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Characterizing the vibration of an elastically point supported rectangular plate using eigensensitivity analysis

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ABSTRACT

Normalized frequencies are computed for a rectangular, isotropic plate resting on elastic supports. The normalized frequencies are determined using eigensensitivity analysis, which approximates the eigenparameters in a Mauclarin series, yielding an approximate closed-form expression. One benefit of the approximate closed-form expression is its computational efficiency and yet another is its application of re-analysis. Accuracy of the approximate expression is assessed by comparing results with the widely used Rayleigh–Ritz method using orthogonal polynomials and beam shape functions in both approaches. Consideration for a variety of edge conditions is given through a combination of simply supported, clamped and free boundary conditions. Results indicate that the accuracy of higher frequencies computed by the sensitivity approach is highly dependent upon choice of basis function.

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1. Background

Point supported plates are plates that have prescribed displacements at a number of discrete locations within its domain. Rectangular, circular as well as elliptical plates represent geometries of interest. Rigid point supports have a prescribed displacement of zero while elastically point supported have displacements dependent of the stiffness at the support. Electronic circuit boards, solar panels, and concrete slabs represent applications that can be effectively modeled as point supported plates. A brief overview of recent contributions is presented. Altekin [1] consider both bucking and free vibration of elliptical plates which have point supports along the symmetric diagonals. A Ritz approach is used to solve for the fundamental frequency and critical buckling load. Huang et al. [2] studied the free vibration of tapered isotropic plates on rigid point supports. A Mindlin plate theory is used and a Green's function approach is used to generate the governing equations. Yu [3] used the method of superposition established by Gorman [4] to analyze the free vibration of cantilever plates containing an attached mass. Zhou et al. [5] utilized a three-dimensional theory to study the frequency analysis of both isotropic and composites plates. A finite layer formulation is used to model the structure and a hybrid basis function is introduced to adequately satisfy the displacement constraints at the point supports.

Sensitivity analysis seeks to assess the effect of a parameter on the response of a system. Application of sensitivity analysis covers all fields of economics, business, science, mathematics, and engineering. Structural eigensensitivity analysis provides a direct method to assess the effect of system parameters on the eigenvalues, typically the frequencies and buckling loads. An early contributor in this area of research was Hearmon [6] who developed a one-term formula to approximate the fundamental frequency of orthotropic plates. Bert [7] studied the optimal design of composite plates for maximum fundamental frequency. Barton and Reiss [8] provided approximate closed-formed formula for uni-axial and bi-axial buckling of symmetric composite plates. Recently Barton [9][10] has applied this technique to investigate the bucking of isotropic plates subject to combined in-plane loading and the thermal buckling of composite plates with clamped-free boundary conditions.

In this paper, eigensensitivity analysis is used to determine an approximate closed-form expression which is used to compute frequencies of a square isotropic plate resting on four elastic point supports. A combination of boundary conditions is considered including simply supported, clamped and free. The article is organized into three sections beginning the problem formulation, an overview of the eigensensivity approach, and the results and discussion section.

2. Problem formulation

There are many ways to formulate the equations that govern the free vibration of the elastically point supported plate including the Newtonian mechanics and the principle of work-

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energy. Here the principle of minimum potential energy is preferred.

The total potential energy Π for the system containing elastic supports and discrete masses is given as

$$\Pi = U - T \tag{1}$$

Here U contains the strain energy of the plate and the supporting springs given by

$$U = \frac{D}{2}\iint_R \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\upsilon)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dx\,dy \right\}$$
$$+ \left\{ \iint_R \sum_{p=1}^{N} k_p \delta(x-x_p)\delta(y-y_p) w^2(x,\ y)\,dx\,dy \right\} \tag{2}$$

where D is the flexural stiffness, k_p is the stiffness of the pth spring, υ is Poisson's ratio and $w(x,y)$ is the transverse displacement. The kinetic energy consists of energy from the plate and from added discrete masses. Here T is given by

$$T_{plate} = \frac{\rho h \omega^2}{2}\iint_R w^2(x,\ y)\,dx\,dy + \frac{\omega^2}{2}\iint_R \sum_{q=0}^{N} m_q \delta(x-x_q)\delta(y-y_q) w^2(x,\ y)\,dx\,dy \tag{3}$$

Above m_q is the qth discrete mass. In the paper, all stiffnesses are taken to be equal as k_p and all masses are taken to be equal as m.

The displacement $w(x,y)$ may be expressed as

$$w(x,\ y) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\phi_i\left(\frac{x}{a}\right)\theta_j\left(\frac{y}{b}\right) \tag{4}$$

where $\phi_i(x/a)$ and $\theta_j(y/b)$ are shape function in the x- and y-directions, respectively. The shape functions are selected to satisfy kinematic and static boundary conditions. The class of functions can be beam shape or polynomials. In fact, Smith et al. [11] considered the buckling of isotropic plates under shear loading and solved the corresponding problem using the Rayleigh–Ritz method. These authors considered a number of orthogonal polynomials as basis functions including Chebyshev type-1 and type-2, Legendre, Hermite, and Laguerre.

The governing differential equation comes by extremizing Eq. (1) with respect to the kinematic variable $w(x,y)$. Substituting Eq. (4) into Eq. (1) and minimizing the total potential energy with respect to the coefficients w_{ij}, results in the eigenvalue problem of

$$\sum_m \sum_n (K_{ijmn} - \omega^2 M_{ijmn}) w_{mn} = 0 \tag{5}$$

where

$$K_{ijmn} = \frac{abD}{a^4}(A_{im}b_{jn} + \upsilon R^2(C_{im}c_{jn} + C_{mi}c_{nj}) + 2(1-\upsilon)R^2 E_{im}e_{jn} + R^4 B_{im}a_{jn}) + k\Gamma_{ijmn}$$
$$M_{ijmn} = MB_{im}b_{jn} + m\Gamma_{ijmn} \tag{6}$$

and $\Gamma_{ijmn} = \sum_{p=1}^{N} \phi_i(x_p)\theta_j(y_p)\phi_m(x_p)\theta_n(y_p)$ is the product of the basis functions and is evaluated where the springs and masses are located. Several parameters are used in the above definitions including R which is the aspect ratio given as a/b and a set of matrices called boundary condition matrices. These boundary condition matrices are defined by

$$\begin{array}{ll} A_{pm} = (\phi''_p,\ \phi''_m) & a_{qn} = (\theta''_q,\ \theta''_n) \\ B_{pm} = (\phi_p,\ \phi''_m) & b_{qn} = (\theta_q,\ \theta_n) \\ C_{pm} = (\phi''_p,\ \phi_m) & c_{qn} = (\theta''_q,\ \theta_n) \\ E_{pm} = (\phi'_p,\ \phi'_m) & e_{pm} = (\theta'_p,\ \theta'_m) \end{array} \tag{7}$$

where the prime represents the spatial derivative and $(\cdot,\cdot)$ is used to represent the L_2 inner product on [0,1]. The above equations for K_{ijmn} and M_{ijmn} are the quite general. They are independent of any particular set of basis functions. Therefore they can be used if ϕ_p and θ_q are kinematically admissible polynomials, beam shape functions, or any other set of kinematically admissible functions.

3. Sensitivity analysis

A complete presentation of the sensitivity approach used in this paper can be referenced in Barton and Reiss [8]. Eq. (5), written as

$$[K]\{\alpha_{ij}\} = \lambda_{ij}[M]\{\alpha_{ij}\} \tag{8}$$

provides the basis to apply the sensitivity approach. Here we define $\{\alpha_{ij}\}$ as the (i,j)th component of the eigenvector and λ_{ij} is its corresponding eigenvalue. An approximate expression for the eigenvalue λ_{ij} can be determined by introducing parameters S_1 and S_2 into Eq. (8) and considering

$$[\hat{K}(S_1)]\{\hat{\alpha}_{ij}(S_1,S_2)\} = \hat{\lambda}_{ij}(S_1,S_2)[\hat{M}(S_2)]\{\hat{\alpha}_{ij}(S_1,S_2)\} \tag{9}$$

where

$$\left[\begin{array}{l} [\hat{K}(S_1)] = [K_D] + S_1[\Delta K] \\ [\hat{M}(S_2)] = [M_D] + S_2[\Delta M] \end{array}\right.$$

Here $[K_D]$ and $[M_D]$ are a diagonal matrices obtained from $[K]$ and $[M]$ by deleting all off-diagonal elements; $[\Delta K]$ and $[\Delta M]$ are matrices which have zeros on the diagonal and contain only the off-diagonal elements of $[K]$ and $[M]$, respectively. The parameters S_1 and S_2 take on values of either 0 or 1. If both S_1 and S_2=0, the solution to Eq. (9) becomes the ratio of the diagonal elements of the stiffness matrix $[K_D]$ and matrix $[M_D]$. If both S_1 and S_2=1, then the original eigenvalue problem, Eq. (9) is recovered. The desired eigenvalue λ_{mn} is obtained by expanding $\hat{\lambda}_{mn}$ in a Maclaurin series about (S_1,S_2)=(0,0) and evaluating at (S_1,S_2)=(1,1). Thus

$$\lambda_{ij} = \hat{\lambda}_{ij}(1,1) \cong \hat{\lambda}_{ij}(0,0) + \delta\hat{\lambda}_{ij}(0,0) + \frac{1}{2}\delta^2\hat{\lambda}_{ij}(0,0) \tag{10}$$

The expressions appearing on the right-hand side of Eq. (10) were presented in [8] and are

$$\begin{aligned} \hat{\lambda}_{ij}(0,\ 0) &= \frac{K_{ijij}}{M_{ijij}} \\ \delta\hat{\lambda}_{ij}(0,\ 0) &= 0 \\ \delta^2\hat{\lambda}_{ij}(0,\ 0) &= -\frac{2}{M_{ijij}{}^2}\sum_{k\neq i}\sum_{l\neq j}\left\{\frac{\left[K_{ijij}\Delta M_{klij} - M_{ijij}\Delta K_{klij}\right]^2}{K_{klkl}M_{ijij} - K_{ijij}M_{klkl}}\right\} \end{aligned} \tag{11}$$

Substituting Eq. (11) into Eq. (10) provides the desired quadratic approximate closed-form expression of

$$\lambda_{ij} = \frac{K_{ijij}}{M_{ijij}} - \frac{1}{M_{ijij}{}^2}\sum_{k\neq i}\sum_{l\neq j}\left\{\frac{\left[K_{ijij}\Delta M_{klij} - M_{ijij}\Delta K_{klij}\right]^2}{K_{klklq}M_{ijij} - K_{ijijn}M_{klkl}}\right\} \tag{12}$$

4. Discussion

In this section, the general form of the approximate closed-form expression, Eq. (11), is presented for combinations of simply supported, clamped or free boundary conditions and for two types of basis function. Boundary conditions specific results are then evaluated based upon chosen basis functions.

Eq. (12) is specialized for a plate consisting of four springs of equal stiffness and up to four equal masses can be determined by first evaluating the stiffness and mass matrix elements given through Eq. (6) and then substituting these results into Eq. (12). Evaluating Eq. (6) requires determining the diagonal matrices for

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both [K] and [M] which are

$$K_{ijij} = (A_{ii}b_{jj} + 2\upsilon R^2 C_{ii}c_{jj} + 2(1-\upsilon)R^2 E_{ii}e_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2 R}{D}\Gamma_{ijij}$$
$$M_{ijij} = B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij} \tag{13}$$

and computing the off-diagonal matrices for both [K] and [M] are

$$\Delta K_{ijkl} = \upsilon R^2(C_{ik}c_{jl} + C_{ki}c_{lj}) + 2(1-\upsilon)R^2 E_{ik}e_{jl} + \frac{k_p a^2 R}{D}\Gamma_{ijkl}$$
$$\Delta M_{ijkl} = \frac{m}{M}\Gamma_{ijkl} \tag{14}$$

Substituting both Eqs. (13) and (14) into the general expression given in Eq. (12) yields

$$\begin{aligned}\frac{\omega_{ij}^2 a^4 \rho h}{D} = {} & \left[A_{ii}b_{jj} + 2\upsilon R^2 C_{ii}c_{jj} + 2(1-\upsilon)R^2 E_{ii}e_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right] \times \left[B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right]^{-1} \\ & - \frac{1}{\left(B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right)^2} \times \sum_{k\neq i}^{\infty}\sum_{l\neq j}^{\infty}\left\{\left\{\left[A_{ii}b_{jj} + 2\upsilon R^2 C_{ii}c_{jj} + 2(1-\upsilon)R^2 E_{ii}e_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right]^2\right.\right. \\ & \times \left[\frac{m}{M}\Gamma_{klij}\right]^2 - 2 \times \left[A_{ii}b_{jj} + 2\upsilon R^2 C_{ii}c_{jj} + 2(1-\upsilon)R^2 E_{ii}e_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right] \\ & \times \left[\frac{m}{M}\Gamma_{klijn}\right] \times \left[B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right] \times \left[\upsilon R^2(C_{ik}c_{jl} + C_{ki}c_{lj}) + 2(1-\upsilon)R^2 E_{ik}e_{jl} + \frac{k_p a^2}{D}\Gamma_{klij}^{(j)}\right] \times \left[B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right]^2 \\ & \left.\times \left[\upsilon R^2(C_{ik}c_{jl} + C_{ki}c_{lj}) + 2(1-\upsilon)R^2 E_{ik}e_{jl} + \frac{k_p a^2}{D}\Gamma_{klij}\right]^2\right\} \\ & \times \left\{\left[A_{kk}b_{ll} + 2\upsilon R^2 C_{kk}c_{ll} + 2(1-\upsilon)R^2 E_{kk}e_{ll} + R^4 B_{kk}a_{ll} + \frac{k_p a^2}{D}\Gamma_{klkl}\right]\right. \\ & \left.\left.\times \left[B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right] - \left[A_{ii}b_{jj} + 2\upsilon R^2 C_{ii}c_{jj} + 2(1-\upsilon)R^2 E_{ii}e_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right] \times \left[B_{kk}b_{ll} + \frac{m}{M}\Gamma_{klkl}\right]\right\}^{-1}\right\}\end{aligned} \tag{15}$$

Eq. (15) is valid for any combination of simply supported, clamped or free boundaries. The formula requires identifying an admissible basis and evaluating the boundary matrices given in Eq. (7). Selecting orthogonal polynomials as the basis requires numerical evaluation of the boundary matrices and, as a result, no simplification occurs in the form of Eq. (15). If free boundary conditions are excluded, then the following results holds $E_{pm} = -C_{pm}$ and $e_{pm} = -c_{pm}$. Eq. (15) then becomes

$$\begin{aligned}\frac{\omega_{ij}^2 a^4 \rho h}{D} = {} & \left[A_{ii}b_{jj} + 2R^2 C_{ii}c_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right] \\ & \times \left[B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right]^{-1} - \frac{1}{\left(B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right)^2} \\ & \times \sum_{k\neq i}^{\infty}\sum_{l\neq j}^{\infty}\left\{\left\{\left[A_{ii}b_{jj} + 2R^2 C_{ii}c_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right]^2\right.\right. \\ & \times \left[\frac{m}{M}\Gamma_{klij}\right]^2 - 2 \times \left[A_{ii}b_{jj} + 2R^2 C_{ii}c_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right] \\ & \times \left[\frac{m}{M}\Gamma_{klijn}\right] \times \left[B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right] \times \left[2R^2 C_{ki}c_{lj} + \frac{k_p a^2}{D}\Gamma_{klij}^{(j)}\right] \\ & \left. + \left[B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right]^2 \times \left[2R^2 C_{ki}c_{lj} + \frac{k_p a^2}{D}\Gamma_{klij}\right]^2\right\} \\ & \times \left\{\left[A_{kk}b_{ll} + 2R^2 C_{kk}c_{ll} + R^4 B_{kk}a_{ll} + \frac{k_p a^2}{D}\Gamma_{klkl}\right]\right. \\ & \times \left[B_{ii}b_{jj} + \frac{m}{M}\Gamma_{ijij}\right] - \left[A_{ii}b_{jj} + 2R^2 C_{ii}c_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right] \\ & \left.\left.\times \left[B_{kk}b_{ll} + \frac{m}{M}\Gamma_{klkl}\right]\right\}^{-1}\right\}\end{aligned} \tag{16}$$

Simplification to Eq. (16) for the case of no added discrete masses occurs by setting $m=0$ resulting in

$$\begin{aligned}\frac{\omega_{ij}^2 a^4 \rho h}{D} = {} & \left[A_{ii}b_{jj} + 2R^2 C_{ii}c_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right] \\ & \times [B_{ii}b_{jj}]^{-1} - \sum_{k\neq i}^{\infty}\sum_{l\neq j}^{\infty}\left\{\left\{\left[2R^2 C_{ki}c_{lj} + \frac{k_p a^2}{D}\Gamma_{klij}\right]^2\right\}\right. \\ & \times \left\{\left[A_{kk}b_{ll} + 2R^2 C_{kk}c_{ll} + R^4 B_{kk}a_{ll} + \frac{k_p a^2}{D}\Gamma_{klkl}\right]\right. \\ & \left.\left.\times [B_{ii}b_{jj}] - \left[A_{ii}b_{jj} + 2R^2 C_{ii}c_{jj} + R^4 B_{ii}a_{jj} + \frac{k_p a^2}{D}\Gamma_{ijij}\right] \times [B_{kk}b_{ll}]\right\}^{-1}\right\}\end{aligned} \tag{17}$$

and if both added discrete mass, $m=0$, and elastic support stiffness, $k_p=0$, Eq. (17) reduces to

$$\begin{aligned}\frac{\omega_{ij}^2 a^4 \rho h}{D} = {} & [A_{ii}b_{jj} + 2R^2 C_{ii}c_{jj} + R^4 B_{ii}a_{jj}] \times [B_{ii}b_{jj}]^{-1} \\ & - \sum_{k\neq i}^{\infty}\sum_{l\neq j}^{\infty}\{[2R^2 C_{ki}c_{jli}]^2\} \times \{[A_{kk}b_{ll} + 2R^2 C_{kk}c_{ll} + R^4 B_{kk}a_{ll}] \\ & \times [B_{ii}b_{jj}] - [A_{ii}b_{jj} + 2R^2 C_{ii}c_{jj} + R^4 B_{ii}a_{jj}] \times [B_{kk}b_{ll}]\}^{-1}\}\end{aligned} \tag{18}$$

Specific results can now be evaluated for a variety of support conditions including simply support on all sides, simply supported on opposite sides, and clamped on all sides.

5. Simply supported plates (S–S–S–S)

The first plate to be considered is one containing any number of discrete masses and equal elastic supports such that the ith spring coordinates lie within $0 < x_i < a$ and $0 < y_i < b$. For a plate simply supported on all sides, the boundary condition in the x and y-direction is given by

$$w = 0 \quad M_{xx} = 0 \quad x = 0,\ a$$
$$w = 0 \quad M_{yy} = 0 \quad y = 0,\ b$$

Selecting orthonormal polynomials [11] as the basis functions requires numerically evaluating the boundary matrices. These results are recalled when needed to evaluate Eq. (16). The

Gramm-Schmidt process is used to construct an ortho-normal set after assuming the initial element to be

$$\phi_0(x)=x-2x^3+x^4 \quad \theta_0(y)=y-2y^3+y^4$$

Choosing beam shape functions allows for an analytical evaluation of the boundary matrices which can be directly substituted into Eq. (16). To this end, select the basis function as

$$\phi_m\left(\frac{x}{a}\right)=\sqrt{2}\,\mathrm{Sin}\left(\frac{m\pi x}{a}\right) \quad \theta_n\left(\frac{y}{b}\right)=\sqrt{2}\,\mathrm{Sin}\left(\frac{n\pi y}{b}\right) \tag{19}$$

Evaluating the boundary matrices given by Eq. (7) provides

$$\begin{aligned} A_{im}&=i^4\pi^4\delta_{im} & a_{jn}&=j^4\pi^4\delta_{jn}\\ B_{im}&=\delta_{im} & b_{jn}&=\delta_{jn}\\ C_{im}&=-i^2\pi^2\delta_{im} & c_{jn}&=-i^2\pi^2\delta_{im}\end{aligned} \tag{20}$$

and substituting these into Eq. (16) yields the desired approximate closed-form expression for simply supported boundary conditions on all sides.

$$\begin{aligned}\frac{\omega_{ij}^2a^4\rho h}{D}&=\pi^4\left[i^4+2R^2i^2j^2+R^4n^4+\frac{k_pa^2}{D}\Gamma_{ijij}\right]\\ &\times\left[1+\frac{m}{M}\Gamma_{ijij}\right]^{-1}-\frac{\pi^4}{(1+(m/M)\Gamma_{ijij})^2}\\ &\times\sum_{k\neq i}^{\infty}\sum_{l\neq j}^{\infty}\left\{\left\{\left[i^4+2R^2i^2j^2+R^4j^4+\frac{k_pa^2}{\pi^4D}\Gamma_{ijij}\right]^2\right.\right.\\ &\times\left[\frac{m}{M}\Gamma_{klij}\right]^2-2\times\left[i^4+2R^2i^2j^2+R^4j^4+\frac{k_pa^2}{\pi^4D}\Gamma_{ijij}\right]\\ &\times\left[\frac{m}{M}\Gamma_{klij}\right]\times\left[1+\frac{m}{M}\Gamma_{ijij}\right]\times\left[\frac{k_sa^2}{\pi^4D}\Gamma_{klij}\right]+\left[1+\frac{m}{M}\Gamma_{ijij}\right]^2\\ &\times\left.\left[\frac{k_pa^2}{\pi^4D}\Gamma_{klij}\right]^2\right\}\times\left\{\left[k^4+2R^2k^2l^2+R^4l^4+\frac{k_pa^2}{\pi^4D}\Gamma_{klkl}\right]\right.\\ &\times\left[1+\frac{m}{M}\Gamma_{ijij}\right]-\left[i^4+2R^2i^2j^2+R^4j^4+\frac{k_pa^2}{\pi^4D}\Gamma_{ijij}\right]\\ &\times\left.\left.\left[1+\frac{m}{M}\Gamma_{klkl}\right]\right\}^{-1}\right\}\end{aligned} \tag{21}$$

A simplification to Eq. (21) for the case of no added discrete masses produces

$$\begin{aligned}\frac{\omega_{ij}^2a^4\rho h}{D}&=\pi^4\left[i^4+2R^2i^2j^2+R^4j^4+\frac{k_pa^2}{D}\Gamma_{mnmn}\right]\\ -4\pi^4&\times\sum_{k\neq i}^{\infty}\sum_{l\neq j}^{\infty}\frac{[(k_pa^2/\pi^4D)\Gamma_{klij}]^2}{[(k^4-i^4)+2R^2(k^2l^2-i^2j^2)+R^4(l^4-j^4)+(k_sa^2/\pi^4D)(\Gamma_{klkl}-\Gamma_{ijij})]}\end{aligned} \tag{22}$$

and if both added discrete mass m=0 and elastic support stiffness k_p=0, Eq. (22) reduces to the well known equation for the frequency of a simply supported, isotropic plate.

$$\frac{\omega_{ij}^2a^4\rho h}{D}=\pi^4\left[i^4+2R^2i^2j^2+R^4j^4\right]$$

6. Opposite edges simply supported

In a similar manner, an expression for a plate with two opposite edges simply supported in the x-direction and clamped in the y-direction requires identifying the following beam shape functions

$$\phi_m\left(\frac{x}{a}\right)=\sqrt{2}\,\mathrm{Sin}\left(\frac{m\pi x}{a}\right)$$

$$\theta_n\left(\frac{y}{b}\right)=\mathrm{Cosh}\left(\frac{\lambda_ny}{b}\right)-\mathrm{Cos}\left(\frac{\lambda_ny}{b}\right)-\sigma_n\left(\mathrm{Sinh}\left(\frac{\lambda_ny}{b}\right)-\mathrm{Sin}\left(\frac{\lambda_ny}{b}\right)\right) \tag{23}$$

and

$$\mathrm{Cos}(\lambda_n)\mathrm{Cosh}(\lambda_n)=1$$

$$\sigma_n=\frac{\mathrm{Cosh}(\lambda_n)-\mathrm{Cos}(\lambda_n)}{\mathrm{Sinh}(\lambda_n)-\mathrm{Sin}(\lambda_n)}$$

then the boundary matrices, Eq. (7) become

$$\begin{aligned} A_{im}&=i^4\pi^4\delta_{im} & a_{jn}&=\lambda_j^4\delta_{jn}\\ B_{im}&=\delta_{im} & b_{jn}&=\delta_{jn}\\ C_{im}&=-i^2\pi^2\delta_{im} & c_{jn}&=c_{jn}\end{aligned}$$

and Eq. (16) becomes

$$\begin{aligned}\frac{\omega_{ij}^2a^4\rho h}{D}&=\left[i^4\pi^4-2\pi^2i^2R^2c_{jj}+R^4\lambda_j^4+\frac{k_pa^2}{D}\Gamma_{ijij}\right]\\ &\times\left[1+\frac{m}{M}\Gamma_{ijij}\right]^{-1}-\frac{1}{(1+(m/M)\Gamma_{ijij})^2}\\ &\times\sum_{k\neq i}^{\infty}\sum_{l\neq j}^{\infty}\left\{\left\{\left[i^4\pi^4-2\pi^2i^2R^2c_{jj}+R^4\lambda_j^4+\frac{k_sa^2}{D}\Gamma_{ijij}\right]^2\right.\right.\\ &\times\left[\frac{m}{M}\Gamma_{klij}\right]^2-2\times\left[i^4\pi^4-2\pi^2i^2R^2c_{jj}+R^4\lambda_j^4+\frac{k_pa^2}{D}\Gamma_{ijij}\right]\\ &\times\left[\frac{m}{M}\Gamma_{klij}\right]\times\left[1+\frac{m}{M}\Gamma_{ijij}\right]\times\left[-2R^2k^2\pi^2\delta_{ki}c_{lj}+\frac{k_pa^2}{D}\Gamma_{klij}\right]\\ &+\left.\left[1+\frac{m}{M}\Gamma_{ijij}\right]^2\times\left[-2R^2k^2\pi^2\delta_{ki}c_{lj}+\frac{ka^2}{D}\Gamma_{klij}\right]^2\right\}\\ &\times\left\{\left[k^4\pi^4-2\pi^2k^2R^2c_{ll}+R^4\lambda_l^4+\frac{k_pa^2}{D}\Gamma_{klkl}\right]\right.\\ &\times\left[1+\frac{m}{M}\Gamma_{ijij}\right]-\left[i^4\pi^4-2\pi^2i^2R^2c_{jj}+R^4\lambda_j^4+\frac{k_{ps}a^2}{D}\Gamma_{ijij}\right]\\ &\times\left.\left.\left[1+\frac{m}{M}\Gamma_{klkl}\right]\right\}^{-1}\right\}\end{aligned} \tag{24}$$

Plates which are simply supported in the x-direction and free in y-directions have boundary conditions given by

$$\begin{aligned}&w=0 && M_{xx}=0 && x=0,\ a\\ &\frac{\partial^2w}{\partial y^2}+\upsilon\frac{\partial^2w}{\partial x^2} && \frac{\partial^3w}{\partial y^3}+(2-\upsilon)\frac{\partial^3w}{\partial x^2\partial y} && y=0,\ b\end{aligned}$$

and following beam shape functions of

$$\phi_m\left(\frac{x}{a}\right)=\sqrt{2}\,\mathrm{Sin}\left(\frac{m\pi x}{a}\right)$$

$$\theta_n\left(\frac{y}{b}\right)=\mathrm{Cosh}\left(\frac{\lambda_ny}{b}\right)+\mathrm{Cos}\left(\frac{\lambda_ny}{b}\right)-\sigma_n\left(\mathrm{Sinh}\left(\frac{\lambda_ny}{b}\right)+\mathrm{Sin}\left(\frac{\lambda_ny}{b}\right)\right)$$

are used to satisfy them. As in the case of a plate with clamped conditions along the y-direction, the coefficients defined above hold for this case of free boundary in along the y-direction. The boundary matrices give through Eq. (8) become

$$\begin{aligned} A_{im}&=i^4\pi^4\delta_{im} & a_{jn}&=\lambda_j^4\delta_{jn}\\ B_{im}&=\delta_{im} & b_{jn}&=\delta_{jn}\\ C_{im}&=-i^2\pi^2\delta_{im} & c_{jn}&=(\theta''_j,\ \theta_n)\\ E_{im}&=im\pi^2\delta_{im} & e_{jn}&=(\theta'_j,\ \theta'_n)\end{aligned}$$

As before substituting the boundary matrices into Eq. (16) provides

$$\frac{\omega_{ij}^2a^4\rho h}{D}=\left[i^4\pi^4-2\upsilon R^2i^2\pi^2(e_{jj}+c_{jj})+2R^2i^2\pi^2e_{jj}+R^4\lambda_j^4+\frac{k_pa^2}{D}\Gamma_{ijij}\right]$$

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$$
\times\left[b_{jj}+\frac{m}{M}\Gamma_{ijij}\right]^{-1}-\frac{1}{(b_{jj}+(m/M)\Gamma_{ijij})^2}
\times\sum_{k\neq i}^{\infty}\sum_{l\neq j}^{\infty}\left\{\left\{\left[i^4\pi^4-2\upsilon R^2 i^2\pi^2(e_{jj}+c_{jj})+2R^2 i^2\pi^2 e_{jj}+R^4\lambda_j^4+\frac{k_p a^2}{D}\Gamma_{ijij}\right]^2\right.\right.
\times\left[\frac{m}{M}\Gamma_{klij}\right]^2-2\times\left[i^4\pi^4-2\upsilon R^2 i^2\pi^2(e_{jj}+c_{jj})+2R^2 i^2\pi^2 e_{jj}+R^4\lambda_j^4+\frac{k_p a^2}{D}\Gamma_{ijij}\right]
\times\left[\frac{m}{M}\Gamma_{klijn}\right]\times\left[b_{jj}+\frac{m}{M}\Gamma_{ijij}\right]\times\left[\frac{ka^2}{D}\Gamma^{(j)}_{klij}\right]+\left[b_{jj}+\frac{m}{M}\Gamma_{ijij}\right]^2\times\left.\left[\frac{ka^2}{D}\Gamma_{klij}\right]^2\right\}
\times\left\{\left[k^4\pi^4-2\upsilon R^2 k^2\pi^2(e_{ll}+c_{ll})+2R^2 k^2\pi^2 e_{ll}+R^4\lambda_l^4+\frac{k_p a^2}{D}\Gamma_{klkl}\right]\right.
\times\left[b_{jj}+\frac{m}{M}\Gamma_{ijij}\right]-\left[i^4\pi^4-2\upsilon R^2 i^2\pi^2(e_{jj}+c_{jj})+2R^2 i^2\pi^2 e_{jj}+R^4\lambda_j^4+\frac{k_p a^2}{D}\Gamma_{ijij}\right]
\left.\left.\times\left[b_{ll}+\frac{m}{M}\Gamma_{klkl}\right]\right\}^{-1}\right\} \quad (25)
$$

For all other combination of boundary conditions, clamped in the x-direction and free in y-direction for instance, Eq. (16) will be used since no simplification occurs after evaluating it with the boundary matrices.

7. Results

In this section, numerical results are presented for the derived approximate closed-form results and compared to results generated using the Rayleigh–Ritz method for the elastically supported plate. Under consideration is a square plate 30 in $\times$ 30 in $\times$ 0.5 in with Young's modulus E and Poisson's ratio υ of 10×10^6 psi and 0.315, respectively. Each of the fours springs has the same stiffness of 130 lb/in. Choice of spring placement and configuration is infinite. Three possible plate configurations are shown in Fig. 1 with corresponding spring coordinates given in Table 1. Configuration A has springs placed along the main dialogues, B has springs centered on each edge and C has springs lumped at the center of the plate. Tabulated results are presented for configuration A.

Accuracy of the approximate formula is compared with the Ritz method. Frequencies are first computed using orthogonal polynomials as basis function and then compared to results using beam shape functions. For convenience, introduce the normalized frequency parameter

$$
k_i=\frac{\omega_i a^2}{2\pi}\sqrt{\frac{\rho h}{D}}
$$

which identifies the ith normalized Ritz frequency and $\hat{k}_{ij}$ is used to identify (i,j)th normalized frequency computed using the quadratic approximate expression.

Table 2 presents convergence results for k_1 and $\hat{k}_{11}$ for all boundary conditions using orthogonal polynomials as basis functions and Table 3 presents the same data using beam shape functions instead as the basis functions. Eq. (16) was used to generate the approximate closed-form results. Configuration A without any added discrete masses was used to generate results in all the tables. In general the closed-form expression accurately predicts the fundamental frequency for all boundary conditions using either type of basis function. The largest percent difference in the two results occurs for the SFSF boundary condition. Although negligible, these are 0.5% and 0.73% using the orthogonal polynomials and beam shape functions, respectively.

Tables 4–8 present the first five frequencies computed by both methods. Again both orthogonal polynomials and beam shape

Table 1
Coordinates for spring placement (inches).

Spring	Configuration A		Configuration B		Configuration C	
	X	Y	X	Y	X	Y
1	7.5	7.5	15	5	15	15
2	22.5	7.5	25	15	15	15
3	22.5	22.5	15	25	15	15
4	7.5	22.5	5	15	15	15

Table 2
Convergence results for Ritz and closed-form expression using orthogonal polynomials.

	SSSS		CCCC		SCSC		SFSF		CFCF	
N	Ritz	Quad	Ritz	Quad	Ritz	Quad	Ritz	Quad	Ritz	Quad
1	3.149	3.149	5.730	5.730	4.609	4.609	–	–	–	–
3	3.148	3.148	5.728	5.728	4.608	4.608	2.565	2.565	4.214	4.214
5	3.147	3.147	5.727	5.727	4.608	4.608	2.564	2.564	4.213	4.213
7	3.147	3.147	5.727	5.727	4.608	4.608	2.550	2.553	4.194	4.196
9	3.147	3.147	5.727	5.727	4.608	4.608	2.550	2.543	4.190	4.176
11	3.147	3.147	5.727	5.727	4.608	4.608	2.550	2.535	4.188	4.189

Table 3
Convergence results for Ritz and closed-form expression using beam shape functions.

	SSSS		CCCC		SCSC		SFSF		CFCF	
N	Ritz	Quad	Ritz	Quad	Ritz	Quad	Ritz	Quad	Ritz	Quad
1	3.158	3.158	5.752	5.752	4.624	4.624	2.858	2.858	4.382	4.382
3	3.158	3.158	5.736	5.735	4.617	4.617	2.779	2.773	4.323	4.318
5	3.158	3.158	5.733	5.732	4.617	4.617	2.754	2.757	4.305	4.306
7	3.158	3.158	5.733	5.732	4.616	4.616	2.742	2.752	4.297	4.303
9	3.158	3.158	5.732	5.732	4.616	4.616	2.735	2.750	4.292	4.301
11	3.158	3.158	5.732	5.731	4.616	4.616	2.729	2.749	4.288	4.300

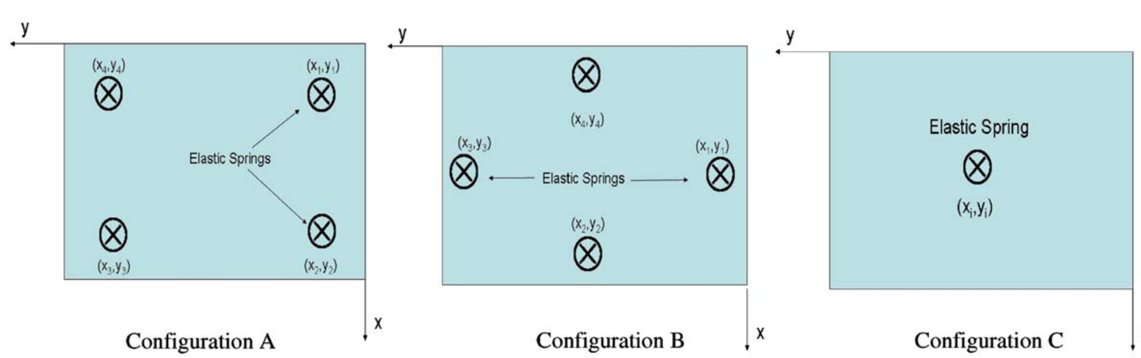

Fig. 1. Spring locations.

Table 4
Comparison of normalized Ritz frequencies with closed-formed approximate frequencies for SSSS boundary condition (Hertz).

j	k_j		$\hat{k}_{1j}$		$\hat{k}_{2j}$		$\hat{k}_{3j}$	
	Poly	Beam	Poly	Beam	Poly	Beam	Poly	Beam
1	3.147	3.158	3.147	3.158	7.254	7.867	9.423	15.71
2	7.854	7.867	7.254	7.867	11.80	12.58	15.68	20.42
3	7.859	7.867	9.424	15.71	15.68	20.42	21.54	28.27
4	12.56	12.58	–	26.70	8.622	31.42	20.66	39.27
5	15.71	15.71	–	40.84	–	45.55	–	53.41

Table 5
Comparison of normalized Ritz frequency with quadratic approximate equation for CCCC boundary condition, N=11, k=1612.5.

j	k_j		$\hat{k}_{1j}$		$\hat{k}_{2j}$		$\hat{k}_{3j}$	
	Poly	Beam	Poly	Beam	Poly	Beam	Poly	Beam
1	5.727	5.732	5.727	5.731	11.67	11.85	20.89	20.99
2	11.68	11.85	11.67	11.85	17.22	17.58	26.21	26.43
3	11.68	11.85	20.89	20.99	26.21	26.43	34.97	34.94
4	17.22	17.58	32.92	33.52	38.23	38.80	46.95	47.05
5	21.94	20.94	46.83	49.20	53.30	54.41	61.22	62.50

Table 6
Comparison of normalized Ritz frequency with quadratic approximate equation for SCSC boundary condition.

j	k_j		$\hat{k}_{1j}$		$\hat{k}_{2j}$		$\hat{k}_{3j}$	
	Poly	Beam	Poly	Beam	Poly	Beam	Poly	Beam
1	4.608	4.616	4.608	4.616	8.168	8.724	10.20	16.28
2	8.713	8.722	11.03	11.213	14.74	15.26	18.35	22.52
3	11.03	11.21	20.43	20.55	24.39	24.63	29.09	31.77
4	15.05	15.26	32.57	33.17	36.80	37.32	42.38	44.47
5	16.26	16.27	46.53	48.91	51.11	53.14	57.48	60.33

Table 7
Comparison of normalized Ritz frequency with quadratic approximate equation for SFSF boundary condition using beam shape functions.

j	k_j	$\hat{k}_{1j}$	$\hat{k}_{2j}$	$\hat{k}_{3j}$
1	2.729	2.749	7.363	15.17
2	7.352	7.204	11.48	18.98
3	7.545	10.25	13.80	20.67
4	12.06	17.61	20.32	25.65
5	14.27	22.20	24.61	29.63

Table 8
Comparison of normalized Ritz frequency with quadratic approximate equation for CFCF boundary condition using beam shape functions.

j	k_j	$\hat{k}_{1j}$	$\hat{k}_{2j}$	$\hat{k}_{3j}$
1	4.29	4.30	10.81	20.09
2	8.41	8.06	14.21	23.30
3	10.80	10.87	16.16	24.70
4	14.76	17.94	21.92	28.88
5	14.84	22.46	25.97	32.53

functions are used as basis functions. All results are generated for configuration A. The number of terms taken in the displacement expansion corresponds to N=11. Frequencies from the Ritz analysis are ordered beginning with the fundamental frequency, the lowest numerical result. In order to compare frequencies computed using the closed-form expression, several modes corresponding combinations of indices i and j must be computed. For the most part, $k_1=\hat{k}_{11}$, $k_2=\hat{k}_{12}$, $k_3=\hat{k}_{21}$, $k_4=\hat{k}_{22}$, and $k_5=\hat{k}_{31}$.

Table 4 provides normalized frequencies for the simply supported plate. The first column report frequencies computed using the Ritz approach using both types of basis functions while columns two through four report frequencies using the closed-form expressions given in Eq. (16), for orthogonal polynomials and Eq. (21) for beam shape functions. Some frequencies computed using the closed-form equation with orthogonal polynomials, such as $\hat{k}_{14}$, are not real. This is due to two factors. First $\delta^2\lambda_{14}$, which contains products of the off-diagonal elements of $[K]$, are not small and are greater than the zeroth-order term. Secondly, the boundary matrix (ϕ''_1, ϕ''_4), which contributes to the off-diagonal components of K, is not small. When beam shape functions are used necessarily $(\phi''_i, \phi''_j)=\mu_i^4\delta_{ij}$ where μ_i is the beam's ith natural frequency. For this boundary condition, the approximate closed-form expression Eq. (21) which uses beam shape functions compute more accurately frequencies when compared with the Ritz approach using beam shape functions than does the closed-form expression Eq. (16) which uses orthogonal polynomials. The difference in results is more exaggerated when considering higher modes. From Table 4 using orthogonal polynomials, the fourth Ritz mode, k_4, is 12.56 Hz and the fourth mode $\hat{k}_{22}$ predicted by Eq. (18) is 11.80 Hz resulting in a 6.1% difference. Comparing the same cases using beam shape functions shows no difference.

Table 5 presents results for a plate fully clamped on all sides. Eq. (16) was used to generate results for both types of basis functions. As before, the closed-form expression better predicts the Ritz frequencies using beam shape function than when using orthogonal polynomials. A percent difference of 4.7% occurs in the comparison of the fifth frequency using polynomials and only 0.24% occurs when using beam shape functions. Table 6 shows the first of three results for mixed boundary conditions, simply supported in the x-direction and clamped in the y-direction. Eq. (23) is used to generate all numerical results. The largest percent difference occurs again when using the polynomials as basis. Here a 12.9% difference occurs in the computation of the fifth frequency. Percent differences of 6.3% and 2.1%, respectively occur in the second and fourth frequencies. The maximum percent difference of 0.06% occurs in the fifth frequency using beam shape functions.

Tables 7 and 8 show results for the remaining two mixed boundary condition cases which includes a free edge in the y-direction. Previous results indicate that beam shape functions are better suited for the closed-form expression, especially when considering higher frequencies. Therefore only beam shape function results are shown. For the SFSF boundary condition, Table 7, k_1 is approximated by $\hat{k}_{11}$ and differs by 0.73%, k_2 is approximate by $\hat{k}_{12}$ and differs by 2.01%, k_3 is approximated by $\hat{k}_{21}$ differing by 2.4%, k_4 is approximated by $\hat{k}_{22}$ differing by 4.8%, and k_5 is approximated by $\hat{k}_{23}$ and differs by 3.3%. For the SFSF boundary condition, Table 7, k_1 is approximated by $\hat{k}_{11}$ and differs by 0.23%, k_2 is approximate by $\hat{k}_{12}$ and differs by 4.2%, k_3 is approximated by $\hat{k}_{21}$ differing by 0.09%, k_4 is approximated by $\hat{k}_{22}$ differing by 3.7%, and k_5 is approximated by $\hat{k}_{23}$ and differs by 8.9%.

8. Conclusion

In this paper approximate closed-form formulas were developed to determine the vibration of elastically supported plates. A combination of edge support conditions were investigated indcluding simply supported, clamped and free. Both orthogonal polynomials and beam shape functions were used as admissible basis functions. When compared to Rayleigh–Ritz results, the computed fundamental frequency, for all boundary conditions, using the approximate closed-form expressions are excellent. The accuracy of higher frequencies, computed using the approximated closed-form expression is dependent upon boundary condition and choice of basis functions. A maximum percent difference of 0.06% occurs using beam shape functions while 12.9% occurs using the orthogonal polynomials for the SCSC boundary condition. For other mixed boundary conditions, which includes a free edge, percent differences in the fifth mode of 4.8% and 8.9% occurs for the SFSF and CFCF boundary conditions, respectively.

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