High Frequency Acoustic Propagation using Level Set Methods

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[*] ***Abstract* – This paper describes a new approach to numerical solution of the high frequency approximation to the wave equation. Traditional solutions to the Eikonal equation in high frequency acoustics are obtained via ray tracing. In ray tracing, the numerical grid on which solutions are computed becomes distorted over time as rays diverge from the initial wavefront, reducing the accuracy of the solution. The level set method is a fixed grid method, whereby the user controls the underlying grid, and hence the accuracy, of the solution.**

I. Introduction

Accurate and computationally efficient simulation of physical processes is key to reducing the need for expensive and environmentally risky at-sea system level experiments and data collection. Level Set Methods (LSM) are generic, computational techniques developed by Osher and Sethian [1] for tracking the evolution of moving curves and surfaces. Traditionally, propagation models use ray tracing to solve via method of characteristics. When rays (characteristics) diverge, eventually they do not cover enough of the physical space so that accurate, well-resolved solutions are not available on any uniform grid. For this reason, there has been interest in developing methods for solving general propagation problems in a fixed frame of reference. The advantage of this approach is that standard partial differential equation (PDE) solvers, e.g., finite difference approximations, can be employed to solve the problem on uniform grids which may then be refined to produce higher accuracy. LSM achieve this by representing the propagating surface as the zero level set of a higher dimensional function. This function, say $\phi(\mathbf{x},t)$, is designed so that $\phi(\mathbf{x},0)=0$ for **x** on the initial propagating surface in phase space. This zero level set is then transported via the underlying velocity field. In the case of acoustic propagation in isotropic media, the propagation direction is normal to the propagating surface (wavefront). LSM are attractive due to their robustness – they easily handle the evolving topology of the surface being tracked, the normal vector and curvature can be extracted at any point of the front from the level set function (provided the normal and curvature are well-defined at that point), and it is straightforward to extend the theory to higher dimensions.

Software packages based on several computational approaches in addition to ray tracing already exist which can accurately solve the equations of acoustic propagation. For instance, to compute the full wave equation solution, one can use normal modes, or Parabolic Equation (PE) methods. However, when computational expense is at a premium, such as for real-time simulations, these are not appropriate methods especially at high frequencies where required grid sizes become very large (in one dimension, we need at least 2 points per wavelength to resolve the wave). Ray tracing is therefore the current standard for high frequency propagation modeling. LSM may provide an alternative to ray tracing for solving the high frequency approximation to the wave equation that allows the simulation user greater control over the accuracy of the solutions.

This work builds upon the foundation established by Osher, Cheng, Kang, Shim, and Tsai in [2] in which a basic level set method for geometric optics is introduced. Eulerian geometric optics has been a topic of intense research in the scientific computing community for quite some time. Benamou [3] provides an overview of approaches to this problem. The most similar to the level set method is the segment projection method [4] in which wavefronts are tracked in phase space as projections onto each two dimensional subspace of the three dimensional phase space. This method is effective, but requires complicated bookkeeping in order to reconstruct wavefronts. The approach of Osher, et al. propagates the wavefront in the full phase space. Reference [5] builds upon [2] by extending the method to propagation in anisotropic materials. In [6], an efficient method for incorporating reflecting boundaries is introduced.

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In this paper, we apply these foundations to the specific problem of high frequency acoustics. In section II, we provide an overview of the method and a description of the implementation. In section III, we present some preliminary results demonstrating the algorithm's performance in a few sample cases including varying sound speed profiles and reflecting boundaries.

II. Description of approach

A. Level Set Methods

LSM were originally developed for solving Hamilton-Jacobi type equations. A Hamilton-Jacobi equation is a first-order nonlinear PDE having the form $u_t(\mathbf{x},t)+H(\mathbf{x},\nabla u)=0$, where H is a nonlinear function. In the case of the Eikonal equation derived from the high frequency approximation to the wave equation, $H(\mathbf{x},\nabla u)=\pm c(\mathbf{x})|\nabla u|$, with the nonnegative function $c(\mathbf{x})$ specifying the propagation speed of the medium; the sign ambiguity refers to the direction of propagation (inward or outward from the initial condition). For simplicity, we only consider the case $H(\mathbf{x},\nabla u)=+c(\mathbf{x})|\nabla u|$.

Two difficulties must be addressed when solving the Eikonal equation in a fixed frame of reference: first, the solutions may be multi-valued (e.g., caustics), and second, solving over the entire spatial grid implies substantially increased requirements for storage and processing. The first issue is resolved by considering the wavefront in a higher dimensional reduced phase space. In two spatial dimensions, for instance, the phase space is four-dimensional, but this can be reduced to three dimensions since only the direction of propagation is important. This means that the wavefront is represented in space, time, and a local phase direction. The wavefront is embedded in two level set functions, φ and ψ, which evolve in phase space according to a first order system of transport equations

$$\begin{cases}\phi_t + V\cdot\nabla\phi = 0\\ \psi_t + V\cdot\nabla\psi = 0\end{cases}, \tag{1}$$

where the velocity field for the motion is given by

$$V=[c\cdot\cos\theta,\, c\cdot\sin\theta,\, \frac{\partial c}{\partial x}\sin\theta-\frac{\partial c}{\partial z}\cos\theta]^T \quad , \tag{2}$$

as derived in [4], and $c = c(\mathrm{x,z})$ is the medium sound speed. The phase space variable θ is the angle of the local ray direction (for an isotropic medium). The wavefront is propagated in this higher dimensional space as a system of evolving surfaces, where solutions are single-valued (Fig. 1).

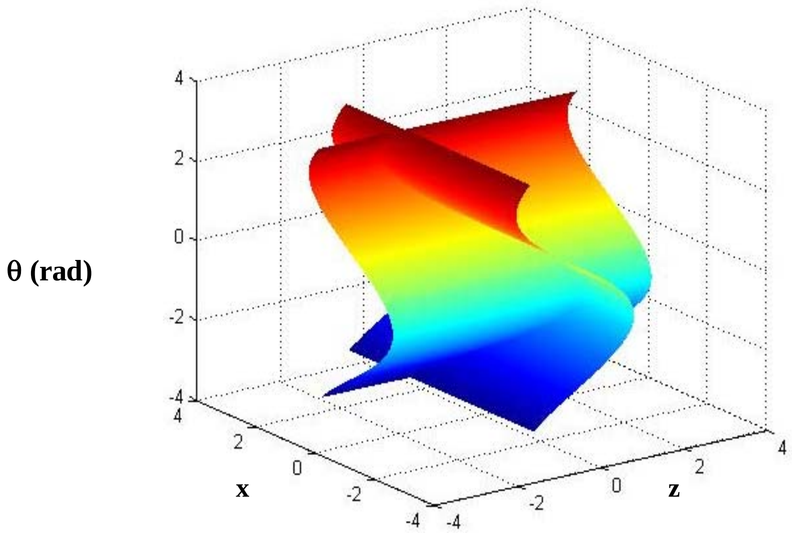

Figure 1: Implicit representation of higher-dimensional wavefront – the wavefront is found at the intersection of the evolving zero-level-set surfaces; the horizontal axes represent physical space while the vertical corresponds to an extended phase space dimension.

The resulting physical wavefront is recovered by finding the intersection of the zero level set surfaces of φ and ψ and, and then projecting the resulting curve onto the horizontal (physical) plane as in Fig. 2. Since higher dimensional spaces are required, memory issues must be carefully addressed. However, by selectively updating the system only locally near the wavefront and by making use of efficient numerical methods, the burden on both storage and processor time can be greatly reduced. This is the basis for the local level set method [7].

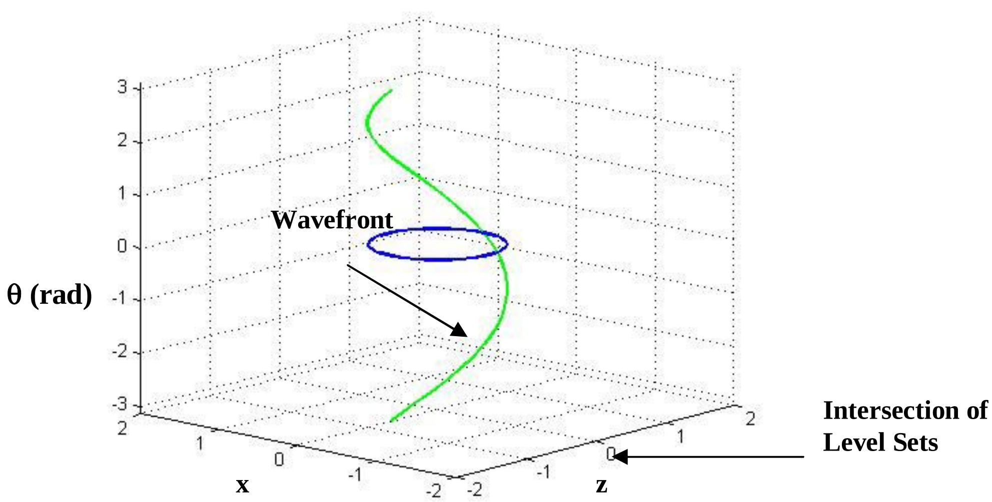

Figure 2: Construction of the wavefront – the physical wavefront is found by projecting the curve of intersection of the two zero level set surfaces (as can be seen in Fig. 1) from the higher dimensional representation onto the two-dimensional spatial plane. As can be seen here, the wavefront is circular since this simple case uses an isospeed medium.

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B. Implementation

The advantage of fixed-grid methods is that they allow for ready application of standard partial differential equation solvers which have well-established convergence properties, e.g., finite differences or finite elements. The level set equations themselves comprise a decoupled system of simple first order hyperbolic problems. For such problems, we focus on upwind schemes, which take advantage of knowledge of the direction of propagation given by the known velocity field *V*. Here, we employ a semi-Lagrangian method to solve the transport equation. Cheng uses a similar method in [9] to implement his generic three-dimensional level set code.

There are two steps involved in our implementation. In the first step, a standard ordinary differential equation (ODE) solver, such as a Runge-Kutta method, is used to step backward along the characteristics, which flow according to the known velocity field V given in (2), one time step at each grid point in phase space. This gives an approximate solution to where the value on the grid at the current time came from. Since these points are not necessarily in the grid itself, an interpolation method is used to find the value of the function at that point, which provides the solution at the current time on the grid. For example in our case, using backward Euler for the time stepping, assuming a constant sound speed *c* and time step Δt, we solve for $\phi^n_{ijk} \approx \phi(x_i, z_j, \theta_k, t^n)$ on the grid $\left(\{x_i\}_{i=1}^{N_x}, \{z_j\}_{j=1}^{N_z}, \{\theta_k\}_{k=1}^{N_\theta}\right)$ where *x* and *z* are the physical space variables, with θ as the phase space variable, as follows:

$$\text{Step1: Compute } \left\langle \begin{array}{l} x^* = x_i - c\Delta t\cos(\theta_k) \\ z^* = z_j - c\Delta t\sin(\theta_k) \\ \theta^* = \theta_k \end{array} \right. .$$

$$\text{Step 2: Set } \phi^n_{ijk} = \phi^{n-1}(x^*, z^*, \theta^*)$$

In step 2, the quantity $\phi^{n-1}(x^*, z^*, \theta^*)$ is estimated via linear interpolation on the data $\{\phi^{n-1}_{ijk}\}$. These steps are applied to both level set functions ϕ and ψ. Further interpolation is then used to find the intersection of the zero level sets and recover the wavefront. Higher order methods can be applied here, however the stability result may not hold. Note also that if the speed function $c(x,z)$ is nonlinear, an iterative method such as Newton-Raphson is required to solve step 1.

This approach has two important benefits. One is that it effectively decouples the grid points from one another in the time stepping portion, making this part of the code readily parallelizable, as well as allowing great flexibility in the choice of points at which to solve. This becomes critical for a three dimensional application where reduced phase space has five dimensions and computation must be focused near the wavefront. The other significant advantage is that this method is unconditionally stable – although this result depends on the interpolation scheme used in step 2. That is, it does not suffer from a time-step restriction (i.e., the Courant-Friedrichs-Levy stability condition) as do standard Eulerian finite difference methods, and thus larger time steps can be taken without sacrificing stability; this could be critical for a real-time implementation as accuracy will have to be sacrificed for speed.

For boundary conditions, if using finite difference methods, one-sided formulas can be derived at the boundaries to the desired order of accuracy. For the semi-Lagrangian method however, an inflow condition must be defined for the case when information is transported into the domain from outside. If the boundary is purely reflecting, all information is within the domain. We currently use this type of condition for the bottom and surface boundaries; since it is a high frequency algorithm, transmission into the bottom is not of significant concern. The reflection condition is implemented by setting $\phi^n(x_i, z_b, \theta_{refl}) = \phi^n(x_i, z_b, \theta_{inc})$ on the boundary points, z_b, where θ_{inc} is the incident angle computed via Snell's law from the reflected angles, θ_{refl}, in the grid. It is not physical to assume reflection on the left and right domain boundaries, so in order to compute a solution on the boundary at incoming angles, we find the time and location at which the boundary was crossed, and assume the solution is constant beyond that point. This is a numerical scheme that works as long as all of the important physical action is sufficiently far away from that boundary. One nice feature of the level set method is that what happens to the level set functions away from the zero level sets themselves is largely irrelevant. As long as the source is inside the domain, approximations made at inflow do not have a significant effect.

III. Results

We next present some preliminary results obtained with our algorithm. All of the plots were generated from our Matlab® prototype code, using the semi-Lagrangian method to solve the relevant differential equations. The first case (Fig. 3) uses an isospeed profile using a sound speed c = 1500 m/s and no boundaries. The circular wavefront gradually expands and exits the domain. Fig. 3a displays snapshots of the wavefronts at a few time steps and Fig. 3b shows the rays extracted from the level set results. As expected in an isospeed medium, the rays appear as straight lines.

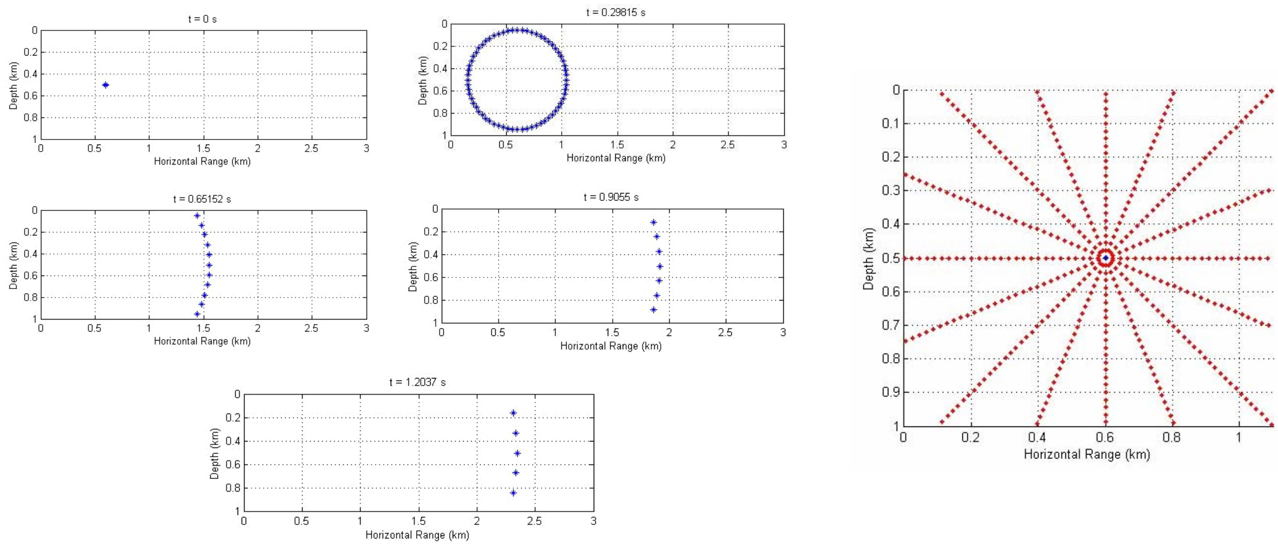

Figure 3: a) Wavefronts in an isospeed medium with no boundaries; b) Rays extracted from a)

In Fig. 4a, the wavefronts are depicted for a similar case, but this time with a linear sound speed profile that gradually increases from 1500 m/s to 2000 m/s (see Fig. 5). The propagation effects are subtle, but we observe the wavefront expanding faster at greater depths thus distorting the wavefront from its original circular form. The ray trace for this case, Fig. 4b, shows the rays curving slightly downwards in the direction of increasing propagation speed. A few “outlier” points appear on the ray trace; we suspect that these transient errors are due to inaccuracies in the contour finding routine we are using. They have not affected the overall solution, so are not indicative of any underlying instability.

The next examples include reflecting boundaries at the surface and bottom of the domain for a simplistic shallow water model. Again we present cases with an isospeed medium (1500 m/s) and with the linear profile depicted in Fig. 5. In Fig. 6a, the initial wavefront expands and eventually reflects off of the surface and bottom boundaries, presenting a nice symmetry. In Fig. 6b, the ray trace shows the correct pattern with straight lines reflected back symmetrically. The linear profile example is more interesting; again the subtle bending of the rays is observable as is a small distortion in the expanding wavefront pattern in Fig. 7.

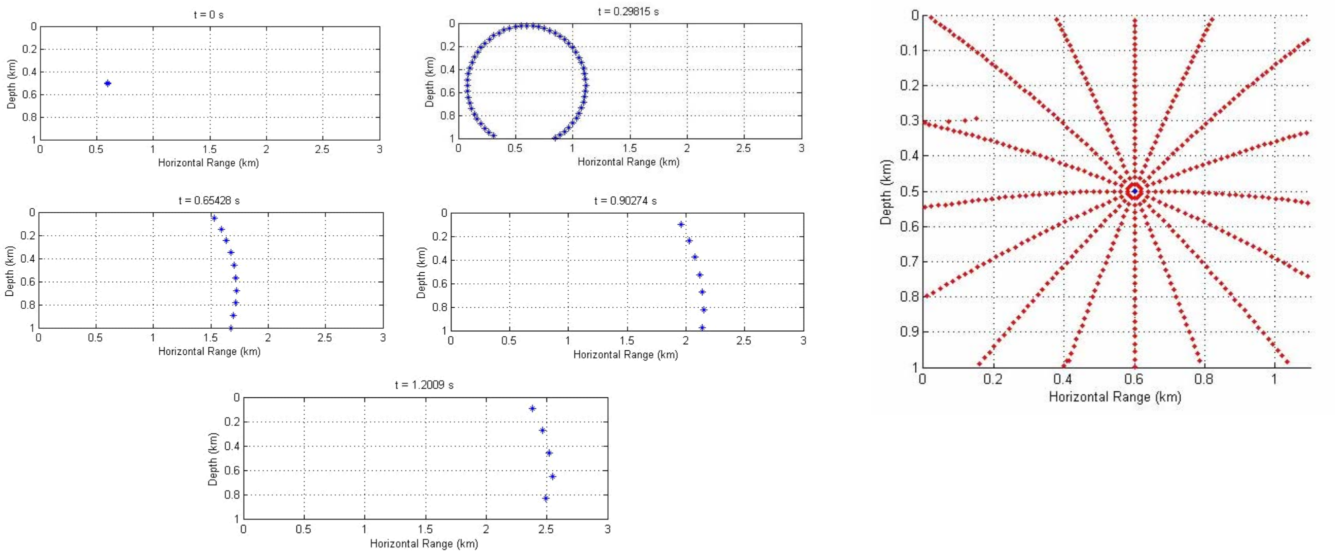

Figure 4: a)Wavefronts in medium with linear sound speed profile and no boundaries; b) Corresponding ray trace

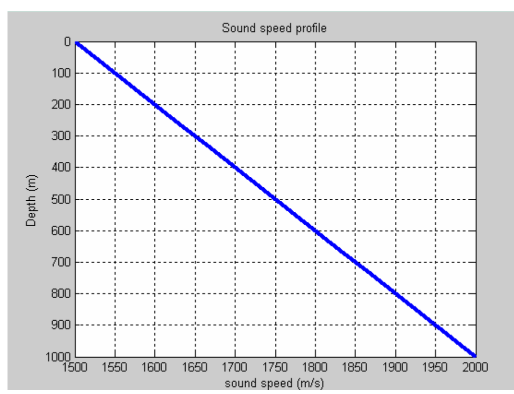

Figure 5: Sound speed profile for the linear sound speed case (Figs. 4,7)

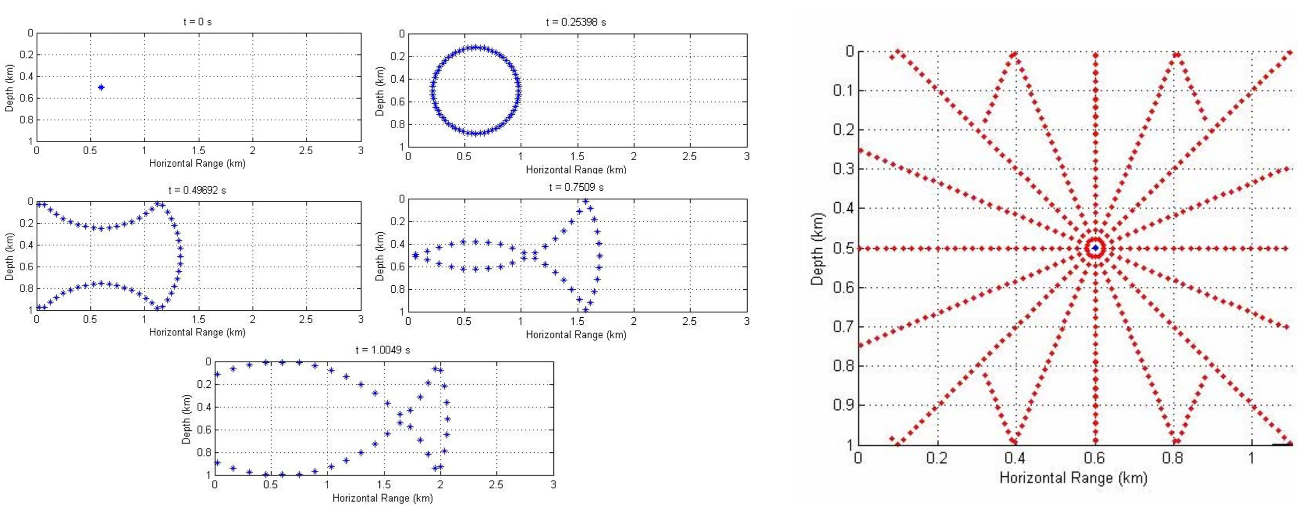

Figure 6: a) Wavefronts in isospeed medium and reflecting boundaries at the surface and bottom; b) Corresponding ray trace

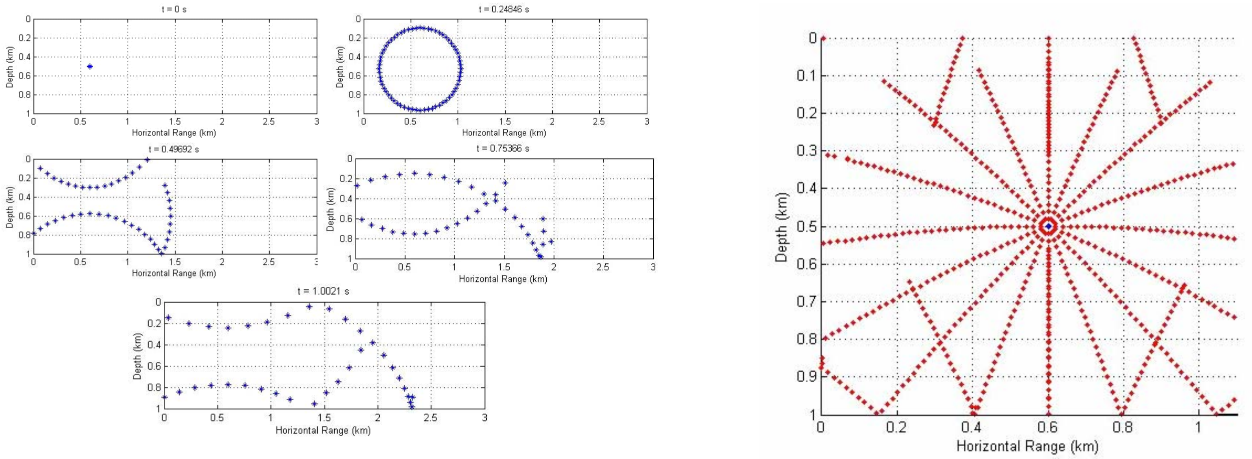

Figure 7: a) Wavefronts in medium with linear sound speed profile and reflecting boundaries at the surface and bottom; b) Corresponding ray trace

IV. CONCLUSION AND FUTURE PLANS

In this work, we have outlined a novel approach to solving the high frequency wave equation of acoustic propagation. The demonstrations of simple propagation models in section III above suggest that this method holds promise. There still is a need to address the errors that appear in the non-constant sound speed case; future plans include incorporating more specialized interpolation algorithms for identifying the level set surfaces and their intersections. This will also be key to obtaining good results with more general geometry. Once this portion of work is complete, a thorough validation against known results will ensue.

The goal of this project is to build a prototype software package that will combine the basic framework of LSM with realistic surface and bottom models, appropriate loss models, and will be optimized for computational speed. It is also desirable to extend this to allow for stochastic boundary conditions (e.g., the ocean surface), and propagation through random media. Existing simulations involving underwater acoustics assume a deterministic acoustic response, and if any variability is added, it is often post-processed, hence not capturing its true nature. Since the variability originates in the environment, this is where it should be accounted for.

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