}(\gamma_k)},$$

as $\ell \to \infty$.

IV. NUMERICAL VALIDATION

A. INPUT PARAMETERS

We now illustrate the results of the previous sections via three examples with up to 12 area-cells. In the first case, we consider 4 area-cells, with just the first 2 area-cells containing a threat object. The "baseline" sensor parameters are a = (0.8, 0.7, 0.6, 0.9), and b = (0.3, 0.1, 0.2, 0.2). These parameters lead to the thresholds γ = (0.5609, 0.3608, 0.3869, 0.5803) and to the optimal allocations θ^* = (0.2773, 0.1659, 0.4401, 0.1167).

B. ILLUSTRATION OF PROPOSITION 1 AND 2

In the first example, we portray the results of propositions 1 and 2. For each area-cell, we analytically compute the exact probability of incorrect determination, given by

$$\sum_{k=0}^{\lfloor \ell\gamma_i\theta_i^* \rfloor} \binom{\lfloor \ell\theta_i^* \rfloor}{k} a_i^k (1-a_i)^{\ell-k}$$

for $i = 1, \ldots, r$, and by

$$\sum_{k=\lfloor \ell\gamma_i\theta_i^* \rfloor+1}^{\ell} \binom{\lfloor \ell\theta_i^* \rfloor}{k} b_i^k (1-b_i)^{\ell-k}$$

for $i = r+1, \ldots, m$. The expected number of incorrectly determined area-cells is the sum of these two expressions over all the area-cells. This is shown in Figure 1, for three cases:

- Optimal allocation, θ_i^* (cf.(5) and (6));
- Constant allocation, with $\theta^* = \ell / m$;
- Simple allocation, where the determination thresholds are the midpoints between a_i and b_i, $(a_i - b_i)/2$. If the allocations are inversely proportional to the distance $a_i - (a_i - b_i)/2$, the allocation becomes $\theta_i = \ell\left(a_i - b_i\right)^{-1} / \sum_{k=1}^{m}\left(a_k - b_k\right)^{-1}$.

As expected from Proposition 1, the logarithm of the expected number of incorrect determinations for the three types of allocation, decays linearly as the number of looks increases. Also, in agreement with Proposition 2, the slope of the line

corresponding to the optimal allocations is more negative than the slope of the line corresponding to the constant and simple allocations. Another take-away from Figure 1, which we also observed through numerical experimentation, is that the sub-exponential terms appear when computing the expected number of incorrectly determined area-cells become unnoticeable for relatively small values of ℓ; i.e., we get a straight line in Figure 1 even for ℓ small. This suggests that, although all our results apply in the limits as $\ell \to \infty$, the transient dies out fairly quickly.

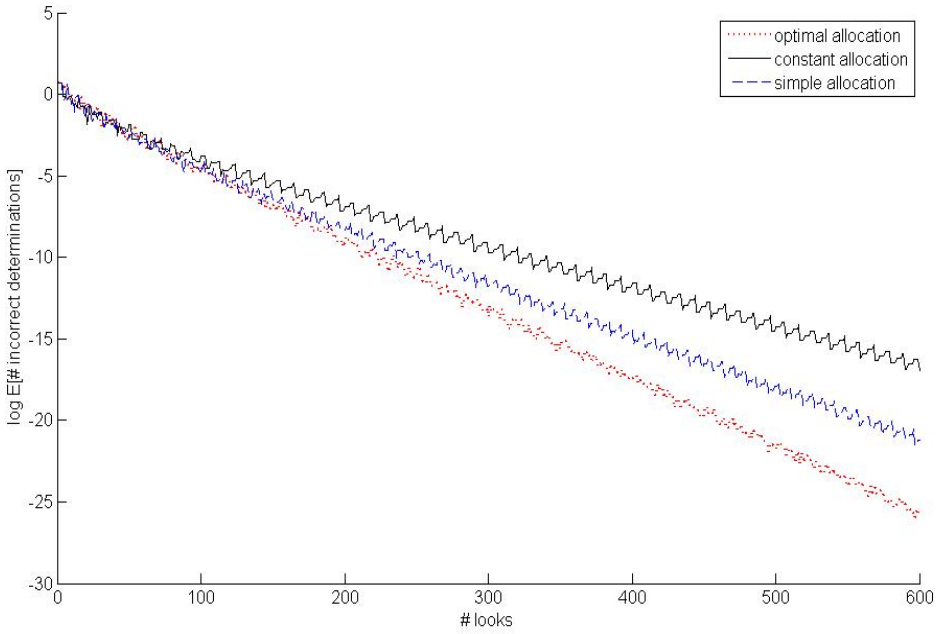

Figure 1. Exponential decay rate of expected number of wrongly determined area-cells for optimal allocation, constant allocation and simple allocation.

In Figure 2, we illustrate the effect of increasing the number of area-cells for the optimal allocation, and of having the difference between a and b being small. Specifically, we consider the case a = (0.5, 0.7, 0.6, 0.4) and b = (0.4, 0.55, 0.55, 0.2).

We plot the case of 4, 8 and 12 area-cells where the parameters and number of threat objects for the 8 and 12 area-cells scenario are copies of those for the 4 area-cell scenario. As can be seen, the performance decreases with the number of area-cells. This can be justified as follows. Recall that the optimal decay rate is

$$\left(\sum_{k=1}^{m} I_k^{-1}(\gamma_k) \right)^{-1}.$$

Clearly, the above expression is decreasing in m, meaning that the expected number of incorrectly determined area-cells approaches zero at a slower rate; i.e., we are bound to make more erroneous determinations. This is the reason why the slope of the lines corresponding to $m = 8$ and $m = 12$ becomes closer to zero.

We observe that the expected number of incorrectly determined area-cells is much larger with the current a and b than with the baseline parameters. This illustrates the potential operational value of our approach, since now we are working with a much larger proportion of incorrectly determined area-cells.

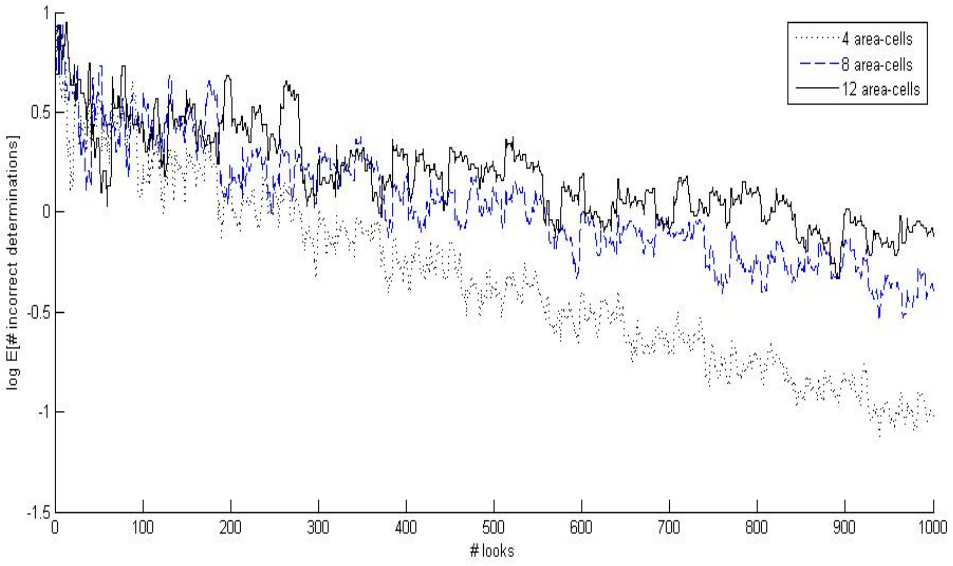

Figure 2. Exponential decay rate of expected number of wrongly determined area-cells for 4, 8 and 12 area-cells scenario using optimal allocation.

C. ILLUSTRATION OF CONJECTURE 1

In the last example, we implement the stochastic approximation algorithm, without presuming knowledge of the presence or absence of the threat object in each cell. The goal is to illustrate conjecture 1. We generate a single replication and plot the fractional allocations generated by the adaptive algorithm. Figure 3 illustrates how the

allocations approach the optimal allocations (6). As can be seen, the convergence to the optimal allocation is already fairly good for 100 looks, and becomes even better as the number of looks increases.

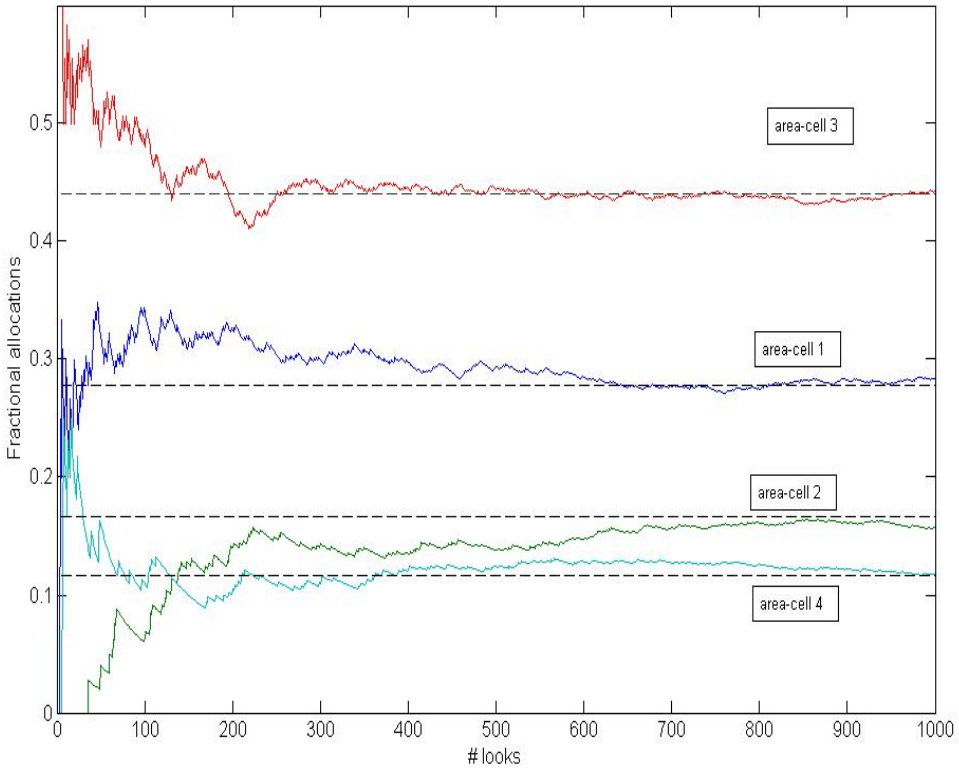

Figure 3. Sample allocations versus optimal allocations.

V. CONCLUSIONS

A. CONCLUDING REMARKS AND FUTURE WORKS

In this thesis, we developed a model for a single sensor that searches for threat objects, when the sensor is subject to false positive and false negative errors. The space is discrete, with each area-cell containing at most one stationary threat object. We employed the large deviations theory to characterize the optimal search effort allocations, and developed a stochastic approximation algorithm. The fractions of time spent in each area-cell, generated by the algorithm, converge almost surely to the optimal search effort allocations. This causes the number of incorrectly determined area-cells to decay at the fastest possible exponential rate. Our results were illustrated via 3 numerical examples, all of which verified our conjectures. Our methodology is appropriate when the number of looks is large; when the number of looks is small, other techniques, such as numerical optimization and dynamic programming may be appropriate.

The models developed in this thesis may be extended in several directions: dealing with multiple sensors, consideration of an arbitrary number of targets in each area-cell or of a single target in the area of interest, etc.

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