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**PROBABILLISTIC SENSITIVITY ANALYSIS IN LIFE-
PREDICTION OF AN $\alpha+\beta$ TITANIUM ALLOY
(PREPRINT)**

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Metals Branch

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Probabilistic Sensitivity Analysis in Life-Prediction of an $\alpha+\beta$ Titanium Alloy

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Abstract

Probabilistic sensitivities using the score function method are developed for a lifing analysis of an $\alpha+\beta$ titanium alloy in a round bar under axial fatigue load. Sensitivities with respect to the statistical inputs of the crack initiation size (a), and Paris crack growth coefficient (C) and exponent (m) are developed with consideration of the correlation between the C and m . The sensitivities are obtained using a single Monte Carlo sampling analysis and do not involve finite difference approximations. The sensitivities indicate the importance of the random variable input parameters on the mean life and standard deviation of life and can be used as a basis for determining constructive data collection efforts. For this example, the crack growth intercept (C) is the dominant variable that affects mean-life and standard deviation of life, indicating that improved confidence in the results can be obtained most efficiently by improving the statistical characterization of C .

Keywords: Life Prediction, Fatigue variability, Ti-6Al-2Sn-4Zr-6Mo, probabilistic sensitivities, score function method

Nomenclature

C crack growth coefficient

m crack growth exponent

a	initial crack size
\tilde{a}	median of initial crack size
COV	coefficient of variation
$E[\cdot]$	expected value operator over entire sample space
f	probability density function
$f_{\mathbf{x}}$	joint probability density function
F	cumulative distribution function
I	indicator function (one in the failure region, zero otherwise)
n	number of random variables
N	number of Monte Carlo samples
PDF	probability density function
CDF	cumulative density function
$S_{\theta_i}^{\mu_z}$	sensitivity of μ_z with respect to θ_i
$S_{\theta_i}^{\sigma_z}$	sensitivity of σ_z with respect to θ_i
$\bar{S}_{\theta_i}^{\mu_z}$	nondimensionalized sensitivity of μ_z with respect to θ_i
$\bar{S}_{\theta_i}^{\sigma_z}$	nondimensionalized sensitivity of σ_z with respect to θ_i
X	random variable
\mathbf{X}	vector of random variables
κ_{θ_i}	kernel function with respect to arbitrary distributional parameter θ_i
κ_{μ_i}	kernel function with respect to μ_i
κ_{σ_i}	kernel function with respect to distributional parameter σ_i
$\kappa_{\rho_{ij}}$	kernel function with respect to correlation coefficient ρ_{ij}

z	response (cycles-to-failure)
μ_i	mean of random variable i
μ_z	mean of life (cycles-to-failure)
σ_i	standard deviation of random variable i
σ_z	standard deviation of life (cycles-to-failure)
θ	arbitrary pdf parameter

1. Introduction

Development of probabilistic sensitivities is frequently considered an essential component of a probabilistic analysis and often critical towards understanding the important physical mechanisms underlying failure and perhaps identifying constructive testing for further data collection. Significant progress has been made over the past few decades in developing methods such that the sensitivity information is provided as a by-product of the analysis using the score function (SF) method with improved accuracy and computational efficiency relative to a finite difference approach.

The score function (SF) method was popularized by Rubinstein in a series of publications with applications to discrete event simulation (Kleijnen and Rubinstein [1], Rubinstein [2], Rubinstein and Shapiro [3]). The attraction of the method is that only one set of Monte Carlo samples is needed to estimate the moments of any response (μ_z and σ_z), e.g., the fatigue life, and the sensitivities of the moments with respect to the parameters of the input probability density functions (PDF), e.g., $\partial\mu_z/\partial\theta_i$ and $\partial\sigma_z/\partial\theta_i$, where θ_i represents any parameter of the input distributions such as the mean, standard deviation, shape factor, median, etc.

Wu and Mohanty proposed use of the sensitivity $\partial\mu_z/\partial\theta_i$ as a metric for screening problems with a large number of random variables (Wu and Mohanty[4]). In their approach, all random variables are mapped to standard normal before obtaining the sensitivities. The sensitivities with respect to the mean and standard deviation then follow t and chi-squared distributions, respectively. Hypothesis testing is used to identify significant variables.

In many engineering problems the input model variables are not independent and require construction and sampling from a joint probability density (see for e.g., Helton, et al. [5], Jacques, et al. [6]). With respect to fatigue, it is known that the parameters C (coefficient) and m (exponent) of the Paris equation are correlated [7 – 9]. Annis [7] analyzed a statistically significant number of crack growth curves by Virkler [10] under nominally the same microstructure and test conditions and showed that C and m are strongly negatively correlated, and treating these quantities as independent significantly overestimated the variance of the simulated response. The correlation can also have a strong influence on the sensitivities and should therefore be incorporated in the sensitivity analysis where applicable [11]. Millwater et al. [12] have developed the required “kernel” functions needed to compute the sensitivities for correlated normal variables using the score function method. As a result, probabilistic sensitivities with respect to means, standard deviations, and correlation coefficients for any number of variables are available.

In this paper, we apply the score function-based sensitivity analysis method for correlated normal variables by Millwater et al. [12] to probabilistic life-prediction of the $\alpha+\beta$ titanium alloy, Ti-6Al-2Sn-4Zr-6Mo (Ti-6-2-4-6). The lifing analysis used here was recently reported by Jha et al. [13]. The probabilistic sensitivity measures with respect to parameters of all model random variables were obtained in a single Monte Carlo analysis run. A ranking of these

parameters in terms of their degree of influence on the response mean and standard deviation is presented. Based on the analysis results, recommendations are made towards constructive data collection for improving the accuracy of life prediction. The analysis also revealed some interesting insights into the dependence of the response sensitivities on the correlation coefficient between C and m .

2. Methodology

2.1 Lifing analysis

The life prediction methodology presented here is based on a recently proposed description of the fatigue variability behavior (Jha, et al. [13-15]). In these publications, the major contribution to the lifetime variability is suggested to arise from the separation (or overlap), with respect to microstructure and loading variables, of a life-limiting mechanism and a mean-lifetime dominating response. Jha et al. [13] showed that microstructure, temperature, and loading variables have different degrees of influence on the life-limiting and the mean-dominating behavior affecting their separation, and therefore, the total lifetime variability. In several materials, the life-limiting mechanism was shown to be controlled by the crack growth lifetime beginning from a material-dependent microstructural unit size [13, 14]. The mean-dominating behavior was largely governed by the crack initiation regime, which diverged from the life-limiting crack-growth-controlled mechanism as the stress level or the temperature was decreased [13]. The description of fatigue variability in terms of the dual responses permitted a probabilistic lifing method based on the life-limiting, i.e., the crack growth controlled mechanism. In the following, the probabilistic model of the life-limiting behavior is described. This paper is concerned with determining the sensitivity of the life-limiting response moments, mean and standard deviation, with respect to the input variable parameters.

The microstructure of the Ti-6-2-4-6 alloy and the experimental procedure considered in this paper has been described elsewhere [13]. In the life-limiting mechanism, crack initiation occurred across an (or a few) equiaxed- α grain(s) at the surface of the specimen. This produced a crack initiation facet (or a few facets) at the crack origin, as illustrated by Fig. 1(a). The crack initiation size distribution, in terms of the crack initiation facet area, is compared to the nominal equiaxed- α sizes in Fig. 1(b). The crack initiation size, a , represented the radius of an equivalent circle with the same area as the faceted crack nucleation area measured on the fracture surface. As shown, the lognormal probability distribution provided a good description of the distribution in both the nominal equiaxed- α and the crack initiation size. Figure 1(b) also indicates that the crack initiation size distribution was slightly displaced to the right of the equiaxed- α distribution but clearly not to the extreme-right tail. It was shown [16] that the crack initiation size, alone, does not correlate with lifetime, since the lifetime is also partly governed by the variability in the small crack growth rate.

The variability in growth rates of naturally initiating small cracks at 860 MPa is shown in Fig. 2 (5 small crack growth curves, each from a different sample, are shown). The small crack growth curves were represented by power-law fits to the data which has been shown to be a reasonable approximation by, among others, Luo and Bowen [8, 9]. The small-crack growth rate is therefore given by:

$$\frac{da}{dN} = e^C \Delta K^m \quad (1)$$

where a is the half crack length, ΔK is the stress intensity factor range, and C and m are the Paris-type [17] crack growth coefficient and the crack growth exponent, respectively.

Foreman and Shivakumar [18] developed a K-solution for an elliptical surface crack in a solid round bar under tension. Their solution is based on the assumption of crack extension

according to the shape that produces the maximum K. In this solution, for mode-I loading, K_I is expressed as:

$$K_I = \sigma F(\lambda) \sqrt{\pi a} \quad (2)$$

where σ is the applied stress level, a is the maximum crack depth, and $F(\lambda)$ is similar to the shape factor which is given by [18]:

$$F(\lambda) = g(\lambda) [0.752 + 2.02\lambda + 0.37(1 - \sin \frac{\pi\lambda}{2})^3] \quad (3)$$

where $g(\lambda)$ is expressed as:

$$g(\lambda) = 0.92 \left(\frac{\lambda}{\pi} \right) \frac{\left(\frac{\tan \frac{\pi\lambda}{2}}{\frac{\pi\lambda}{2}} \right)^{1/2}}{\cos \frac{\pi\lambda}{2}} \quad (4)$$

Here λ is the normalized crack depth, a/D , where D is the specimen diameter. The lifetime, N_{FCG} can be calculated as the sum of the time spent in the small and the long-crack growth regime:

$$N_{FCG} = \int_{a_i}^{a_o} \frac{da}{f(\Delta K)} \Big|_{small-crack} + \int_{a_o}^{a_f} \frac{da}{f(\Delta K)} \Big|_{long-crack} \quad (5)$$

where a_i is the crack initiation size, a_o is the size corresponding to the intersection point of the small and the long crack growth curves and a_f is the crack depth at the onset of fast fracture. The final crack depth, a_f is governed by the cyclic fracture toughness and was measured in the fatigue fracture surfaces from the fractured specimens [see [13] for further details].

Input Random Variables

The random variables in the life prediction analysis and their distribution types are listed in Table I. These are: (i) the crack initiation size, a , (ii) the small-crack growth coefficient, C , and (iii) the small-crack growth exponent, m . When compared to the small crack growth regime, the long-crack growth behavior exhibits significantly less variability. Furthermore, a majority of the

crack growth lifetime of naturally initiated microscale cracks is spent in small-crack growth. The contribution of the long-crack growth regime to the lifetime variability was therefore neglected and the long-crack behavior was treated as deterministic. The size a_f , which is dependent on the cyclic fracture toughness, does not show much variability based on the fractured specimens and was also assumed to be deterministic.

The crack nucleation area distribution is shown in Fig. 1(b). The crack initiation size, a , was taken as the diameter of an equivalent circle with the same area. The distribution in a was modeled by the lognormal density function.

Based on an analysis of a relatively large number of crack growth experiments, Annis [7] showed that the coefficient, C , and the exponent, m , could reasonably be modeled by the normal distribution function. Luo and Bowen [8, 9] applied the power-law approximation to the small-crack growth behavior and experimentally determined that C and m were normally distributed. Given these studies, the small-crack growth variables (C and m) were modeled by a correlated joint normal density function. In order to obtain the parameters of these distributions, power-law fits to the small crack growth curves representing the fastest and the slowest growth rates were taken to correspond to the $\pm 3\sigma$ limits on C and m . This is illustrated in Fig. 2. The resulting distributions are shown in Fig. 3 (a) and (b) respectively.

Correlation

As discussed earlier, the input model variables can be correlated in many real-life problems, and it is critical to incorporate this correlation in the probabilistic model for an accurate prediction of the response parameters and their sensitivities. The sensitivity analysis method discussed here can be used to assess the influence of the correlation coefficient on the response. In the study cited above [7], it was demonstrated that the long-crack growth variables (C and m)

are highly correlated, which, when ignored, resulted in a gross overestimation of the variance in the response. In the present study, the available small-crack growth experiments indicated a strong negative correlation between C and m as shown in Fig. 4. The correlation coefficient, ρ , was determined to be about -0.99. In the subsequent lifetime analysis, C and m were therefore sampled from their joint probability density.

Simulated Lifetime Variability

The lifetimes were simulated by the Monte Carlo analysis method according to Eqn. (5). A comparison of the Monte Carlo results is made to the experiment in Fig. 5 at the stress level of 860 MPa. 10,000 Monte Carlo samples were used after confirming that this number of samples was sufficient to ensure convergence of the Monte Carlo results. A more detailed discussion of the results has been provided in [13]. The experimental points in Fig. 5 show that the lifetime varied by up to about 250x in magnitude. The step-like behavior of the data with respect to the Cumulative Distribution Function (CDF) indicates the superposition of at least two mechanisms, which were earlier referred to as the life-limiting (labeled Type I) and the mean-dominating behaviors (Type II) [13]. The simulation seems to describe the left part of the step, i.e., the life-limiting response, reasonably well, indicating that the underlying Type I mechanism is controlled solely by the variability in crack growth lifetimes from the equiaxed- α scale crack initiation size. Since failure can occur by either of the two superimposing behaviors, the life-prediction analysis is based on the worst-case mechanism [13, 14]. The following sensitivity analysis is therefore aimed at determining the critical random variable parameters (Table 1) that control the mean and the variance of the life-limiting mechanism.

2.2 Score function method

The sensitivity of the mean response and standard deviation of the response with respect to the parameters of the input distributions can be obtained using the SF method, described briefly here. Further information is contained in refs (Kleijnen and Rubinstein [1], Rubinstein [2], Rubinstein and Shapiro [3]). The attractiveness of the SF method is that the same Monte Carlo samples used to estimate the response mean and standard deviation can also be used to compute the sensitivities. Thus, the sensitivities are obtained at negligible cost.

The sensitivity of the mean response, μ_z , with respect to any parameter of the input distributions ($S_{\theta}^{\mu_z}$) can be determined from

$$S_{\theta}^{\mu_z} = \frac{\partial \mu_z}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \cdot \int_{-\infty}^{\infty} z \cdot f_{\mathbf{x}}(\mathbf{x}) \cdot d\mathbf{x} = \int_{-\infty}^{\infty} z \cdot \left(\frac{f_{\mathbf{x}}(\mathbf{x})}{\partial \theta_i} \frac{1}{f_{\mathbf{x}}(\mathbf{x})} \right) f_{\mathbf{x}}(\mathbf{x}) \cdot d\mathbf{x} = \int_{-\infty}^{\infty} z \cdot \kappa_{\theta_i} \cdot f_{\mathbf{x}}(\mathbf{x}) \cdot d\mathbf{x} = E[z \cdot \kappa_{\theta_i}] \quad (6)$$

where θ_i denotes a parameter of the input variable i , z represents the response (cycles-to-failure), $f_{\mathbf{x}}(\mathbf{x})$ represents the joint probability density function, $E[\cdot]$ denotes the expected value operation with respect to the joint probability density function, and $\kappa_{\theta} = \frac{\partial f_{\mathbf{x}}(\mathbf{x})}{\partial \theta} \frac{1}{f_{\mathbf{x}}(\mathbf{x})}$.

Similarly,

$$\frac{\partial V_z}{\partial \theta} = E[z^2 \kappa_{\theta}] - 2\mu_z E[z \kappa_{\theta}] \quad (7)$$

and

$$S_{\theta}^{\sigma_z} = \frac{\partial \sigma_z}{\partial \theta} = (E[z^2 \kappa_{\theta}] - 2\mu_z E[z \kappa_{\theta}]) / (2\sigma_z) \quad (8)$$

The expected value operations and hence the sensitivities can be approximated using sampling methods as

$$E[z \boldsymbol{\kappa}_\theta] \approx \frac{1}{N} \sum_{i=1}^N z(\mathbf{x}_i) \kappa_{\theta_i}(\mathbf{x}_i) \quad (9)$$

and

$$E[z^2 \boldsymbol{\kappa}_\theta] \approx \frac{1}{N} \sum_{i=1}^N z^2(\mathbf{x}_i) \kappa_{\theta_i}(\mathbf{x}_i) \quad (10)$$

where the vector \mathbf{x}_i denotes the i th realization of the random variables.

$\boldsymbol{\kappa}_\theta$ is called the kernel function (or score function) and is specific to each probability distribution type. The kernel functions for the distribution types used in the numerical example are presented in Appendix A.

2.3 Identifying important variables

The probabilistic sensitivities can be used to estimate the change in the life moments μ_z , σ_z without reanalysis, given small changes in the parameters. The approximate percent change can be determined from a first order Taylor Series:

$$\frac{\Delta \mu_z}{\mu_z} \approx \sum_{i=1}^n \frac{\partial \mu_z}{\partial \theta_i} \frac{\Delta \theta}{\mu_z} = \sum_{i=1}^n \bar{S}_{\theta_i}^{\mu_z} \quad (11)$$

$$\frac{\Delta \sigma_z}{\sigma_z} \approx \sum_{i=1}^n \frac{\partial \sigma_z}{\partial \theta_i} \frac{\Delta \theta}{\sigma_z} = \sum_{i=1}^n \bar{S}_{\theta_i}^{\sigma_z} \quad (12)$$

where n is the number of random variables.

Hence, the nondimensionalized sensitivities can be used to assess the relative change in the life moments as a function of the same percentage change in each parameter. In effect, \bar{S}_θ is a first order transfer function between changes in the parameters and moments.

This information can be used to identify important parameters with the caveat that this information is based on the “same percentage change” in each parameter, however, each parameter may not be equally weighted with respect to cost, schedule, etc.

3. Numerical Results

3.1 Finite Difference Estimates

The score function method based results were compared against the estimates by the finite difference method. The finite difference estimates required a separate Monte Carlo analysis run for each input parameter. Also, a significantly larger number of Monte Carlo samples are required, not only to minimize the contribution of sampling variance, but, especially under strong correlation between variables, to allow for sufficient sampling from the joint PDF. Here, in each analysis run, an input parameter was perturbed by 2% and 10^5 Monte Carlo samples were employed. The random number seed was held constant for the finite difference runs. The results are compared to the score function method in Table II. As shown, the values by the score function method were in reasonably good agreement with the finite difference estimates providing validation for the calculated sensitivities. The agreement may further improve with an increase in the number of Monte Carlo samples in the finite difference runs.

3.2 Score Function Results

The sensitivities of the mean (μ_z) and standard deviation (σ_z) of lifetime were determined using Monte Carlo sampling and the SF method. An example of the convergence study on the sensitivities is presented in Fig. 6. The figure shows the nondimensionalized sensitivities of the response (or the lifetime) mean with respect to the parameters μ_c and μ_m ($\bar{S}_{\mu_c}^{\mu_z}$ and $\bar{S}_{\mu_m}^{\mu_z}$ respectively). The COV of $\bar{S}_{\mu_c}^{\mu_z}$ and $\bar{S}_{\mu_m}^{\mu_z}$ at 10,000 Monte Carlo analysis steps, determined from

5 independent runs, were about 0.08 and 0.12 respectively. The convergence behaviors of sensitivities with respect to other parameters were also similar and indicated convergence of the variance at approximately 10,000 Monte Carlo samples. Therefore, all sensitivities were based on 10,000 samples.

The sensitivity results by the score function method (\bar{S}_θ) are presented in Table II and graphically illustrated in Fig. 7 for the parameters of all variables. As stated before, sensitivities with respect to each input parameter were obtained in a single run. The nondimensionalized numbers provide an indication of the relative strength of the input parameters in terms of their effect on the response parameters, compared for a given percentage change. The relative significance of the input parameters on the response mean and standard deviation are shown in Fig. 8 (a) and (b) respectively where the nondimensionalized sensitivities have been normalized with respect to the maximum value.

Figures 7 and 8 show that the crack growth parameter μ_C has the maximum effect, both on μ_z and σ_z . The negative value indicates that an increase in μ_C (which represents an increase in the small crack growth rate) will decrease the mean response. As an example, a 1% increase in μ_C will evoke about 24 % decrease in the mean and the standard deviation of lifetime, with all other parameters held constant. The second tier parameters of significance in the mean response are μ_m and the median crack initiation size, \tilde{a} . An increase in the slope (m) of the small crack growth curve under constant y-intercept (C), once again, represents an increase in the growth rate resulting in the negative sensitivity. However, μ_m does not play as strong a role as μ_C . Not surprisingly, σ_C and σ_m are relatively ineffective in influencing the response mean (Fig. 8(a)). The analysis also reveals that in the present case, the crack initiation size parameters (\tilde{a} and a_{COV}) have a significantly weaker influence on the response mean than the mean of the small

crack growth intercept, μ_c (Fig. 8(a)). This would imply that an accurate characterization of the small crack growth behavior may be most critical for accurate mean life-prediction.

With respect to the standard deviation of the response, Table II and Figs. 7 and 8 indicate that the parameter μ_c , again, has the strongest influence followed by ρ , μ_m , and σ_c , which have only a moderate effect. The crack initiation size parameters, \tilde{a} and a_{COV} appear to have a relatively insignificant role in the response standard deviation, as indicated by Fig. 8(b). These results again suggest the crucial role of the small crack growth behavior in fatigue variability, which may overwhelm any effect arising from the crack initiation size variability. It is to be noted, once again, that in Figs. 7 and 8 the sensitivities are being compared for the same percentage change in the parameters of C, m, and a .

Notwithstanding that the sensitivity of σ_z to the parameters σ_c and σ_m is higher by more than an order of magnitude than the sensitivity of μ_z to the same parameters, it is somewhat surprising that the response standard deviation (σ_z) is more sensitive to the mean parameters, μ_c and μ_m , than to σ_c and σ_m . This highlights the highly non-linear nature of the lifing moments with respect to the input PDF parameters, especially with the consideration of correlation between variables.

Another interesting result in Fig. 7 is the negative sensitivity of σ_z to the parameter σ_m . This may seem counterintuitive since one expects the response standard deviation to increase with an increase in the variability of an input variable. However, as shown in the following section, this is another manifestation of the strong correlation between C and m.

The sensitivity results of Table II and Fig. 8(b) also underscore the importance of the correlation coefficient, ρ , in determining the response standard deviation. This shows that an inaccurate evaluation of ρ may have significant effect on accuracy of the lifing predictions. For

example, a decrease in ρ may significantly increase the lifetime variability, and its effect on σ_z can be greater than the influence of several other parameters including the crack initiation size.

Here it is useful to be reminded that this paper is concerned with the life-limiting mechanism, which is modeled as being controlled exclusively by crack growth. As shown by Fig. 5, this is one of the contributing mechanisms to the total variability. Given the lifing analysis and the regimes of the input variables, the results indicate stronger roles of the small crack growth parameters and the correlation coefficient in the lifetime response than that of the crack initiation size. However, this is not to suggest that crack initiation is not an important factor in the total fatigue variability behavior. It should be emphasized that although \tilde{a} and a_{COV} appear to play a weaker role in the life-limiting mechanism (Type I), the crack initiation regime has a very significant influence on the mean-dominating behavior (Type II). Therefore, the crack initiation regime affects the so called separation of the two mechanisms as the stress level or temperature is decreased [13-15], therefore, potentially playing a major role in the total lifetime variability.

3.3 Effect of the Correlation Coefficient, ρ

A parametric study examining the effect of ρ , varied from -0.99 to 0, on the sensitivity of the response mean and standard deviation to the other parameters is presented in Fig. 9 (a) and (b) respectively. The parametric study of ρ is hypothetical since the values are physically not realizable for this problem. However, the goal here is to evaluate the effects of error in determining ρ on the sensitivities.

As indicated previously, under strong negative correlation between C and m ρ proved to be the second most important parameter in terms the response standard deviation σ_z . Figure 9 (b) shows that the sensitivity with respect to ρ becomes less significant with a decrease in the magnitude of ρ . However, the sensitivity is non-zero even at $\rho = 0$ which is also indicated by the

kernel function, κ_ρ (Appendix A), for $\rho = 0$. Depending on the problem, the effect of perturbations about $\rho = 0$ on the response may be significant. In the present case, the non-dimensionalized sensitivity $\bar{S}_\rho^{\sigma_z}$ was calculated to be about 0.22 at $\rho = 0$ (compared to about 3.64 at $\rho = -0.99$) which is not very significant, relative to other sensitivities.

The sensitivities with respect to the independent variable, i.e., the crack initiation size parameters, remained almost unaffected by ρ . It should be pointed out that, although this appears to be an expected result, this need not be necessarily true. This is because the expansion point in the derivatives is being changed by changing ρ , which may produce some variance in the sensitivity even with respect to an independent variable. Another important effect of the correlation between C and m, evident from Fig. 9, is an increase in sensitivities of both $\bar{S}_\theta^{\mu_z}$ and $\bar{S}_\theta^{\sigma_z}$ with an increase in the degree of correlation (Fig. 9). Figure 9 also indicates an increase in the variance of sensitivity values of the correlated variables with an increase in the correlation.

For independent variables, an increase in the standard deviation of an input always produces an increase in the response standard deviation. However, this is not necessarily true for negatively correlated variables. For example, for a linear response function of the form

$$Z = a_0 + \sum_{i=1}^n a_i x_i, \text{ where } a_i \text{ are the coefficients and } x_i \text{ are the random variables, the exact}$$

sensitivity can be determined as [12], $\frac{\partial \sigma_z}{\partial \sigma_i} = \frac{a_i}{\sigma_z} \sum_{j=1}^n a_j \rho_{ij} \sigma_j$ which indicates that $\partial \sigma_z / \partial \sigma_i$ can be

positive or negative depending upon the signs of the a_i and ρ_{ij} coefficients. Although the sensitivity $\partial \sigma_z / \partial \sigma_m$ was very weak, negative sensitivity was produced, given a strong negative correlation between C and m (see Table 2). As shown in Fig. 9(b), with a decrease in the magnitude of ρ , $\partial \sigma_z / \partial \sigma_m$ increases to a positive value as expected for independent variables.

3.4 Focused Data Collection

The analysis reveals that the small crack growth parameters, especially μ_C , play the most significant role in the life-limiting mechanism. In a physical sense, the input C represents the y-intercept on the log-log scale of the line describing the small crack growth behavior. The strong sensitivity of both the lifetime mean and standard deviation to μ_C suggests the need for an accurate determination of this parameter for robust probabilistic life-prediction. In this regard, it becomes very important to focus efforts at accurate small crack growth experiments.

4. Conclusions

This study was focused at applying the score function method of probabilistic sensitivity analysis to determine the critical parameters in life prediction of the $\alpha+\beta$ titanium alloy, Ti-6-2-4-6. The main conclusions that can be drawn are:

- (i) Small crack growth parameters played a significantly greater role than the crack initiation size parameters in the lifetime mean and standard deviation for the life limiting mechanism.
- (ii) In terms of the mean lifetime response, μ_C had the strongest influence, followed by μ_m and \tilde{a} . This may be an expected trend, given that small crack growth behavior dominates the calculation of total lifetime.
- (iii) The parameter, μ_c also played the largest role in the response standard deviation followed by the correlation coefficient, ρ , σ_C , and σ_m . Relative to these parameters, a_{COV} did not have a significant influence on σ_z .

- (iv) Besides strongly influencing the response standard deviation, an increase in the magnitude of ρ increased the sensitivity of response parameters, particularly σ_z , to the parameters of the correlated variables.
- (v) The analysis suggests the crucial role of accurately determining the small crack growth behavior in the probabilistic life-prediction.

Appendix A – kernel functions

Correlated normals

The derivation of the kernel functions for an arbitrary number of correlated normal random variables is described in [12]. For two random variables (X_i and X_j), the explicit forms are

$$\kappa_{\mu_i} = \frac{1}{\sigma_i(1-\rho^2)}(U_i - \rho U_j) \quad (\text{A-1})$$

$$\kappa_{\sigma_i} = \frac{1}{\sigma_i(1-\rho^2)}(U_i(U_i - \rho U_j) - (1-\rho^2)) \quad (\text{A-2})$$

$$\kappa_{\rho} = \frac{-\rho U_1^2 + (1+\rho^2)U_1 \cdot U_2 - \rho U_2^2 + \rho(1-\rho^2)}{(1-\rho^2)^2} \quad (\text{A-3})$$

where $U_i = (X_i - \mu_i) / \sigma_i$

The kernel functions for uncorrelated variables can be recovered by inserting $\rho = 0$.

Lognormal

The lognormal distribution is described in terms of the median (\tilde{X}) and coefficient of variation (COV) parameters. The kernel function is

$$\kappa_{\tilde{X}} = \frac{\ln(x) - \ln(\tilde{x})}{\tilde{x} \ln(1 + COV^2)} \quad (\text{A-4})$$

$$\kappa_{\text{cov}} = \frac{COV \cdot (-\ln(1 + COV^2) + (\ln(x) - \ln(\tilde{x}))^2)}{(1 + COV^2) \cdot \ln(1 + COV^2)^2} \quad (\text{A-5})$$

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Table I: Input random variables

Random Variable (θ)	Description	Distribution type	Parameters
C	Small crack growth coefficient	Correlated normal*	$\mu_C = -25.24$ $\sigma_C = 0.60$
m	Small crack growth exponent	Correlated normal*	$\mu_m = 3.36$ $\sigma_m = 0.30$
a	Crack initiation size	Lognormal	$\tilde{a} = 8.47$ $a_{COV} = 0.26$

* $\rho = -0.99$

Table II: Comparison of the score function method against the finite difference estimates

θ_i	$\frac{\partial \mu_z}{\partial \theta_i} \frac{\theta_i}{\mu_z}$		$\frac{\partial \sigma_z}{\partial \theta_i} \frac{\theta_i}{\sigma_z}$	
	Score function method	Finite difference estimate (2% perturbation, 100,000 Monte Carlo samples)	Score function method	Finite difference estimate (2% perturbation, 100,000 Monte Carlo samples)
μ_C	-23.51	-18.85	-24.72	-19.09
σ_C	0.14	0.10	1.75	1.02
μ_m	-3.53	-3.31	-2.83	-1.94
σ_m	-0.07	-0.1	-0.88	-0.29
\tilde{a}	-1.29	-0.79	0.36	0.64
a_{COV}	0.23	0.006	0.10	0.329
ρ	-0.24	0.064	3.64	1.43

Figures:

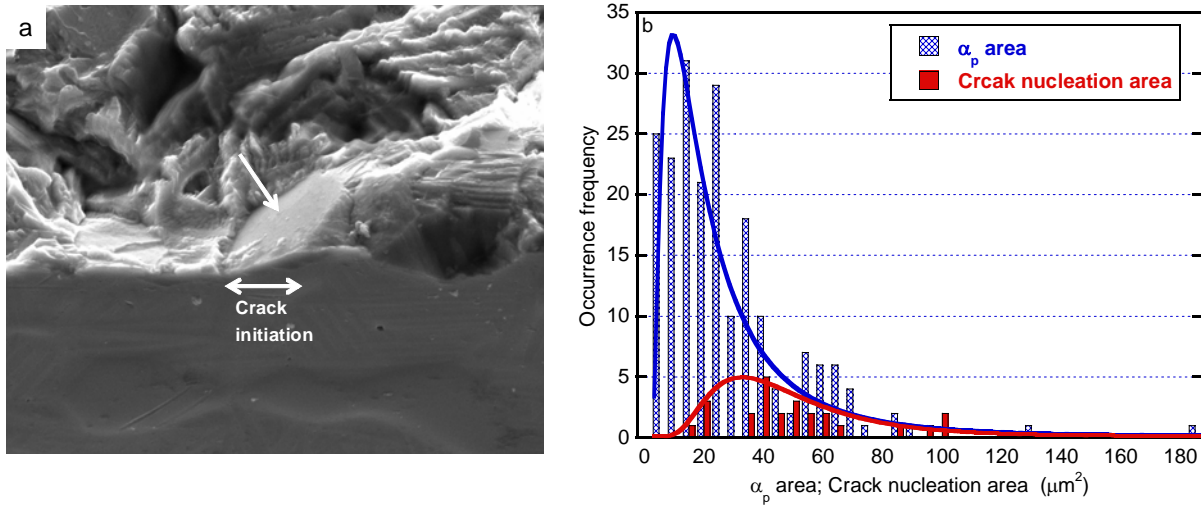


Fig. 1: Fatigue crack initiation characteristics in Ti-6-2-4-6; (a) Crack initiation facet on the fracture surface of a sample loaded at 860 MPa with lifetime $N_f = 39,864$ and (b) crack initiation size distribution compared to the nominal equiaxed- α size distribution.

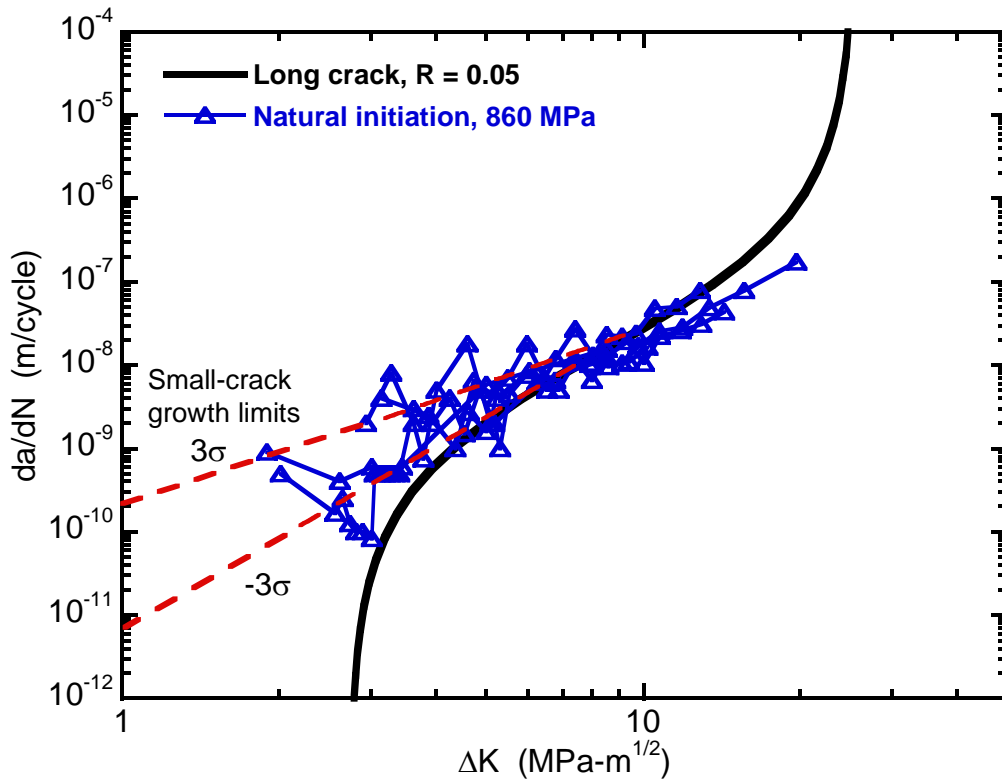


Fig. 2: Small crack growth variability in Ti-6-2-4-6 measured at $\sigma_{max} = 860$ MPa, $\nu = 20$ Hz, and $R = 0.05$.

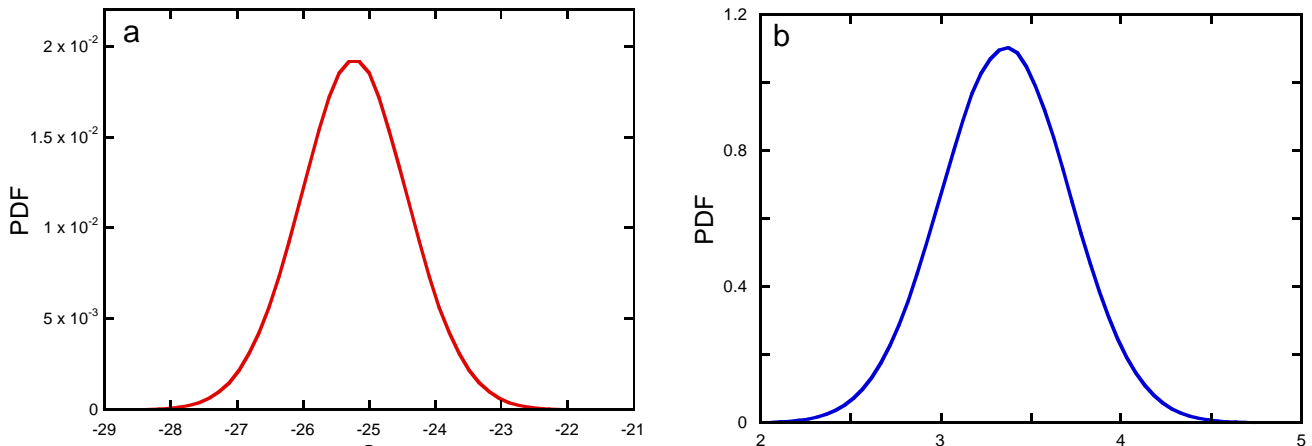


Fig. 3: Distribution in the small crack growth variables derived by assuming $\pm 3\sigma$ limits at the small crack curves representing the fastest and the slowest growth rate; (a) the coefficient C and (b) the exponent m.

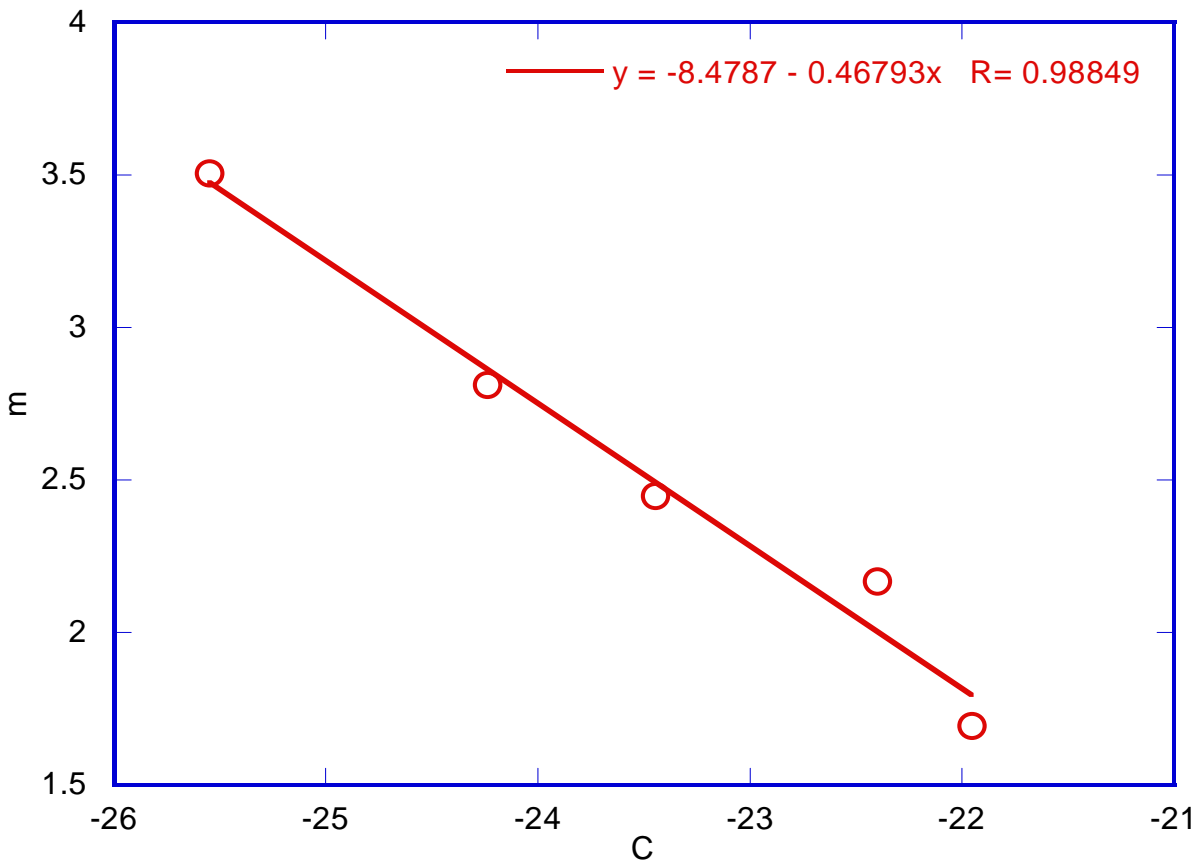


Fig. 4: Illustration of the strong negative correlation between C and m.

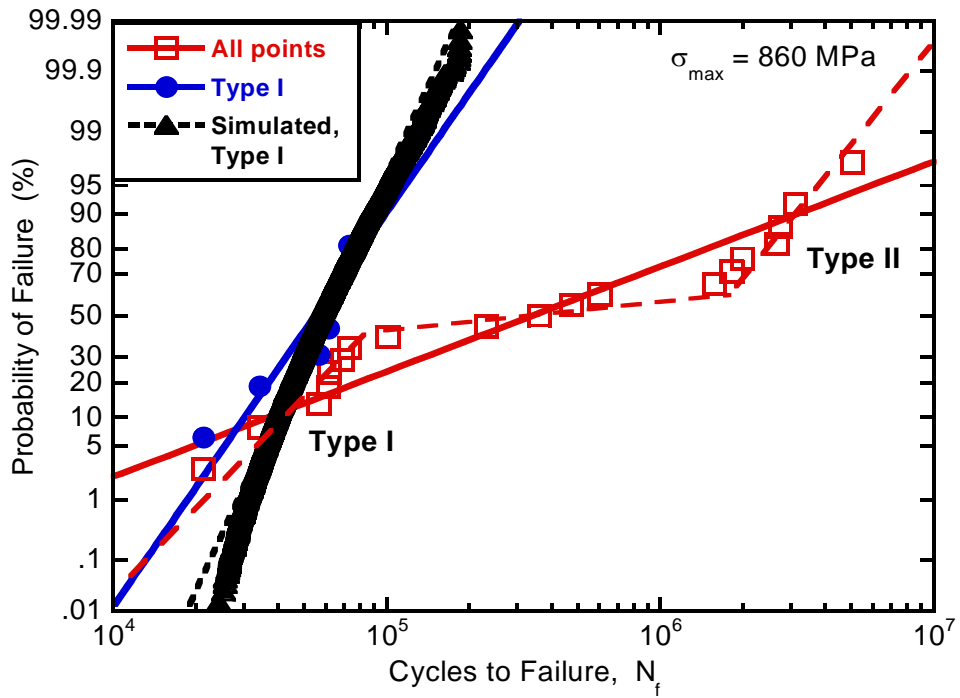


Fig. 5: Comparison of the crack growth based simulation of life-limiting mechanism (Type I) in Ti-6-2-4-6 and the experiment.

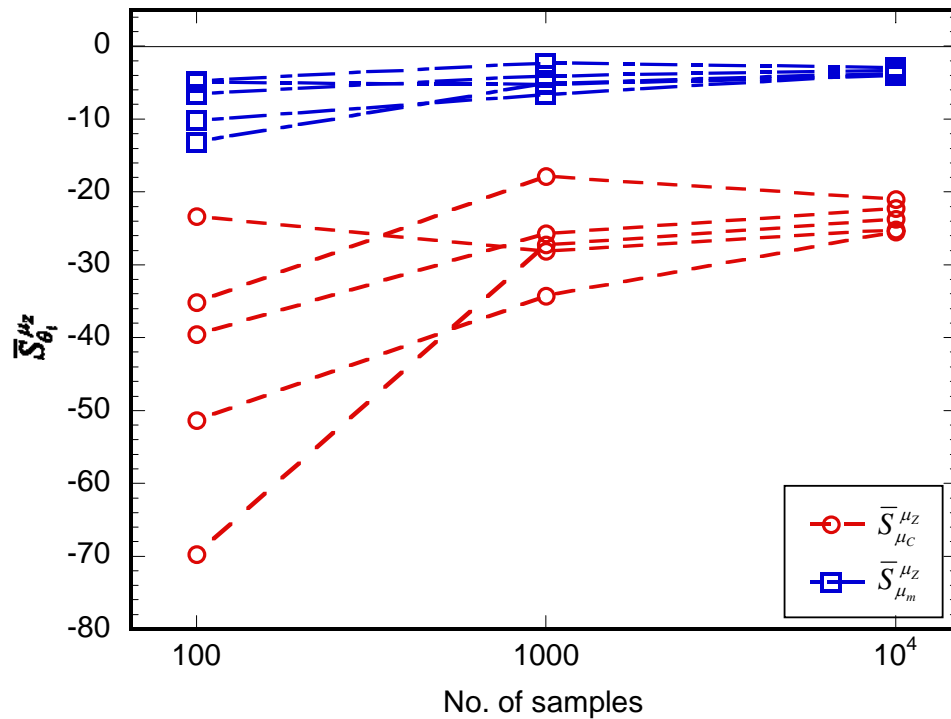


Fig. 6: An example of the convergence study on the sensitivity measures showing a significant reduction in variance at 10,000 Monte Carlo samples.

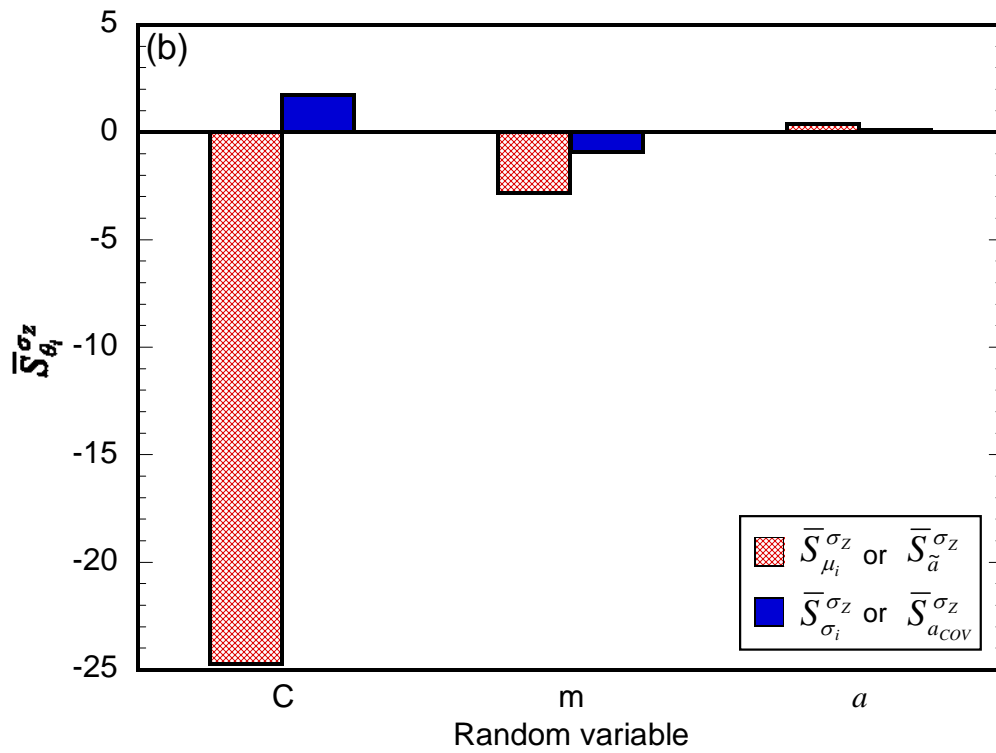
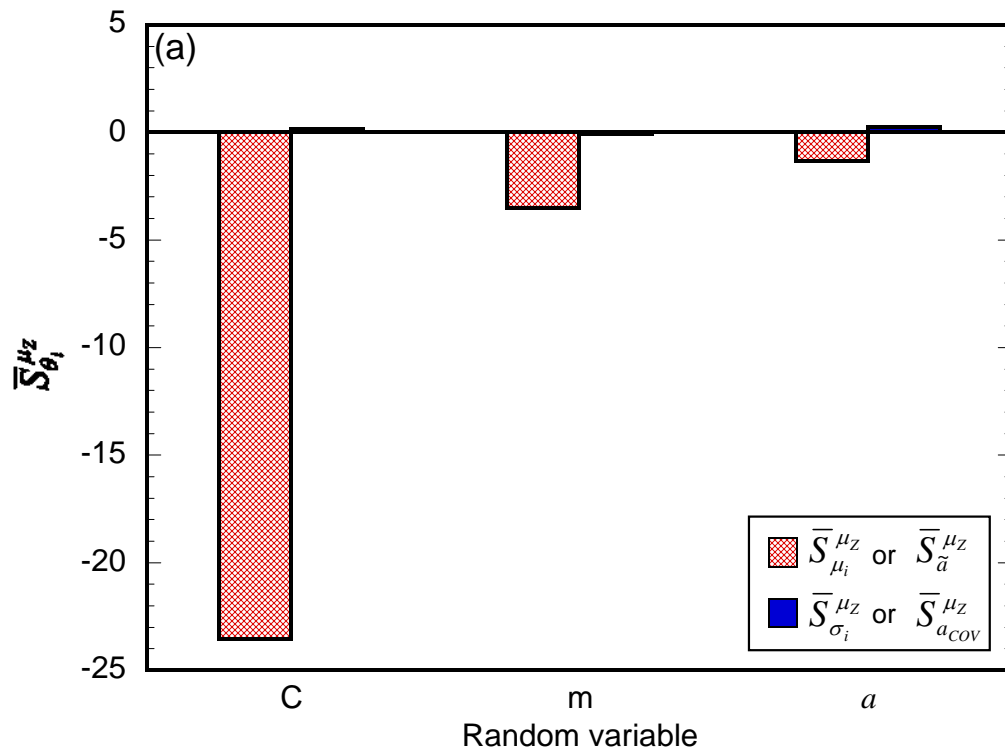


Fig. 7: Results by the score function method; (a) sensitivity of the response mean to the input parameters and (b) sensitivity of the response standard deviation.

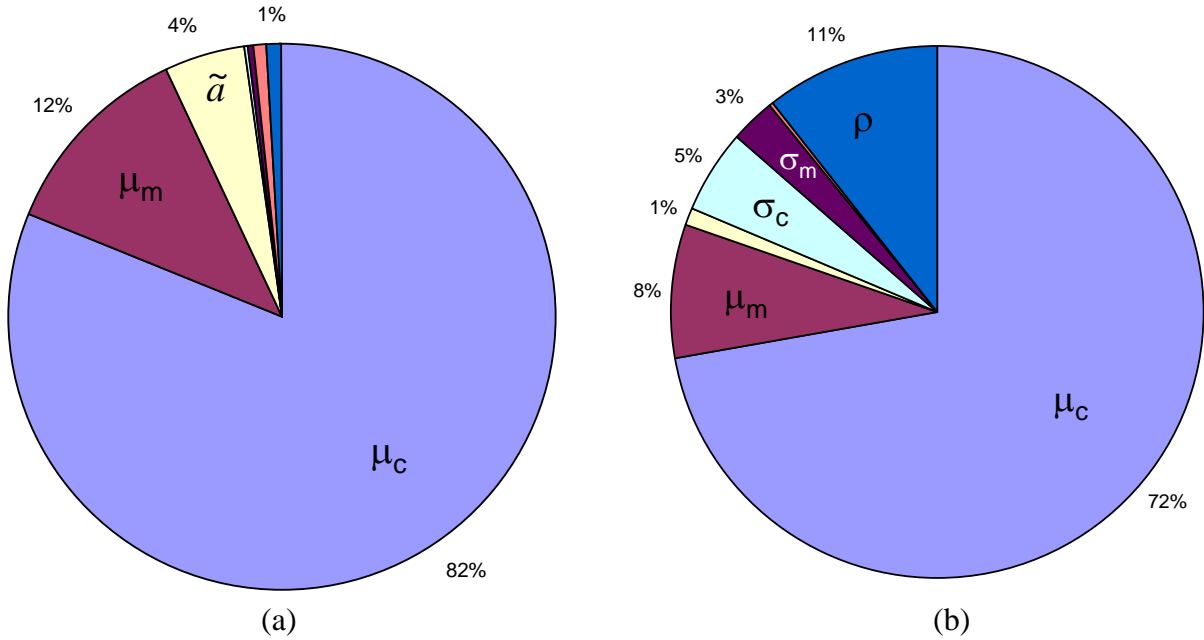
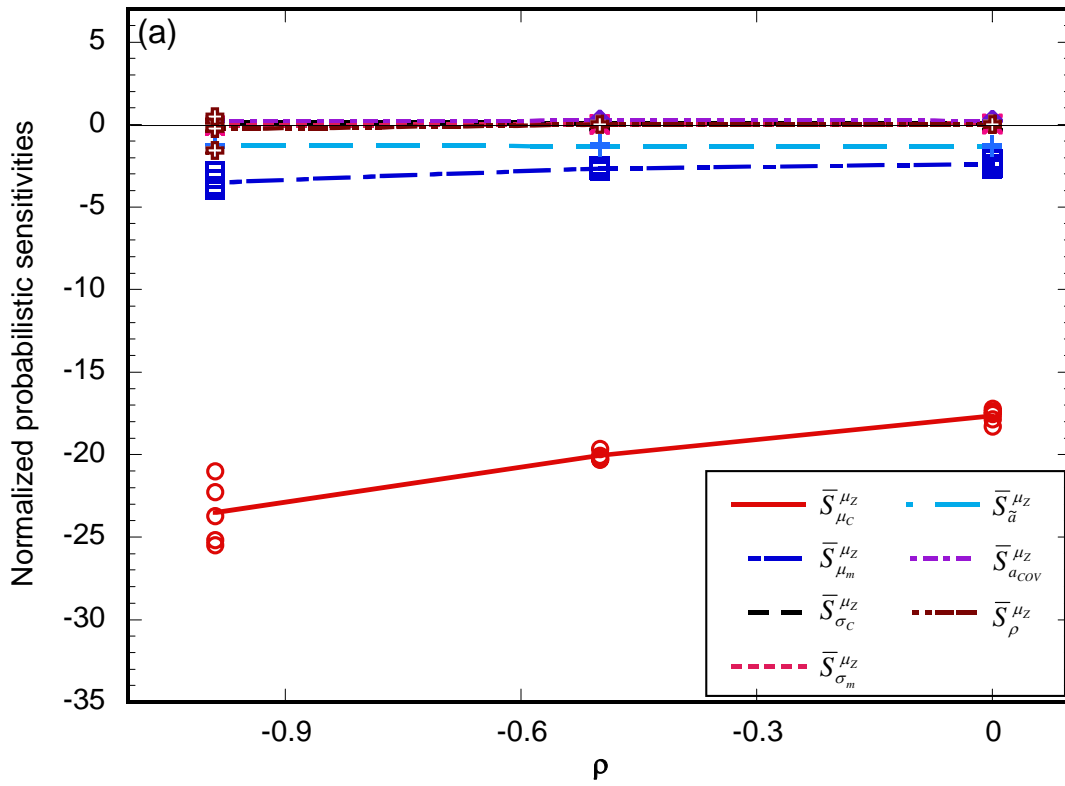


Fig. 8: Illustration of the relative importance of the parameters in (a) the response mean and (b) the response standard deviation.



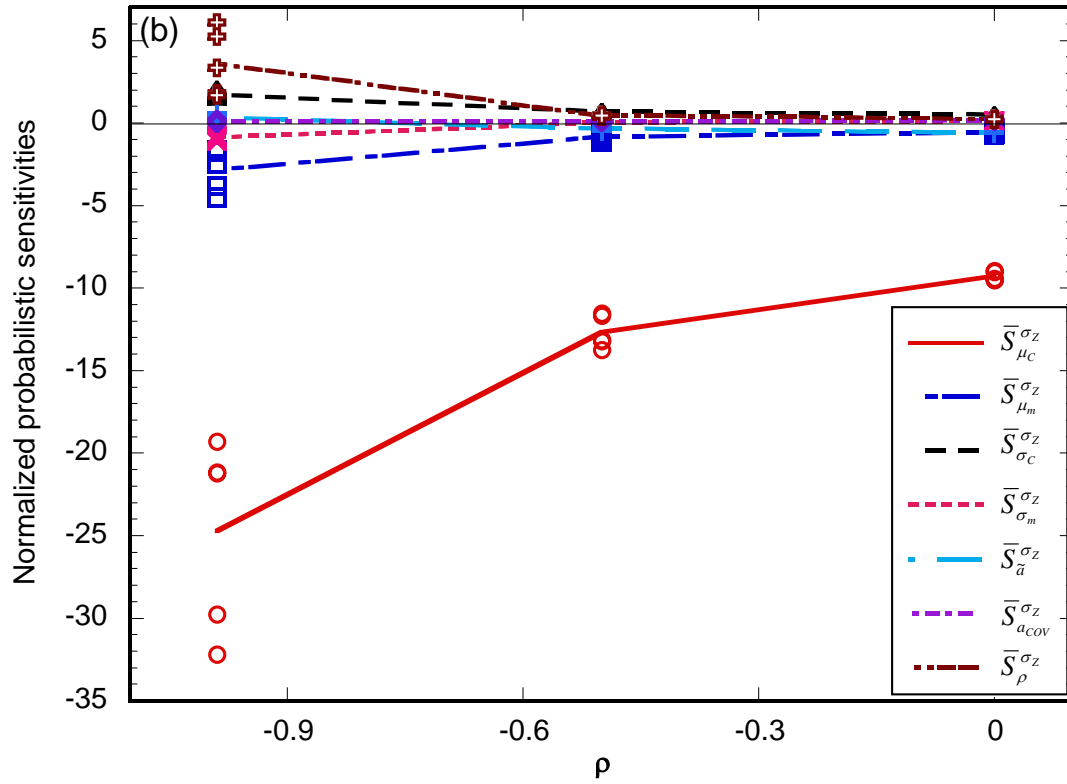


Fig. 9: Effect of the correlation coefficient, ρ on the (a) sensitivity of response mean and (b) sensitivity of response standard deviation to the input parameters.