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Analytic Solutions of the Klein-Gordon Equation
in a Semi-infinite Channel

by

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Abstract

In this report we show how to construct analytic solutions of the Klein-Gordon equation in a semi-infinite channel. The Klein-Gordon equation can be derived from the shallow water equations. The analytic solutions are given for various choices of initial and boundary conditions.

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1 Statement of the Problem

Consider the shallow water equations (SWEs) in a semi-infinite channel. For simplicity we assume that the channel has a flat bottom and that there is no advection, although these assumptions may be removed in future studies. We do take into account rotation (Coriolis) effects. A Cartesian coordinate system (x, y) is introduced such that the channel is parallel to the x direction, as shown in the figure. The width of the channel is denoted b .

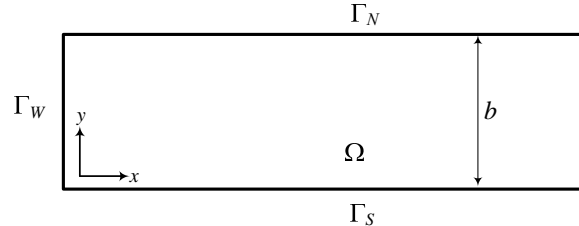


Figure 1: Setup for the wave-guide problem in a semi-infinite wave guide

The SWEs are (see [1]):

$$\begin{aligned}
 \partial_t u + \mu u \partial_x u + \mu v \partial_y u - f v &= -g \partial_x \eta , \\
 \partial_t v + \mu u \partial_x v + \mu v \partial_y v + f u &= -g \partial_y \eta , \\
 \partial_t \eta + \mu u \partial_x \eta + \mu v \partial_y \eta + (h_0 + \mu \eta) (\partial_x u + \partial_y v) &= 0 .
 \end{aligned}
 \tag{1}$$

Here t is time, $u(x, y, t)$ and $v(x, y, t)$ are the unknown velocities in the x and y directions, h_0 is the given water layer thickness (in the direction normal to the xy plane), $\eta(x, y, t)$ is the unknown water elevation above h_0 , f is the Coriolis parameter, and g is the gravity acceleration. We use the following shorthand for partial derivatives

$$\partial_a^i = \frac{\partial^i}{\partial a^i}$$

The parameter μ is 1 for the nonlinear SWEs, and is 0 for the linearized SWEs with vanishing mean flow. We shall only consider the latter in the sequel.

It can be shown (see [2]) that a *single* boundary condition must be imposed along the entire boundary to obtain a well-posed problem. On the south and north channel walls Γ_S and Γ_N we have $v = 0$ (no normal flow). On the west boundary Γ_W we prescribe η using the Dirichlet condition $\eta(0, y, t) = \eta_W(y, t)$, where $\eta_W(y, t)$ is a given function (incoming wave). At $x \rightarrow \infty$ the solution is known to be bounded and not to include any incoming waves. To complete the statement of the problem, initial values for u , v and η are given at time $t = 0$ in the entire domain.

It is easy to see (e.g., [3]) that the system (1) is equivalent to

$$\partial_t^2 \eta - C_0^2 \nabla^2 \eta + f^2 \eta = 0
 \tag{2}$$

where $C_0 = \sqrt{gh_0}$. The boundary conditions are

$$\begin{aligned}\eta(0, y, t) &= F(y, t), \\ \partial_n \eta(x, 0, t) &= 0, \\ \partial_n \eta(x, b, t) &= 0, \\ \lim_{x \rightarrow \infty} \eta(x, y, t) &\text{ is bounded}\end{aligned}\tag{3}$$

and the initial conditions are

$$\begin{aligned}\eta(x, y, 0) &= G(x, y), \\ \partial_t \eta(x, y, 0) &= 0\end{aligned}\tag{4}$$

The general solution to the problem can be found by taking the Fourier sine transform in x and then solve the resulting PDE in t and y . This is very messy and will not be pursued here. In the following sections we will discuss the possible choices for $F(y, t)$ and $G(x, y)$. One can also take a non vanishing second initial condition.

2 Construction of Solutions

Since the problem (2)-(4) has a non-homogeneous boundary condition, we will decompose η

$$\eta = v + w\tag{5}$$

where v will satisfy the PDE with a homogeneous boundary condition on the west boundary and w satisfies the non-homogeneous boundary condition with a simplified PDE. We will choose w to satisfy the following problem

$$\begin{aligned}\partial_t^2 w - C_0^2 \partial_y^2 w + f^2 w &= 0 \\ w(0, y, t) &= F(y, t), \\ \partial_n w(x, 0, t) &= 0, \\ \partial_n w(x, b, t) &= 0, \\ \lim_{x \rightarrow \infty} w(x, y, t) &\text{ is bounded} \\ w(x, y, 0) &= H(y), \\ \partial_t w(x, y, 0) &= 0\end{aligned}\tag{6}$$

The simplification is the fact that w is independent of x .

Remark: If the boundary condition at $x = 0$ is a function of y only, then the above equation (6) will not work. We will consider that case in the next section.

Problem (6) can be solved using separation of variables, i.e. by assuming that $w = Y(y)T(t)$.

It is easy to see that the solution is

$$w = \sum_{m=0}^{\infty} A_m \cos(\sqrt{\nu_0(m)} t) \cos\left(\frac{m\pi}{b} y\right)\tag{7}$$

where

$$\nu_0(m) = f^2 + \left(\frac{m\pi C_0}{b}\right)^2\tag{8}$$

Since w is independent of x , this is also $F(y, t)$. Using the initial condition

$$H(y) = \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi}{b}y\right) \quad (9)$$

we can find the Fourier coefficients A_m . Let's choose (for simplicity)

$$H(y) = \cos\left(\frac{\pi}{b}y\right). \quad (10)$$

This choice will simplify the computation of the Fourier coefficients A_m , i.e.

$$A_m = \begin{cases} 1 & m = 1 \\ 0 & m \neq 1 \end{cases} \quad (11)$$

Therefore

$$F(y, t) = \cos(\sqrt{\nu_0(1)} t) \cos\left(\frac{\pi}{b}y\right) \quad (12)$$

Clearly if we choose a different $H(y)$, we get a different function $F(y, t)$. The solution w is then given by

$$w = \cos(\sqrt{\nu_0(1)} t) \cos\left(\frac{\pi}{b}y\right) \quad (13)$$

Now we take v given by (5) and substitute in (2), we have

$$\partial_t^2 v + \partial_t^2 w - C_0^2 \nabla^2 v - C_0^2 \nabla^2 w + f^2 v + f^2 w = 0$$

Move all the terms with w to the right and use (13), we have

$$\partial_t^2 v - C_0^2 \nabla^2 v + f^2 v = 0 \quad (14)$$

which is identical to (2). Now the boundary conditions become

$$\begin{aligned} v(0, y, t) &= \underbrace{\eta(0, y, t)}_{F(y,t)} - \underbrace{w(0, y, t)}_{F(y,t)} = 0 \\ \partial_n v(x, 0, t) &= 0 \\ \partial_n v(b, y, t) &= 0 \\ \lim_{x \rightarrow \infty} v(x, y, t) &\text{ is bounded} \end{aligned} \quad (15)$$

The initial conditions are

$$\begin{aligned} v(x, y, 0) &= G(x, y) - \underbrace{w(x, y, 0)}_{\cos\left(\frac{\pi}{b}y\right)} \\ \partial_t v(x, y, 0) &= 0 \end{aligned} \quad (16)$$

To solve (14)-(16), we use separation of variables

$$v(x, y, t) = T(t)\phi(x, y) \quad (17)$$

Substituting in (14), we have two differential equations

$$\ddot{T} + \nu T = 0 \quad (18)$$

with

$$\dot{T}(0) = 0 \quad (19)$$

and

$$\nabla^2 \phi + \frac{\nu - f^2}{C_0^2} \phi = 0 \quad (20)$$

with the boundary conditions

$$\begin{aligned} \phi(0, y) &= 0 \\ \partial_y \phi(x, 0) &= 0 \\ \partial_y \phi(x, b) &= 0 \\ \lim_{x \rightarrow \infty} \phi(x, y) &\text{ is bounded} \end{aligned} \quad (21)$$

To solve (20), we will separate the variables again, assuming $\phi(x, y) = X(x)Y(y)$ to get

$$\begin{aligned} Y'' + \mu Y &= 0 & X'' + \left(\frac{\nu - f^2}{C_0^2} - \mu \right) X &= 0 \\ Y'(0) &= 0 & X(0) &= 0 \\ Y'(b) &= 0 & \lim_{x \rightarrow \infty} X(x) &\text{ is bounded} \end{aligned} \quad (22)$$

The solution for the Y equation is

$$\begin{aligned} \mu_m &= \left(\frac{m\pi}{b} \right)^2 \\ Y_m(y) &= \cos \left(\frac{m\pi}{b} y \right) \\ m &= 0, 1, 2, \dots \end{aligned} \quad (23)$$

In order for the X equation to have a non trivial solution, we must have $\nu > \nu_0(m)$ where $\nu_0(m)$ is given by (8). In this case the solution will be

$$X(x) = \sin \left(\frac{\sqrt{\nu - \nu_0(m)}}{C_0} x \right) \quad (24)$$

Therefore the solution for (20) is

$$\phi(x, y) = \sum_{m=0}^{\infty} \left[\int_{\nu_0(m)}^{\infty} A_m(\nu) \sin \left(\frac{\sqrt{\nu - \nu_0(m)}}{C_0} x \right) d\nu \right] \cos \left(\frac{m\pi}{b} y \right) \quad (25)$$

The solution of the T equation is

$$T(t) = \cos(\sqrt{\nu} t) \quad (26)$$

and therefore

$$v(x, y, t) = \sum_{m=0}^{\infty} \left[\int_{\nu_0(m)}^{\infty} A_m(\nu) \cos(\sqrt{\nu} t) \sin \left(\frac{\sqrt{\nu - \nu_0(m)}}{C_0} x \right) d\nu \right] \cos \left(\frac{m\pi}{b} y \right) \quad (27)$$

The only condition left to satisfy is $v(x, y, 0) = G(x, y) - \cos\left(\frac{\pi}{b}y\right)$. Before we do that, let us transform our general solution, by taking

$$\Lambda = \frac{\sqrt{\nu - \nu_0(m)}}{C_0} \quad (28)$$

We have

$$v(x, y, t) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} B_m(\Lambda) \cos\left(\sqrt{C_0^2 \Lambda^2 + \nu_0(m)t}\right) \sin(\Lambda x) 2C_0^2 \Lambda d\Lambda \right] \cos\left(\frac{m\pi}{b}y\right) \quad (29)$$

At $t = 0$, we have

$$G(x, y) - \cos\left(\frac{\pi}{b}y\right) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} B_m(\Lambda) 2C_0^2 \Lambda \sin(\Lambda x) d\Lambda \right] \cos\left(\frac{m\pi}{b}y\right) \quad (30)$$

Let us choose for simplicity

$$G(x, y) = g(x) \cos\left(\frac{\pi}{b}y\right) + \cos\left(\frac{\pi}{b}y\right) \quad (31)$$

then

$$g(x) \cos\left(\frac{\pi}{b}y\right) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} B_m(\Lambda) 2C_0^2 \Lambda \sin(\Lambda x) d\Lambda \right] \cos\left(\frac{m\pi}{b}y\right) \quad (32)$$

and therefore $m = 1$ and

$$g(x) = 2C_0^2 \int_0^{\infty} \Lambda B_1(\Lambda) \sin(\Lambda x) d\Lambda \quad (33)$$

This means that $2C_0^2 \Lambda B_1(\Lambda)$ is the Fourier sine transform of $g(x)$. Let us choose $g(x)$ as (the choice should be such that the Fourier sine transform of this, yield a convergent integral in (29))

$$g(x) = \frac{x}{x^2 + \alpha^2} \quad (34)$$

then

$$B_1(\Lambda) = \frac{e^{-\Lambda\alpha}}{2C_0^2 \Lambda} \quad (35)$$

Now we substitute this into (29) to get

$$v(x, y, t) = \left[\int_0^{\infty} e^{-\Lambda\alpha} \cos\left(\sqrt{C_0^2 \Lambda^2 + \nu_0(1)t}\right) \sin(\Lambda x) d\Lambda \right] \cos\left(\frac{\pi}{b}y\right) \quad (36)$$

Combining this with the solution for $w(x, y, t)$ given in (13) we have

$$\eta(x, y, t) = \left[\cos\left(\sqrt{\nu_0(1)t}\right) + \int_0^{\infty} e^{-\Lambda\alpha} \cos\left(\sqrt{C_0^2 \Lambda^2 + \nu_0(1)t}\right) \sin(\Lambda x) d\Lambda \right] \cos\left(\frac{\pi}{b}y\right) \quad (37)$$

This solution assumes

$$\begin{aligned} G(x, y) &= \left(\frac{x}{x^2 + \alpha^2} + 1 \right) \cos\left(\frac{\pi}{b}y\right) \\ F(y, t) &= \cos\left(\sqrt{\nu_0(1)t}\right) \cos\left(\frac{\pi}{b}y\right) \end{aligned} \quad (38)$$

3 The case that $F(y, t)$ is independent of t

In this case, one cannot use (6) because the solution (13) depends on time. Instead of (6), we should take

$$-C_0^2 (\partial_x^2 w + \partial_y^2 w) + f^2 w = 0 \quad (39)$$

subject to

$$\begin{aligned} w(0, y) &= F(y), \\ \partial_n w(x, 0) &= 0, \\ \partial_n w(x, b) &= 0, \\ \lim_{x \rightarrow \infty} w(x, y) &\text{ is bounded} \end{aligned} \quad (40)$$

The solution is given by

$$w(x, y) = \sum_{m=0}^{\infty} A_m e^{-(\nu_0(m)/C_0)x} \cos\left(\frac{m\pi}{b}y\right) \quad (41)$$

where $\nu_0(m)$ is given by (8) and

$$F(y) = \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi}{b}y\right) \quad (42)$$

If we choose

$$F(y) = \cos\left(\frac{\pi}{b}y\right) \quad (43)$$

then

$$w(x, y) = e^{-(\nu_0(1)/C_0)x} \cos\left(\frac{\pi}{b}y\right) \quad (44)$$

This $w(x, y)$ will give the same PDE for v . The only condition affected is (16) which is now

$$v(x, y, 0) = G(x) - e^{-(\nu_0(1)/C_0)x} \cos\left(\frac{\pi}{b}y\right) \quad (45)$$

For simplicity, we assume that

$$G(x) = g(x) \cos\left(\frac{\pi}{b}y\right) \quad (46)$$

and therefore

$$v(x, y, 0) = \left(g(x) - e^{-(\nu_0(1)/C_0)x}\right) \cos\left(\frac{\pi}{b}y\right) \quad (47)$$

This condition is only used when we reach (30) where we now have

$$\left(g(x) - e^{-(\nu_0(1)/C_0)x} - 1\right) \cos\left(\frac{\pi}{b}y\right) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} B_m(\Lambda) 2C_0^2 \Lambda \sin(\Lambda x) d\Lambda \right] \cos\left(\frac{m\pi}{b}y\right) \quad (48)$$

Therefore $m = 1$ and

$$g(x) - e^{-(\nu_0(1)/C_0)x} - 1 = \int_0^{\infty} B_1(\Lambda) 2C_0^2 \Lambda \sin(\Lambda x) d\Lambda \quad (49)$$

If we now take

$$g(x) = \frac{x}{x^2 + \alpha^2} + e^{-(\nu_0(1)/C_0)x} + 1 \quad (50)$$

then $B_1(\Lambda)$ is given by (35) and $v(x, y, t)$ is given by (36) as before. In this case we chose

$$\begin{aligned} G(x, y) &= \left(\frac{x}{x^2 + \alpha^2} + e^{-(\nu_0(1)/C_0)x} + 1 \right) \cos\left(\frac{\pi}{b}y\right) \\ F(y) &= \cos\left(\frac{\pi}{b}y\right) \end{aligned} \quad (51)$$

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