

DECONSTRUCTING SPATIOTEMPORAL CHAOS USING LOCAL SYMBOLIC DYNAMICS

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ABSTRACT

We find that the global symbolic dynamics of a diffusively coupled map lattice (CML) is well-approximated by a very small local model for weak to moderate coupling strengths. A local symbolic model is a truncation of the full symbolic model to one that considers only a single element and a few neighbors. Using interval analysis we give rigorous results for a range of coupling strengths and different local model widths. Examples are presented of extracting a local symbolic model from data and of controlling spatiotemporal chaos.

1. INTRODUCTION

In this Letter, we approximate the global symbolic dynamics of a coupled map lattice (CML) with a set of local models each using only a small number of symbols. Coupled map lattices [Kaneko, 1990] are popular models of spatiotemporal chaos in reaction diffusion systems and their description via symbolic dynamics may provide an efficient and rigorous basis for understanding them. Indeed, the topology of unimodal maps has been completely elucidated in terms of 2-symbol alphabets [Kitchens, 1998], and recently it has been conjectured that these results extend simply to the CML case [Pethel, 2006]. Nevertheless, one still has to contend with the issue of dimensionality in lattices. While a single logistic map is fully described by 2 symbols, an N -element lattice of them requires an alphabet of 2^N symbols. Manipulation of the global symbolic dynamics of a large lattice would therefore appear impractical. In the case of diffusive coupling, however, we find that the symbolic dynamics at a particular site is largely determined by a local neighborhood, at least for some range of coupling strengths. Previously, it has been shown that the symbol statistics at a single site can indicate degrees of global synchronization [Jalan, 2006]. Here we propose using symbolic information from a small neighborhood to reconstruct the dynamics of the entire lattice. In what follows we use interval analysis to quantify this idea and to show that the global symbolic dynamics can be well-approximated by a compact local model for weak to moderate coupling strengths.

2. SYMBOLIC DYNAMICS OF CMLS

We consider a map lattice with N sites labeled $i=1\dots N$. Each site is described by a state x_t^i in the interval I^i and a unimodal local dynamic $f_i : I^i \rightarrow I^i$. Denote by F the product function of f_i onto each site, and by A an $N \times N$ coupling matrix, then the map lattice can be written as $\bar{x}_{t+1} = H(\bar{x}_t)$, where $H = A \circ F$. Models of this type have been extensively studied with regard to turbulence and pattern formation [Kaneko, 1985]. Here we wish to introduce an alternate formulation in terms of local symbolic dynamics.

As reported previously [Pethel, 2006], it is conjectured that for A nonsingular a homeomorphism exists between the spatiotemporal sequence $\{x_t^i : i=1\dots N, t \geq 0\}$ and the equivalently sized set of binary symbols s_t^i defined by

$$s_t^i = \begin{cases} 0 & \text{if } x_t^i \in I_0^i \\ 1 & \text{if } x_t^i \in I_1^i \end{cases} \quad (1)$$

where I_0^i and I_1^i are the two sub-intervals over which the unimodal map f_i is monotonic. Figure 1a,b illustrates this mapping for a CML of logistic maps. The particular state x_t^i is homeomorphic to s_t^i plus the set $\{s_l^j : j=1\dots N, l > t\}$. This relationship can be understood in the following way. Consider that the symbol vector $\bar{s}_{t+n}$ indicates which sub-interval the components of $\bar{x}_{t+n}$ lie in. Provided one knows which preimage to use, an estimate for $\bar{x}_{t+n-1}$ can be obtained by applying H^{-1} onto this vector of sub-intervals. The symbol vector $\bar{s}_{t+n-1}$ identifies the correct preimage because the two preimages of f_i lie on different monotonic segments. We repeat this process until an estimate of $\bar{x}_t$ is reached. At the last step only the symbol s_t^i is required to estimate x_t^i . Because the inverse of a chaotic map is, on average, contracting, the estimate converges to x_t^i as $n \rightarrow \infty$.

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Let P_{s_t} be the vector of sub-intervals containing the CML state at time t . The first n terms of the above algorithm can be expressed as

$$\begin{aligned} \{x_0\}_{s_0 \dots s_{n-1}} &= I^N \cap H_{s_0}^{-1} \circ \dots \circ H_{s_{n-2}}^{-1} P_{s_{n-2}}, \\ &= I^N \cap (F_{s_0}^{-1} A^{-1}) \circ \dots \circ (F_{s_{n-2}}^{-1} A^{-1}) P_{s_{n-2}} \end{aligned} \quad (2)$$

where I^N is the N dimensional domain of H and the subscript on F^{-1} specifies a particular inverse branch. The l.h.s. is the set of all states that produce the sequence $s_0 \dots s_{n-1}$. An empty set implies a forbidden sequence.

Exact evaluation of Eq.2 is difficult; however, it is amenable to *interval analysis* [Galias, 2001]. This method utilizes interval arithmetic for the purpose of achieving rigorous bounds on the solution space. In this case, interval computations result in guaranteed, tight bounds on the components of the vector x_0 given some symbol sequence. Tightness is a consequence of the monotonicity of F_s^{-1} and is significant because it implies that the interval approximation converges with symbol depth n at the same rate as Eq.2.

In practice one uses a finite n and deals with a space of truncated symbol sequences. This truncation is analogous to using a fixed number of bits to represent the real line in a digital computer. Symbols that occur further in the future are exponentially less significant than symbols that occur earlier in the temporal sequence. The total number of symbols representing x_t^i is, then, $Nn+1$. Thus to construct an explicit mapping between the CML site value and the symbolic states one has to account for as many as 2^{Nn+1} possibilities, which is impractical for large lattices.

3. THE LOCAL SYMBOLIC MODEL

Using interval arithmetic the local symbolic dynamics can be investigated rigorously. We define the local symbolic representation of a CML site x_t^i to be the symbols in a neighborhood of spatial width m and temporal depth n . Because the symbols s_t^j , $j \neq i$ are irrelevant to x_t^i we are effectively left with the paddle-shaped region illustrated in Figure 1b. Restricting Eq.2 to the local neighborhood requires using a full interval for each unspecified component of $P_{s_{n-1}}$ and taking the union of the two preimages at each of these sites. In this way we

can compute guaranteed bounds on x_t^i given only knowledge of symbols in a local neighborhood.

As mentioned earlier, we expect the local symbolic model to be suitable for diffusively coupled systems. From Eq.2 we see that the symbols are related to the CML state via repeated applications of the inverse mapping, $H^{-1} = F^{-1} \circ A^{-1}$. In the case that the coupling matrix A is tridiagonal---i.e. the CML employs nearest neighbor coupling---the elements of A^{-1} can be found analytically [Yamani, 1997] and shown to fall off exponentially in magnitude with increasing offset from the diagonal. A major point here is that the symbolic dynamics at site i is largely determined by a small neighborhood $m \ll N$, at least for some range of coupling strengths. Importantly, the local model restricted in this way requires only $mn - m + 1$ symbols and is therefore independent of the lattice size N .

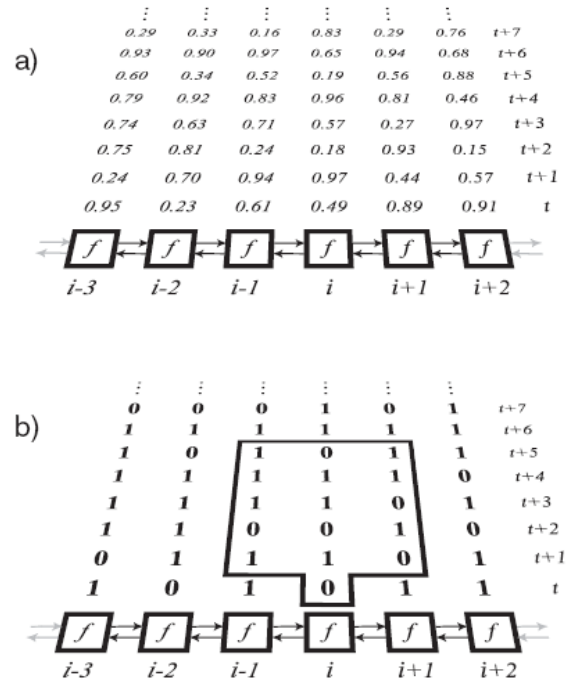


Fig. 1 A time-space segment of a 1-d CML of logistic maps. (a) Real valued CML states shown with 2 digits of precision, (b) the equivalent symbolic representation. A the local symbolic model using $n = 5$ is outlined.

The local models give us a tractable means of describing the global symbolic dynamics of the CML. If

we define the complexity of a set to be its cardinality, then the maximum complexity of the entire collection of local models is $N \cdot 2^{3n+1}$. This extreme assumes no symmetry is present in the CML and a different local model must be used at each site. The difficulty, then, of approximating the global symbolic dynamics using truncated local models scales at worst linearly with N , whereas the exact symbolic description scales exponentially.

4. EXAMPLE CASE

Having defined our local model approach, we wish to investigate its efficacy in terms of the commonly studied CML written as

$$x_{t+1}^i = (1 - \varepsilon) f(x_t^i) + \frac{\varepsilon}{2} [f(x_t^{i-1}) + f(x_t^{i+1})] \quad (3)$$

along with rules for the boundary sites [Kaneko, 1990]. The coupling parameter $\varepsilon \in [0, 0.5]$ sets the diffusion rate. Staying with convention we use the logistic map $f(x) = 4x(1 - x)$ as the local dynamic; therefore, the symbol “0” corresponds to a state value in the interval $I_0 = [0, 0.5)$ and the symbol “1” to the interval $I_1 = [0.5, 1]$. For large N the off-diagonal terms of A^{-1} decay in magnitude as $(\varepsilon / (2 - 2\varepsilon))^{|\delta|}$, where δ is the offset from the diagonal [Yamani, 1997]. The restriction of the local model to nearest neighbor symbols is tantamount to neglecting the $\delta \geq 2$ elements of A^{-1} , which are less than 1% of the diagonal terms.

We gauge the fidelity of the local symbolic model by its mean error over a test CML trajectory. The test data is symbolized and Eq.2 used to produce an interval estimate at each CML site x_t^i for various local model sizes and coupling strengths. We define the expected value $\tilde{x}_t^i$ to be the mid-point of this interval and compute a mean absolute error $E = \langle |x_t^i - \tilde{x}_t^i| \rangle$ over the data set. In Figure 2 we plot E versus symbol depth n , and coupling strength ε for an $N = 128$ lattice of logistic maps. The test data is 10^4 iterations of the CML starting from a random initial condition. These errors should be compared to the $[0, 1]$ range of the dynamics at each site.

As seen in Figure 2a, the fidelity of the local model improves smoothly with increasing n . The error is shown fitted to the function, $E(n) \approx \alpha e^{-\lambda n} + \beta$ for $\varepsilon = 0.1$ and for the cases of no neighboring symbolic information ($m = 1$), nearest neighbors only ($m = 3$), and for the global symbolic model ($m = 128$). The value β is the extrapolated limit of $E(n)$ for $n \rightarrow \infty$. The $m = 3$ local model performs well compared to the global model;

for $n \leq 6$ there is very little difference in fidelity. We note that at $n = 6$, the local model requires only 16 symbols ($3 \times 6 - 2$) to achieve an error of approximately 1% of the $[0, 1]$ dynamic range at each site. After $n = 10$, the $m = 3$ model hits an error floor ($\beta = 0.0044$), whereas the global model asymptotes to zero error.

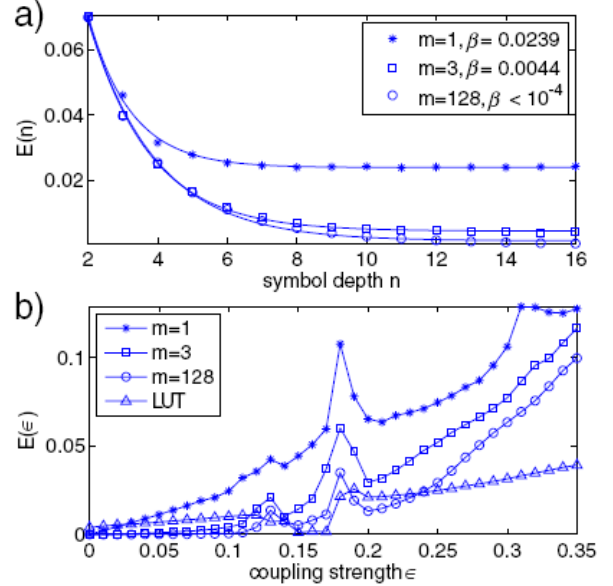


Fig.2 Mean Absolute error E for $m = 1, m = 3$, and the global symbolic model ($m = 128$) of a logistic CML of 128 elements. (a) E versus symbol depth n for $\varepsilon = 0.1$. β is the extrapolated value of $E(n)$ for $n \rightarrow \infty$; (b) E versus coupling strength ε for $n = 16$.

In Figure 2b, n is fixed at 16 and the error is shown versus coupling strength. In the region of weak to moderate coupling strength ($\varepsilon \leq 0.11$) the error is small and there is good agreement between the $m = 3$ local model and the global model. For strong coupling ($\varepsilon \geq 0.2$), the error grows steeply for the global model, and therefore for the local models as well. The error in this case is not so much due to the local symbol approximation, but to slow convergence of Eq.2 with respect to n . The region $\varepsilon \in [0.12, 0.18]$ contains non-chaotic states, for which our definition of $E(n)$ is not intended; we include this region only for completeness.

5. THEORY AND DISCUSSION

Given that there is merit to the local model approach, we now formalize the theory. Define a local

symbolic model as the set S^i of at most 2^{mn-m+1} elements representing all truncated symbol patterns that are allowed by the dynamics at site i . By "pattern" we mean a particular arrangement of symbols within the $m \cdot n$ paddle-shaped window (Fig. 1b). A *global* symbolic state $\Sigma_{k_1 \dots k_N} = S_{k_1}^1 S_{k_2}^2 \dots S_{k_N}^N$ is an overlapping concatenation of local symbolic states. An overlapping concatenation is possible between two local symbolic states only if overlapping symbols match. The approximated global symbolic dynamic can now be defined as the set $\{\Sigma_{k_1 \dots k_N}\}$ of all overlapping concatenations of local symbolic states. Obviously, any incompatibilities that occur outside the spatial and temporal boundaries of the local model are not accounted for in this approximation. The set $\{\Sigma_{k_1 \dots k_N}\}$ as defined above is therefore a superset of the actual global symbolic model.

Local models are a compact means of describing the global symbolic dynamics of the CML. If we define the complexity of a set to be the growth rate of its cardinality, then the maximum complexity of the entire collection of local models is order $N \cdot 2^{mn-m+1}$. This extreme assumes that a different local model must be used at each site. Importantly, the complexity of local model approach scales at worst linearly with N , whereas the exact symbolic description scales as 2^{Nn} .

There are two major features of the local symbolic model that we wish to emphasize and elaborate on here. The first is that local models can be small enough to compile from data and store as a table. As seen earlier, a $m=3$, $n=6$ local model closely approximates the dynamics of our example CML ($N=128, \varepsilon=0.1$). Such a model can contain at most $2^{16} = 65,535$ symbol patterns, whereas the corresponding global model could have up to $2^{128 \times 6}$.

A table of local symbol patterns can effectively replace Eq.2 if we associate with each entry the mean CML state observed for that symbol pattern. The resulting data structure is a look-up table (LUT) mapping symbol patterns to an expected value at that site. A LUT assembled from a finite time data set may not be complete; however we can default to LUTs of lesser symbol depths to fill in missing entries. The tradeoff is that rare symbol patterns will not be resolved as well as common ones.

We find that the $m=3$ LUT assembled from 10^5 iterations of our example CML gives an error which is almost identical to that from interval analysis for $n \leq 7$. In Figure 2b we have plotted the error (triangles) of

the $m=3$, $n=6$ LUT as a function of ε . For very strong coupling ($\varepsilon \geq 0.25$), the LUT is *superior* to the global ($m=128, n=16$) symbolic model, in spite of the lower symbol depth. We note that the expected value produced by the LUT is weighted by the natural invariant density, which is information not used in Eq.2. This extra information results in an improved estimate, at least for typical orbits.

The second point we wish to emphasize is that local symbolic models can be used to construct arbitrary global states and connecting orbits for the purpose of controlling spatiotemporal chaos or for transmitting information [Hayes, 1993]. In sequence space chaos control is straightforward: we may simply append the desired target symbol sequence onto the end of the current symbolic state [Corron, 2003]. Mapping the modified sequence back to state space gives us the connecting orbit to the target state. In this setting the error $E(n)$ of the symbolic model is equivalent to the control signal amplitude needed to steer the CML along this orbit.

6. CONTROLLING TURBULENCE

Take as a control example the challenge of steering a logistic CML ($\varepsilon=0.1$) along the symbolic trajectory represented by the 480×363 image [Carroll, 1865] shown in Fig. 3 (inset). The black and white pixels are interpreted as "0" and "1" symbols, respectively, and each row as the symbolic state of a CML. That is, a CML site is greater than 0.5 only if the corresponding pixel is white. For dynamical reasons we doubled the pixel dimensions to 960×726 and replaced every black pixel with a 2×2 checkerboard of white and black pixels, thereby eliminating blocks of consecutive "0"s. Using Eq.2 we found that all the $m=3$ local symbol patterns in this modified image are allowed by the CML dynamics for $n \leq 6$. The $N=726$, $\varepsilon=0.1$ CML orbit corresponding to this symbol sequence is shown in Fig. 3. The first 100 iterates of Eq.3 are uncontrolled. After iteration 100, the CML state is steered toward the orbit described by the image symbols. The target states were read from the $m=3$, $n=6$ LUT described earlier. The mean perturbation required to force the CML along this orbit was found to be 0.007, which is less than 1% of the $[0,1]$ dynamical range at each site. This assumes a controller that pushes the system state exactly onto the desired orbit at each time step. One-sided limiter control [Corron, 2000] produces the same results, but with slightly higher control perturbations (0.011). We conclude that local symbolic models extracted from data can be used to control spatiotemporal chaos in diffusive CMLs.

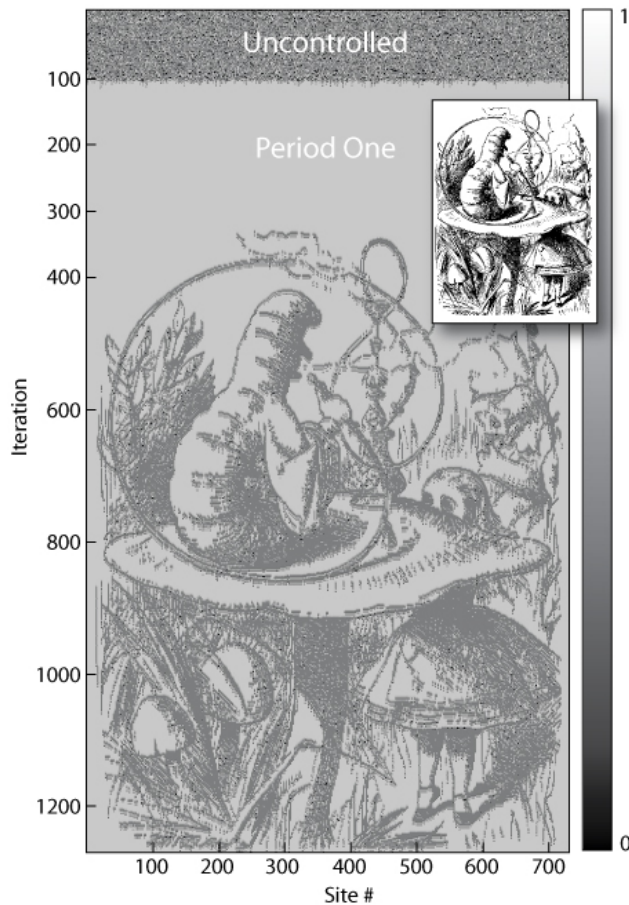


Fig.3 A time-space plot of a 1-d CML of 726 logistic maps. CML site values are shown on a compressed grayscale. Prior to iteration 100 the CML is uncontrolled. At iteration 100 control is initiated. By iteration 112 the CML has reached the uniform period one state. At iteration 300 a complex orbit is targeted that approximated the image shown in the overlay.

7. CONCLUSION

We conclude that complex systems that can be modeled as diffusively coupled lattices of unimodal maps are likely to have a compact description in terms of local

symbolic models. For these systems chaos control is straightforward and novel global states can be predicted and targeted from measured data. The approach discussed here is easily generalized to multi-dimensional lattices of maps with more than two symbols. We think it likely that any small in-degree network of such maps is a good candidate for reduction to a local symbolic model. It remains an open question as to whether networks of invertible maps or multi-dimensional maps can be treated similarly.

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