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This report describes the results of a study of the fundamental issues in electromagnetic scattering theory with an emphasis on pole behaviors of a lossless sphere arising from the singularity expansion method (SEM). Mie theory is used to solve the acoustic and electromagnetic scattering problems for spheres with lossless boundary conditions and an incident plane wave. It is shown that, for certain lossless sheet impedance boundary conditions there exist second order poles for both cases. The general procedure to directly construct lossless sheet impedance boundary conditions that will produce high order poles is discussed, in addition to the difficulties that subsequently arise. For the electromagnetic scattering case, Foster's theorem is imposed on the impedance condition to ensure that a lossless scattering problem is obtained. The validity of the forward scattering theorem associated with the SEM is studied as well.

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Scattering From a Lossless Sphere

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Abstract

We study fundamental issues in electromagnetic scattering theory, with an emphasis on pole behaviors of a lossless sphere arising from the singularity expansion method (SEM). We use Mie Theory to solve the acoustic and electromagnetic scattering problems for spheres with lossless boundary conditions and an incident plane wave. We show that for certain lossless sheet impedance boundary conditions there exist second order poles for both cases. Our general procedure to directly construct lossless sheet impedance boundary conditions which will produce high order poles is discussed as well as the difficulties to which it leads. In the electromagnetic scattering case, Foster's Theorem is imposed on the impedance condition to ensure that a lossless scattering problem is obtained. We also study the validity of the forward-scattering theorem associated with SEM.

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1 Introduction

The singularity expansion method (SEM) [2] was introduced in 1971 as a way to represent the solution of electromagnetic interaction or scattering problems in terms of the singularities in the complex-frequency (s of two-sided-Laplace-transform) plane. Particularly for the pole terms associated with a scatterer (natural frequencies), their factored form separates the dependencies on various parameters of the incident field, observer location, and scatterer characteristics, with an equally simple form in both frequency (poles) and time (damped sinusoids) domains.

The forward-scattering theorem is a classical result in electromagnetic theory. A recent paper [4] has generalized the forward-scattering theorem with particular application to lossless bodies. From the forward-scattering theorem we have relations between the absorption and scattering cross sections, and the forward scattering. The scattered fields are represented by a scattering dyadic times the incident plane wave. This allows one to reformulate the results in terms of the scattering dyadic, exhibiting some general characteristics of this dyadic. It is extended (for lossless scatterers) by analytic continuation away from the $j\omega$ axis out into the complex s plane and applied to poles in the singularity expansion method. In particular this gave new insight into the scattering natural frequencies and modes, also implying new ways to calculate them from the scattering operator in the right half of the complex $s-$plane. This gives new insight into the properties of the poles (natural frequencies, s_α) in the left-half plane and the associated natural scattering modes.

Although in practice, we only encounter the first order scattering poles, an interesting question concerning the SEM and the forward-scattering theorem concerns the existence of higher order scattering poles. Carl Baum showed that 2nd order poles can be constructed for a transmission line problem [6]. Since the transmission line problem is finitely dimensioned, we can actually use the scattering matrix to find the poles (i.e. the eigenvalues). However, in general the problem is infinitely dimensioned. Thus, we consider a classical model problem, scattering from a sphere with an incident plane wave. We compute the exact solution using Mie Theory. In [2], Carl Baum showed that for a perfectly conductor sphere, there only exist first order poles. Sancer [12] also proved some similar results for a general shape scatterer.

We show that for the acoustic scattering problem high order poles can be constructed for certain impedance boundary conditions, while for hard and soft spheres there only exist first order scattering poles. The general procedure to construct arbitrary order poles is discussed as well the necessary condition for the conservation of energy to the scattering problem. For electromagnetic scattering from a sheet impedance loading sphere, we show that there exist 2nd order poles. Foster's Theorem is imposed on the impedance condition to ensure the scatterer is lossless. The existence of a 2nd order pole for a transmission line scattering problem is included in section 4. Some analytical and numerical treatment of the forward-scattering theorem is presented in section 5.

2 Scattering from a lossless acoustic sphere

2.1 Introduction

In this section, we are considering the following problem. An incident plane sound wave is propagating in some direction with the scatterer being a sphere. The scattered solution can be written explicitly using spherical harmonics. However, the thing we are really interested in is to explicitly solve the problem using SEM and to study the scattering pole behavior of the solution. By putting different boundary conditions on the scatterer, we are able to construct not only simple poles but also 2nd or even higher order poles. Meanwhile, we always keep the impedance function satisfying the Foster's Theorem, so that it remains a lossless system.

2.2 Formulation of the acoustic scattering problem

Considering the linear acoustic equation [11]

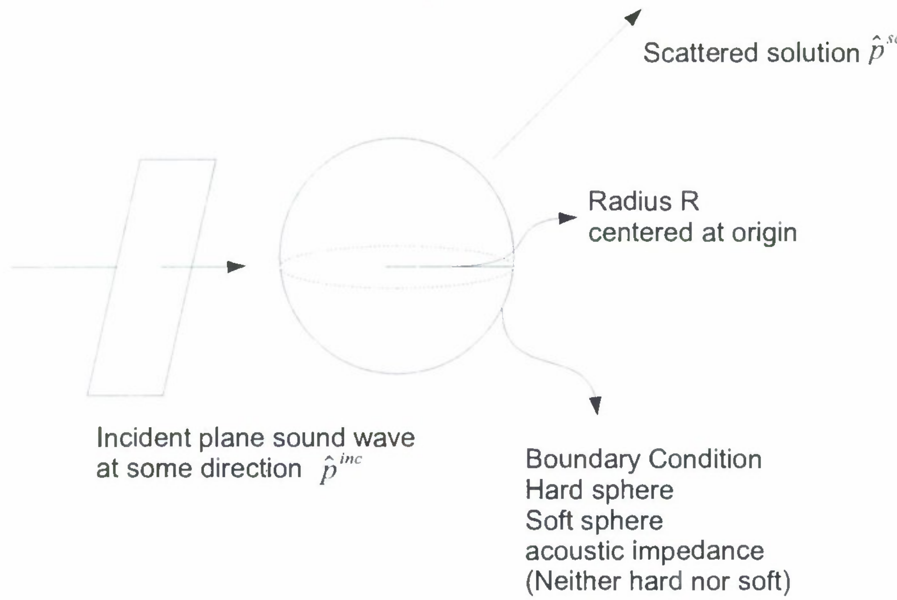
$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot v = 0 \tag{1}$$

$$\rho_0 \frac{\partial v}{\partial t} + \nabla p = 0 \tag{2}$$

$$p = c^2 \rho \quad , \quad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_0$$

where p is for acoustic pressure, ρ is for density, v is for fluid velocity, c is the speed of sound. We want to solve for the scattering solution to the

Figure 1:



above equation with an incident plane wave and some different but lossless boundary conditions on the sphere. We will use the expansion in terms of spherical harmonics to solve the equation in order to locate the poles. In fact, this is just the SEM in the acoustic case. Let's first derive the wave equation from the system above, so that later on we will treat the problem mathematically, temporarily disregarding its physical meaning. Plugging $p = c^2\rho$ into (1) we get

$$\frac{\partial p}{\partial t} + \rho_0 c^2 \nabla \cdot v = 0 \tag{3}$$

$$\rho_0 \frac{\partial v}{\partial t} + \nabla p = 0 \tag{4}$$

if we $\frac{\partial}{\partial t}(3)$ and plug in (4), we get

$$\frac{\partial^2 p}{\partial t^2} + \rho_0 \nabla \cdot (-\frac{1}{\rho_0} \nabla p) = 0$$

$$\Rightarrow \quad \frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p = 0 \tag{5}$$

similarly for v, we $\frac{\partial}{\partial t}$ (4), and plug in (3) we get

$$\frac{\partial^2 v}{\partial t^2} + \frac{1}{\rho_0} \nabla(-\rho_0 c^2 \nabla \cdot v) = 0$$

$$\Rightarrow \quad \frac{\partial^2 v}{\partial t^2} - c^2 \nabla(\nabla \cdot v) = 0$$

assume v is irrotational i.e. $\nabla \times v = 0$, then $\nabla(\nabla \cdot v) = \nabla^2 v$. We get

$$\frac{\partial^2 v}{\partial t^2} - c^2 \nabla^2 v = 0 \tag{6}$$

Thus if u is either p or v,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$$

Taking the Laplace transformation we get

$$(\nabla^2 - \gamma^2)\hat{u} = 0, \quad where \quad \gamma = \frac{s}{c} \tag{7}$$

Assume the scatterer is a ball with radius 1 and impedance boundary condition of the following form

$$\frac{\partial \hat{u}}{\partial n} + \alpha(s)\hat{u} = 0 \quad , \quad at \quad r = 1 \tag{8}$$

where n is the outward normal.

Assume the incident plane wave is in the z-direction i.e.

$$\hat{u}^{(inc)} = e^{-\gamma(0,0,1)\cdot(x,y,z)} = e^{-\gamma z} \tag{9}$$

We want to solve scattered solution $\hat{u}^{(sc)}$explicitly according to (7)(8)(9) and study the pole behavior of the scattered solution $\hat{u}^{(sc)}$.

2.3 Hard and soft spherical scatterer

First of all, we want to relate the mathematical impedance function $\alpha(\gamma)$ to the actual acoustic impedance $Z_a(s)$. Units of acoustic impedance are $Pa \cdot s/m$ or $kg/(m^2 \cdot s)$ By definition

$$Z_a(s) = \frac{\hat{p}(r,s)}{\hat{v}(r,s) \cdot n_{in}} = \rho_a \hat{v}(s) = \left(\frac{\rho_a}{\hat{k}_a(s)}\right)^{\frac{1}{2}}$$

where $\hat{p}, \hat{v}$ are the Laplace transforms of p and v, and n_{in} is the inward normal. Take the Laplace transform of (2), we get $s\hat{v} + \frac{1}{\rho_0}\nabla\hat{p} = 0$. And then take the inner product with outward normal direction n , we get

$$\begin{aligned} s\hat{v} \cdot n + \frac{1}{\rho_0}\frac{\partial \hat{p}}{\partial n} &= 0 \\ \Rightarrow \frac{\partial \hat{p}}{\partial n} &= -\rho_0 s \hat{v} \cdot n \end{aligned} \tag{10}$$

Then acoustic impedance can be written as

$$Z(s) = \frac{\hat{p}}{\hat{v} \cdot n_{in}} = \frac{\hat{p}}{\frac{\partial \hat{p}}{\partial n}}\rho_0 s$$

In our formulation, if we plug (10) into (8), assuming we put the impedance on pressure(i.e. $u = p$ here), we derive

$$\frac{\hat{p}}{\hat{v} \cdot n_{in}} = -\frac{\rho_0 s}{\alpha(s)} = Z(s)$$

which relates the mathematical impedance condition to the real acoustic impedance boundary condition.

The infinite specific-acoustic-impedance limit $|Z| \to \infty$ corresponds to a hard(rigid) surface, the limit $|Z| \to 0$ corresponds to a soft(pressure-release) surface. Thus, for a hard sphere the mathematical impedance boundary conditions become $\frac{\partial \hat{u}}{\partial n} = 0$ and $\hat{u} = 0$ for the soft sphere.

For detailed computation of the following please see section 5.

$$\hat{u}^{(inc)} = e^{-\gamma z} = e^{-\gamma r \cos\theta} = \sum_{n=0}^{\infty}(2n+1)(-1)^n i_n(\gamma r) P_n(\cos\theta)$$

$$\begin{aligned}\hat{u}^{(sc)} &= \sum_{n=0}^{\infty}\sum_{m=-n}^{m=n} a_{nm}(\gamma)k_n(\gamma r)Y_n^m(\theta,\varphi) \\ &= \sum_{n=0}^{\infty} a_n(\gamma)k_n(\gamma r)P_n(\cos\theta)\end{aligned}$$

Where $k_n(s)$ is the modified Bessel's function, $a_n(\gamma)$ is a coefficient to be determined.

Applying the impedance boundary condition for hard and soft spheres respectively, at $r = 1$

$$\frac{\partial \hat{u}^{(sc)}_{hard}}{\partial n} = -\frac{\partial \hat{u}^{(inc)}_{hard}}{\partial n}$$

$$\hat{u}^{(sc)}_{soft} = -\hat{u}^{(inc)}_{soft}$$

n is the outward normal, i.e. r in our case(sphere).

We derive the scattered solutions as follows

$$\hat{u}^{(sc)}_{hard} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)[\gamma i'_n(\gamma)]}{\gamma k'_n(\gamma)}k_n(\gamma r)P_n(\cos\theta)$$

$$\hat{u}^{(sc)}_{soft} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)[\gamma i_n(\gamma)]}{k_n(\gamma)}k_n(\gamma r)P_n(\cos\theta)$$

Since the modified spherical Bessel functions k_n and their derivatives only have simple zeros [2], for both the hard and soft sphere scatterer, the solution only has simple poles.

2.4 Lossless impedance loading of a sphere

Consider the total energy in the usual mathematical way

$$E_D = \int\int_D\int \frac{1}{2}(u_t^2 + |\nabla u|^2)$$

$$
\begin{aligned}
\frac{dE_D}{dt} &= \iiint\limits_D u_t u_{tt} + \nabla u^T \cdot \nabla u_t \\
&= \iiint\limits_D u_t(u_{tt} - \nabla^2 u) + \int\limits_{\partial D} u_t u_\nu \\
&= \iint\limits_{\partial D} u_t u_\nu
\end{aligned}
$$

and the net energy flux is

$$
\iint\limits_{\partial D} s\hat{u}(s) \cdot \overline{\frac{\partial \hat{u}(y,s)}{\partial r}} \tag{11}
$$

where in our case D is a sphere. Plugging in our impedance boundary condition, the net energy flux becomes

$$
-\iint\limits_{\partial D} s\overline{\alpha(s)}|\hat{u}(y,s)|^2
$$

For a lossless case, the total energy change should be zero, which implies $\int_{-i\infty}^{i\infty}(11)=0 \Rightarrow (11)$ should be odd $\Rightarrow \alpha(s)$ should be even, $Z(s)$ odd. Mathematically, this condition is necessary to guarantee that energy is conserved. However, in order to extend our results later on, we need more constraints on $Z(s)$ or equivalently on $\alpha(s)$. Namely, we want $Z(s)$ to satisfy the Foster reactance theorem which we will discuss a little bit later.

2.5 High order poles

2.5.1 2nd order poles

The general expansion (see section 5 for more details about the calculations) of the scattered solution can for our impedance condition can been written as

$$
\hat{u}^{(sc)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\alpha(s)i_n(s) + si'_n(s)]}{\alpha(s)k_n(s) + sk'_n(s)} k_n(sr)P_n(\cos\theta)
$$

Assume $c = 1$ here, so $\gamma = s$

The far field pattern of the solution is

$$\hat{u}_{\infty}^{(sc)} = \frac{e^{-sr}}{r} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\alpha(s)i_n(s) + si_n'(s)]}{s(\alpha(s)k_n(s) + sk_n'(s))} P_n(\cos\theta)$$

In order to construct a second order pole we need both the denominator and the derivative of the denominator for the scattered solution to be zero at some specific s.

That is, the denominator=0

$$s\alpha(s)k_n(s) + s^2 \frac{d}{ds} k_n(s) = 0 \tag{12}$$

and the derivative of the denominator=0

$$\alpha(s)\, k_n(s) + s \left(\frac{d}{ds} \alpha(s) \right) k_n(s) + s\alpha(s) \frac{d}{ds} k_n(s) + 2\, s \frac{d}{ds} k_n(s) + s^2 \frac{d^2}{ds^2} k_n(s) = 0 \tag{13}$$

Note that we do not want to solve the above system of ODEs, because we only need the equations to hold at one specific s for some pre-chosen impedance boundary condition $\alpha(s)$

By using Bessel's equation itself we can replace

$$s^2 \frac{d^2}{ds^2} k_n(s) = 2\, s \frac{d}{ds} k_n(s) - \left(s^2 + n(n+1)\right) k_n(s)$$

Solve (12) for $\frac{d}{ds} k_n(s)$ and plug into (13), we get

$$k_n(s)\,(s \left(\frac{d}{ds} \alpha(s) \right) + a(s) - \alpha(s)^2 + s^2 + n + n) = 0$$

for $k_n(s) \neq 0$ we want

$$s\alpha' + s^2 + \alpha + n(n+1) - \alpha^2 = 0$$

Choose $n = 1, k_1(s) = \frac{\pi}{2} \frac{s+1}{s^2} e^{-s}$

Solve (12) for $\alpha(s)$, we get

$$\alpha(s) = \frac{s^2 + 2\, s + 2}{s + 1}$$

Solve (13) for $\alpha'(s)$, we get

$$\alpha'(s) = \frac{s\,(s+2)}{s^2+2\,s+1}$$

Note: We can not differentiate $\alpha(s)$ here, since it's just the value evaluated at s, not a function.

The solution we derived above simply means that, if we want to construct a 2nd order pole at s, the impedance function $\alpha(s)$ should be chosen

1. $\alpha(s)$ Even function
2. Value of $\alpha(s)$ at s equals $\frac{s^2+2\,s+2}{s+1}$
3. Value of $\alpha'(s)$ at s equals $\frac{s(s+2)}{s^2+2\,s+1}$.

Let's build a concrete example to verify the above results.

Choose $s=-2$, then $\alpha(s)=-2$, $\alpha'(s)=0$

Assume $\alpha(s)$ has the following form

$$\alpha(s) = \frac{c_1+c_2s^2}{1+c_3s^2}$$

after some calculation we get $\alpha(s)\equiv -2$

The denominator of the scattered solution $\hat{u}^{(sc)}$ for $n=1$ is

$$-2sk_1(s)+s^2k_1'(s) = \frac{\pi}{2}\frac{(s+2)^2}{s}e^{-s}$$

Thus, we have a 2nd order pole at $s=-2$.

2.5.2 3rd order poles

The process is similar to the 2nd order pole case. We want denominator, denominator',denominator"=0 at some s.

$$s\alpha\,(s)\,k_n\,(s)+s^2\frac{d}{ds}k_n\,(s)=0 \tag{14}$$

$$\alpha\,(s)\,k_n\,(s)+s\left(\frac{d}{ds}\alpha\,(s)\right)k_n\,(s)+s\alpha\,(s)\,\frac{d}{ds}k_n\,(s)+2s\frac{d}{ds}k_n\,(s)+s^2\frac{d^2}{ds^2}k_n\,(s)=0 \tag{15}$$

$$2\left(\frac{d}{ds}\alpha(s)\right)k_n(s)+2\alpha(s)\frac{d}{ds}k_n(s)+s\left(\frac{d^2}{ds^2}\alpha(s)\right)k_n(s)+2s\left(\frac{d}{ds}\alpha(s)\right)\frac{d}{ds}k_n(s)$$
$$+s\alpha(s)\frac{d^2}{ds^2}k_n(s)+2\frac{d}{ds}k_n(s)+4s\frac{d^2}{ds^2}k_n(s)+s^2\frac{d^3}{ds^3}k_n(s)=0 \tag{16}$$

By computing the derivative of Bessel's equation we get

$$s^2\frac{d^3}{ds^3}k_n(s)=-4s\frac{d^2}{ds^2}k_n(s)-2\frac{d}{ds}k_n(s)+2sk_n(s)+\left(s^2+n(n+1)\right)\frac{d}{ds}k_n(s)$$

together with

$$s^2\frac{d^2}{ds^2}k_n(s)=2s\frac{d}{ds}k_n(s)-\left(s^2+n(n+1)\right)k_n(s)$$

and

$$\frac{d}{ds}k_n(s)=-\frac{\alpha(s)k_n(s)}{s}$$

equation (15), (16) reduce to

$$k_n(s)\left(\alpha(s)+s\frac{d}{ds}\alpha(s)-(\alpha(s))^2+s^2+n^2+n\right)=0 \tag{17}$$
$$k_n(s)\left(2\frac{d}{ds}\alpha(s)+s\frac{d^2}{ds^2}\alpha(s)-2\left(\frac{d}{ds}\alpha(s)\right)\alpha(s)+2s\right)=0 \tag{18}$$

solve (14), (17), (18) for $\alpha, \alpha', \alpha''$ respectively, we get

$$\alpha(s)=-\frac{s\frac{d}{ds}k_n(s)}{k_n(s)}$$

$$\alpha'(s)=-\frac{-s\left(\frac{d}{ds}k_n(s)\right)k_n(s)-s^2\left(\frac{d}{ds}k_n(s)\right)^2+s^2(k_n(s))^2+n^2(k_n(s))^2+n(k_n(s))^2}{s(k_n(s))^2}$$

$$
\begin{aligned}
\alpha''(s) &= 2\{-s\left(\frac{d}{ds}k_n(s)\right)(k_n(s))^2 - 2\,k_n(s)\,s^2\left(\frac{d}{ds}k_n(s)\right)^2 + n^2(k_n(s))^3 \\
&+ n(k_n(s))^3 - s^3\left(\frac{d}{ds}k_n(s)\right)^3 + s^3\left(\frac{d}{ds}k_n(s)\right)(k_n(s))^2 \\
&+ s\left(\frac{d}{ds}k_n(s)\right)n^2(k_n(s))^2 + s\left(\frac{d}{ds}k_n(s)\right)n(k_n(s))^2\}/s^2(k_n(s))^3
\end{aligned}
$$

Choose $n = 2$, $k_2(s) = \frac{1}{2}\frac{\pi e^{-s}\left(s^2+3s+3\right)}{s^3}$, we get

$$
\begin{aligned}
\alpha(s) =& \frac{s^3+4s^2+9s+9}{s^2+3s+3} \\
\alpha'(s) =& \frac{s(s^3+6s^2+12s+6)}{(s^2+3s+3)^2} \\
\alpha''(s) =& \frac{6(3+s^3+6s^2+9s)}{(s^2+3s+3)^3}
\end{aligned}
$$

Again, the solution we derived above simply means that, if we want to construct a third order pole at s, the impedance function $\alpha(s)$ should be chosen

1. $\alpha(s)$ Even function
2. Value of $\alpha(s)$ at s equals $\frac{s^3+4s^2+9s+9}{s^2+3s+3}$
3. Value of $\alpha'(s)$ at s equals $\frac{s(s^3+6s^2+12s+6)}{(s^2+3s+3)^2}$
4. Value of $\alpha''(s)$ at s equals $\frac{6(3+s^3+6s^2+9s)}{(s^2+3s+3)^3}$

Let test our results again with $s = -4$. Thus, we require $\alpha(-4) = -\frac{27}{7}$, $\alpha'(-4) = \frac{40}{49}$, $\alpha''(-4) = -\frac{6}{343}$

Assume $\alpha(s)$ still has the following form

$$
\alpha(s) = \frac{c_1 + c_2 s^2}{1 + c_3 s^2}
$$

we get

$$\alpha(s) = \frac{-\frac{47}{27} - \frac{11}{54}s^2}{1 + \frac{1}{54}s^2} = \frac{-11(s^2 + \frac{94}{11})}{s^2 + 54}$$

The denominator of the scattered solution $\hat{u}^{(sc)}$ for $n = 2$ is

$$\frac{\pi}{2} \frac{e^{-s}\,(s^2 + 3\,s + 12)\,(s + 4)^3}{s^2\,(54 + s^2)}$$

Thus, we have a 3rd order pole at $s = -4$.

2.5.3 Aribitrary order poles

In principle, we can follow this procedure to get poles of any high order at any specific location. However, the computation will become messier and messier. It will become more clear to construct high order poles if we try to view this process using Taylor expansion around the pole s_p.

First, let's introduce some very nice properties of the modified Bessel function [9].

$$-s\frac{k_l'(s)}{k_l(s)} = s + 1 + \hat{S}_l(s)$$

where, for $l \neq 0$,

$$\hat{S}_l(z) = \frac{P_l(z)}{Q_l(z)}$$

$$P_l(z) = \sum_{k=0}^{l-1} \frac{(2l-k)!}{k!(l-k-1)!}(2z)^k$$

$$Q_l(z) = \sum_{k=0}^{l} \frac{(2l-k)!}{k!(l-k)!}(2z)^k$$

We also have the following continued fraction representation for $\hat{S}_l$:

$$\hat{S}_l(z) = \frac{l(l+1)}{2} \frac{1}{z + 1 + \frac{l(l+1)-1\cdot 2}{4(z+2+\frac{l(l+1)-2\cdot 3}{4(z+3+\dots)})}}$$

Thus, it is possible for us to rewrite

$$k_l(s) = \frac{k_l^{(1)}(s)}{k_l^{(2)}(s)}\frac{e^{-s}}{s}$$

where, $k_l^{(1)}(s)$, $k_l^{(2)}(s)$ are just polynomials in s.

Suppose impedance function $\alpha(s)$ has the following form

$$\alpha(s) = -c_0\frac{(s^2+a_0)(s^2+a_2)\cdots(s^2+a_k)}{(s^2+a_1)(s^2+a_3)\cdots(s^2+a_{2n-1})} = -c_0\frac{\alpha_1(s)}{\alpha_2(s)}$$

where, $c_0 > 0$, $0 \leq a_0 < a_1 < a_2 < \cdots < a_{2n-2} < a_{2n-1} < a_{2n} < \infty$, $k = 2n-2\,or\,2n$, and $\alpha_1(s)$, $\alpha_2(s)$ are of course polynomials.

For example, for 2nd order pole we need 2 free parameters, thus $\alpha(s)$ can been chosen

$$\alpha(s) = -c_0(s^2+a_0)$$

For 3rd order pole we need 3 free parameters, thus $\alpha(s)$ can been chosen

$$\alpha(s) = -c_0\frac{(s^2+a_0)}{(s^2+a_1)}$$

For 4th order pole we need 4 free parameters, thus $\alpha(s)$ can been chosen

$$\alpha(s) = -c_0\frac{(s^2+a_0)(s^2+a_2)}{(s^2+a_1)}$$

The denominator to the scattered solution $\hat{u}^{(sc)}$ can now be written as

$$-\frac{k_n^{(1)}(s)}{k_n^{(2)}(s)}\frac{e^{-s}}{s}\Big(c_0\frac{\alpha_1(s)}{\alpha_2(s)} + s + 1 + \hat{S}_n(s)\Big) \tag{19}$$

If we want to construct a $j-$th order pole at s_p, we just need to choose the correct power of $\alpha_1(s)$, $\alpha_2(s)$ and let $n = j-1$, and rewrite the numerator using Taylor expansion around s_p.

$$k_n^{(1)}(s)\Big(c_0\alpha_1(s)\hat{Q}_n(s) + \alpha_2(s)\hat{P}_n(s)\Big) = \sum_{i=0}^{j-1}\beta_i(s-s_p)^i + O((s-s_p)^j) \tag{20}$$

where $s + 1 + \hat{S}_n(s) = \frac{\hat{P}_n(s)}{\hat{Q}_n(s)}$

Choose c_0, a_0, a_1, $\ldots$ a_{j-2} so that $\beta_i = 0$ for $i = 0, \ldots j-1$. Note that we always want c_0 and a_j satisfy our assumptions.

Thus we construct a lossless impedance function $\alpha(s)$, which will produce a $j-$th order pole at s_p. One might ask: Can we always get a solution which satisfies all of our assumptions for arbitrary $s_p < 0$? At least, in our case, the answer is NO! There will be some restriction on s_p.

Let's go over the 3rd pole again using the Taylor method to get a better picture of the procedure and the restriction.

We are constructing 3rd order pole at $s = s_p$, so $j = 3$, $n = 2$,

$$\alpha(s) = -c\frac{(s^2+a)}{(s^2+b)}$$

$$k_2(s) = \frac{e^{-s}}{s}\frac{(s^2+3s+3)}{s^2}$$

$$s+1+\hat{S}_n(s) = \frac{\hat{P}_n(s)}{\hat{Q}_n(s)} = \frac{s^3+4\,s^2+9\,s+9}{s^2+3\,s+3}$$

According to (20), the zeros are contained in

$$(s^2+3s+3)\Big(c(s^2+a)(s^2+3\,s+3)+(s^2+b)(s^3+4\,s^2+9\,s+9)\Big) \qquad (21)$$

expand (21) around $(s-s_p)$, we get the coefficients

$\beta_0 =$

${s_p}^7 + 18\,c{s_p}^3 + 7\,b{s_p}^4 + b{s_p}^5 + c{s_p}^6 + 6\,ca{s_p}^3 + 9\,c{s_p}^2 + 24\,{s_p}^5 + 7\,{s_p}^6 + 15\,ca{s_p}^2 + ca{s_p}^4 + 48\,{s_p}^4 + 6\,c{s_p}^5 + 54\,{s_p}^3 + 24\,b{s_p}^3 + 15\,c{s_p}^4 + 54\,b{s_p} + 18\,ca{s_p} + 27\,{s_p}^2 + 27\,b + 48\,b{s_p}^2 + 9\,ca$

$\beta_1 =$

$54\,c{s_p}^2 + 7\,{s_p}^6 + 30\,ca{s_p} + 4\,ca{s_p}^3 + 6\,c{s_p}^5 + 162\,{s_p}^2 + 18\,ca{s_p}^2 + 120\,{s_p}^4 + 30\,c{s_p}^4 + 18\,ca + 72\,b{s_p}^2 + 28\,b{s_p}^3 + 18\,c{s_p} + 42\,{s_p}^5 + 54\,s_p + 96\,b{s_p} + 192\,{s_p}^3 + 5\,b{s_p}^4 + 54\,b + 60\,c{s_p}^3$

$\beta_2 =$

$72\,b{s_p} + 6\,ca{s_p}^2 + 162\,s_p + 54\,c{s_p} + 15\,c{s_p}^4 + 18\,ca{s_p} + 21\,{s_p}^5 + 105\,{s_p}^4 + 27 + 15\,ca + 60\,c{s_p}^3 + 48\,b + 240\,{s_p}^3 + 9\,c + 10\,b{s_p}^3 + 288\,{s_p}^2 + 90\,c{s_p}^2 + 42\,b{s_p}^2$

Solve $\beta_0 = \beta_1 = \beta_2 = 0$ for a, b, c in terms of s_p we get

$$a = \frac{{s_p}^6 + 12\, {s_p}^5 + 81\, {s_p}^4 + 315\, {s_p}^3 + 648\, {s_p}^2 + 648\, s_p + 216}{3\, {s_p}^4 + 28\, {s_p}^3 + 99\, {s_p}^2 + 153\, s_p + 96}$$

$$b = 3\, \frac{{s_p}^5 + 9\, {s_p}^4 + 35\, {s_p}^3 + 72\, {s_p}^2 + 72\, s_p + 24}{{s_p}^3 + 9\, {s_p}^2 + 27\, s_p + 24}$$

$$c = -\frac{3\, {s_p}^4 + 28\, {s_p}^3 + 99\, {s_p}^2 + 153\, s_p + 96}{{s_p}^3 + 9\, {s_p}^2 + 27\, s_p + 24}$$

If we choose $s_p = -4$, then $a = \frac{94}{11}$, $b = 54$, $c = 11$, clearly $c > 0$, and $0 \le a < b < \infty$

$$\alpha(s) = \frac{-11(s^2 + \frac{94}{11})}{s^2 + 54}$$

which is the same as we showed before. In fact, if we choose $s_p = -3$, then a, b, c will not satisfy our assumptions. If we plot a, b, c in terms of s_p, we will see that in order to satisfy all the assumptions, s_p can only be chosen approximately $s_p < -3.2$. At $s_p = -3$, a will be negative. For the more general case, we can also use this graph to determine the range of s_p. For example, in this case, the range of s_p will be the interval where the graph of b is above a and all of the graphs a, b, c are above the x-axis.

2.6 Interpretation of the Results

Our prototype here for scattering is a constant-frequency plane wave proceeding in direction e_z. The overall acoustic pressure is written

$$\hat{p}(s, r, \theta) = \hat{p}^{(inc)}(s, r, \theta) + \hat{p}^{(sc)}(s, r, \theta)$$

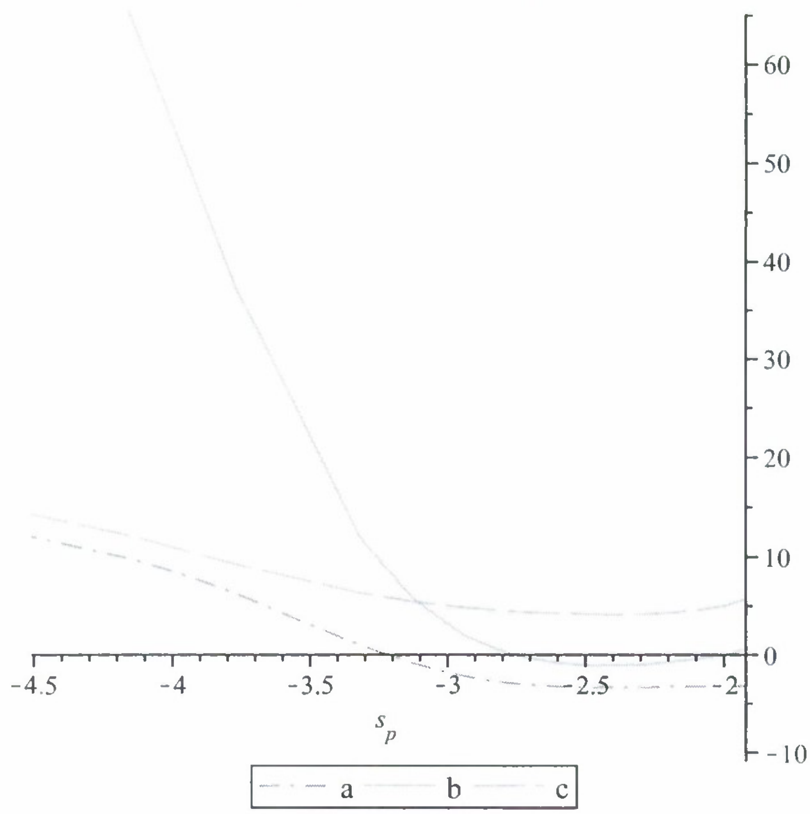
where $\hat{p}^{(inc)}(s, r, \theta)$ is the incident plane wave and $\hat{p}^{(sc)}(s, r, \theta)$ is the scattered solution(wave's complex amplitude). The function $\hat{p}^{(sc)}(s, r, \theta)$ satisfies the Helmholtz equation and the Sommerfeld radiation condition.

For a hard surface scatterer we require

$$\nabla \hat{p} = 0 \qquad \Rightarrow \qquad \nabla \hat{p}^{(sc)} \cdot n = -\nabla \hat{p}^{(inc)} \cdot n$$

For a soft surface scatterer we require

Figure 2: plots of a,b,c



$$\hat{p} = 0 \qquad \Rightarrow \quad \hat{p}^{(sc)} \cdot n = -\hat{p}^{(inc)} \cdot n$$

where n is a unit normal vector pointing into fluid.

If we assume the scatterer is a sphere centered at the origin with radius R. We have

$$\hat{p}^{(sc)}_{hard} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\gamma i'_n(\gamma R)]}{\gamma k'_n(\gamma R)} k_n(\gamma r) P_n(\cos\theta)$$

$$\hat{p}^{(sc)}_{soft} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\gamma i_n(\gamma)]}{k_n(\gamma R)} k_n(\gamma r) P_n(\cos\theta)$$

The poles of $\hat{p}^{(sc)}_{hard,soft}$ are all simple as discussed above. Since $\hat{v}^{(sc)}$ have exactly the same zeros as $\hat{p}^{(sc)}$does, we will just talk about $\hat{p}^{(sc)}$ here.

If the sphere scatterer has a specific-acoustic-impedance (neither soft nor hard) condition (equivalent to $\alpha(s) = -2$)

$$Z(s) = \frac{\rho_0 s}{2}$$

then

$$\hat{p}^{(sc)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\alpha(\gamma R) i_n(\gamma R) + \gamma R i'_n(\gamma R)]}{\alpha(\gamma R) k_n(\gamma R) + \gamma R k'_n(\gamma R)} k_n(\gamma r) P_n(\cos\theta)$$

Thus $\hat{p}^{(sc)}$ will have a second order pole at $\gamma R = -2$ or $s = -\frac{2c}{R}$.

If the sphere scatterer has a specific-acoustic-impedance (neither soft nor hard) condition (equivalent to $\alpha(s) = \frac{-11(s^2+\frac{94}{11})}{s^2+54}$)

$$Z(s) = \frac{\rho_0 s(s^2+54)}{11(s^2+\frac{94}{11})}$$

then $\hat{p}^{(sc)}$ will have a third order pole at $\gamma R = -4$ or $s = -\frac{4c}{R}$.

In [12], Sancer showed that for acoustic scattering SEM poles are simple, but there is no contradiction. In that reference, the impedance condition is not considered in the proof and Sancer used a hard acoustic scatterer to derive the simple pole behavior for arbitrary shape of scatterer, with which our results agree.

3 Scattering from a lossless electromagnetic sphere

3.1 Introduction

In this section, we are considering the problem of a plane wave incident on a sphere (with perfectly conducting surface and lossless sheet impedance loading respectively) as illustrated in figure 3. An E wave has been chosen as an incident electromagnetic plane wave propagating in $\vec{1}_1$ direction. The scattered solution as well as the surface current density can be written explicitly using vector spherical harmonics. In the case 1, a perfectly conducting surface, we will briefly summarize the work done by Carl Baum, which shows that there exist only first order scattering poles. In the case 2, a lossless sheet impedance loading sphere, there exist 2nd order scattering poles for some mathematically chosen boundary conditions. Foster's Theorem is enforced on the impedance function $\hat{Z}_s(s)$ to guarantee it is a realizable physical boundary condition.

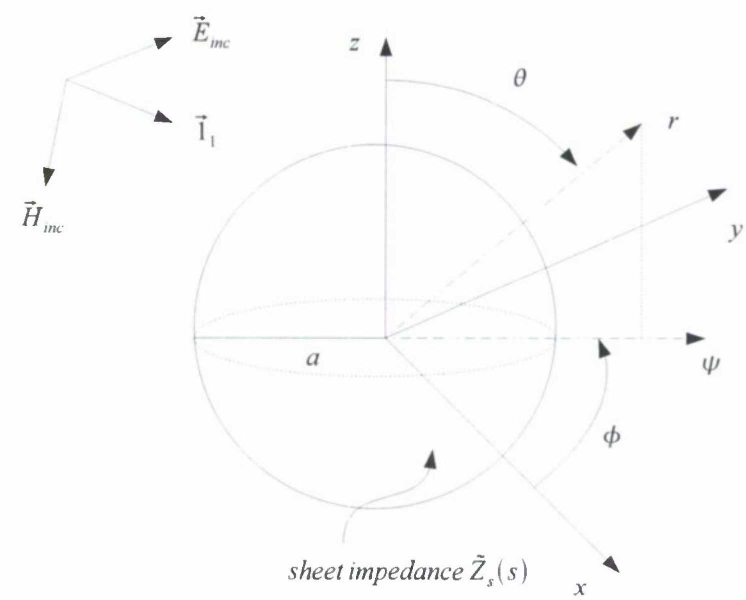
3.2 Formulation of the electromagnetic scattering problem

Define a set of orthogonal (right-handed) unit vectors by

$$
\begin{aligned}
\vec{1}_1 &= \sin(\theta_1)\cos(\phi_1)\vec{1}_x + \sin(\theta_1)\sin(\phi_1)\vec{1}_y + \cos(\theta_1)\vec{1}_z \\
\vec{1}_2 &= -\cos(\theta_1)\cos(\phi_1)\vec{1}_x - \cos(\theta_1)\sin(\phi_1)\vec{1}_y + \sin(\theta_1)\vec{1}_z \\
\vec{1}_3 &= \sin(\phi_1)\vec{1}_x - \cos(\phi_1)\vec{1}_y
\end{aligned}
$$

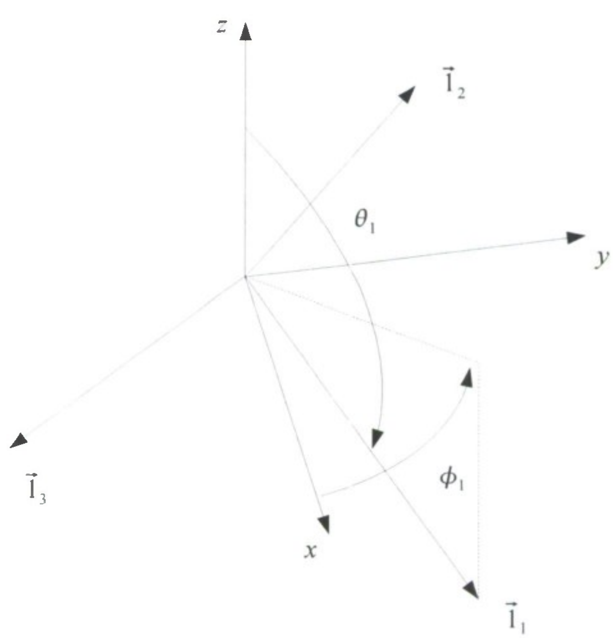
As shown in figure 4, $\vec{1}_1$ is the direction of propagation and $\vec{1}_2$ and $\vec{1}_3$ are mutually orthogonal unit vectors, each orthogonal to $\vec{1}_1$ to indicate the polarization of the electromagnetic fields in the incident plane wave. For convenience $\vec{1}_2$ is chosen in a plane parallel to $\vec{1}_1$ and the z axis (E or TM polarization if the electric field is parallel to $\vec{1}_2$) while $\vec{1}_3$ is parallel to the x, y plane (H or TE polarization if the electric field is parallel to $\vec{1}_3$). In free space, electromagnetic plane waves have both electric and magnetic fields orthogonal to $\vec{1}_1$. Thus only $\vec{1}_2$ and $\vec{1}_3$ are concerned. This removes the $\vec{L}$ functions (details are shown later) in the expansion (plane waves have zero-divergence fields).

Figure 3:



$\vec{E}_{inc}$
z
θ
r
$\vec{1}_1$
$\vec{H}_{inc}$
y
a
ψ
ϕ
sheet impedance $\tilde{Z}_s(s)$
x

Figure 4:



We can use the relations between Cartesian and spherical coordinates

$$\begin{aligned} x &= r\sin(\theta)\cos(\phi) \\ y &= r\sin(\theta)\sin(\phi) \\ z &= r\cos(\theta) \end{aligned}$$

$$\begin{aligned} \vec{1}_x &= \sin(\theta)\cos(\phi)\vec{1}_r + \cos(\theta)\cos(\phi)\vec{1}_\theta - \sin(\phi)\vec{1}_\phi \\ \vec{1}_y &= \sin(\theta)\sin(\phi)\vec{1}_r + \cos(\theta)\sin(\phi)\vec{1}_\theta + \cos(\phi)\vec{1}_\phi \\ \vec{1}_z &= \cos(\theta)\vec{1}_r - \sin(\theta)\vec{1}_\theta \end{aligned}$$

to express the incident-wave unit vectors in terms of (θ_1, ϕ_1) and (θ, ϕ) as

$$\begin{aligned} \vec{1}_1 &= [\cos(\theta_1)\cos(\theta) + \sin(\theta_1)\sin(\theta)\cos(\phi - \phi_1)]\vec{1}_r \\ &+ [-\cos(\theta_1)\sin(\theta) + \sin(\theta_1)\cos(\theta)\cos(\phi - \phi_1)]\vec{1}_\theta \\ &+ [-\sin(\theta_1)\sin(\phi - \phi_1)]\vec{1}_\phi \end{aligned}$$

$$\begin{aligned} \vec{1}_2 &= [\sin(\theta_1)\cos(\theta) - \cos(\theta_1)\sin(\theta)\cos(\phi - \phi_1)]\vec{1}_r \\ &- [\sin(\theta_1)\sin(\theta) + \cos(\theta_1)\cos(\theta)\cos(\phi - \phi_1)]\vec{1}_\theta \\ &+ [\cos(\theta_1)\sin(\phi - \phi_1)]\vec{1}_\phi \end{aligned}$$

$$\begin{aligned} \vec{1}_3 &= -\sin(\theta)\sin(\phi - \phi_1)\vec{1}_r \\ &- \cos(\theta)\cos(\phi - \phi_1)\vec{1}_\theta \\ &- \cos(\phi - \phi_1)]\vec{1}_\phi \end{aligned}$$

Having the direction of incidence and two polarizations expressed in spherical coordinates we can go on to express the response to some delta plane wave functions. For an incident delta function plane wave we need spherical harmonics and vector wave function in which to express the expansion in spherical coordinates. In spherical coordinates we have the common differential operators as

$$\begin{aligned} \nabla F &= \vec{1}_r\frac{\partial}{\partial r}F + \vec{1}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}F + \vec{1}_\phi\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}F \\ \nabla\cdot\vec{F} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)F_\theta) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}F_\phi \\ \nabla\times\vec{F} &= \vec{1}_r[\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)F_\phi) - \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}F_\theta] \\ &+\vec{1}_\theta[\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)F_r) - \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi)] + \vec{1}_\phi[\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}F_r] \end{aligned}$$

$$\nabla_s F = \vec{1}_\theta \frac{\partial}{\partial\theta} F + \vec{1}_\phi \frac{1}{\sin(\theta)} \frac{\partial}{\partial\phi} F$$
$$\nabla_s \cdot \vec{F} = \frac{1}{\sin(\theta)} \frac{\partial}{\partial\theta}(\sin(\theta) F_\theta) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial\phi} F_\phi$$
$$\nabla_s \times \vec{F} = \vec{1}_r [\frac{1}{\sin(\theta)} \frac{\partial}{\partial\theta}(\sin(\theta) F_\phi) - \frac{1}{\sin(\theta)} \frac{\partial}{\partial\phi} F_\theta] + \vec{1}_\theta [\frac{1}{\sin(\theta)} \frac{\partial}{\partial\phi} F_r]$$

Spherical Harmonics

The scalar spherical harmonics can be written as

$$Y_{n,m,{}^e_o}(\theta, \phi) = P_n^{(m)}(cos(\theta)) \left\{ \begin{array}{c} cos(m\phi) \\ sin(m\phi) \end{array} \right\}$$

where $P_n^{(m)}(x)$ is the Legendre function defined as

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x), \quad P_n(x) = P_n^0(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

Vector spherical harmonics are defined as follows

$$\begin{aligned} \vec{P}_{n,m,p}(\theta,\phi) = &\quad Y_{n,m,p}(\theta,\phi)\vec{1}_r & \\ \vec{Q}_{n,m,p}(\theta,\phi) = &\quad \nabla_s Y_{n,m,p}(\theta,\phi) & = \vec{1}_r \times \vec{R}_{n,m,p} \\ \vec{R}_{n,m,p}(\theta,\phi) = &\quad \nabla_s \times \vec{P}_{n,m,p}(\theta,\phi) & = -\vec{1}_r \times \vec{Q}_{n,m,p} \end{aligned}$$

They also are mutually orthogonal in an integral sense on the unit sphere. The spherical scalar wave functions are defined as

$$\Xi^{(l)}_{n,m,p}(\gamma\vec{r}) = f_n^{(l)}(\gamma r) P_n^{(m)}(\theta,\phi)$$

where $f_n^{(1)}(\gamma r) = i_n(\gamma r)$, $f_n^{(2)}(\gamma r) = k_n(\gamma r)$ are modified Bessel functions. They satisfy the Wronskian relation

$$W\{si_n(s), sk_n(s)\} = si_n(s)[sk_n(s)]' - [si_n(s)]' sk_n(s) = -1$$

$\gamma = [s\mu(\sigma + s\epsilon)]^{1/2}$ with μ, σ, ϵ are permeability, conductivity, permittivity, respectively. s is the variable of the two-sided Laplace transformation. Coefficients times the scalar wave function $\Xi^{(l)}_{n,m,p}(\gamma\vec{r})$ when summed over all possible indices satisfy the scalar wave equation which for each function we can write in operator form as

$$[\nabla^2 - \gamma^2]\Xi^{(l)}_{n,m,p}(\gamma\vec{r}) = 0$$

From the solution of the scalar wave equation one constructs as usual the solutions of the vector wave equation of three kinds.

$$\begin{aligned}
\hat{L}^{(l)}_{n,m,p}(\gamma\vec{r}) &= \tfrac{1}{\gamma}\nabla\Xi^{(l)}_{n,m,p}(\gamma\vec{r}) \\
\hat{M}^{(l)}_{n,m,p}(\gamma\vec{r}) &= \nabla\times[\vec{r}\Xi^{(l)}_{n,m,p}(\gamma\vec{r})] \\
\hat{N}^{(l)}_{n,m,p}(\gamma\vec{r}) &= \tfrac{1}{\gamma}\nabla\times\hat{M}^{(l)}_{n,m,p}(\gamma\vec{r})
\end{aligned}$$

Note that all three kinds of vector wave functions satisfy the vector wave equation in Laplacian form which we can summarize as

$$[\nabla^2-\gamma^2]\left\{\begin{array}{c}\hat{L}^{(l)}_{n,m,p}\\ \hat{M}^{(l)}_{n,m,p}\\ \hat{N}^{(l)}_{n,m,p}\end{array}\right\}=0$$

We can also write a curl curl wave equation for only the second and third kinds of vector wave functions as

$$[\nabla\times\nabla+\gamma^2]\left\{\begin{array}{c}\hat{M}^{(l)}_{n,m,p}\\ \hat{N}^{(l)}_{n,m,p}\end{array}\right\}=0$$

The three kinds of vector wave functions have some interrelations as

$$\begin{aligned}
\hat{M}^{(l)}_{n,m,p}(\gamma\vec{r}) &= -\gamma\vec{r}\times\hat{L}^{(l)}_{n,m,p}(\gamma\vec{r}) \\
\hat{M}^{(l)}_{n,m,p}(\gamma\vec{r}) &= -\tfrac{1}{\gamma}\nabla\times\hat{N}^{(l)}_{n,m,p}(\gamma\vec{r}) \\
\hat{N}^{(l)}_{n,m,p}(\gamma\vec{r}) &= \tfrac{1}{\gamma}\nabla\times\hat{M}^{(l)}_{n,m,p}(\gamma\vec{r})
\end{aligned}$$

It is also useful to write them as

$$\begin{aligned}
&\hat{L}^{(l)}_{n,m,p}(\gamma\vec{r}) = [f^{(l)}_n(\gamma r)]'\vec{P}_{n,m,p}(\theta,\phi)+[f^{(l)}_n(\gamma r)]\vec{Q}_{n,m,p}(\theta,\phi)/\gamma r \\
&\hat{M}^{(l)}_{n,m,p}(\gamma\vec{r}) = [f^{(l)}_n(\gamma r)]\vec{R}_{n,m,p}(\theta,\phi) \\
&\hat{N}^{(l)}_{n,m,p}(\gamma\vec{r}) = \{n(n+1)[f^{(l)}_n(\gamma r)]\vec{P}_{n,m,p}(\theta,\phi)+[\gamma r f^{(l)}_n(\gamma r)]'\vec{Q}_{n,m,p}(\theta,\phi)\}/\gamma r
\end{aligned}$$

Plane wave in spherical coordinates

As shown in figure 3, the delta function plane waves (transformed) can be written as

$$\vec{1}_2 e^{-\gamma\vec{1}_1\cdot\vec{r}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{p=e,o}[a'_{n,m,p}\hat{M}^{(1)}_{n,m,p}(\gamma\vec{r}) + b'_{n,m,p}\hat{N}^{(1)}_{n,m,p}(\gamma\vec{r})]$$

$$\vec{1}_3 e^{-\gamma\vec{1}_1\cdot\vec{r}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{p=e,o}[b'_{n,m,p}\hat{M}^{(1)}_{n,m,p}(\gamma\vec{r}) - a'_{n,m,p}\hat{N}^{(1)}_{n,m,p}(\gamma\vec{r})]$$

where

$$a'_{n,m,p} = [2 - 1_{0,m}](-1)^{n+1}\tfrac{2n+1}{n(n+1)}\tfrac{(n-m)!}{(n+m)!}m\tfrac{P_n^{(m)}(\cos(\theta_1))}{\sin(\theta_1)}\left\{\begin{array}{c} -\sin(m\phi_1) \\ \cos(m\phi_1)\end{array}\right\}$$

$$b'_{n,m,p} = [2 - 1_{0,m}](-1)^{n}\tfrac{2n+1}{n(n+1)}\tfrac{(n-m)!}{(n+m)!}\tfrac{dP_n^{(m)}(\cos(\theta_1))}{d\theta_1}\left\{\begin{array}{c} \cos(m\phi_1) \\ \sin(m\phi_1)\end{array}\right\}$$

Note that we have

$$\frac{1}{\gamma}\nabla \times [\vec{1}_2 e^{-\gamma\vec{1}_1\cdot\vec{r}}] = \vec{1}_3 e^{-\gamma\vec{1}_1\cdot\vec{r}}$$

$$\frac{1}{\gamma}\nabla \times [\vec{1}_3 e^{-\gamma\vec{1}_1\cdot\vec{r}}] = -\vec{1}_2 e^{-\gamma\vec{1}_1\cdot\vec{r}}$$

which is associated with the curl relations between the $\hat{M}^{(l)}_{n,m,p}$ and $\hat{N}^{(l)}_{n,m,p}$ functions. Furthermore any divergenceless electric field expansion ($\vec{E}$) can be converted to a magnetic field expansion ($\vec{H}$) by dividing by the wave impedance Z of the medium and changing $\hat{M}^{(l)}_{n,m,p}$ to $-\hat{N}^{(l)}_{n,m,p}$ and $\hat{N}^{(l)}_{n,m,p}$ to $\hat{M}^{(l)}_{n,m,p}$. To go from $\vec{H}$ to $\vec{E}$ multiply by Z and change $\hat{M}^{(l)}_{n,m,p}$ to $\hat{N}^{(l)}_{n,m,p}$ and $\hat{N}^{(l)}_{n,m,p}$ to $-\hat{M}^{(l)}_{n,m,p}$.

Solution of the scattered field

Define our incident plane wave as an E wave (TM wave)

$$\tilde{\vec{E}}_{inc}(\vec{r}, s) = E_0\vec{1}_2 e^{-\gamma\vec{1}_1\cdot\vec{r}}$$

$$\tilde{\vec{H}}_{inc}(\vec{r}, s) = \frac{E_0}{Z_0}\vec{1}_3 e^{-\gamma\vec{1}_1\cdot\vec{r}}$$

Expand the fields for $r < a$ as

$$\tilde{\vec{E}}_{in}(\vec{r}, s) = E_0 \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [a''_{n,m,p} \hat{\vec{M}}^{(1)}_{n,m,p}(\gamma\vec{r}) + b''_{n,m,p} \hat{\vec{N}}^{(1)}_{n,m,p}(\gamma\vec{r})]$$

$$\tilde{\vec{H}}_{in}(\vec{r}, s) = \frac{E_0}{Z_0} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [b''_{n,m,p} \hat{\vec{M}}^{(1)}_{n,m,p}(\gamma\vec{r}) - a''_{n,m,p} \hat{\vec{N}}^{(1)}_{n,m,p}(\gamma\vec{r})]$$

The solution of the scattered fields for $r > a$ can be written as

$$\tilde{\vec{E}}_{sc}(\vec{r}, s) = E_0 \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [a'''_{n,m,p} \hat{\vec{M}}^{(2)}_{n,m,p}(\gamma\vec{r}) + b'''_{n,m,p} \hat{\vec{N}}^{(2)}_{n,m,p}(\gamma\vec{r})]$$

$$\tilde{\vec{H}}_{sc}(\vec{r}, s) = \frac{E_0}{Z_0} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [b'''_{n,m,p} \hat{\vec{M}}^{(2)}_{n,m,p}(\gamma\vec{r}) - a'''_{n,m,p} \hat{\vec{N}}^{(2)}_{n,m,p}(\gamma\vec{r})]$$

3.3 Perfectly conducting sphere

Carl Baum showed in [2] that there only exist simple poles for perfectly conductor sphere. Here we will just briefly repeat the same argument in our context. Constrain the tangential electric field to be zero on $r = a$, we have $\vec{1}_r \times [\tilde{\vec{E}}_{inc}(\vec{r}, s) + \tilde{\vec{E}}_{sc}(\vec{r}, s)] = 0$. The we get

$$\begin{aligned} \vec{1}_r \times [a'_{n,m,p} \hat{\vec{M}}^{(1)}_{n,m,p}(\gamma a \vec{1}_r) + a'''_{n,m,p} \hat{\vec{M}}^{(2)}_{n,m,p}(\gamma a \vec{1}_r)] &= \vec{0} \\ \vec{1}_r \times [b'_{n,m,p} \hat{\vec{N}}^{(1)}_{n,m,p}(\gamma a \vec{1}_r) + b'''_{n,m,p} \hat{\vec{N}}^{(2)}_{n,m,p}(\gamma a \vec{1}_r)] &= \vec{0} \end{aligned}$$

This give equations for the coefficient as

$$\begin{aligned} a'''_{n,m,p} &= -\frac{i_n(\gamma a)}{k_n(\gamma a)} a'_{n,m,p} \\ b'''_{n,m,p} &= -\frac{[\gamma a i_n(\gamma a)]'}{[\gamma a k_n(\gamma a)]'} b'_{n,m,p} \end{aligned}$$

To see that the poles must be simple poles we just need to show that all the zeros of $k_n(s)$ and $sk_n(s)$ are simple zeros. Since $k_n(s)$ satisfies spherical Bessel function we have

$$s^2 \frac{d^2}{ds^2} k_n(s) + 2s \frac{d}{ds} k_n(s) - [s^2 + n(n+1)] k_n(s) = 0$$

Suppose the zero is higher than first order, say a 2nd order zero at $s_\alpha \neq 0$. Since both $k_n(s)$ and $k'_n(s)$ have to be zero at s_α, so does $k''_n(s)$. Thus, the zero must be at least a third order zero. Repeat the same process, we will eventually have all the derivatives at s_α to be zero, thus the function must be identically zero. So there exist only simple poles for $a'''_{n,m,p}$. The argument for $b'''_{n,m,p}$ is similar as $[sk_n(s)]$ satisfies the Riccati-Bessel equation

$$\frac{s^2}{s^2+n(n+1)}\frac{d^2}{ds^2}[sf_n^{(l)}(s)] - sf_n^{(l)}(s) = 0$$

For more details of the perfectly conducting sphere including surface current and charge densities please see [2].

3.4 Scatterer of spherical sheet impedance loading

3.4.1 Lossless sheet impedance

Spherical coordinates (r, θ, ϕ) as in figure 3 are one of the few coordinate systems in which solutions of Maxwell's equations are separable. In particular let us assume a sheet impedance $\tilde{Z}_s(s)$ (a scalar) which is located on a spherical surface give by $r = a$ and which is independent of θ, ϕ. This sheet impedance relates tangential electric field and surface current density as in [3], we have

$$\begin{aligned}
\overleftrightarrow{1}_t \cdot \tilde{\vec{E}}(a, \theta, \phi, s) &= \tilde{Z}_s(s)\tilde{J}_s(\theta, \phi, s) \\
\overleftrightarrow{1}_t &= \overleftrightarrow{1} - \vec{1}_r\vec{1}_r = transverse \;\; dyad \\
\overleftrightarrow{1} &\equiv identity \;\; dyad
\end{aligned}$$

$\sim$ stands for two-sided Laplace transform. The surface current density is in turn related to the magnetic field via

$$\vec{1}_r \times [\tilde{\vec{H}}(a+, \theta, \phi, s) - \tilde{\vec{H}}(a-, \theta, \phi, s)] = \tilde{J}_s(\theta, \phi, s)$$

The sheet impedance function $\tilde{Z}_s(s)$ also has to satisfy Foster's Theorem to guarantee lossless boundary conditions.

Foster Theorem

In [1], a *positive real* function F(s) is an analytic function of the complex variable $s = \sigma + j\omega$, which has the following properties:

1.$F(s)$ is regular for $\sigma > 0$

2.$F(\sigma)$ is real

3.$\sigma \geq 0$ implies $\Re[F(s)] > 0$

A *reactance function* is a positive real function that maps the imaginary axis into the imaginary axis.

Theorem: A *real rational function* of s is a *reactance function* if and only if all of its poles and zeros are simple, lie on the $j\omega$-axis, and alternate with each other. In other words

$$\psi(s) = K\frac{s(s^2+\omega_1^2)(s^2+\omega_3^2)\cdots(s^2+\omega_{2n-1}^2)}{(s^2+\omega_0^2)(s^2+\omega_2^2)\cdots(s^2+\omega_{2n}^2)}$$

is a reactance function, where $k = 2n-2$ or $2n$, $K > 0$, $0 \leq \omega_0 < \omega_1 < \cdots < \omega_{2n-1} < \omega_{2n} < \infty$

Theorem: A *rational function* of s is a *reactance function* if and only if it is the driving-point impedance or admittance of a lossless network.

3.4.2 Solving the scattering problem

Matching the boundary condition on $r = a$, with the sheet impedance and continuity of the tangential electric field gives

$$\begin{aligned}
&\overleftrightarrow{1_t} \cdot [\tilde{\vec{E}}_{inc}(a+,\theta,\phi,s) + \tilde{\vec{E}}_{sc}(a+,\theta,\phi,s)] = \overleftrightarrow{1_t} \cdot \tilde{\vec{E}}_{in}(a-,\theta,\phi,s) \\
&= \tilde{Z}_s(s)\tilde{J}_s(\theta,\phi,s) \\
&= \tilde{Z}_s(s) \times [\tilde{\vec{H}}_{inc}(a+,\theta,\phi,s) + \tilde{\vec{H}}_{sc}(a+,\theta,\phi,s) - \tilde{\vec{H}}_{in}(a-,\theta,\phi,s)]
\end{aligned}$$

Plugging in the expansion we derive a system of equations involving $a''_{n,m,p}$, $b''_{n,m,p}$, $a'''_{n,m,p}$, $b'''_{n,m,p}$. Solve for $a''_{n,m,p}$ and $b''_{n,m,p}$ we get

$$a''_{n,m,p} = \frac{a'_{n,m,p}}{1 + \frac{Z0}{\tilde{Z}_s(s)}(\gamma a)^2 i_n(\gamma a)k_n(\gamma a)}$$

$$b''_{n,m,p} = \frac{b'_{n,m,p}}{1 - \frac{Z0}{\tilde{Z}_s(s)}[\gamma a i_n(\gamma a)]'[\gamma a k_n(\gamma a)]'}$$

Now the surface current density is

$$\tilde{J}_s(\theta,\phi,s) = \frac{1}{\tilde{Z}_s(s)} \overleftrightarrow{1}_t \cdot \tilde{\vec{E}}(a,\theta,\phi,s)$$
$$= \frac{E_0}{\tilde{Z}_s(s)} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [a''_{n,m,p} i_n(\gamma a) \vec{R}_{n,m,p}(\theta,\phi) + b''_{n,m,p} \frac{[\gamma a i_n(\gamma a)]'}{\gamma a} \vec{Q}_{n,m,p}(\theta,\phi)]$$

3.4.3 Existence of 2nd order poles

The coefficients we care about concerning the existence of a second pole are $c_1 = \frac{i_n(\gamma a)}{\tilde{Z}_s(s)} a''_{n,m,p}$ and $c_2 = \frac{[\gamma a i_n(\gamma a)]'}{\tilde{Z}_s(s)\gamma a} b''_{n,m,p}$. Let's give a simple example showing that a second pole does exist for coefficient c_2. For simplicity, let's assume $\gamma a = s$. Consider the sheet impedance function

$$\tilde{Z}_s(s) = \frac{(\frac{1}{2}e^2 + e + \frac{1}{2})s}{s^2 + \frac{1}{2}e + \frac{1}{4}}$$

Clearly $\tilde{Z}_s(s)$ satisfies Foster's Theorem with $K > 0$ and $\omega_0 > 0$. The expansions of $i_n(s)$ and $k_n(s)$ are

$$k_n(s) = \frac{e^{-s}}{s} \sum_{j=0}^{n} \frac{(n+j)!\, 2^{-j} s^{-j}}{j!\, (n-j)!}$$
$$i_n(s) = \frac{1}{2}[(-1)^{n+1} k_n(s) - k_n(-s)]$$

For $n = 0$, the denominator of c_2 is

$$De(s) = \left(4\,e^{-2\,s} + 4\right) s^2 + \left(4\,e^2 + 8\,e + 4\right) s + 1 + 2\,e^{1-2\,s} + 2\,e + e^{-2\,s}$$

It is easy to see that $De(-\frac{1}{2}) = 0$ and $\frac{d}{ds} De(s)|_{s=-\frac{1}{2}} = 0$ or in Taylor expansion around $-\frac{1}{2}$

$$De(s) = (\left(16\,e + 4 + 4\,e^2\right) \left(s + \frac{1}{2}\right)^2 + \left(-\frac{56}{3}\,e - \frac{8}{3}\,e^2\right) \left(s + \frac{1}{2}\right)^3 + O\left(\left(s + \frac{1}{2}\right)^4\right))$$

Thus we derive a second order pole at $s = -\frac{1}{2}$.

In general, we want to construct a sheet impedance function $\tilde{Z}_s(s) = \frac{Ks}{(s^2+\omega)}$ such that c_2 have a second order pole in the left half plane of s mean while $K > 0$ and $\omega > 0$. The denominator of c_2 has the following form

$$De(s) = Ks + \left(-s^2 i_n(s) - s^3 \frac{d}{ds} i_n(s) - \omega\, i_n(s) - \omega\, s \frac{d}{ds} i_n(s)\right) k_n(s) + \left(-s^3 i_n(s) - s^4 \frac{d}{ds} i_n(s) - s\omega\, i_n(s) - s^2 \omega \frac{d}{ds} i_n(s)\right) \frac{d}{ds} k_n(s)$$

We want to solve $De(s) = 0$ and $\frac{d}{ds}De(s) = 0$ for K, ω in terms of s. The solution s_α must satisfy $s_\alpha < 0$, $K(s_\alpha) > 0$ and $\omega(s_\alpha) > 0$. For $n = 0$, that is to solve

$$2\,Ks + s^2 e^{-2\,s} + s^2 + \omega\, e^{-2\,s} + \omega = 0$$
$$2\,K + 2\,se^{-2\,s} - 2\,s^2 e^{-2\,s} + 2\,s - 2\,\omega\, e^{-2\,s} = 0$$

The solutions are

$$K = -\frac{s\left(e^{-4\,s} + 2\,e^{-2\,s} + 1\right)}{2\,se^{-2\,s} + e^{-2\,s} + 1}$$
$$\omega = -\frac{s^2\left(-e^{-2\,s} + 2\,se^{-2\,s} - 1\right)}{2\,se^{-2\,s} + e^{-2\,s} + 1}$$

From figure 5,6, approximately when s is chosen from -0.64 to 0, both K and ω will be positive. Pushing the poles to even a higher order is not done here. In order to construct a high order pole (including the 2nd order case), a transcendental equation has to be solved analytically which in general is not possible. This is different from the perfectly conductor sphere case, where only a system of linear equations need to be solved.

It seems to us that it only works for coefficient c_2 with $n = 0$. For $n > 0$, there is no region in the left half plane of s. Although we don't have a rigorous proof of that, it seems to be the case. We tested with many different n, K and ω. K and ω will have either different signs or both will be negative. For coefficient c_1, it doesn't work either. We tried to include more terms in the $\tilde{Z}_s(s)$ according to Foster's Theorem (i.e. more ω_i) with larger n, but it is not helpful for this case. Figure 7,8 show some results of different cases with different n , K and ω_i. There are no regions for which K and ω are positive simultaneously. It is likely that for lossless sheet impedance loading boundary condition, most scattering poles are first order.

Figure 5:

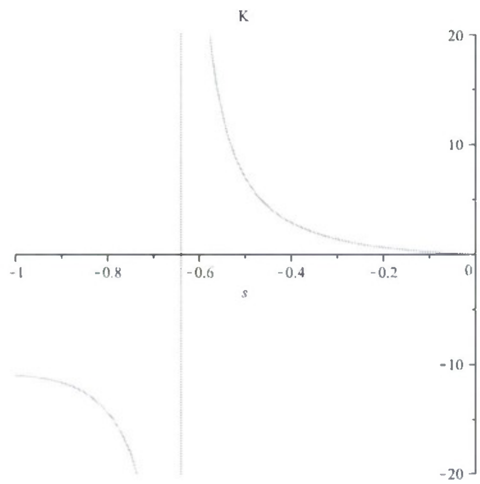


Figure 6:

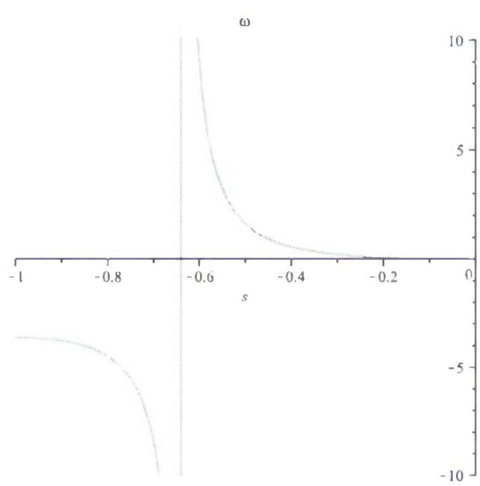


Figure 7:

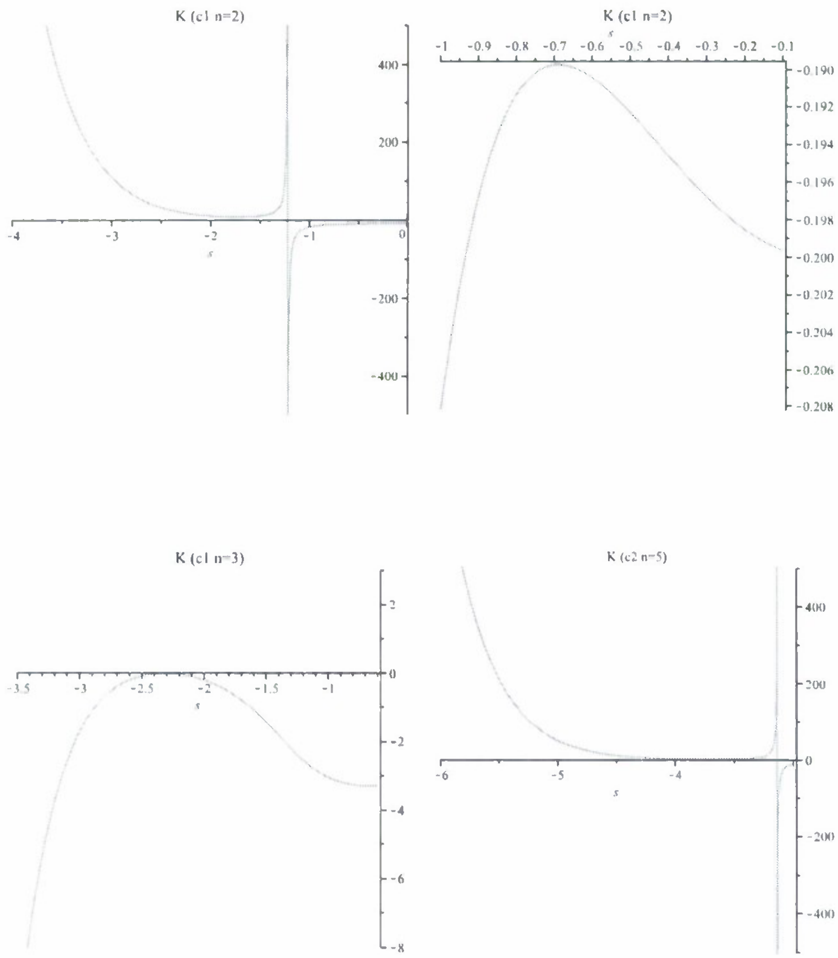
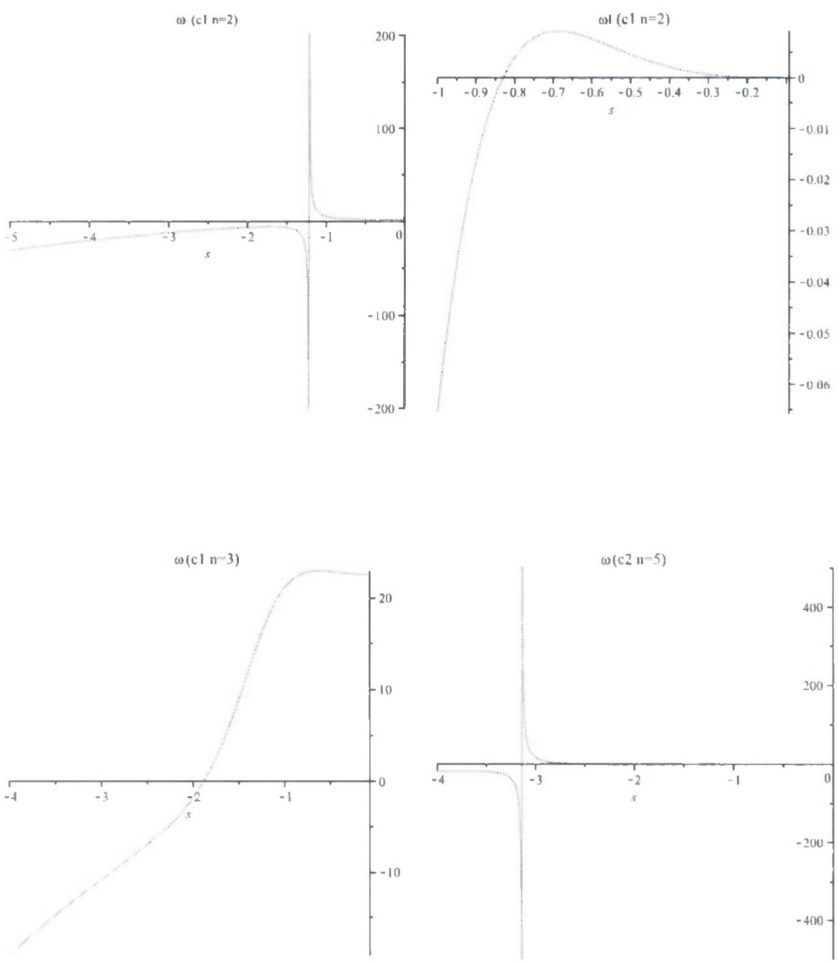


Figure 8:



4 Other scattering results

In [6], a canonical problem consisting of a junction in a transmission-line (single conductor plus reference) of a common characteristic impedance Z_c on both sides of the junction has been considered. Scattering from such a junction is only forward and backward. It is shown that scattering from a lossless, reciprocal network at a transmission-line junction can have a second order s-plane pole. In [5], Carl extend the previous results to a more complicated scattering at a junction of transmission lines. A transmission-line structure which mimics polarization has been considered to make the problem even closer to the electromagnetic case. An incident wave is propagated as two orthogonal polarizations on a four wire transmission line. This, in turn, scatters forward into a similar four-wire transmission line supporting two orthogonal polarizations. Those work gives an closer simulation of the electromagnetic-scattering case, contribute to the discussion concerning 3-dimensional electromagnetic scattering from lossless, as well as perfectly conducting targets.

5 Some analysis on the forward-scattering theorem on the scalar wave equation

5.1 Formulation of the scattering problem from a scalar wave equation

We start from the simple scalar wave equation in a 3-D region Ω, and suppose $c = 1$

$$u_{tt} = \nabla^2 u$$

$$\alpha(x)u_t + \beta(x)u_n + \gamma(x)u = 0$$

Let's call $Bu = 0$ the boundary condition, u_n is the outward normal. Simply applying a Laplace transform we get

$$s^2\hat{u} = \nabla^2\hat{u} \qquad \hat{B}\hat{u} = 0$$

From now on, all the computations will be done in the Laplace s domain, except when otherwise indicated. We want to explicitly compute the scattered solution $\hat{u}^{(sc)}(r, s)$, assuming an incident plane wave $\hat{u}^{(inc)}(r, s)$ coming in the z-direction ($\hat{u} = \hat{u}^{(sc)} + \hat{u}^{(inc)}$). Let $\mathcal{S}_1$ denote the unit sphere, parameterized by the usual angles θ, φ. Suppose $\vec{\omega}$ is the incident direction , $\vec{\omega} \in \mathcal{S}_1$

$$\begin{cases} \omega_1 = \sin\theta' \cos\varphi' \\ \omega_2 = \sin\theta' \sin\varphi' \\ \omega_3 = \cos\theta' \end{cases}$$

Suppose the incident plane wave in the time-domain is $u^{(inc)}(\vec{x}, t) = A(t - \vec{\omega} \cdot \vec{x})$ propagating in the $\vec{\omega}$ direction. Then $\hat{u}^{(inc)}(\vec{x}, s) = \hat{A}(s) \cdot e^{-s\vec{\omega}\cdot\vec{x}}$ (just the Laplace transform of $u^{(inc)}$). The boundary condition becomes $\hat{B}(s)\hat{u}^{(sc)}(\vec{x}, s) = -\hat{B}(s)\hat{u}^{(inc)}(\vec{x}, s)$, $\hat{B}$ is the operator for the boundary condition. Suppose we write the scattered solution $\hat{u}^{(sc)}(\vec{x}, s) = \hat{A}(s) \cdot \hat{U}(\vec{x}, s; \vec{\omega})$. Then $\hat{B}(s)\hat{U}(\vec{x}, s; \vec{\omega}) = -\hat{B}(s)e^{-s\vec{\omega}\cdot\vec{x}}$ on the boundary. We can expand $\hat{U}$ in spherical harmonics outside of a sphere containing Ω.

$$\hat{U}(\vec{x}, s; \vec{\omega}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \hat{U}_{mn}(s; \vec{\omega}) Y_n^m(\theta, \varphi) k_n(rs)$$

When $r \to \infty$, $k_n(rs) \sim \frac{\pi}{2}\frac{e^{-rs}}{rs}$, k_n is the modified spherical Bessel's function (good at infinity). let $\vec{x} = r\vec{q}$, $r > 0$, $\vec{q} \in \mathcal{S}_1$ $\vec{q}$ is parameterized by (θ, φ), thus $\hat{U}(\vec{x}, s; \vec{\omega}) = \hat{U}(r, \theta, \varphi, s; \theta', \varphi') \sim \frac{e^{-rs}}{r} v(\theta, \varphi, \theta', \varphi', s)$ and

$$v(\theta, \varphi, \theta', \varphi', s) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} s^{-1}\hat{U}_{mn}(\theta', \varphi', s) Y_n^m(\theta, \varphi) = v(\vec{q}, \vec{\omega}, s)$$

5.2 Representation Theorem

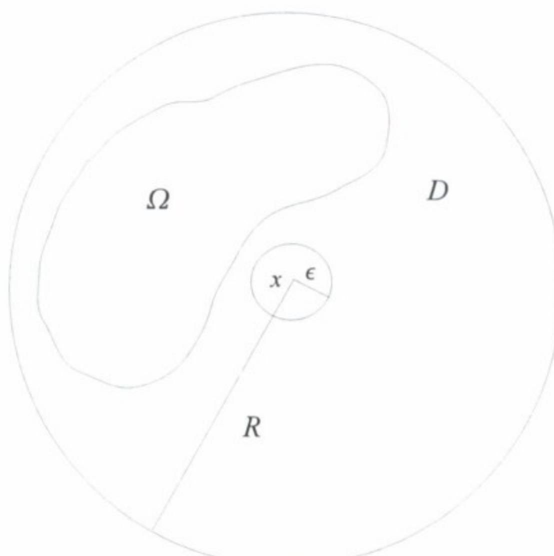
$$\hat{u}^{(sc)}(x) = -\frac{1}{4\pi}\int\!\!\int_{\partial\Omega} [\hat{u}^{(sc)}(y)(\frac{\partial}{\partial\nu_y}\frac{e^{-|x-y|s}}{|x-y|}) - \frac{e^{-|x-y|s}}{|x-y|}(\frac{\partial}{\partial\nu_y}\hat{u}^{(sc)}(y))]dA_y$$

Here, the parameters x, y are vectors, $\frac{\partial}{\partial\nu_y}$ is the outward normal derivative, $\vec{\omega}$, s have been omitted for simplicity (always there), dA_y means integrating over y-area.

Proof:

$$\begin{aligned} \nabla^2\frac{e^{-rs}}{r} &= (\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r})\frac{e^{-rs}}{r} \\ &= s^2\frac{e^{-rs}}{r} \end{aligned}$$

Figure 9:



Ball of radius R centered at x

r is the radius centered at x

$$\Rightarrow \iiint_D \hat{u}^{(sc)} \nabla^2 \frac{e^{-rs}}{r} - (\nabla^2 \hat{u}^{(sc)}) \frac{e^{-rs}}{r} dv = 0$$

by Green's identity, the left hand side also equals to

$$\iint_{\partial\Omega} [\hat{u}^{(sc)}(y)(\frac{\partial}{\partial \nu_y} \frac{e^{-|x-y|s}}{|x-y|}) - \frac{e^{-|x-y|s}}{|x-y|}(\frac{\partial}{\partial \nu_y} \hat{u}^{(sc)}(y))] dA_y + \tag{22}$$

$$\iint_{|x-y|=R} 4\pi R^2 [\hat{u}^{(sc)}(y)(-s\frac{e^{-Rs}}{R} - \frac{e^{-Rs}}{R^2}) - \frac{e^{-Rs}}{R}(\frac{\partial}{\partial \nu_y} \hat{u}^{(sc)}(y))] d\hat{A}_y - \tag{23}$$

$$\iint_{|x-y|=\epsilon} 4\pi \epsilon^2 [\hat{u}^{(sc)}(y)(-s\frac{e^{-\epsilon s}}{\epsilon} - \frac{e^{-\epsilon s}}{\epsilon^2}) - \frac{e^{-\epsilon s}}{\epsilon}(\frac{\partial}{\partial \nu_y} \hat{u}^{(sc)}(y))] d\hat{A}_y \tag{24}$$

$d\hat{A}_y$ means integrate over a sphere with surface area = 1. In equation(22) $\frac{e^{-Rs}}{R}(\frac{\partial}{\partial \nu_y}\hat{u}^{(sc)}(y))$ is $\sim s\hat{u}^{(sc)}\frac{e^{-Rs}}{R} + o(\frac{1}{R^2})$ and it's easy to show that (23) $\rightarrow 0$ as $R \rightarrow \infty$ for $\Re s \geq 0$. Equation (24) $\rightarrow 4\pi\hat{u}^{(sc)}(x)$ as $\epsilon \rightarrow 0$ (behaves like a δ function).

Thus, we derived the representation for

$$\hat{u}^{(sc)}(x) = -\frac{1}{4\pi} \iint_{\partial\Omega} [\hat{u}^{(sc)}(y)(\frac{\partial}{\partial \nu_y} \frac{e^{-|x-y|s}}{|x-y|}) - \frac{e^{-|x-y|s}}{|x-y|}(\frac{\partial}{\partial \nu_y} \hat{u}^{(sc)}(y))] dA_y$$

For the far field pattern, we want to look at terms like $\lim_{r\to\infty} r e^{rs} \hat{u}^{(sc)}(x)$, because that's what v looks like. (Note fixed origin $x = r\hat{x}$, $\hat{x} \in \mathcal{S}_1$)

$$|x-y|^2 = |r\hat{x} - y|^2 = r^2 - 2r\hat{x}\cdot y + |y|^2$$

$$|x-y| = r\sqrt{1 - \frac{2}{r}\hat{x}\cdot y + \frac{|y|^2}{r^2}} = r - \hat{x}\cdot y + O(\frac{1}{r})$$

$$re^{rs} \cdot \frac{e^{-|x-y|s}}{|x-y|} = \frac{e^{(\hat{x}\cdot y)s + o(\frac{1}{r})}}{1 + o(\frac{1}{r})} \rightarrow e^{(\hat{x}\cdot y)s} \quad as\ r \rightarrow \infty$$

Thus , we have the representation for v in the far field

$$v(\hat{x},\hat{\omega},s) = -\frac{1}{4\pi}\int\int_{\partial\Omega}[\hat{u}^{(sc)}(y)(\frac{\partial}{\partial\nu_y}e^{(\hat{x}\cdot y)s}) - (\frac{\partial}{\partial\nu_y}\hat{u}^{(sc)}(y))e^{(\hat{x}\cdot y)s}]dA_y$$

Note $e^{(\hat{x}\cdot y)s} = \hat{u}^{(inc)}(y;-\hat{x})$
Apply Green's identities inside Ω (for incident wave, good at the origin)

$$\begin{aligned} 0 &= \int\int\int_{\Omega}\hat{u}^{(inc)}(y;\hat{\omega})\nabla^2\hat{u}^{(inc)}(y;-\hat{x}) - \nabla^2\hat{u}^{(inc)}(y;\hat{\omega})\hat{u}^{(inc)}(y;-\hat{x})dv_y \\ &= \int\int_{\partial\Omega}[\hat{u}^{(inc)}(y;\hat{\omega})(\frac{\partial\hat{u}^{(inc)}(y;-\hat{x})}{\partial\nu_y}) - (\frac{\partial\hat{u}^{(inc)}(y;\hat{\omega})}{\partial\nu_y})\hat{u}^{(inc)}(y;-\hat{x})]dA_y \end{aligned}$$

similarly, apply outside Ω and use the radiation condition ($\Re s > 0$)

$$\int\int_{\partial\Omega}[\hat{u}^{(sc)}(y;\hat{\omega})(\frac{\partial\hat{u}^{(sc)}(y;-\hat{x})}{\partial\nu_y}) - (\frac{\partial\hat{u}^{(sc)}(y;\hat{\omega})}{\partial\nu_y})\hat{u}^{(sc)}(y;-\hat{x})]dA_y = 0$$

5.3 Reciprocity from boundary conditions

Let us assume general boundary conditions like

$$\alpha(x)s\hat{u} + \beta(x)\frac{\partial\hat{u}}{\partial\nu_y} + \gamma(x)\hat{u} = 0$$

rewrite as $\beta\frac{\partial\hat{u}^{(sc)}}{\partial\nu_y} + \lambda(s)\hat{u}^{(sc)} = -\beta\frac{\partial\hat{u}^{(inc)}}{\partial\nu_y} - \lambda(s)\hat{u}^{(inc)}$
Consider 2 cases:
i) Dirichlet: $\beta = 0, \lambda = 1$
ii) mixed: $\beta = 1; \Re\lambda(s) > 0$

case i):

$$
\begin{aligned}
v(\hat{x},\hat{\omega},s) &= -\frac{1}{4\pi}\int\int_{\partial\Omega}[\hat{u}^{(sc)}(y)(\frac{\partial}{\partial\nu_y}e^{(\hat{x}\cdot y)s}) - (\frac{\partial}{\partial\nu_y}\hat{u}^{(sc)}(y))e^{(\hat{x}\cdot y)s}]dA_y \\
&= -\frac{1}{4\pi}\int\int_{\partial\Omega}[\hat{u}^{(sc)}(y;\hat{\omega})(\frac{\partial\hat{u}^{(inc)}(y;-\hat{x})}{\partial\nu_y}) - \frac{\partial\hat{u}^{(sc)}(y;\hat{\omega})}{\partial\nu_y}\hat{u}^{(inc)}(y;-\hat{x})]dA_y \\
&= -\frac{1}{4\pi}\int\int_{\partial\Omega}[(\frac{\partial\hat{u}^{(sc)}(y;\hat{\omega})}{\partial\nu_y})\hat{u}^{(sc)}(y;-\hat{x}) - \hat{u}^{(inc)}(y;\hat{\omega})\frac{\partial\hat{u}^{(inc)}(y;-\hat{x})}{\partial\nu_y}]dA_y \\
&= -\frac{1}{4\pi}\int\int_{\partial\Omega}[(\frac{\partial\hat{u}^{(sc)}(y;-\hat{x})}{\partial\nu_y})\hat{u}^{(sc)}(y;\hat{\omega}) - \hat{u}^{(inc)}(y;-\hat{x})\frac{\partial\hat{u}^{(inc)}(y;\hat{\omega})}{\partial\nu_y}]dA_y \\
&= -\frac{1}{4\pi}\int\int_{\partial\Omega}[\hat{u}^{(sc)}(y;-\hat{x})\frac{\partial\hat{u}^{(inc)}(y;\hat{\omega})}{\partial\nu_y} - (\frac{\partial\hat{u}^{(sc)}(y;-\hat{x})}{\partial\nu_y})\hat{u}^{(inc)}(y;\hat{\omega})]dA_y \\
&= v(-\hat{\omega},-\hat{x},s)
\end{aligned}
$$

case ii):

$$
\begin{aligned}
v(\hat{x},\hat{\omega},s) &= -\frac{1}{4\pi}\int\int_{\partial\Omega}[\hat{u}^{(sc)}(y;\hat{\omega})(\frac{\partial\hat{u}^{(inc)}(y;-\hat{x})}{\partial\nu_y}) - \frac{\partial\hat{u}^{(sc)}(y;\hat{\omega})}{\partial\nu_y}\hat{u}^{(inc)}(y;-\hat{x})] \\
&= -\frac{1}{4\pi}\int\int_{\partial\Omega}[(\frac{\partial\hat{u}^{(inc)}(y;\hat{\omega})}{\partial\nu_y})\hat{u}^{(inc)}(y;-\hat{x}) - \hat{u}^{(sc)}(y;\hat{\omega})\frac{\partial\hat{u}^{(sc)}(y;-\hat{x})}{\partial\nu_y} \\
&\quad +\lambda(s)\hat{u}^{(inc)}(y,-\hat{x})[\hat{u}^{(inc)}(y;\hat{\omega}) + \hat{u}^{(sc)}(y;\hat{\omega})] \\
&\quad -\lambda(s)\hat{u}^{(sc)}(y,\hat{\omega})[\hat{u}^{(inc)}(y;-\hat{x}) + \hat{u}^{(sc)}(y;-\hat{x})] \\
&= -\frac{1}{4\pi}\int\int_{\partial\Omega}[\hat{u}^{(inc)}(y;\hat{\omega})(\frac{\partial\hat{u}^{(inc)}(y;-\hat{x})}{\partial\nu_y}) - \frac{\partial\hat{u}^{(sc)}(y;\hat{\omega})}{\partial\nu_y}\hat{u}^{(sc)}(y;-\hat{x})] \\
&\quad +\lambda(s)[\hat{u}^{(inc)}(y,-\hat{x})\hat{u}^{(inc)}(y;\hat{\omega}) - \hat{u}^{(sc)}(y,\hat{\omega})\hat{u}^{(sc)}(y;-\hat{x})] \\
&= -\frac{1}{4\pi}\int\int_{\partial\Omega}[\frac{\partial\hat{u}^{(inc)}(y;\hat{\omega})}{\partial\nu_y}\hat{u}^{(sc)}(y;-\hat{x}) - \hat{u}^{(inc)}(y;\hat{\omega})(\frac{\partial\hat{u}^{(sc)}(y;-\hat{x})}{\partial\nu_y})] \\
&= v(-\hat{\omega},-\hat{x},s)
\end{aligned}
$$

There are more general cases in references [8] [7].

5.4 Forward scattering theorem based on energy theory

Define the total energy to be

$$E_D = \int\int\int \frac{1}{2}(u_t^2 + |\nabla u|^2)$$

$$\frac{dE_D}{dt} = \int\int_{\partial D} u_t u_\nu$$

Integrate the outer flux term $\int\int_S \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial r}$ over time, we have

$$\int_0^\infty \int\int_S \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial r}$$

(by Parseval , assume real u)

$$= \int_{-i\infty}^{i\infty} \int\int_S s\hat{u}(s) \cdot \overline{\frac{\partial \hat{u}(y,s)}{\partial r}} = \int_{-i\infty}^{i\infty} \int\int_S s\hat{u}(y,s) \cdot \frac{\partial \hat{u}(y,-s)}{\partial r}$$

$$\int\int_S s\hat{u}(y,s) \cdot \frac{\partial \hat{u}(y,-s)}{\partial r} = \int\int_S s\hat{u}^{(inc)}(y,s) \cdot \frac{\partial \hat{u}^{(inc)}(y,-s)}{\partial r} \tag{25}$$

$$+\int\int_S s\hat{u}^{(inc)}(y,s) \cdot \frac{\partial \hat{u}^{(sc)}(y,-s)}{\partial r} \tag{26}$$

$$+\int\int_S s\hat{u}^{(sc)}(y,s) \cdot \frac{\partial \hat{u}^{(inc)}(y,-s)}{\partial r} \tag{27}$$

$$+\int\int_S s\hat{u}^{(sc)}(y,s) \cdot \frac{\partial \hat{u}^{(sc)}(y,-s)}{\partial r} \tag{28}$$

term (28) when R is large and s purely imaginary,

$$\sim -\int\int_{\mathcal{S}_1} R^2 (s\frac{e^{-Rs}}{R} v(\hat{y},\hat{\omega};s))(s\frac{e^{Rs}}{R} v(\hat{y},\hat{\omega};-s)) d\hat{A}$$

$$= -s^2 \int\int_{\mathcal{S}_1} v(\hat{y},\hat{\omega};s) v(\hat{y},\hat{\omega};-s) d\hat{A}$$

term (25) $= 0$, since $\hat{u}^{(inc)} = e^{-Rs\hat{\omega}\cdot\hat{y}} \Rightarrow R^2 s^2 \int\int_{\mathcal{S}_1} \hat{\omega}\cdot\hat{y} = 0$

term (26) + (27) $\sim$

$$R^2 \int\int_{\mathcal{S}_1} (se^{-Rs\hat{\omega}\cdot\hat{y}} \cdot \frac{se^{Rs}}{R} v(\hat{y},\hat{\omega};-s) + \frac{se^{Rs}}{R} v(\hat{y},\hat{\omega};s) \cdot s\hat{\omega}\cdot y e^{Rs\hat{\omega}\cdot\hat{y}}$$

$$= Rs^2 \int\int_{\mathcal{S}_1} e^{Rs(1-\hat{\omega}\cdot\hat{y})} \cdot v(\hat{y},\hat{\omega};-s) + \hat{\omega}\cdot\hat{y} e^{-Rs(1-\hat{\omega}\cdot\hat{y})} v(\hat{y},\hat{\omega};s)$$

Suppose $\hat{\omega} = e_3$, in coordinates $\hat{\omega}\cdot\hat{y} = \cos\theta$

$$\begin{aligned}
\int\int_{\mathcal{S}_1} (e^{Rs(1-\hat{\omega}\cdot y)} \cdot v(\hat{x},\hat{\omega};-s) &= \int_0^{2\pi}\int_0^{\pi} e^{Rs(1-\cos\theta)} \cdot v(\theta,\varphi;-s)\sin\theta d\theta d\varphi \\
&= \frac{1}{Rs}\int_0^{2\pi}\int_0^{\pi} v(\theta,\varphi;-s)\frac{d}{d\theta} e^{Rs(1-\cos\theta)} d\varphi \\
&= \frac{2\pi}{Rs}[v(\pi,(),-s)e^{2Rs} - v(0,(),-s)] \\
&\quad -\frac{1}{Rs}\int_0^{2\pi}\int_0^{\pi} \frac{d}{d\theta} v(\theta,\varphi;-s) e^{Rs(1-\cos\theta)} d\theta d\varphi \\
&= \frac{2\pi}{Rs}[v(\pi,(),-s)e^{2Rs} - v(0,(),-s)] + o(\frac{1}{R})
\end{aligned}$$

$$\iint\limits_{S_1} \hat{\omega}\cdot ye^{-Rs(1-\hat{\omega}\cdot y)}v(\hat{x},\hat{\omega};s) = \int_0^{2\pi}\int_0^{\pi} e^{-Rs(1-\cos\theta)}\cdot v(\theta,\varphi;s)\cos\theta\sin\theta d\theta d\varphi$$
$$= -\frac{1}{Rs}\int_0^{2\pi}\int_0^{\pi} v(\theta,\varphi;s)\cos\theta\frac{d}{d\theta}e^{-Rs(1-\cos\theta)}d\varphi$$
$$= \frac{2\pi}{Rs}[v(\pi,(),-s)e^{-2Rs}+v(0,(),-s)]$$
$$+\frac{1}{Rs}\int_0^{2\pi}\int_0^{\pi}\frac{d}{d\theta}[v(\theta,\varphi;s)\cos\theta]e^{Rs(1-\cos\theta)}d\theta d\varphi$$
$$= \frac{2\pi}{Rs}[v(\pi,(),s)e^{-2Rs}+v(0,(),s)]+o(\frac{1}{R})$$

While, the last step is not completely obvious here. We've used the method of stationary phase to determine the asymptotic behavior for R large. For example,

$$\int_0^{\pi}\frac{d}{d\theta}[v(\theta,\varphi;s)\cos\theta]e^{Rs(1-\cos\theta)}d\theta \sim \frac{d}{d\theta}[v(\theta,\varphi;-s)\cos\theta]\Big|_{\theta=0}\left[\frac{2\pi}{R|s|}\right]^{1/2}e^{i\pi/4}$$

combine (25) ~ (28) energy flux for R large , we get

$$\iint\limits_{S} s\hat{u}(y,s)\cdot\frac{\partial\hat{u}(y,-s)}{\partial r} = s^2\iint\limits_{S_1} v(\hat{y},\hat{\omega};s)v(\hat{y},\hat{\omega};-s)d\hat{A}_y$$
$$+2\pi s[v(-\hat{\omega},\hat{\omega};-s)e^{2Rs}-v(\hat{\omega},\hat{\omega};-s)]$$
$$+2\pi s[v(-\hat{\omega},\hat{\omega};s)e^{-2Rs}+v(\hat{\omega},\hat{\omega};s)]$$

since $2\pi s[v(-\hat{\omega},\hat{\omega};-s)e^{2Rs}+v(-\hat{\omega},\hat{\omega};s)e^{-2Rs}]$ is odd in s, so $\int 2\pi s[v(-\hat{\omega},\hat{\omega};-s)e^{2Rs}+v(-\hat{\omega},\hat{\omega};s)e^{-2Rs}]=0$

Energy flux:

$$s^2\iint\limits_{S_1} v(\hat{y},\hat{\omega};s)v(\hat{y},\hat{\omega};-s)d\hat{A}_y + 2\pi s[v(\hat{\omega},\hat{\omega};s)-v(\hat{\omega},\hat{\omega};-s)]$$

For a lossless scatterers the net energy flux=0 i.e.

$$\int\!\!\int_{S_1} v(\hat{y},\hat{\omega};s)v(\hat{y},\hat{\omega};-s)d\hat{A}_y = -\frac{2\pi}{s}[v(\hat{\omega},\hat{\omega};s) - v(\hat{\omega},\hat{\omega};-s)] \qquad (29)$$

which is the same as the ones in [4] analogous to (2.4) (5.6) (6.6).

5.5 Numerical results

Suppose we have the following wave equation and Dirichlet boundary condition.

$$s^2\hat{u} = \nabla^2\hat{u}$$

$$\hat{u} = 0 \ , \ r = 1$$

Assume an incoming plane wave is in the z-direction i.e.

$$\hat{u}^{(inc)} = e^{-s(0,0,1)\cdot(x,y,z)} = e^{-sz}$$

We want to compute the exact scattered solution $\hat{u}^{(sc)}$.

5.5.1 Background information

Legendre's polynomial and spherical harmonics use Rodriguez' formula, the *Legendre polynomial* can be written as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Orthogonality relation:

$$\int_{-1}^{1} P_{n'}(x)P_n(x)dx = \frac{2}{2n+1}\delta_{n'n}$$

If the problem does not possess azimuthal symmetry, we will need associated Legendre functions. For $m = 0, \pm1, \pm2, \ldots, \pm n$

Associated Legendre functions are

$$P_n^m(x) = \frac{(-1)^m}{2^n n!}(1-x^2)^{m/2}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n$$

Properties and Orthogonality relation:

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

$$\int_{-1}^{1} P_{n'}^m(x) P_n^m(x) dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{n'n}$$

Notes:

1. $P_n(x)$ forms a complete orthogonal set for the expansion of functions of the variable x

2. $P_n^m(\cos(\theta)) \cdot e^{im\varphi}$ forms such a set for the expansion of arbitrary functions on the surface of a sphere.

Spherical harmonics are defined as

$$Y_n^m(\theta, \varphi) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos(\theta)) \cdot e^{im\varphi}$$

Orthogonality relation:

$$\int_{4\pi} Y_n^m(\theta, \varphi) Y_{n'}^{m'}(\theta, \varphi) d\Omega = \int_0^{2\pi} \int_0^{\pi} Y_n^m(\theta, \varphi) \overline{Y_{n'}^{m'}(\theta, \varphi)} \sin\theta d\theta d\varphi = \delta_{nn'} \delta_{mm'}$$

Modified spherical Bessel's functions *Hankel's symbol*

$$(n, k) = \frac{\Gamma(\frac{1}{2} + n + k)}{k! \Gamma(\frac{1}{2} + n - k)}$$

Modified spherical Bessel's functions are the solution to the equation

$$z^2 \omega'' + 2z\omega' - [z^2 + n(n+1)]\omega = 0$$

$$\begin{aligned} i_n(z) &= \sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z) \\ &= \begin{cases} e^{-n\pi i/2} j_n(ze^{\pi i/2}) & \arg z \in (-\pi, \pi/2] \\ e^{3n\pi i/2} j_n(ze^{-3\pi i/2}) & \arg z \in (\pi/2, \pi] \end{cases} \end{aligned}$$

Expansion of $i_n(z)$

$$i_n(z) = (2z)^{-1}[R(n+\frac{1}{2}, -z)e^z - (-1)^n R(n+\frac{1}{2}, z)e^{-z}]$$

Where
$R(n+\frac{1}{2}, z) = \sum_{k=0}^{n}(n+\frac{1}{2}, k)(2z)^{-k}$

$$\begin{aligned} k_n(z) &= \sqrt{\frac{\pi}{2z}}K_{n+\frac{1}{2}}(z) \\ &= \pi(2z)^{-1}e^{-z}\sum_{k=0}^{n}(n+\frac{1}{2}, k)(2z)^{-k} \\ &= \frac{\pi}{2z}e^{-z}R(n+\frac{1}{2}, z) \end{aligned}$$

Notes:
i_n, k_n are the particular solutions we will be using later on. i_n behaves good at origin, while k_n behaves good at infinity.

Some useful expansions

$$\begin{aligned} e^{z\cos\theta} &= \sum_{n=0}^{\infty}(2n+1)[\sqrt{\frac{\pi}{2z}}I_{n+\frac{1}{2}}(z)]P_n(\cos\theta) \\ &= \sum_{n=0}^{\infty}(2n+1)i_n(z)P_n(\cos\theta) \end{aligned}$$

$$\begin{aligned} e^{-z\cos\theta} &= \sum_{n=0}^{\infty}(2n+1)(-1)^n[\sqrt{\frac{\pi}{2z}}I_{n+\frac{1}{2}}(z)]P_n(\cos\theta) \\ &= \sum_{n=0}^{\infty}(2n+1)(-1)^n i_n(z)P_n(\cos\theta) \end{aligned}$$

5.5.2 Computation of the solution

We already have the expansion for the incident wave

$$\hat{u}^{(inc)} = e^{-sz} = e^{-sr\cos\theta} = \sum_{n=0}^{\infty}(2n+1)(-1)^n i_n(sr)P_n(\cos\theta)$$

The general scattered solution(in this case) has the form

$$\begin{aligned}\hat{u}^{(sc)} &= \sum_{n=0}^{\infty}\sum_{m=-n}^{m=n} a_{nm}(s)k_n(sr)Y_n^m(\theta,\varphi) \\ &= \sum_{n=0}^{\infty} a_n(s)k_n(sr)P_n(\cos\theta)\end{aligned}$$

The second line comes from azimuthal symmetry. In our case, the system is symmetric about φ, thus $m = 0$ i.e. no m dependence, then we have a simplified form.

Applying the boundary condition, at $r = 1$

$$\alpha(s)\hat{u}^{(sc)} + \beta\frac{\partial\hat{u}^{(sc)}}{\partial n} = -\alpha(s)\hat{u}^{(inc)} - \beta\frac{\partial\hat{u}^{(inc)}}{\partial n}$$

n is the outward normal, i.e. r in our case (sphere). We get

$$a_n(s)[\alpha(s)k_n(s) + s\beta k_n'(s)] = (2n+1)(-1)^{n+1}[\alpha(s)i_n(s) + s\beta i_n'(s)]$$

$$a_n(s) = \frac{(-1)^{n+1}(2n+1)[\alpha(s)i_n(s) + s\beta i_n'(s)]}{\alpha(s)k_n(s) + s\beta k_n'(s)}$$

Then

$$\hat{u}^{(sc)} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)[\alpha(s)i_n(s) + s\beta i_n'(s)]}{\alpha(s)k_n(s) + s\beta k_n'(s)}k_n(sr)P_n(\cos\theta)$$

Let's look at the far field pattern $r \to \infty$, $k_n(sr) \to \frac{\pi}{2}\frac{e^{-sr}}{sr}$ (asymptotic behavior)

$$\hat{u}_{\infty}^{(sc)} = \frac{e^{-sr}}{r}\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)[\alpha(s)i_n(s) + s\beta i_n'(s)]}{s[\alpha(s)k_n(s) + s\beta k_n'(s)]}P_n(\cos\theta)$$

By our previous definition

$$\begin{aligned}
v(\hat{x},\hat{\omega};s) &= \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)[\alpha(s)i_n(s)+s\beta i_n'(s)]}{s[\alpha(s)k_n(s)+s\beta k_n'(s)]}P_n(\cos\theta) \\
&= \frac{\pi}{2}\sum_{n=0}^{\infty} v_n(s)P_n(\cos\theta)
\end{aligned}$$

For this v, (29) should hold.

For simplicity, let's assume a Dirichlet boundary condition i.e. $\alpha(s) = 1, \beta = 0$ to compute the singularity expansion and check the numerical results.

5.5.3 Singularity Expansions

We will perform a singularity expansion as we did in [4], finally we will see it holds (numerically verified). By using the expansion for i_n, k_n we have

$$\begin{aligned}
v_n(s) &= \frac{(-1)^{n+1}(2n+1)i_n(s)}{sk_n(s)} \\
&= (-1)^{n+1}(2n+1)\frac{(2s)^{-1}[R(n+\frac{1}{2},-s)e^{s}-(-1)^n R(n+\frac{1}{2},s)e^{-s}]}{\pi(2s)^{-1}e^{-s}R(n+\frac{1}{2},s)s} \\
&= \frac{(2n+1)}{\pi}\Big[\frac{(-1)^{n+1}R(n+\frac{1}{2},-s)}{R(n+\frac{1}{2},s)s}e^{2s}+\frac{1}{s}\Big]
\end{aligned}$$

Consider

$$\frac{(-1)^{n+1}R(n+\frac{1}{2},-s)}{R(n+\frac{1}{2},s)s} = \frac{Q_1^n(s)}{Q_2^n(s)}$$

It is easy to see that Q_1^n, Q_2^n are polynomials of degree $n, n+1$ respectively. From the property of i_n, k_n, we can conclude Q_1^n, Q_2^n do not share zeros and zeros are simple.

Suppose $Q_2^n(s_j^n) = 0$ for $j = 0, 1, \ldots n$

$$\frac{Q_1^n(s)}{Q_2^n(s)} = \sum_{j=0}^{n}\frac{C_j^n}{s-s_j^n}$$

$$C_j^n = \frac{Q_1^n(s_j^n)}{\frac{d}{ds}Q_2^n(s_j^n)}$$

Proof of C_j^n:

Suppose $Q_2^n(s) = B\prod_{j=0}^n (s - s_j^n)$, we multiply $\frac{Q_1^n(s)}{B\prod_{j=0}^n(s-s_j^n)} = \sum_{j=0}^n \frac{C_j^n}{s-s_j^n}$ by $(s - s_{j'}^n)$ and set $s = s_{j'}^n$. We get

$$C_{j'}^n = \frac{Q_1^n(s_{j'}^n)}{B\prod_{j=0,j\neq j'}^n (s_{j'}^n - s_j^n)} = \frac{Q_1^n(s_{j'}^n)}{\frac{d}{ds}Q_2^n(s_{j'}^n)}$$

Thus, we have the singularity expansion

$$v_n(s) = \frac{2n+1}{\pi}\left[\sum_{j=0}^n \frac{C_j^n}{s - s_j^n} e^{2s} + \frac{1}{s}\right]$$

$$v = \frac{\pi}{2}\sum_{n=0}^\infty v_n(s) P_n(\cos\theta) \tag{30}$$

5.5.4 Connection with the forward-scattering theorem

In [4], Dr. Carl Baum suggests the following form for the scattering dyadic,

$$\overleftrightarrow{\tilde{\Lambda}}(\vec{1}_r, \vec{1}_i; s) = \sum_\alpha \frac{e^{-[s-s_\alpha]t_i}}{s - s_\alpha}\vec{c}_\alpha(-\vec{1}_r)\vec{c}_\alpha(\vec{1}_i) + entire function$$

(In our case, it's just (30)) and derives the following equation,

$$\int_{S_1} \vec{c}_\alpha(-\vec{1}_r)\cdot \overleftrightarrow{\tilde{\Sigma}}(\vec{1}_r, \vec{1}_i'; -s_\alpha) = -\frac{2\pi}{\gamma_\alpha}\vec{c}_\alpha(-\vec{1}_i') \tag{31}$$

which we are going to verify in the next part. In order to derive an analogue of the above equation from (29), we will follow the same procedure as in [4]. Since there is an infinite sum in the integral, we will also use the orthogonality to simplify the equation.

In (29) , let's multiply both sides by $s - s_\alpha^n$, and take the limit $s \to s_\alpha^n$, (s_α^n is one zero of Q_2^n , $s_\alpha^n \neq 0$) , using the orthogonality of P_n we get

$$\frac{\pi}{2}\int_{4\pi} \frac{2n+1}{\pi} C_\alpha^n e^{2s_\alpha^n} P_n(\cos\theta)\cdot \sum_{n=0}^\infty v_n(-s_\alpha^n) P_n(\cos\theta) = -\frac{2\pi}{s_\alpha^n}\frac{2n+1}{\pi} C_\alpha^n e^{2s_\alpha^n} P_n(\cos 0)$$

Table 1: Coefficient C_j^n for $n = 1, ..., 5$ $j = 0, ...n$

n

	-1	-1	-1	-1	-1
	2	2.564e-016 - 3.4641i	-5.1583 + 0.67542i	2.2877 + 6.9041i	8.3088 - 5.0428i
j		2.564e-016 + 3.4641i	-5.1583 - 0.67542i	2.2877 - 6.9041i	8.3088 + 5.0428i
			12.317	-2.2877 - 30.596i	111.19
				-2.2877 + 30.596i	-62.905 + 12.511i
					-62.905 - 12.511i

Table 2: roots of $Q_2^n(s)$ for $n = 1, ..., 5$ $j = 1, ...n$ $(s_0^n = 0)$

n

	-1	-1.5 + 0.86603i	-1.8389 + 1.7544i	-2.1038 + 2.6574i	-2.3247 + 3.571i
		-1.5 - 0.86603i	-1.8389 - 1.7544i	-2.1038 - 2.6574i	-2.3247 - 3.571i
j			-2.3222	-2.8962 + 0.86723i	-3.6467
				-2.8962 - 0.86723i	-3.352 + 1.7427i
					-3.352 - 1.7427i

$$\Rightarrow \frac{\pi}{2}\int_{4\pi} v(-s_\alpha^n)[P_n(\cos\theta)]^2 = -\frac{2\pi}{s_\alpha^n}P_n(\cos 0)$$

$$\Rightarrow \frac{\pi}{2}\frac{2n+1}{\pi}\Big[\sum_{j=0}^{n}\frac{C_j^n}{-s_\alpha^n - s_j^n}e^{-2s_\alpha^n}+\frac{1}{-s_\alpha^n}\Big]\cdot\int_0^{2\pi}\int_0^{\pi}[P_n(\cos\theta)]^2\sin\theta d\theta d\varphi = -\frac{2\pi}{s_\alpha^n}P_n(\cos 0)$$

Simplifying we get

$$\sum_{j=0}^{n}\frac{C_j^n}{-s_\alpha^n - s_j^n}e^{-2s_\alpha^n} + \frac{1}{-s_\alpha^n} = -\frac{1}{s_\alpha^n}P_n(1) \tag{32}$$

Finally, this is the equation analogous to (31), which we are going to verify numerically.

C_j^n are the coefficients of the singularity expansion. s_j^n are the zeros of $Q_2^n(s)$, which are also the zeros of $k_n(s)$, s_α^n is anyone of s_j^n, $s_\alpha^n \neq 0$

For your reference, I computed all the C_j^n , s_j^n, error=|LHS-RHS| of (32)

See Table 1, 2, 3

Table 3: Error for each s_α^n and for $n = 1, ..., 5$

n

	0	1.1525e-014	2.1245e-013	4.1905e-012	5.1391e-011
		1.2012e-014	2.1245e-013	4.1905e-012	5.1391e-011
α			3.6948e-013	1.9759e-011	6.9491e-010
				1.9723e-011	3.9091e-010
					3.9096e-010

5.5.5 Legendre's addition theorem

If we apply Legendre's addition theorem [10] to the above results, we can get a more general solution for arbitrary incidence direction.

let $\hat{r}_1, \hat{r}_2$ be two unit vectors with components given by

$$\hat{r}_1 = (\sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1, \cos\theta_1)$$

$$\hat{r}_2 = (\sin\theta_2\cos\phi_2, \sin\theta_2\sin\phi_2, \cos\theta_2)$$

The angle between $\hat{r}_1$ and $\hat{r}_2$ is γ , defined by

$$\hat{r}_1 \cdot \hat{r}_2 = \cos\gamma = \sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2) + \cos\theta_1\cos\theta_2$$

$$P_n(\cos\gamma) = \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta_1) P_n^m(\cos\theta_2)\cos m(\phi_1 - \phi_2)$$

$\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m > 0$

In terms of spherical harmonics , we have

$$P_n(\hat{r}_1 \cdot \hat{r}_2) = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_n^m(\hat{r}_1)\overline{Y_n^m(\hat{r}_2)}$$

Thus, the incident plane wave can be written as

$$\hat{u}^{(inc)} = \sum_{n=0}^{\infty}(2n+1)(-1)^n i_n(sr) \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta_1) P_n^m(\cos\theta_2)\cos m(\phi_1 - \phi_2)$$

or in terms of spherical harmonics

$$\hat{u}^{(inc)} = \sum_{n=0}^{\infty}(2n+1)(-1)^n i_n(sr)\frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_n^m(\hat{r}_1)\overline{Y_n^m(\hat{r}_2)}$$

in which, we can define $\hat{r}_1$ as the incidence direction and $\hat{r}_2$ as the viewing direction for appropriate choice of γ.

Thus, the scattered solution(30) becomes

$$v = \frac{\pi}{2}\sum_{n=0}^{\infty} v_n(s)\frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_n^m(\hat{r}_1)\overline{Y_n^m(\hat{r}_2)}$$

which has almost the same form as appears in the [4]. This form can be used to determine any incident direction $\hat{r}_1$ and scattered direction $\hat{r}_2$ for appropriate choice of γ.

6 Conclusion

We show that for acoustic scattering problems arbitrary order scattering poles can be constructed with certain lossless impedance boundary condition imposed on the sphere. While for the hard and soft sphere, there only exist simple poles. For the electromagnetic scattering problems for spheres with lossless sheet impedance loading boundary conditions, 2nd order scattering poles can be constructed. There only exist first order poles for the perfectly conductor sphere. The validity of the acoustic forward-scattering theorem (scalar wave equation) have been analyzed as well as some numerical experiment of the theorem.

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