

Randomized Algorithms for Systems and Control: Theory and Applications

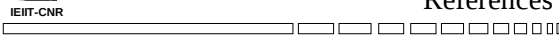
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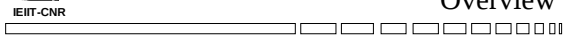
- Additional documents, papers, etc, please consult
<http://staff.polito.it/roberto.tempo/>

- Questions may be sent to
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References

- R. Tempo, G. Calafiore and F. Dabbene, "Randomized Algorithms for Analysis and Control of Uncertain Systems," Springer-Verlag, London, 2005
- R. Tempo and H. Ishii, "Monte Carlo and Las Vegas Randomized Algorithms for Systems and Control: An Introduction," European Journal of Control, Vol. 13, pp. 189-203, 2007
- RACT: Randomized Algorithms Control Toolbox for Matlab
<http://ract.sourceforge.net>

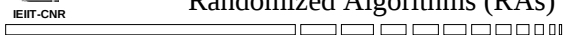


Overview

- Preliminaries
- Randomized Algorithms for Analysis
- Probabilistic Robust Synthesis
- Randomized Algorithms for Optimal Control (LQR)
- Extensions
- Applications: Probabilistic Control of Mini UAVs



Preliminaries



Randomized Algorithms (RAs)

- Randomized algorithms are frequently used in many areas of engineering, computer science, physics, finance, optimization,...but their appearance in systems and control is mostly limited to Monte Carlo simulations...
- Main objective of this mini-course: Introduction to rigorous study of RAs for uncertain systems and control, with specific applications

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Randomized Algorithms (RAs)

- Combinatorial optimization, computational geometry
- Examples: Data structuring, search trees, graph algorithms, sorting (RQS), ...
- Motion and path planning problems
- Mathematics of finance: Computation of path integrals
- Bioinformatics (string matching problems)



Uncertainty

- Uncertainty has been always a critical issue in control theory and applications
- First methods to deal with uncertainty were based on a stochastic approach
- Optimal control: LQG and Kalman filter
- Since early 80's alternative deterministic approach (worst-case or robust) has been proposed



Robustness

- Major stepping stone in 1981: Formulation of the $\mathcal{H}_\infty$ problem by George Zames
- Various "robust" methods to handle uncertainty now exist: Structured singular values, Kharitonov, optimization-based (LMI), l -one optimal control, quantitative feedback theory (QFT)



Robustness

- Late 80's and early 90's: Robust control theory became a well-assessed area
- Successful industrial applications in aerospace, chemical, electrical, mechanical engineering, ...
- However, ...



Limitations of Robust Control - 1

- Researchers realized some drawbacks of robust control
- Consider uncertainty Δ bounded in a set $\mathcal{B}$ of radius ρ . Largest value of ρ such that the system is stable for all $\Delta \in \mathcal{B}$ is called (worst-case) robustness margin
- Conservatism: Worst case robustness margin may be small
- Discontinuity: Worst case robustness margin may be discontinuous wrt problem data



Limitations of Robust Control - 2

- Computational Complexity: Worst case robustness is often $\mathcal{NP}$ -hard (not solvable in polynomial time unless $\mathcal{P} = \mathcal{NP}$)^[1]
- Various robustness problems are $\mathcal{NP}$ -hard
 - static output feedback
 - structured singular value
 - stability of interval matrices

[1] V. Blondel and J.N. Tsitsiklis (2000)



Conservatism and Complexity Trade-Off

- Uncertain or control design parameters often enter into the system in a nonlinear/nonconvex fashion
- To avoid complexity issues (or just to find a solution of the problem) relaxation techniques such as SOS are used
- Study issues about the accuracy of the approximation introduced and related complexity



Different Paradigm Proposed

- New paradigm proposed is based on uncertainty randomization and leads to randomized algorithms for analysis and synthesis
- Within this setting a different notion of problem tractability is needed
- Objective: Breaking the curse of dimensionality^[1]

[1] R. Bellman (1957)



Probability and Robustness

- The interplay of Probability and Robustness for control of uncertain systems
- Robustness: Deterministic uncertainty bounded
- Probability: Random uncertainty (pdf is known)
- Computation of the probability of performance
- Controller which stabilizes *most* uncertain systems



Key Features

- We obtain larger robustness margins at the expense of a small risk
- We study the probability degradation *beyond* the robustness margins
- Computational complexity is generally not an issue: Randomized algorithms are low complexity



Uncertain Systems

$M(s)$ System

Δ Uncertainty

- Δ belongs to a structured set $\mathcal{B}_D$
 - Parametric uncertainty q
 - Nonparametric uncertainty Δ_i
 - Mixed uncertainty



Worst Case Model

- Worst case model: Set membership uncertainty
- The uncertainty Δ is bounded in a set $\mathcal{B}_D$

$$\Delta \in \mathcal{B}_D$$

- Real parametric uncertainty $q = [q_1, \dots, q_\ell] \in \mathbb{R}^\ell$
 $q_i \in [q_i^-, q_i^+]$

- Nonparametric uncertainty

$$\Delta_i \in \{\Delta_i \in \mathbb{R}^{n,n} : \|\Delta_i\| \leq 1\}$$

Robustness

- Uncertainty Δ is bounded in a structured set $\mathcal{B}_D$
- $z = \mathcal{F}_u(M, \Delta) w$, where $\mathcal{F}_u(M, \Delta)$ is the upper LFT

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Objective of Robustness

- Objective of robustness: To guarantee stability and performance for all

$$\Delta \in \mathcal{B}_D$$

- Different probabilistic paradigm based on uncertainty randomization of Δ within $\mathcal{B}_D$

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Example: Flexible Structure - 1

- Mass spring damper model
- Real parametric uncertainty affecting stiffness and damping
- Complex unmodeled dynamics (nonparametric)

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Flexible Structure - 2

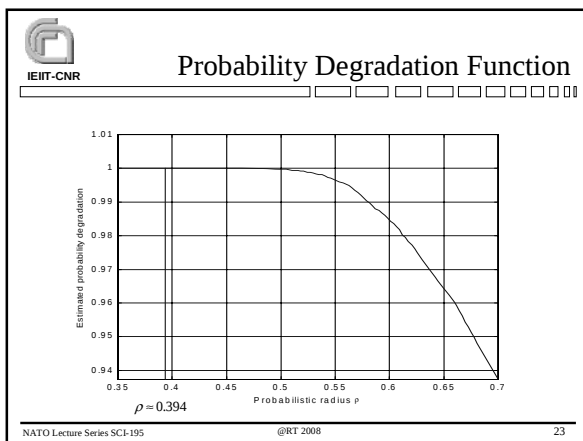
- M - Δ configuration for controlled systems and study stability of

$$M(s) = C(sI - A)^{-1}B$$

$$\Delta = \begin{bmatrix} q_1 I_5 & 0 & 0 \\ 0 & q_2 I_5 & 0 \\ 0 & 0 & \Delta_1 \end{bmatrix}$$

$q_1, q_2 \in \mathbb{R}$
 $\Delta_1 \in \mathbb{C}^{4,4}$
 $\Delta \in \mathcal{B}_D = \{ \Delta \in D : \sigma(\Delta) < \rho \}$

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Probabilistic Model

- Probability density function associated to $\mathcal{B}_D$
- We now assume that Δ is a random matrix with given density function $f_\Delta(\Delta)$ and support $\mathcal{B}_D$
- Example: Δ is uniform in $\mathcal{B}_D$

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Uniform Density

- Take $f_{\Delta}(\Delta) = \mathcal{U}[\mathcal{B}_D]$ (uniform density within $\mathcal{B}_D$)

$$\mathcal{U}[\mathcal{B}_D] = \begin{cases} \frac{1}{\text{vol}(\mathcal{B}_D)} & \text{if } \Delta \in \mathcal{B}_D \\ 0 & \text{otherwise} \end{cases}$$

- In this case, for a subset $\mathcal{S} \subseteq \mathcal{B}_D$

$$\Pr\{\Delta \in \mathcal{S}\} = \frac{\int_{\mathcal{S}} d\Delta}{\text{vol}(\mathcal{B}_D)} = \frac{\text{vol}(\mathcal{S})}{\text{vol}(\mathcal{B}_D)}$$



Performance Function

- In classical robustness we guarantee that a certain performance requirement is attained for all $\Delta \in \mathcal{B}_D$
- This can be stated in terms of a performance function

$$J = J(\Delta)$$

- Examples: $\mathcal{H}_{\infty}$ performance and robust stability



Example: $\mathcal{H}_{\infty}$ Performance - 1

- Compute the $\mathcal{H}_{\infty}$ norm of the upper LFT $\mathcal{F}_u(M, \Delta)$

$$J(\Delta) = \|\mathcal{F}_u(M, \Delta)\|_{\infty}$$

- For given $\gamma > 0$, check if

$$J(\Delta) < \gamma$$

for all Δ in $\mathcal{B}_D$



Example: $\mathcal{H}_{\infty}$ Performance - 2

- Continuous time SISO systems with real parametric uncertainty q with upper LFT

$$\mathcal{F}_u(M, \Delta) = \mathcal{F}_u(M, q) =$$

$$\frac{0.5q_1 q_2 s + 10^{-5} q_1}{(10^{-5} + 0.05 q_2) s^2 + (0.00102 + 0.5 q_2) s + (2 \cdot 10^{-5} + 0.5 q_1^2)}$$

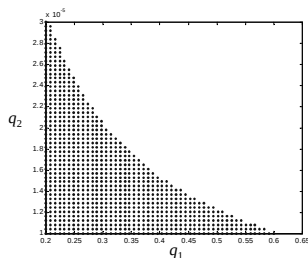
where $q_1 \in [0.2, 0.6]$ and $q_2 \in [10^{-5}, 3 \cdot 10^{-5}]$

- Letting $J(q) = \|\mathcal{F}_u(M, q)\|_{\infty}$, we choose $\gamma = 0.003$
- Check if $J(q) < \gamma$ for all q in these intervals



Example: $\mathcal{H}_{\infty}$ Performance - 3

- The set of q_1, q_2 for which $J(q) < \gamma$ is shown below



Example^[1]: Robust Stability - 1

- Consider the closed loop uncertain polynomial

$$p(s, q) =$$

$$(1 + r^2 + 6q_1 + 6q_2 + 2q_1 q_2) + (q_1 + q_2 + 3)s + (q_1 + q_2 + 1)s^2 + s^3$$

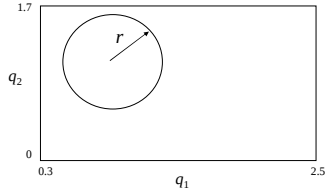
where $q_1 \in [0.3, 2.5]$, $q_2 \in [0, 1.7]$ and $r = 0.5$

- Check stability for all q in these intervals

[1] G. Truxal (1961)

Example: Robust Stability - 2

- Set of unstable polynomials



- Taking $r=0$ the unstable set reduces to a singleton

Good and Bad Sets

- We define two subsets of $\mathcal{B}_D$

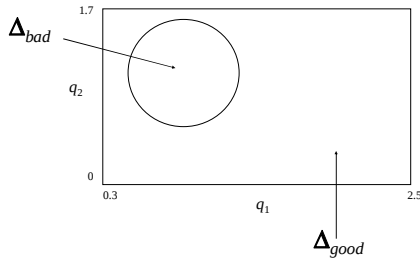
$$\Delta_{good} = \{\Delta: J(\Delta) \leq \gamma\} \subseteq \mathcal{B}_D$$

$$\Delta_{bad} = \{\Delta: J(\Delta) > \gamma\} \subseteq \mathcal{B}_D$$

- Δ_{good} is the set of Δ 's satisfying performance
- Measure of robustness is

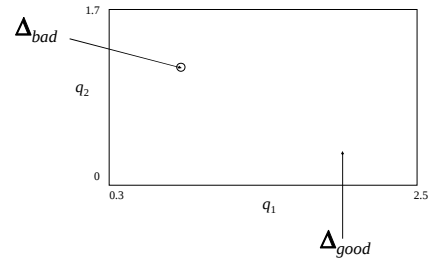
$$vol(\Delta_{good}) = \int_{\Delta_{good}} d\Delta$$

Example of Good and Bad Sets



Example of Good and Bad Sets - 2

Taking small r



Probabilistic Robustness Measure

- In worst-case analysis we compute γ such that all Δ satisfy performance. Equivalently, we evaluate γ such that

$$\Delta_{good} = \mathcal{B}_D$$

- In a probabilistic setting, we are satisfied if the ratio

$$\frac{vol(\Delta_{good})}{vol(\mathcal{B}_D)}$$

is close to one

Probability of Performance^[1]

- We define the probability of performance as

$$p_\gamma = \Pr\{J(\Delta) \leq \gamma\}$$

- Notice that, if $f_\Delta(\Delta)$ is uniform, then

$$p_\gamma = \frac{vol(\Delta_{good})}{vol(\mathcal{B}_D)}$$

[1] R.F. Stengel (1980)



Example: Closed-Form Computation

- For Truxal's example, we compute p_γ in closed-form
- For uniform distribution, we have

$$\text{vol}(\Delta_{\text{good}}) = 3.74 - \pi r^2$$

$$\text{vol}(\mathcal{B}_D) = 3.74$$



P1: Performance Verification

- For given performance level γ , check whether

$$J(\Delta) \leq \gamma$$

for all Δ in $\mathcal{B}_D$

- Compute the probability of performance p_γ



P2: Worst-Case Performance

- Find $J_{\max}$ such that

$$J_{\max} = \max_{\Delta \in \mathcal{B}_D} J(\Delta)$$

- Compute the worst case performance (or its probabilistic counterpart)



Randomized Algorithms for Analysis



Randomized Algorithm: Definition

- Randomized Algorithm (RA): An algorithm that makes random choices during its execution to produce a result
- Example of a "random choice" is a coin toss

heads

or

tails



Randomized Algorithm: Definition

- Randomized Algorithm (RA): An algorithm that makes random choices during its execution to produce a result
- For hybrid systems, "random choices" could be switching between different states or logical operations
- For uncertain systems, "random choices" require (vector or matrix) random sample generation



Monte Carlo Randomized Algorithm

- Monte Carlo Randomized Algorithm (MCRA): A randomized algorithm that may produce incorrect results, but with bounded error probability



Las Vegas Randomized Algorithm

- Las Vegas Randomized Algorithm (LVRA): A randomized algorithm that always produces correct results, the only variation from one run to another is the running time



Randomization of Uncertain Systems

- Consider random uncertainty Δ , associated pdf and bounding set B
- Δ is a (real or complex) random vector (parametric uncertainty) or matrix (nonparametric uncertainty)
- Consider a performance function

$$J(\Delta): B \rightarrow \mathbf{R}$$
and level $\gamma > 0$
- Define worst case and average performance

$$J_{\max} = \max_{\Delta \in B} J(\Delta) \quad J_{\text{ave}} = E_{\Delta}(J(\Delta))$$



Example: H_{∞} Performance - 1

- H_{∞} performance of sensitivity function

$$B = \{\Delta: \Delta = \text{bdiag}(\Delta_1, \dots, \Delta_q) \in \mathbf{F}^{n,m}, \sigma_{\max}(\Delta) \leq \rho\}$$

$$S(s, \Delta) = 1/(1 + P(s, \Delta) C(s))$$

$$J(\Delta) = \|S(s, \Delta)\|_{\infty}$$



Example: H_{∞} Performance - 2

- H_{∞} performance of sensitivity function

$$B = \{\Delta: \Delta = \text{bdiag}(\Delta_1, \dots, \Delta_q) \in \mathbf{F}^{n,m}, \sigma_{\max}(\Delta) \leq \rho\}$$

$$S(s, \Delta) = 1/(1 + P(s, \Delta) C(s))$$

$$J(\Delta) = \|S(s, \Delta)\|_{\infty}$$
- Objective: Check if

$$J_{\max} \leq \gamma \quad \text{and} \quad J_{\text{ave}} \leq \gamma$$
- These are uncertain decision problems



Two Problem Instances

- We have two problem instances for worst case performance

$$J_{\max} \leq \gamma \quad \text{and} \quad J_{\max} > \gamma$$
and two problem instances for average case performance

$$J_{\text{ave}} \leq \gamma \quad \text{and} \quad J_{\text{ave}} > \gamma$$
- This leads to one-sided and two-sided MCRA

One-Sided MCRA

- One-sided MCRA: Always provide a correct solution in one of the instances (they may provide a wrong solution in the other instance)

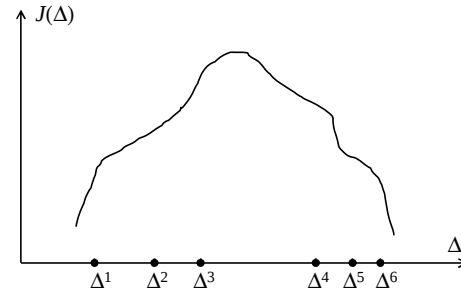
- Consider the empirical maximum

$$\hat{J}_{\max} = \max_{i=1, \dots, N} J(\Delta^i)$$

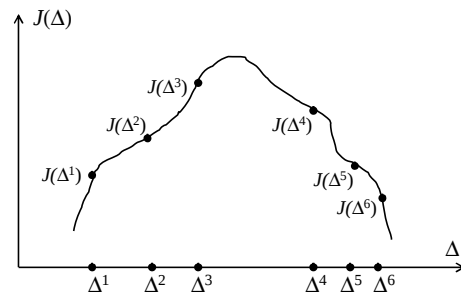
where Δ^i are random samples and N is the sample size

- Check if $\hat{J}_{\max} \leq \gamma$ or $\hat{J}_{\max} > \gamma$

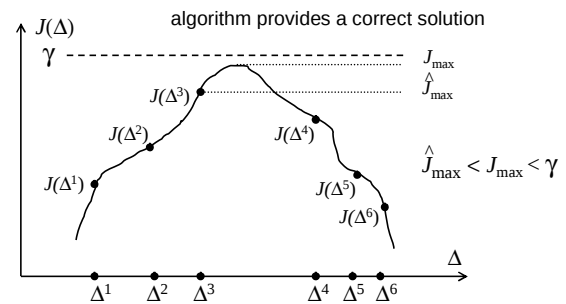
One-Sided MCRA: Case 1



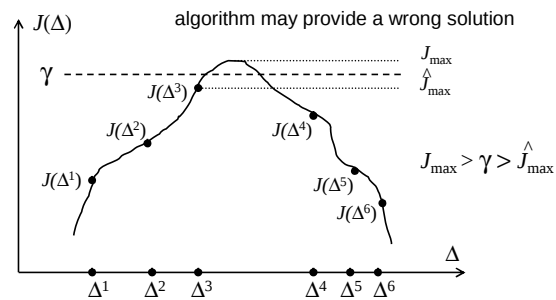
One-Sided MCRA: Case 1



One-Sided MCRA: Case 1



One-Sided MCRA: Case 2



Two-Sided MCRA

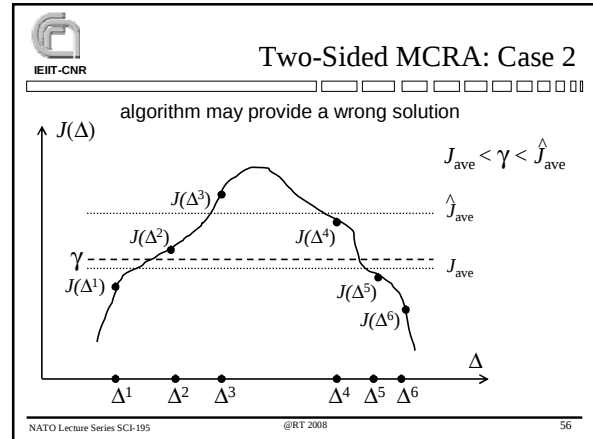
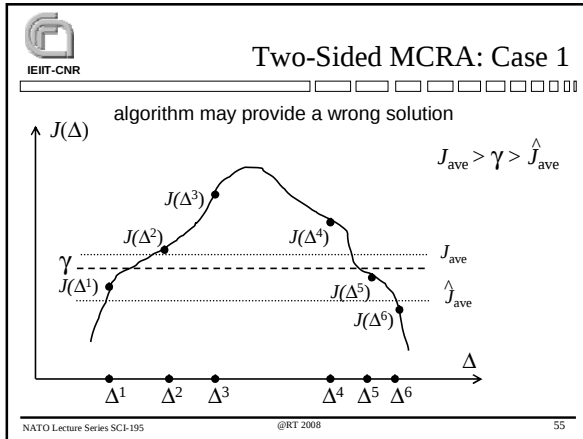
- Two-sided MCRA: They may provide a wrong solution in both instances

- Consider the empirical average

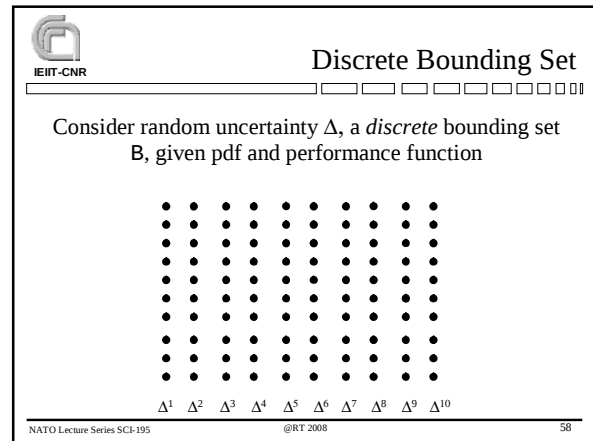
$$\hat{J}_{\text{ave}} = \text{ave}_{i=1, \dots, N} J(\Delta^i)$$

where Δ^i are random samples and N is the sample size

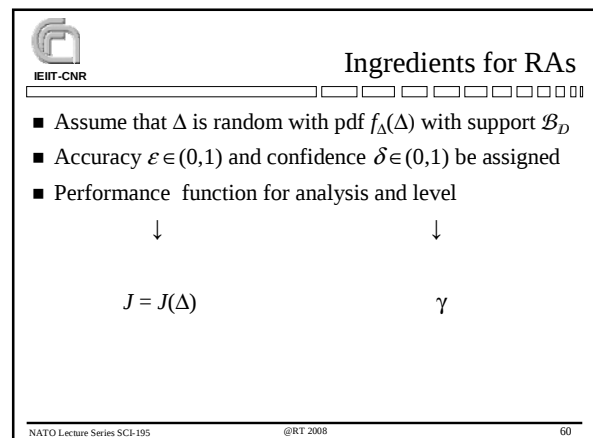
- Check if $\hat{J}_{\text{ave}} \leq \gamma$ or $\hat{J}_{\text{ave}} > \gamma$



- Las Vegas Randomized Algorithms
- We also have zero-sided (Las Vegas) randomized algorithms
 - Las Vegas Randomized Algorithm (LVRA): Always give the correct solution
 - The solution obtained with a LVRA is probabilistic, so “always” means with probability one
 - Running time may be different from one run to another
 - We can study the *average* running time
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- The Las Vegas Viewpoint
- Consider discrete random variables
 - The sample space is discrete and M^N possible choices can be made
 - In the binary case we have 2^N
 - Finding maximum requires ordering the 2^N choices
 - Las Vegas can be used for ordering real numbers
 - Example: Randomized Quick Sort for sorting real numbers (classical in computer science)
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Randomized Algorithms for Analysis

- Two classes of randomized algorithms for probabilistic robust performance analysis
- P1: Performance verification (compute p_γ)
- P2: Worst-case performance (compute $J_{\max}$)
- Both are based on uncertainty randomization of Δ
- Bounds on the sample size are obtained

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Randomized Algorithms - 2

- We estimate p_γ by means of a randomized algorithm
- First, we generate N i.i.d. samples

$$\Delta^1, \Delta^2, \dots, \Delta^N \in \mathcal{B}_D$$

according to the density f_Δ

- We evaluate $J(\Delta^1), J(\Delta^2), \dots, J(\Delta^N)$

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Empirical Probability

- Construct an indicator function

$$I(\Delta^i) = \begin{cases} 1 & \text{if } J(\Delta^i) \leq \gamma \\ 0 & \text{otherwise} \end{cases}$$

- An estimate of p_γ is the empirical probability

$$\hat{p}_N = \frac{1}{N} \sum_{i=1}^N I(\Delta^i) = \frac{N_{\text{good}}}{N}$$

where N_{good} is the number of samples such that $J(\Delta^i) \leq \gamma$

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A Reliable Estimate

- The empirical probability is a reliable estimate if

$$|p_\gamma - \hat{p}_N| = |\Pr\{J(\Delta) \leq \gamma\} - \hat{p}_N| \leq \varepsilon$$

- Find the minimum N such that

$$\Pr\{|p_\gamma - \hat{p}_N| \leq \varepsilon\} \geq 1 - \delta$$

where $\varepsilon \in (0,1)$ and $\delta \in (0,1)$

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Chernoff Bound^[1]

- For any $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, if

$$N \geq \frac{\log \frac{2}{\delta}}{2\varepsilon^2}$$

then

$$\Pr\{|p_\gamma - \hat{p}_N| \leq \varepsilon\} \geq 1 - \delta$$

[1] H. Chernoff (1952)

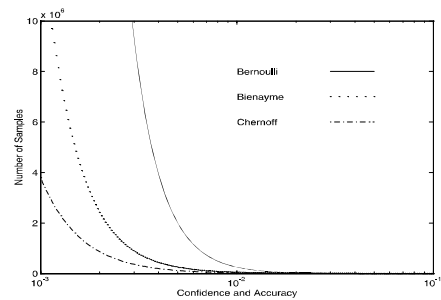
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Comparison Between Bounds



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Chernoff Bound

- Remark: Chernoff bound improves upon other bounds such as Bernoulli (Law of Large Numbers)
- Dependence on $1/\delta$ is logarithmic
- Dependence on $1/\epsilon$ is quadratic

ϵ	0.1%	0.1%	0.5%	0.5%
$1-\delta$	99.9%	99.5%	99.9%	99.5%
N	$3.9 \cdot 10^6$	$3.0 \cdot 10^6$	$1.6 \cdot 10^6$	$1.2 \cdot 10^5$



Computational Complexity of RAs

- RAs are efficient (polynomial-time) because
 1. Random sample generation of Δ^i can be performed in polynomial-time
 2. Cost associated with the evaluation of $J(\Delta^i)$ for fixed Δ^i is polynomial-time
 3. Sample size is polynomial in the problem size and probabilistic levels ϵ and δ



1. Random Sample Generation

- Random number generation (RNG): Linear and nonlinear methods for uniform generation in $[0,1]$ such as Fibonacci, feedback shift register, BBS, MT, ...
- Non-uniform univariate random variables: Suitable functional transformations (e.g., the inversion method)
- The problem is much harder: Multivariate generation of samples of Δ with pdf $f_{\Delta}(\Delta)$ and support $\mathcal{B}_{\Delta}$
- It can be resolved in polynomial-time



2. Cost of Checking Stability

- Consider a polynomial

$$p(s, a) = a_0 + a_1 s + \dots + a_n s^n$$
- To check left half plane stability we can use the Routh test. The number of multiplications needed is

$$\frac{n^2}{4} \text{ for } n \text{ even} \quad \frac{n^2 - 1}{4} \text{ for } n \text{ odd}$$
- The number of divisions and additions is equal to this number
- We conclude that checking stability is $O(n^2)$



3. Bounds on the Sample Size

- Chernoff bound is independent on the size of $\mathcal{B}_{\Delta}$, on the structure $\mathcal{D}$ on the number of blocks, on the pdf $f_{\Delta}(\Delta)$
- It depends only on δ and ϵ
- Same comments can be made for other bounds such as Bernoulli



Worst-Case Performance

- Recall that

$$J_{\max} = \max_{\Delta \in \mathcal{B}_{\Delta}} J(\Delta)$$
- Generate N i.i.d. samples

$$\Delta^1, \Delta^2, \dots, \Delta^N \in \mathcal{B}_{\Delta}$$
 according to the density f_{Δ}
- Compute the empirical maximum

$$\hat{J}_{\max} = \max_{i=1, \dots, N} J(\Delta^i)$$

Worst-Case Bound (Log-over-Log)^[1]

- For any $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, if

$$N \geq \frac{\log \frac{1}{\delta}}{\log \frac{1}{1-\varepsilon}}$$

then

$$\Pr\{J(\Delta) > \hat{J}_N \leq \varepsilon\} \geq 1 - \delta$$

[1] R. Tempo, E. W. Bai and F. Dabbene (1996)

Comparison and Comments

- Number of samples is much smaller than Chernoff
- Bound is a specific instance of the fpras (fully polynomial randomized approximated scheme) theory
- Dependence on $1/\varepsilon$ is basically linear $\left(\log \frac{1}{1-\varepsilon} \approx \varepsilon\right)$

ε	0.1%	0.1%	0.5%	0.5%	0.01%	0.001%
$1-\delta$	99.9%	99.5%	99.9%	99.5%	99.99%	99.999%
N	$6.91 \cdot 10^3$	$5.30 \cdot 10^3$	$1.38 \cdot 10^3$	$1.06 \cdot 10^3$	$9.21 \cdot 10^4$	$1.16 \cdot 10^6$

Volumetric Interpretation

- In the case of $f_\Delta(\Delta)$ uniform, we have

$$\Pr\{J(\Delta) > \hat{J}_N\} = \frac{\text{vol}(\Delta_{\text{bad}})}{\text{vol}(\mathcal{B}_D)}$$

- Therefore

$$\Pr\{\Pr\{J(\Delta) > \hat{J}_N\} \leq \varepsilon\} \geq 1 - \delta$$

is equivalent to

$$\Pr\{\text{vol}(\Delta_{\text{bad}}) \leq \varepsilon \text{vol}(\mathcal{B}_D)\} \geq 1 - \delta$$

Confidence Intervals

- The Chernoff and worst-case bounds can be computed *a-priori* and provide an explicit functional relation

$$N = N(\varepsilon, \delta)$$

- The sample size obtained with the confidence intervals is not explicit
- Given $\delta \in (0,1)$, upper and lower confidence intervals p_L and p_U are such that

$$\Pr\{p_L \leq p_\gamma \leq p_U\} = 1 - \delta$$

Confidence Intervals - 2

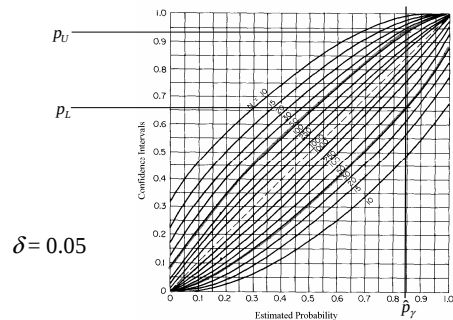
- The probabilities p_L and p_U can be computed *a posteriori* when the value of N_{good} is known, solving equations of the type

$$\sum_{k=N_{\text{good}}}^N \binom{N}{k} p_L^k (1-p_L)^{N-k} = \delta_L$$

$$\sum_{k=0}^{N_{\text{good}}} \binom{N}{k} p_U^k (1-p_U)^{N-k} = \delta_U$$

with $\delta_L + \delta_U = \delta$

Confidence Intervals - 3





Statistical Learning Theory

- The Chernoff Bound studies the problem

$$\Pr\{|p_\gamma - \hat{p}_N| \leq \varepsilon\} \geq 1 - \delta$$

where $p_\gamma = \Pr\{J(\Delta) \leq \gamma\}$

- Performance function J is fixed
- Statistical Learning Theory computes bounds on the sample size for the problem

$$\Pr\{|\Pr(J(\Delta) \leq \gamma) - \hat{p}_N| \leq \varepsilon, \forall J \in \mathcal{J}\} \geq 1 - \delta$$

where $\mathcal{J}$ is a given class of functions

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VC and P-dimension^[1,2]

- Statistical Learning Theory aims at studying uniform Law of Large Numbers
- The bounds obtained depend on quantities called VC-dimension (if J is a binary valued function), or P-dimension (if J is a continuous valued function)
- VC and P-dimension are measures of the problem complexity

[1] M. Vidyasagar (1997)

[2] E.D. Sontag [1998]

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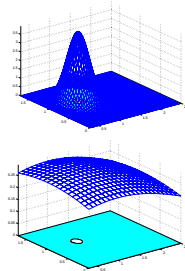
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Choice of the Distribution - 1

- The probability $\Pr\{\Delta \in \mathcal{S}\}$ depends on $f_\Delta(\Delta)$
- It may vary between 0 and 1 depending on the pdf $f_\Delta(\Delta)$



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Choice of the Distribution - 2

- The bounds discussed are independent on the choice of the distribution but for computing $\Pr\{J(\Delta) \leq \gamma\}$ we need to know the distribution $f_\Delta(\Delta)$
- Some research has been done in order to find the worst-case distribution in a certain class^[1]
- Uniform distribution is the worst-case if a certain target is convex and centrally symmetric

[1] B. R. Barmish and C. M. Lagoa (1997)

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Choice of the Distribution - 3

- Minimax properties of the uniform distribution have been studied^[1]

[1] E. W. Bai, R. Tempo and M. Fu (1998)

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Probabilistic Robust Synthesis

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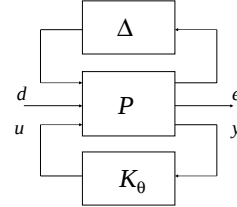


Analysis vs Design with Uncertainty

- Starting point: Worst-case analysis versus design
- Consider an interval family $p(s,q)$, $q \in \mathcal{B}_q = \{q \in \mathbb{R}^n, \|q\|_\infty \leq 1\}$
- Analysis problem:
 - Check if $p(s,q)$ is stable for all $q \in \mathcal{B}_q$
 - Answer: Kharitonov Theorem
- Design Problem:
 - Does there exist a $q \in \mathcal{B}_q$ such that $p(s,q)$ is stable?
 - Answer: *Unknown* in general



Synthesis Paradigm



- Design the parameterized controller K_θ to guarantee stability and performance



Synthesis Performance Function

- Recall that the parameterized controller is K_θ
- We replace $J(\Delta)$ with a synthesis performance function

$$J = J(\Delta, \theta)$$

where $\theta \in \Theta$ represents the controller parameters to be determined and their bounding set



Randomized Algorithms for Synthesis

- Two classes of RAs for probabilistic synthesis
- Average performance synthesis^[1]
- Based on expected value minimization
- Use of Statistical Learning Theory results
- Very general problems can be handled
- Existing bounds are very conservative and controller randomization is required
- Ongoing research aiming at major reduction of sample size

[1] M. Vidyasagar (1998)



Randomized Algorithms for Synthesis

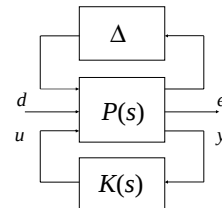
- Robust performance synthesis^[1]
- Problem reformulation as robust feasibility
- Only convex problems can be handled
- Finite-time convergence with probability one is obtained

[1] B. Polyak and R. Tempo (2001)



Robust Performance Synthesis

- Uncertainty randomization of Δ in $\mathcal{B}_D$
- Convex optimization to design the controller $K(s)$



RAs for Optimal Control (LQR)

Uncertain Systems in State Space

- We consider a state space description of the uncertain system

$$\dot{x}(t) = A(\Delta)x(t) + Bu(t)$$

with $x(0)=x_0$; $x \in \mathbb{R}^n$; $u \in \mathbb{R}^m$, $\Delta \in \mathcal{B}_D$

- For example, $A(\Delta)$ is an interval matrix with bounded entries $a_{ij}^- \leq a_{ij} \leq a_{ij}^+$

Interval and Vertex Matrices

- We consider interval uncertainty A (i.e. when $\Delta \in \mathcal{B}_D$)
- That is, a_{ik} ranges in the interval for all i, k

$$|a_{ik} - a_{ik}^*| \leq w_{ik}$$

where a_{ik}^* are nominal values and w_{ik} are weights

- Define the $N = 2^{n^2}$ vertex matrices $A^1, A^2, \dots, A^N$

$$a_{ik} = a_{ik}^* + w_{ik} \quad \text{or} \quad a_{ik} = a_{ik}^* - w_{ik}$$

for all $i, k = 1, 2, \dots, n$

Common Lyapunov Functions

- Given matrices A^* , W and feedback K , find a *common quadratic Lyapunov function* $Q > 0$ for the system

$$\dot{x}(t) = (A + BK)x(t) \quad \text{for all } A \in \mathcal{A}$$

- Find $Q > 0$ such that

$$L(Q, A) = (A+BK)^T Q + Q(A+BK) < 0 \quad \text{for all } A \in \mathcal{A}$$

- Equivalently, find $Q > 0$ such that

$$\lambda_{\max} L(Q, A) < 0 \quad \text{for all } A \in \mathcal{A}$$

Lyapunov Stability of Interval Systems

- Quadratic Lyapunov stability analysis and synthesis of interval systems are NP-hard problems
- In principle, they can be solved in one-shot with convex optimization, but the number of constraints is exponential
- We can use relaxation (e.g. $\pi/2$ Theorem^[1]) or randomization

[1] Yu. Nesterov (1997)

Vertex Solution

- Due to convexity, it suffices to study $L(Q, A) < 0$ for all vertex matrices^[1]
- Question: Do we really need to check all the vertex matrices ($N = 2^{n^2}$)?

[1] H.P. Horisberger, P.R. Belanger (1976)



Vertex Reduction

- Answer: It suffices to check “only” a subset of 2^{2n} vertex matrices^[1]
- This is still exponential (the problem is NP-hard), but it leads to a major computational improvement for medium size problems (e.g. $n = 8$ or 10)
- For example, for $n=8$, N is of the order 10^5 (instead of 10^{19})

[1] T. Alamo, R. Tempo, D. Rodriguez, E.F. Camacho (2007)

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Diagonal Matrices and Generalizations

- Transform the original problem from full square matrices A to diagonal matrices $Z \in \mathbb{R}^{2n, 2n}$
- It suffices to check the vertices of Z
- Extensions for L_2 -gain minimization and other related LMI problems
- Generalizations for multiaffine interval systems

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Las Vegas Randomized Algorithm

- We may perform randomization of the $N = 2^{n^2}$ vertices (in the worst case)
- If we select the vertices in random order according to a given pdf, we have a LVRA

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Probabilistic Solution

- Randomly generate $A^1, \dots, A^N$. Then, check if the Lyapunov equation

$$A^i Q + Q(A^i)^T \leq 0$$

is feasible for $i=1, \dots, N$ and find a common solution $Q = Q^T > 0$

- Critical problem: Even if N is relatively small, this is a hard computational problem

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Sequential Algorithm

- Key point: Sequential algorithm which deals with one constraint at each step
- At step k we have
 - Phase 1: Uncertainty randomization of Δ
 - Phase 2: Gradient algorithm and projection
- Final result: Find a solution $Q = Q^T > 0$ with probability one in a finite number of steps

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Definition

- Let $\mathcal{E}_n$ be an Euclidean space

$$\mathcal{E}_n = \left\{ A = A^T \in \mathbb{R}^n, \|A\| = \sqrt{\sum_{i,k=1}^n a_{ik}^2} \right\}$$

and C be the cone of positive semi-definite matrices

$$C = \{A \in \mathcal{E}_n : A \geq 0\}$$

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Projection on a Cone

- For any real symmetric matrix A we define the projection $[A]^+ \in C$ as

$$[A]^+ = \arg \min_{X \in C} \|A - X\|$$

- The projection can be computed through the eigenvalue decomposition $A = T\Lambda T^T$
- Then

$$[A]^+ = T\Lambda^+ T^T$$

where $\lambda_i^+ = \max\{\lambda_i, 0\}$



Phase 1: Uncertainty Randomization

- Uncertainty randomization: Generate $\Delta^k \in \mathcal{B}_D$
- Then, for guaranteed cost we obtain the Lyapunov equation

$$A(\Delta^k)Q + QA^T(\Delta^k) \leq 0$$



Matrix Valued Function

- Define a matrix valued function

$$V(Q, \Delta^k) = A(\Delta^k)Q + QA^T(\Delta^k)$$

and a scalar function

$$v(Q, \Delta^k) = \| [V(Q, \Delta^k)]^+ \|^2$$

where $\|\cdot\|$ is the Frobenius norm

- We can also take the maximum eigenvalue of $V(Q, \Delta^k)$



Phase 2: Gradient Algorithm

- We write

$$Q^{k+1} = \begin{cases} [Q^k - \mu^k \partial_Q \{v(Q^k, \Delta^k)\}]^+ & \text{if } v(Q^k, \Delta^k) > 0 \\ Q^k & \text{otherwise} \end{cases}$$

where ∂_Q is the subgradient and the stepsize μ^k is

$$\mu^k = \frac{v(Q^k, \Delta^k) + r \|\partial_Q \{v(Q^k, \Delta^k)\}\|}{\|\partial_Q \{v(Q^k, \Delta^k)\}\|^2}$$

and $r > 0$ is a parameter



Closed-form Gradient Computation

- The function $v(Q, \Delta^k)$ is convex in Q and its subgradient can be easily computed in a closed form



Theorem^[1]

- Assumption: Every open subset of $\mathcal{B}_D$ has positive measure
- Theorem: A solution Q , if it exists, is found in a finite number of steps with probability one
- Idea of proof: The distance of Q^k from the solution set decreases at each correction step

[1] B.T. Polyak and R. Tempo (2001)

Example^[1]

- We study a multivariable example for the design of a controller for the lateral motion of an aircraft.
- The model consists of four states and two inputs

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & L_p & L_\beta & L_r \\ \frac{g}{V} & 0 & Y_\beta & -1 \\ N_\beta(\frac{g}{V}) & N_p & N_\beta + N_\beta Y_\beta & N_r - N_\beta \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix} u(t)$$

[1] B.D.O. Anderson and J.B. Moore (1971)

Example - 2

- The state variables are
 - x_1 bank angle
 - x_2 derivative of bank angle
 - x_3 sideslip angle
 - x_4 yaw rate
- The control inputs are
 - u_1 rudder deflection
 - u_2 aileron deflection

Example - 3

- Nominal values: $L_p=-2.93$, $L_\beta=-4.75$, $L_r=0.78$, $g/V=0.086$, $Y_\beta=-0.11$, $N_\beta=0.1$, $N_p=-0.042$, $N_\beta=2.601$, $N_r=-0.29$
- Perturbed matrix $A(\Delta)$: each parameter can take values in a range of $\pm 15\%$ of the nominal value
- Quadratic stability ($\gamma=0$): take $R=I$ and $S=0.01I$
- Remark: $A(\Delta)$ is multi-affine in the uncertain parameters: quadratic stability can be ascertained solving simultaneously $2^9=512$ LMIs

Example - 4

- Sequential algorithm:
 - Initial point Q_0 randomly selected
 - 800 random matrices Δ^k
 - The algorithm converged to

$$Q = \begin{bmatrix} 0.7560 & -0.0843 & 0.1645 & 0.7338 \\ -0.0843 & 1.0927 & 0.7020 & 0.4452 \\ 0.1645 & 0.7020 & 0.7798 & 0.7382 \\ 0.7338 & 0.4452 & 0.7382 & 1.2162 \end{bmatrix}$$

Example - 5

- The corresponding controller

$$K = B^T Q^{-1} = \begin{bmatrix} 38.6191 & -4.3731 & 43.1284 & -49.9587 \\ -2.8814 & -10.1758 & 10.2370 & -0.4954 \end{bmatrix}$$
 satisfies all the 512 vertex LMIs and therefore it is also a quadratic stabilizing controller in a deterministic sense
- The optimal LQ controller computed on the nominal plant satisfies only 240 vertex LMIs

Extensions



Related Literature and Extensions

- Minimization of a measure of violation for problems that are not strictly feasible^[1]
- Uncertainty in the control matrix, $B=B(\Delta)$, $\Delta \in \mathcal{B}_D$

We take the feedback law

$$u = YQ^{-1}x$$

where Y and $Q=Q^T > 0$ are design variables

[1] B.R. Barmish and P. Shcherbakov (1999)

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Related Literature

- Related literature on optimization and adaptive control with linear constraints^[1,2,3,4]
- Stochastic approximation algorithms have been widely studied in the stochastic control and optimization literature^[6,7]

[1] S. Agmon (1954)

[2] T.S. Motzkin and I.J. Schoenberg (1954)

[3] B.T. Polyak (1964)

[4] V.A. Bondarko and V.A. Yakubovich (1992)

[6] H.J. Kushner and G.G. Yin (2003)

[7] J.C. Spall (2003)

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Subsequent Research

- Design of common Lyapunov functions for switched systems^[1]
- From common to piecewise Lyapunov functions^[2]
- Ellipsoidal algorithm instead of gradient algorithm^[3]
- Stopping rule which provides the number of steps^[4]
- Other algorithms have been recently proposed^[5-6]

[1] D. Liberzon and R. Tempo (2004)

[2] H. Ishii, T. Basar and R. Tempo (2005)

[3] S. Kanev, B. De Schutter and M. Verhaegen (2002)

[4] Y. Oishi and H. Kimura (2003)

[5] Y. Fujisaki and Y. Oishi (2007)

[6] T. Alamo, R. Tempo, D. R. Ramirez and E. F. Camacho (2007)

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Optimization Problems^[1]

- Extensions to optimization problems
- Consider convex function $f(x)$ and function $g(x, \Delta)$ convex in x for fixed Δ
- Semi-infinite (nonlinear) programming problem

$$\min f(x)$$

$$g(x, \Delta) \leq 0 \text{ for all } \Delta \in \mathcal{B}$$
- Reformulation as stochastic optimization
- Drawback: Convergence results are only asymptotic

[1] V. B. Tadic, S. P. Meyn and R. Tempo (2003)

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Scenario Approach

- The scenario approach for convex problems^[1]
- Non-sequential method which provides a one-shot solution for general convex problems
- Randomization of $\Delta \in \mathcal{B}$ and solution of a single convex optimization problem
- Derivation of a bound on the sample size^[1]
- A new improved bound based on a pack-based strategy^[2]

[1] G. Calafiore and M. Campi (2004)

[2] T. Alamo, R. Tempo and E.F. Camacho (2007)

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Convex Semi-Infinite Optimization

- The semi-infinite optimization problem is

$$\min c^T \theta \quad \text{subject to } f(\theta, \Delta) \leq 0 \quad \text{for all } \Delta \in \mathcal{B}$$

where $f(\theta, \Delta) \leq 0$ is convex in θ for all $\Delta \in \mathcal{B}$

- We assume that this problem is either unfeasible or, if feasible, it attains a unique solution for all $\Delta \in \mathcal{B}$ (this assumption is technical and may be removed)
- We assume that $\theta \in \Theta \subseteq \mathbb{R}^n$

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Scenario Problem

- Using randomization, we construct a scenario problem
- Taking random samples $\Delta^i, i = 1, 2, \dots, N$, we construct

$$f(\theta, \Delta^i) \leq 0, \quad i = 1, 2, \dots, N$$

and

$$\min c^T \theta \quad \text{subject to } f(\theta, \Delta^i) \leq 0, \quad i = 1, 2, \dots, N$$



Theorem^[1]

- Theorem: For any $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, if

$$N \geq \left\lceil \frac{2}{\varepsilon} \log(1/\delta) + 2n + 2n/\varepsilon \log(2/\varepsilon) \right\rceil$$

then, with probability no smaller than $1 - \delta$

- either the scenario problem is unfeasible and then also the semi-infinite optimization problem is unfeasible
- or, the scenario problem is feasible, then its optimal solution $\hat{\theta}_N$ satisfies

$$\Pr\{\Delta \in \mathcal{B} : f(\theta, \Delta) > 0\} \leq \varepsilon$$

[1] G. Calafiore and M. Campi (2004)



A New Improved Bound^[1]

- A new improved bound (based on a so-called pack-based strategy) has been recently obtained

$$N \geq \left\lceil \frac{2}{\varepsilon} \log(1/2\delta) + 2n + 2n/\varepsilon \log 4 \right\rceil$$

- The main difference with the previous bound is that the factor

$$2n/\varepsilon \log(2/\varepsilon)$$

is replaced with

$$2n/\varepsilon \log 4$$

[1] T. Alamo, R. Tempo and E.F. Camacho (2007)



RACT



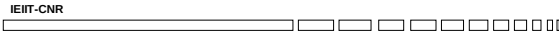
RACT

- RACT: Randomized Algorithms Control Toolbox for Matlab
- RACT has been developed at IEIIT-CNR and at the Institute for Control Sciences-RAS, based on a bilateral international project
- Members of the project
 - Andrey Tremba (Main Developer and Maintainer)
 - Giuseppe Calafiore
 - Fabrizio Dabbene
 - Elena Gryazina
 - Boris Polyak (Co-Principal Investigator)
 - Pavel Shcherbakov
 - Roberto Tempo (Co-Principal Investigator)



RACT

- Main features
- Define a variety of uncertain objects: scalar, vector and matrix uncertainties, with different pdfs
- Easy and fast sampling of uncertain objects of almost any type
- Randomized algorithms for probabilistic performance verification and probabilistic worst-case performance
- Randomized algorithms for feasibility of uncertain LMIs using stochastic gradient, ellipsoid or cutting plane methods (YALMIP needed)



Applications of Randomized Algorithms

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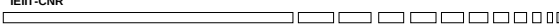
Application of RAs

- Randomized algorithms have been developed for various specific applications
- Control of flexible structures
- Stability and robustness of high speed networks
- Stability of quantized sampled-data systems
- Brushless DC motors
- Control design of Mini UAV

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Probabilistic Control of Mini-UAVs^[1]

[1] L. Lorefice, B. Pralio and R. Tempo (2007)

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Italian National Project for Fire Prevention

- This activity is supported by the Italian Ministry for Research within the National Project

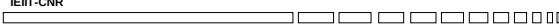
Study and development of a real-time land control and monitoring system for fire prevention

- Five research groups are involved together with a government agency for fire surveillance and patrol located in Sicily
- The aerial platform is based on the MicroHawk configuration, developed at the Aerospace Engineering Department, Politecnico di Torino, Italy

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MH1000 Platform - 1

■ Platform features

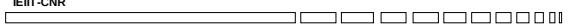
- wingspan 3.28 ft (1 m)
- total weight 3.3 lb (1.5 kg)



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MH1000 Platform - 2

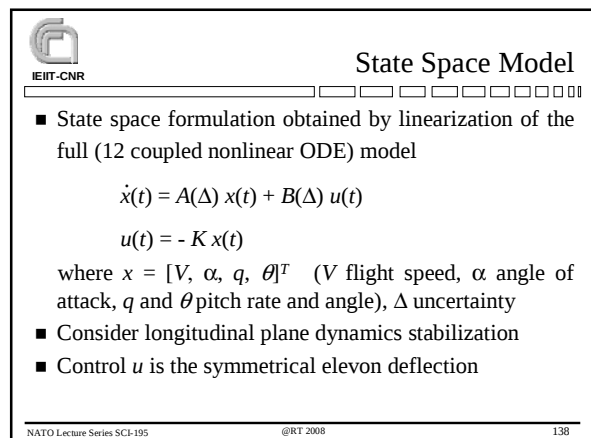
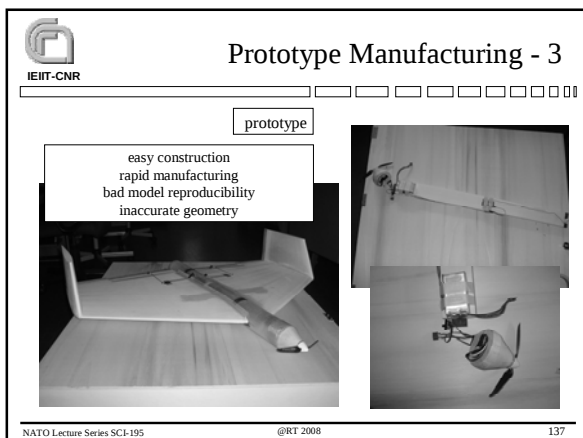
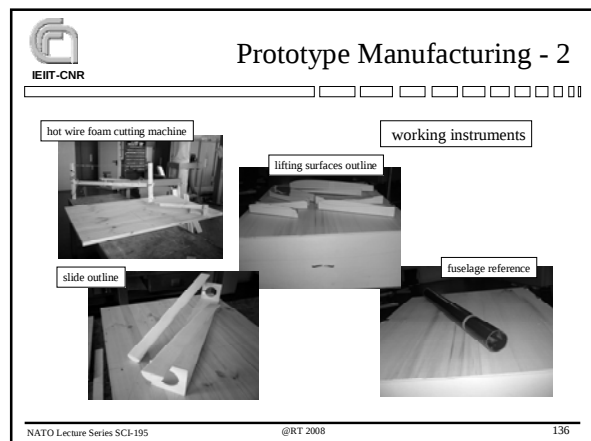
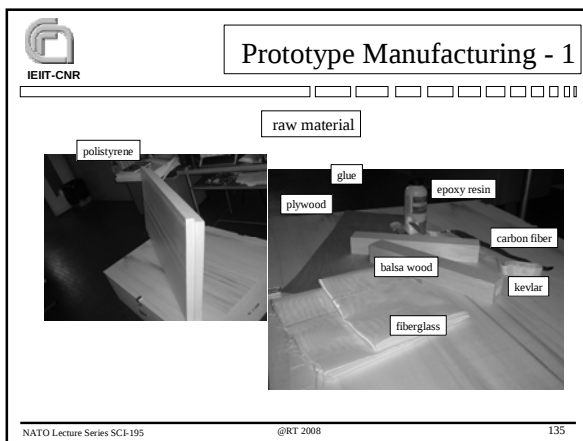
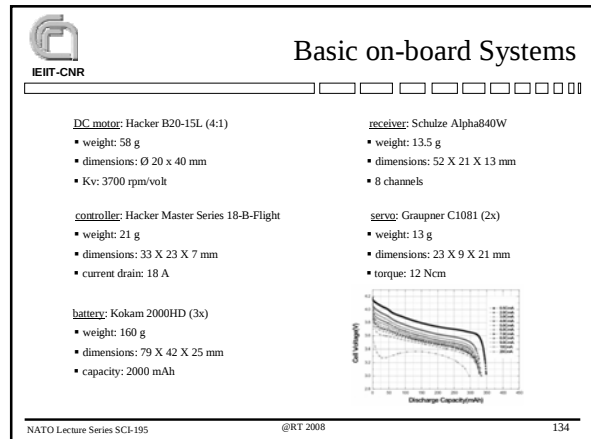
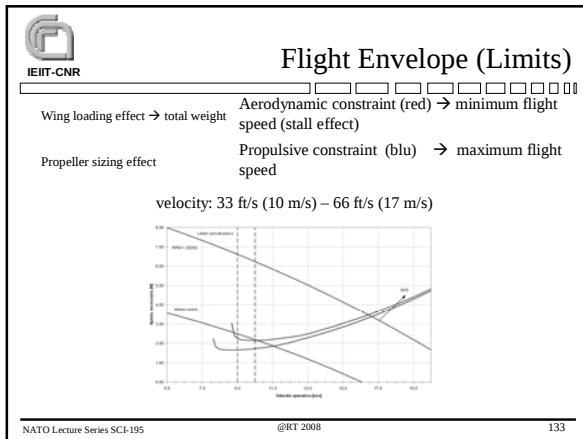
■ Main on-board equipment

- various sensors and two cameras (color and infrared)
- DC motor
- Remote piloting and autonomous flight
- Flight endurance of about 40 min
- Flight envelope
 - min/max velocity: 33 ft/s (10 m/s) – 66 ft/s (17 m/s)
 - average velocity: 43 ft/s (14 m/s)

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Uncertainty Description - 1

- We consider structured parameter uncertainties affecting plant and flight conditions, and aerodynamic database
- Uncertainty vector $\Delta = [\delta_1, \dots, \delta_{16}]$ where $\delta_i \in [\delta_i^-, \delta_i^+]$
- Key point: There is no explicit relation between state space matrices A and B and uncertainty Δ
- This is due to the fact that state space system is obtained through linearization and off-line flight simulator
- The only techniques which could be used in this case are simulation-based which lead to randomized algorithms



Uncertainty Description - 2

- We consider random uncertainty $\Delta = [\delta_1, \dots, \delta_{16}]^T$
- The pdf is either uniform (for plant and flight conditions) or Gaussian (for aerodynamic database uncertainties)
- Flight conditions uncertainties need to take into account large variations on physical parameters
- Uncertainties for aerodynamic data are related to experimental measurement or round-off errors



Plant and Flight Condition Uncertainties

parameter	pdf	$\bar{\delta}_i$	%	δ_i^-	δ_i^+	#
flight speed [ft/s]	U	42.65	± 15	36.25	49.05	1
altitude [ft]	U	164.04	± 100	0	328.08	2
mass [lb]	U	3.31	± 10	2.98	3.64	3
wingspan [ft]	U	3.28	± 5	3.12	3.44	4
mean aero chord [ft]	U	1.75	± 5	1.67	1.85	5
wing surface [ft ²]	U	5.61	± 10	5.06	6.18	6
moment of inertia [lb ft ²]	U	1.34	± 10	1.21	1.48	7

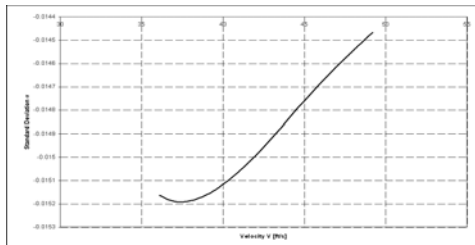


Aerodynamic Database Uncertainties

parameter	pdf	$\bar{\delta}_i$	σ_i	#
C_X [-]	G	-0.01215	0.00040	8
C_Z [-]	G	-0.30651	0.00500	9
C_m [-]	G	-0.02401	0.00040	10
C_{Xq} [rad ⁻¹]	G	-0.20435	0.00650	11
C_{Zq} [rad ⁻¹]	G	-1.49462	0.05000	12
C_{mq} [rad ⁻¹]	G	-0.76882	0.01000	13
C_{Xr} [rad ⁻¹]	G	-0.17072	0.00540	14
C_{Zr} [rad ⁻¹]	G	-1.41136	0.02200	15
C_{mr} [rad ⁻¹]	G	-0.94853	0.01500	16



Standard Deviation and Velocity



Standard deviation is experimentally computed from the velocity



Critical Parameters and Matrices

- We select flight speed (δ_1) and take off mass (δ_3) as critical parameters
- Flight speed is taken as critical parameter to optimize gain scheduling issues
- Take off mass is a key parameter in mission profile definition
- We define critical matrices

$$A_c^1 \ A_c^2 \ A_c^3 \ A_c^4 \ B_c^1 \ B_c^2 \ B_c^3 \ B_c^4$$
- They are constructed setting δ_1 , δ_3 to the extreme values $\delta_1^-, \delta_1^+, \delta_3^-, \delta_3^+$ and all the remaining δ_i are equal to $\bar{\delta}_i$



Phase 1: Random Gain Synthesis (RGS)

- Critical parameters are flight speed and take off mass
- Specification property

$$S_1 = \{K: A_c - B_c K \text{ satisfies the specs below}\}$$

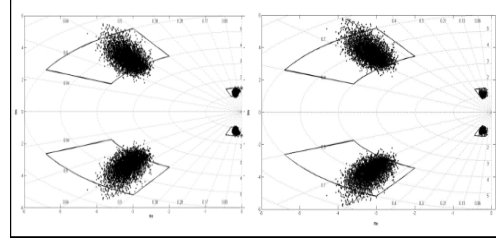
$$\omega_{SP} \in [4.0, 6.0] \text{ rad/s} \quad \zeta_{SP} \in [0.5, 0.9] \quad \omega_{PH} \in [1.0, 1.5] \text{ rad/s}$$

$$\zeta_{PH} \in [0.1, 0.3] \quad \Delta\omega_{SP} < \pm 45\% \quad \Delta\omega_{PH} < \pm 20\%$$

where ω and ζ are undamped natural frequency and damping ratio of the characteristic modes; $_{SP}$ and $_{PH}$ denote short period and phugoid mode



Specs in the Complex Plane



Volume of the Good Set

- Define a bounding set B of gains K

$$B = \{K: k_i \in [k_i^-, k_i^+], i = 1, \dots, 4\}$$

- Define the volume of the good set

$$\text{Vol}_{\text{good}} = \int_A dK$$

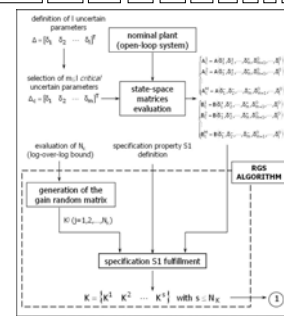
$$\text{where } A = \{K \in B \cap S_1\}$$

- Vol_B is simply the volume of the hyperrectangle B



Randomized Algorithm 1 (RGS)

- Uniform pdf for controller gains K in given intervals
- Accuracy and confidence $\varepsilon = 4 \cdot 10^{-5}$ and $\delta = 3 \cdot 10^{-4}$
- Number of random samples is computed with "Log-over-Log" Bound obtaining $N = 200,000$
- We obtained 5 gains K^i satisfying specification property S_1



Randomized Algorithm 1 (RGS)

Given $\varepsilon, \delta \in (0,1)$, RGS returns the set of gains $\{K^1, \dots, K^5\}$ satisfying S_1

1. Compute N using the Log-over-log Bound;
2. For fixed $j=1, 2, \dots, N$, generate uniformly the gain random matrix $K^j \in B$;
3. Set $C=0$;
4. For fixed $i=1, 2, 3, 4$, compute the closed-loop matrix $A_{cl}^i(K^j) = A_c^i - B_c^i K^j$;
 - if $K^j \in S_1$, set $C = C+1$;
 - otherwise, set $C = C$;
5. End;
6. If $C = 4$, return the gain K^j ;
7. Set $j = j+1$ and return to Step 2;
8. End



Random Gain Set

gain set	K_V	K_α	K_β	K_θ
K^1	0.00044023	0.09465000	0.01577400	-0.00473510
K^2	0.00021450	0.09581200	0.01555500	-0.00323510
K^3	0.00054999	0.09430800	0.01548200	-0.00486340
K^4	0.00010855	0.09183200	0.01530000	-0.00404380
K^5	0.00039238	0.09482700	0.01609300	-0.00417340



Phase 2: Random Stability Robustness Analysis (RSRA)

- Take $K_{rand} = K^i$ obtained in Phase 1
- Randomize Δ according to the given pdf and take N random samples Δ^i
- Specification property

$$S_2 = \{\Delta: A(\Delta) - B(\Delta) K_{rand} \text{ satisfies the specs of } S_1\}$$

- Computation of the empirical probability of stability $\hat{p}_N$



Empirical Probability

- Consider fixed gain K_{rand}
- Define the probability

$$p_{true} = \int_C p(\Delta) d\Delta$$
 where $C = \{\Delta \in B \cap S_2\}$ and $p(\Delta)$ is the given pdf
- Then, we introduce a "success" indicator function

$$I(\Delta) = 1 \text{ if } \Delta \in S_2$$
 or $I(\Delta) = 0$ otherwise
- The empirical probability for S_2 is given by

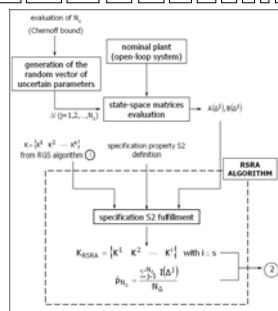
$$\hat{p}_N = N_{good}/N$$
 where N_{good} is equal to the number of successes



Randomized Algorithm 2 (RSRA)

- Take K_{rand} from Phase 1
- Accuracy and confidence

$$\varepsilon = \delta = 0.0145$$
- Number of random samples is computed with Chernoff Bound obtaining $N=5,000$
- Empirical probability is defined using an indicator function



Randomized Algorithm 2 (RSRA)

Given $\varepsilon, \delta \in (0,1)$, RSRA returns the empirical probability $\hat{p}_N$ that S_2 is satisfied for a gain K_{rand} provided by Algorithm 1

- Compute N using the Chernoff Bound;
- Generate N random vectors $\Delta^i \in B$ according to the given pdf;
- For fixed $j=1,2,\dots,N$, compute the closed-loop matrix

$$A_{cl}(\Delta^j) = A(\Delta^j) - B(\Delta^j)K_{rand};$$
 - if $A_{cl}(\Delta^j) \in S_2$, set $I(\Delta^j) = 1$;
 - otherwise, set $I(\Delta^j) = 0$;
- End;
- Return the empirical probability $\hat{p}_N$



Empirical Probability of Stability for Phase 2

gain set	empirical probability
K^1	88.56%
K^2	90.60%
K^3	89.31%
K^4	93.86%
K^5	85.14%



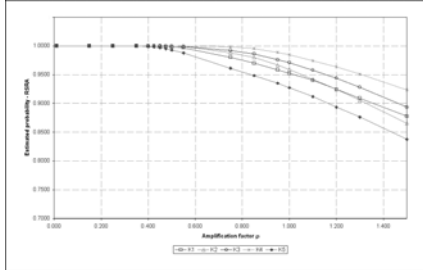
Probability Degradation Function

- Flight condition uncertainties are multiplied by the *amplification factor* $\rho > 0$ keeping the nominal value constant

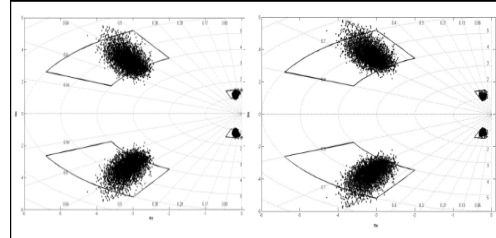
$$\delta_i \in \rho [\delta_i^-, \delta_i^+] \text{ for } i = 1, 2, \dots, 7$$
- No uncertainty affects the aerodynamic database, i.e.

$$\delta_i = \bar{\delta}_i \text{ for } i = 8, 9, \dots, 16$$
- For fixed $\rho \in [0,1.5]$ we compute the empirical probability for different gain sets K^i
- The plot empirical probability vs ρ is the probability degradation function

Probability Degradation Function for Phase 2



Root Locus Plot for Phase 2



Root locus for K^2 (left) and K^4 (right)

Phase 3: Random Performance Robustness Analysis (RPRA)

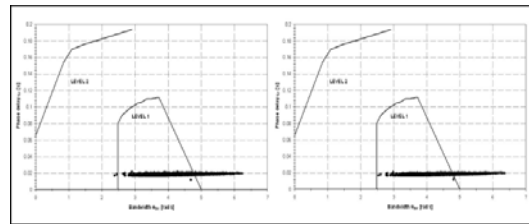
- This phase is similar to Phase 2, but military specs are considered (bandwidth criterion)
- Specification property
 $S_3 = \{\Delta: A(\Delta) - B(\Delta) K_{rand} \text{ satisfies the specs below}\}$

$$\omega_{BW} \in [2.5, 5.0] \text{ rad/s} \quad \tau_p \in [0.0, 0.5] \text{ s}$$

where ω_{BW} and τ_p are bandwidth and phase delay of the frequency response

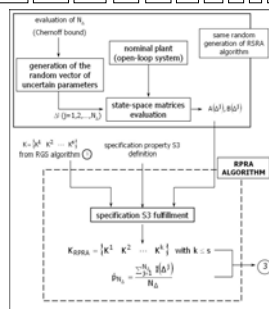
- Computation of the empirical probability that S_3 is satisfied

Bandwidth Criterion



Randomized Algorithm 3 (RPRA)

- Take K_{rand} from Phase 1
- Numer of random samples is computed with the Chernoff Bound obtaining $N=5,000$
- Empirical probability is defined using an indicator function



Randomized Algorithm 3 (RPRA)

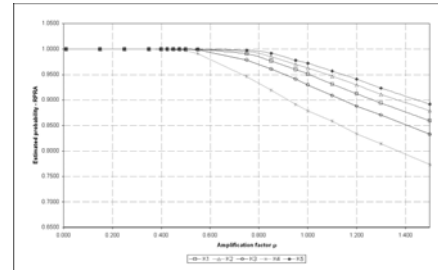
Given N and $A_{cl}(\Delta)$, $j=1,2,\dots,N$, provided by Algorithm 2, RPRA returns the empirical probability $\hat{p}_N$ that S_3 is satisfied for a gain K_{rand} provided by Algorithm 1

- For fixed $j=1,2,\dots,N$
 - if $A_{cl}(\Delta) \in S_3$, set $I(\Delta) = 1$;
 - otherwise, set $I(\Delta) = 0$;
- End;
- Return the empirical probability $\hat{p}_N$

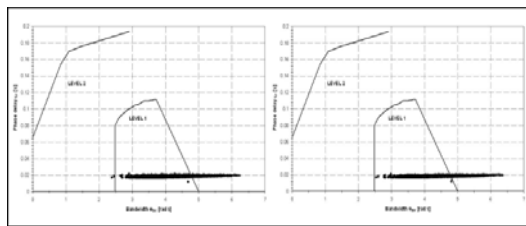
Empirical Probability of Performance for Phase 3

gain set	empirical probability
K^1	93.58%
K^2	95.16%
K^3	90.80%
K^4	84.78%
K^5	96.06%

Probability Degradation Function for Phase 3



Bandwidth Criterion for Phase 3



Bandwidth criterion for K^1 (left) and K^3 (right)

Gain Selection

- Multi-objective criterion as a compromise between different specifications

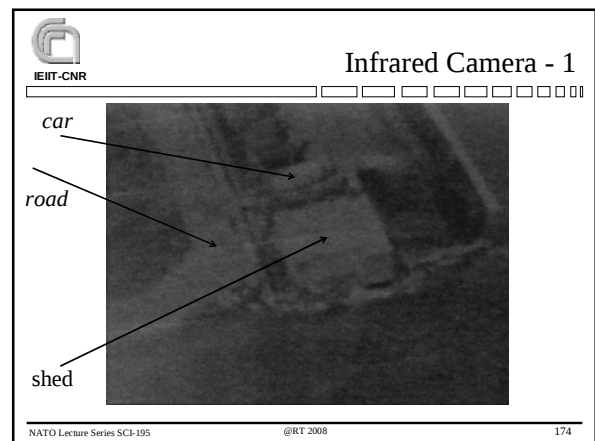
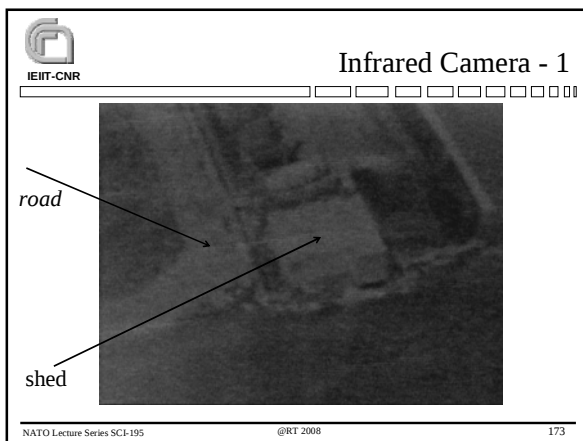
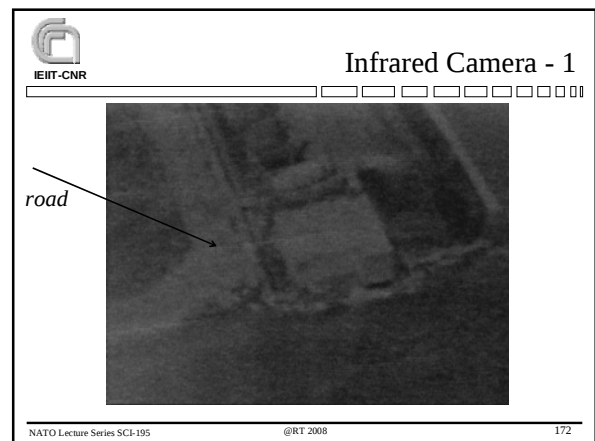
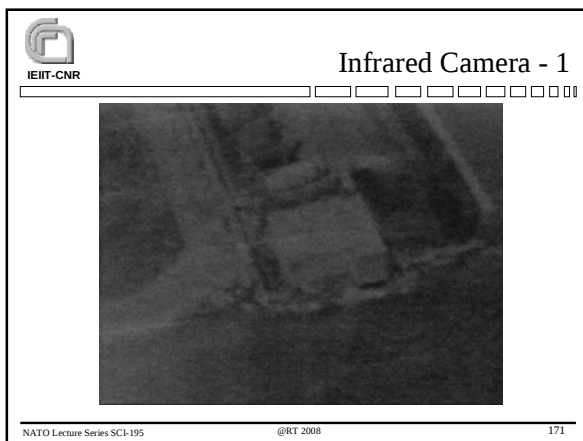
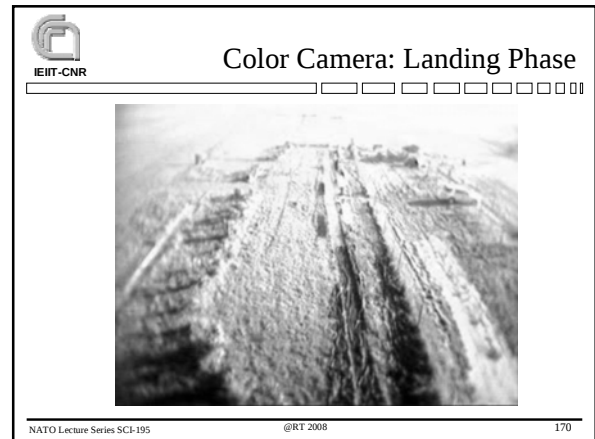
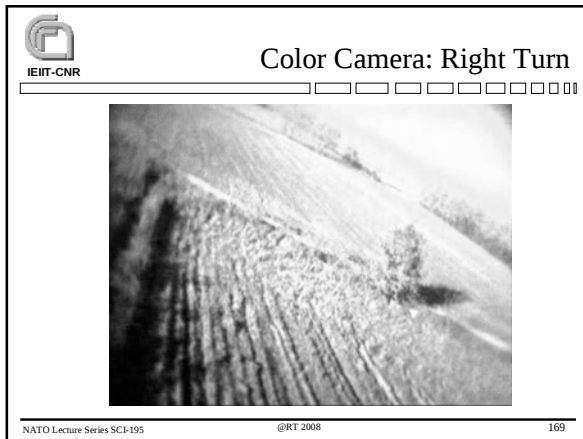
Finally we selected gain K^1 as the best compromise between all the specs and criteria!

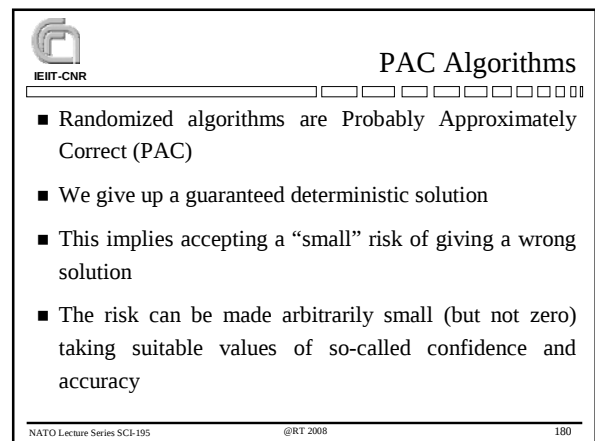
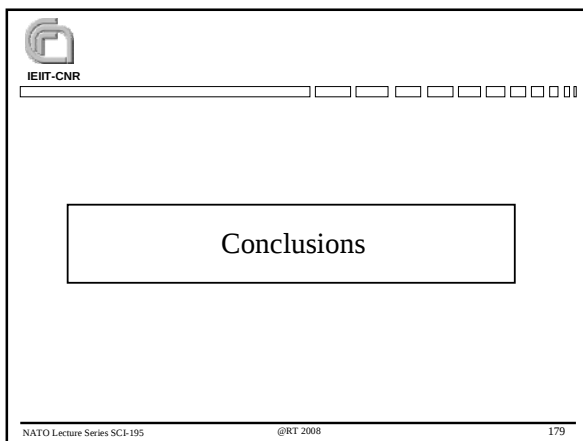
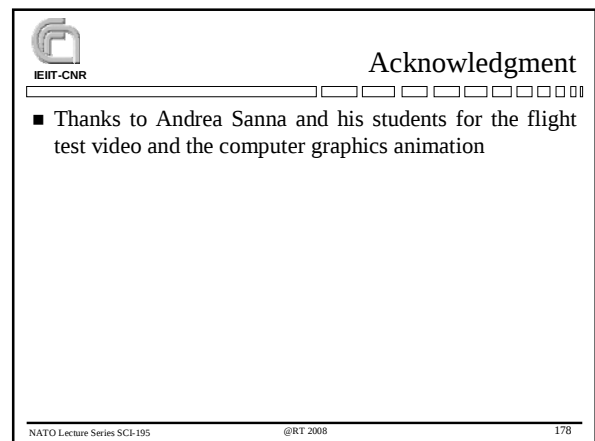
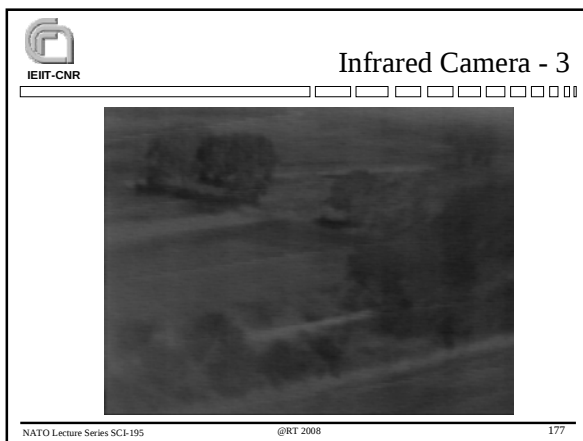
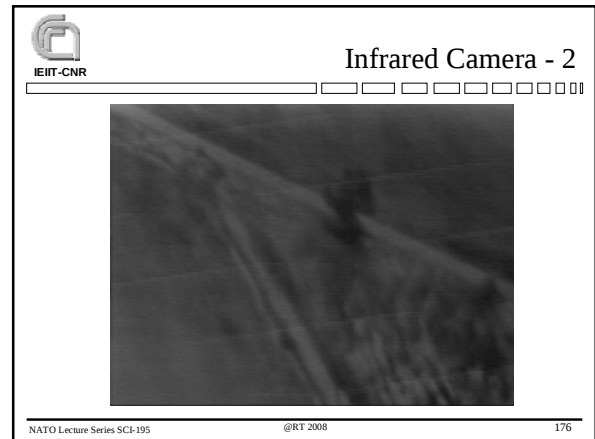
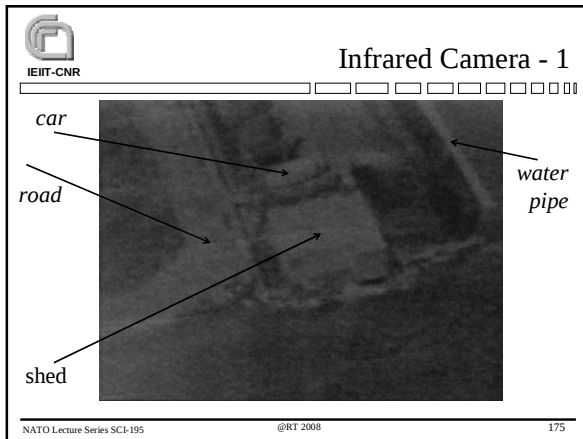
Conclusions: Flight Tests in Sicily - 1

- Evaluation of the payload carrying capabilities and autonomous flight performance
- Mission test involving altitude, velocity and heading changing was performed in Sicily
- Checking effectiveness of the control laws for longitudinal and lateral-directional dynamics
- Flight control design based on RAs for stabilization and guidance

Conclusions: Flight Tests in Sicily - 2

- Satisfactory response of MH1000
- Possible improvements by iterative design procedure
- Stability of the platform is crucial for the video quality and in the effectiveness of the surveillance and monitoring tasks







PAC Algorithms

- Two open problems
- Optimization with sequential methods
- Derive “reasonable” bounds for the statistical learning theory approach