Enumerative Algebraic Geometry of Conics

Andrew Bashelor, Amy Ksir, and Will Traves

1. INTRODUCTION. In 1848 Jakob Steiner, professor of geometry at the University of Berlin, posed the following problem [**19**]: Given five conics in the plane, are there any conics that are tangent to all five? If so, how many are there? Problems that ask for the number of geometric objects with given properties are known as enumerative problems in algebraic geometry. The tools developed to solve these problems have been used in many other situations and reveal deep and beautiful geometric phenomena.

In this expository paper, we describe the solutions to several enumerative problems involving conics, including Steiner's problem. The results and techniques presented here are not new; rather, we use these problems to introduce and demonstrate several of the key ideas and tools of algebraic geometry. The problems we discuss are the following: Given p points, l lines, and c conics in the plane, how many conics are there that contain the given points, are tangent to the given lines, and are tangent to the given conics? It is not even clear *a priori* that these questions are well-posed. The answers may depend on which points, lines, and conics we are given. Nineteenth and twentieth century geometers struggled to make sense of these questions, to show that with the proper interpretation they admit clean answers, and to put the subject of enumerative algebraic geometry on a firm mathematical foundation. Indeed, Hilbert made this endeavor the subject of his fifteenth challenge problem.

Enumerative problems have a long history: many such problems were posed by the ancient Greeks. Enumerative geometry is also currently one of the most active areas of research in algebraic geometry, mainly due to a recent influx of ideas from string theory. For instance, mirror symmetry and Gromov-Witten theory are two hot mathematical topics linked to enumerative geometry; both areas developed rapidly because of their connection to theoretical physics. While we will not discuss these subjects explicitly in the main part of this paper, many of the ideas and techniques we introduce are fundamental to these more advanced topics.

In the next section we give basic definitions of what we mean by a "conic," and introduce a **moduli space** of all conics. For each condition imposed on the conics we are counting, there is a subset of the moduli space consisting of the conics that satisfy this condition. To find the conics satisfying all of the given conditions, we intersect the corresponding subsets. If this intersection consists of a finite number of points, this number is our answer. In section 3 we will carry out this computation in several examples, each of which leads to some key ideas.

Steiner's original answer to his problem, 7776, was incorrect. He probably made the mistake of assuming that the intersection of the five subsets corresponding to the five given conics was finite. In fact, it is not finite, which we show in section 4. The infinite component of the intersection consists of **double lines**, conics whose equation is the square of a linear equation. If we can remove these, we will be left with a finite number of points corresponding to ellipses, hyperbolas, and parabolas. The first to successfully remove the double lines and count the remaining points was the French naval officer de Jonquieres [**18**, p. 469], who in 1859 gave the correct answer to Steiner's problem, 3264. Later, Michel Chasles developed a method for determining the answer 3264 and solving many other similar problems [**19**]. In section 4 we introduce the duality of the

Report Documentation Page					*Form Approved* *OMB No. 0704-0188*
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE **OCT 2008**		2. REPORT TYPE			3. DATES COVERED **00-00-2008 to 00-00-2008**
4. TITLE AND SUBTITLE **Enumerative Algebraic Geometry of Conics**					5a. CONTRACT NUMBER
					5b. GRANT NUMBER
					5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)					5d. PROJECT NUMBER
					5e. TASK NUMBER
					5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **United States Naval Academy (USNA),Mathematics Department,Annapolis,MD,21402**					8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)					10. SPONSOR/MONITOR'S ACRONYM(S)
					11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**28**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

plane and show how it can be used to remove the double lines in some problems. In section 5 we use a tool called "blowing up" to remove the double lines in the remaining problems.

We take a different point of view in section 6, using deformations to look at Steiner's problem and give an intuitive description of where the number 3264 comes from. In section 7 we prove that by removing the double lines, we do indeed get precisely 3264 conics solving Steiner's problem. In the last section we give some exercises and suggestions for further reading.

2. A MODULI SPACE OF PLANE CONICS. A plane conic curve is the set of points $(x, y) \in \mathbb{R}^2$ that satisfy a degree two polynomial relation,

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1}$$

where not all of the coefficients are zero. Circles, ellipses, hyperbolas, and parabolas are common examples of conics. In these examples the polynomial defining the conic is irreducible and the conic is said to be **nondegenerate**. If the polynomial defining the conic factors into a product of linear polynomials, then the conic is just the union of two lines. Such a conic is said to be **degenerate**. When the two lines are the same, or the polynomial defining the curve is a square of a linear polynomial, then the conic should be thought of as a **double line**, a line with some additional algebraic structure. These double lines play a key role in counting problems involving conics.

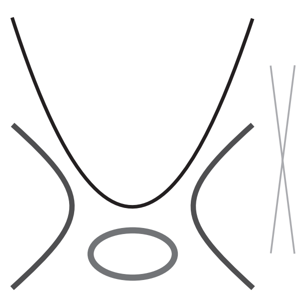

Figure 1. Parabolas, ellipses, and hyperbolas are conics. So are pairs of lines.

Any conic is completely determined by the coefficients a, b, c, d, e, and f of its defining equation (1), but not uniquely so; for example, the equations $x^2 - y = 0$ and $3x^2 - 3y = 0$ describe the same curve. If we consider the point $(a, b, c, d, e, f) \in \mathbb{R}^6$ as representing the conic $ax^2 + bxy + cy^2 + dx + ey + f = 0$, we see that for any nonzero scalar λ, the point $(\lambda a, \lambda b, \lambda c, \lambda d, \lambda e, \lambda f)$ represents the same curve. Therefore any point on the line spanned by the vector $\langle a, b, c, d, e, f\rangle$ gives rise to the same conic. So we have a one-to-one correspondence between lines through the origin in $\mathbb{R}^6$ and the equations defining plane conics (up to scalar multiple).

We'll use **homogeneous coordinates** $[a : b : c : d : e : f]$ to describe the line spanned by $\langle a, b, c, d, e, f\rangle$. This notation reminds us that the values of the coefficients a, b, c, d, e, and f are less important than their ratios to one another. What happens if a, b, c, d, e, and f are all zero? The zero vector does not span a line so $[0 : 0 : 0 : 0 : 0 : 0]$ is not a valid set of homogeneous coordinates. As well, this set of parameters does not correspond to a curve since the equation of the associated conic (1) reduces to $0 = 0$ and places no constraints on our points. Therefore this is not a meaningful set of coefficients to consider.

The set of lines through the origin in $\mathbb{R}^6$ is called the **five-dimensional real projective space** and is denoted $\mathbb{RP}^5$. It serves as our **moduli space** for conics, a space whose points are in one-to-one correspondence with the set of conics.

Why is $\mathbb{RP}^5$ five-dimensional? Well, each point of $\mathbb{RP}^5$ is part of an open set which can be identified with $\mathbb{R}^5$. Given a point in $\mathbb{RP}^5$, one of its homogeneous coordinates $a, \ldots, f$ is not zero. Let us suppose that $f \neq 0$. Then the set $U_f = \{[a : b : c : d : e : f] : f \neq 0\}$ can be identified with $\mathbb{R}^5$ via

$$[a : b : c : d : e : f] = \left[\frac{a}{f} : \frac{b}{f} : \frac{c}{f} : \frac{d}{f} : \frac{e}{f} : 1\right] \sim \left(\frac{a}{f}, \frac{b}{f}, \frac{c}{f}, \frac{d}{f}, \frac{e}{f}\right) \in \mathbb{R}^5.$$

In this sense, $\mathbb{RP}^5$ is a five-dimensional space. More generally, the set of lines through the origin in $\mathbb{R}^{n+1}$ forms n-dimensional real projective space, $\mathbb{RP}^n$.

2.1. The basic counting strategy. We've described the moduli space $\mathbb{RP}^5$ for plane conics. Using certain subsets of this moduli space, we can introduce the basic strategy to count the conics passing through some fixed points and tangent to some fixed lines or conics. For each point p we form the subset $H_p \subset \mathbb{RP}^5$ of conics passing through the point, for each line ℓ we form the subset $H_\ell \subset \mathbb{RP}^5$ of conics tangent to ℓ, and for each given nondegenerate conic Q we form the subset H_Q of conics tangent to Q. The points in the intersection of all of these subsets correspond to conics that pass through all of the points and are tangent to all of the lines and conics. This shift, from counting conics to counting the number of points in an intersection of certain subsets of $\mathbb{RP}^5$, may seem like a sleight of hand but it allows us to use the geometry of $\mathbb{RP}^5$ as well as the geometry of the plane to solve our counting problems.

Let's examine these subsets H_p, H_ℓ, and H_Q in more detail. To be concrete, let's fix a point, say $p\,(2, 3)$. If a conic defined by equation (1) passes through p, then it must be true that $4a + 6b + 9c + 2d + 3e + f = 0$. We see that this is a linear equation in a, b, c, d, e, and f. The set of points in $\mathbb{RP}^5$ satisfying this condition forms a four-dimensional plane, or a **hyperplane** in $\mathbb{RP}^5$. Each point on this hyperplane H_p corresponds to a conic passing through $p\,(2, 3)$. If we chose a different point $q \in \mathbb{R}^2$ we would get a different hyperplane H_q. Points on the intersection of H_p and H_q will correspond to conics passing through both p and q.

Similarly, if we look at all of the conics tangent to a particular line, for example the line $y = 0$, we get a four-dimensional hypersurface H_ℓ in $\mathbb{RP}^5$. This can be seen by first finding the intersection of a general conic, $ax^2 + bxy + cy^2 + dx + ey + f = 0$, and the line $y = 0$. The points of intersection have the form $(x, 0)$, where $ax^2 + dx + f = 0$. Usually we have two different points of intersection and the line $y = 0$ is a secant line to the conic. But when the discriminant $d^2 - 4af$ is zero the two points coincide and the line $y = 0$ is tangent to the conic. So the points in $\mathbb{RP}^5$ that satisfy the equation $d^2 - 4af = 0$ correspond to the conics that are tangent to the line $y = 0$. If we started with a different line ℓ in $\mathbb{R}^2$ we would get a hypersurface H_ℓ defined by a different degree 2 equation.

Lastly, if we look at all of the conics tangent to a particular conic Q, for example the parabola $y = x^2$, we also get a four-dimensional hypersurface in $\mathbb{RP}^5$. To find its equation, we substitute $y = x^2$ into the general conic equation to get $cx^4 + bx^3 + (a + e)x^2 + dx + f = 0$. The two conics will be tangent when this polynomial has a multiple root, which again is when the discriminant is zero. The discriminant of a degree four polynomial has degree six in the coefficients ([**6**, p. 42]), so H_Q is a degree six hypersurface in $\mathbb{RP}^5$. If we started with a different nondegenerate conic Q, we would get a hypersurface defined by a different degree 6 equation.

These three hypersurfaces H_p, H_ℓ, and H_Q are examples of projective algebraic varieties. A **projective algebraic variety** is a subset of a projective space $\mathbb{RP}^n$ consisting of the common zeros of a collection of homogeneous polynomials—polynomials in $n+1$ variables so that for each polynomial all its terms have the same degree.

If we require a conic to pass through several points and be tangent to several lines and conics, we will look at the intersection of the corresponding H_p's, H_ℓ's, and H_Q's. When studying intersections of varieties, it is useful to consider each variety's **codimension** rather than its dimension. In this case, since H_p, H_ℓ, and H_Q are four-dimensional hypersurfaces in a five-dimensional space, their codimension is $5-4=1$. For most pairs of varieties, the codimension of their intersection is the sum of their codimensions (of course, this will not be true if the varieties overlap too much).

Since the parameter space $\mathbb{RP}^5$ is five-dimensional, we expect that if we have five conditions then the intersection of the corresponding five hypersurfaces will have codimension 5 and be a finite collection of points. So it makes sense to ask: How many conics pass through p points and are tangent to ℓ lines and c conics, if $p+\ell+c=5$? In the next section we'll lay the foundation for answering this question by first considering the case where no conics are present.

3. SOME BASIC ENUMERATIVE QUESTIONS. In this section we are going to concentrate on the **point-line enumerative problems**: Find the number of conics through p points and tangent to ℓ lines if $p+\ell=5$.

3.1. Five points. Let's count the number of conics passing through five points in the plane. First, each point p imposes a hyperplane condition H_p on our conic. Thinking of $\mathbb{RP}^5$ as the set of lines through the origin in $\mathbb{R}^6$, each hyperplane H_p corresponds to a hyperplane (a five-dimensional subspace) of $\mathbb{R}^6$. If these hyperplanes are linearly independent then their intersection is a one-dimensional subspace of $\mathbb{R}^6$. This line through the origin in $\mathbb{R}^6$ corresponds to a single point in $\mathbb{RP}^5$. In turn, this single point represents a unique conic passing through all five points. We've shown that when the five points impose independent hyperplane conditions there is a unique conic passing through all five points.

It turns out that the points impose independent hyperplane conditions precisely when no four of the points are collinear. This requires a little argument, as follows. When four or five of the points are collinear, then there are lots of conics that pass through all of the points (and hence the hyperplanes couldn't impose independent conditions): just consider conics consisting of pairs of lines, where one line passes through the four collinear points and the other is any line that passes through the fifth point, as in Figure 2. So we may restrict our attention to the case where no four of the points lie on a line.

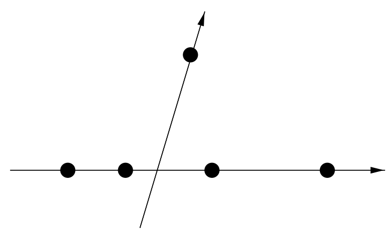

Figure 2. The case of 4 collinear points.

The question is not dependent on our choice of coordinates on the plane, so we choose coordinates such that three of the points are p_1 $(0, 0)$, p_2 $(1, 0)$, and p_3 $(0, 1)$. We'll label the other two points p_4 (s, t) and p_5 (u, v), as in Figure 3.

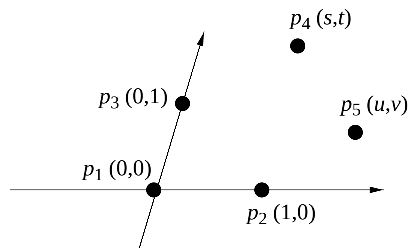

Figure 3. Choosing coordinates.

The system of equations imposed by these five points has coefficient matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ s^2 & st & t^2 & s & t & 1 \\ u^2 & uv & v^2 & u & v & 1 \end{bmatrix}.$$

The five constraints fail to impose linearly independent conditions precisely when this matrix fails to have full rank. The maximal rank of M is 5, so we can detect when the matrix does not have full rank by checking that all of the 5×5-submatrices of M have determinant zero.[1] By deleting one column at a time, we get six polynomials that are simultaneously zero precisely when M fails to have maximal rank. One of these (when the last column is deleted) is always zero; the first and fourth are the same, and the third and fifth are the same, so we end up with three conditions:

$$\begin{aligned} tv(s(v-1) - u(t-1)) &= 0 \\ su(v(s-1) - t(u-1)) &= 0 \\ (s^2 - s)(v^2 - v) - (u^2 - u)(t^2 - t) &= 0. \end{aligned}$$

Some careful case-by-case analysis will show that if all three of these equations hold, then either four of the given points are collinear or two of the points are coincident. For example, if the first equation holds, either $t = 0$, $v = 0$, or $s(v-1) = u(t-1)$. If $t = 0$, then the second equation says that either $s = 0$, in which case $p_4(s, t) = p_1(0, 0)$; $s = 1$, in which case $p_4(s, t) = p_2(1, 0)$; $v = 0$, in which case p_1, p_2, p_4, and p_5 are collinear; or $u = 0$. If $t = u = 0$, then the third equation requires two points to be coincident. Going back to the first equation, the case of $v = 0$ is essentially identical to the case of $t = 0$. The last case is where $s(v-1) = u(t-1)$, or $\frac{s}{t-1} = \frac{u}{v-1}$. This says that p_3, p_4, and p_5 are collinear. Looking again at equation two in this case, either $s = 0$ or $u = 0$ will cause p_1 to be on the same line with p_3, p_4, and p_5; the last possibility, that $v(s-1) = t(u-1)$, causes p_2 to be on this line instead; either way we have four collinear points.

[1] This is a useful characterization of rank: A matrix has rank $< d$ if and only if the determinants of all its $d \times d$ submatrices vanish [**1**, p. 153, ex. 10].

A more high-tech way (both using a computer and some commutative algebra) to see this is the following. Using a handy computer algebra program, like Macaulay2, Singular, or CoCoA, we can check that the ideal generated by the six submatrix determinants in the polynomial ring $\mathbb{R}[s, t, u, v]$ has the following primary decomposition:

$$(t-v, s-u) \cap (t, s-1) \cap (t-1, s) \cap (t, s) \cap (v, u) \cap (v, u-1) \cap (v-1, u) \\ \cap (u+v-1, s+t-1) \cap (u, s) \cap (v, t).$$

This means that the matrix M fails to have maximal rank precisely when the polynomials generating one of the primary ideals vanish. The first seven of these pairs of equations just indicate that two of our five points are equal, while the last three pairs indicate that four of our points are collinear.

The upshot of all this is that the five points impose linearly independent conditions if and only if no four of the points are collinear.

Theorem 1. *Given five points in the plane, no four of which lie on a line, there is a unique conic passing through the five points. The conic is nondegenerate precisely when no three of the points are collinear.*

Proof. It just remains to prove the last statement. If three of the points are collinear but no four are collinear, the unique conic passing through all of the points is a degenerate conic, a pair of lines. This is easy to see: three of the points lie on a line $L : G(x, y) = 0$, and if $\ell : H(x, y) = 0$ is the line through the other two points, then $GH = 0$ is the equation of the unique conic $L \cup \ell$ passing through the five points. If no three of the points are collinear, then the pigeonhole principle tells us that there is no pair of lines containing all five points. Therefore the unique conic passing through all five points cannot be degenerate. ■

3.2. Four points and one line. Now we solve the next problem: How many conics pass through four given points and are tangent to a given line? In answering this question, we'll see that it is necessary to develop a more expansive view of plane conics and tangency.

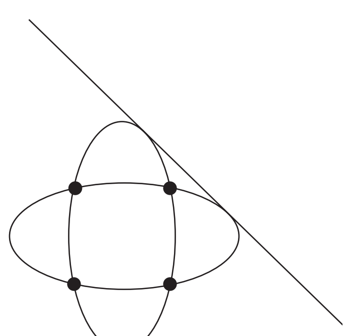

Figure 4. Two conics through four points and tangent to a line.

Each of the four points gives a linear condition, and the reader can check that the four conditions are independent if and only if the four points are not collinear. The intersection of the four hyperplanes is a line in $\mathbb{RP}^5$. Recall that the set of conics tangent to a line ℓ in the plane formed a hypersurface H_ℓ, whose defining equation had degree 2. The intersection of the line with H_ℓ can be found by plugging the parametric equation for the line into the equation for H_ℓ and solving. The resulting equation is quadratic and so in general we get two solutions. Each solution gives a point in $\mathbb{RP}^5$ corresponding to a conic passing through the given four points and tangent to the line ℓ. Therefore in general we expect there to be two such conics as in Figure 4.

Sometimes we need to be a little open-minded to recognize the resulting curves as conics that satisfy our constraints. For example, consider the conics passing through the points $(\frac{1}{2}, 2)$, $(2, \frac{1}{2})$, $(-\frac{1}{2}, -2)$, $(-2, -\frac{1}{2})$ and tangent to the line $y = 0$ as in Figure 5.

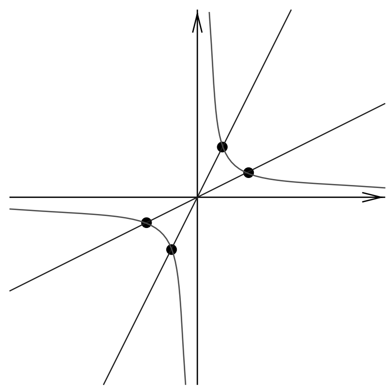

Figure 5. A pathological example.

Following the algebraic procedure described above, we get two solutions. One solution corresponds to the pair of crossed lines $(y - 4x)(x - 4y) = 0$. This pair of lines certainly passes through the four given points, but in what sense is it tangent to the given line? Algebraically, plugging the equation of the line into the equation of the conic gives a quadratic equation with a double root, which is what we associate with tangency. Geometrically, the given line intersects the crossed lines at their crossing point. If the given line were moved slightly, it would intersect each line once in two different points. This also reminds us of tangency. So we will consider this line to be tangent to this conic, even though the given line and the crossed lines are not tangent in the sense of derivatives and do not have the same direction at that point.

The other solution we find is the hyperbola $xy - 1 = 0$. Again, this hyperbola passes through the four given points, but is it tangent to the given line? Algebraically, if we plug $y = 0$ into $xy = 1$, we get no solutions at all! However, the line and hyperbola approach each other as $x \to \infty$. We would like to consider these to be tangent as well, and we can do this by adding some points to the plane "at infinity." One way to do this is to assume that the plane that we have been working in is part of a two-dimensional projective space, similar to the five-dimensional projective space $\mathbb{RP}^5$ that parameterizes all conics. The **projective plane** $\mathbb{RP}^2$ is the set of all lines through the origin in $\mathbb{R}^3$, and has coordinates $[X : Y : Z]$, where $[X : Y : Z] = [\lambda X : \lambda Y : \lambda Z]$ for any nonzero λ. We have been looking at the subset where Z is not zero, so the point $[X : Y : Z]$ is the same as $[\frac{X}{Z} : \frac{Y}{Z} : 1]$, and the variables we have been calling x and y are secretly $\frac{X}{Z}$ and $\frac{Y}{Z}$. The new points we are adding **at infinity** are the points with $Z = 0$.

How does this help us? If we translate our hyperbola $xy - 1 = 0$ into these new coordinates, we get $\frac{X}{Z}\frac{Y}{Z} - 1 = 0$, or $XY - Z^2 = 0$. Now plugging in the (also translated) equation of our given line, $Y = 0$, we get $Z^2 = 0$, which has a double root at $Z = 0$. So in a very concrete way the line and the hyperbola are tangent at infinity, or more specifically at the point $[1 : 0 : 0]$ in the projective plane $\mathbb{RP}^2$.

We obtained our two solutions by solving a quadratic equation with real coefficients, but such equations often have complex solutions. Indeed, if we move the line so that it separates one of the four points from the other three then the two conics that solve our problem both have complex-valued coefficients. We allow our homogeneous coordinates to be complex numbers in order to accommodate such solutions. Our moduli space for conics becomes the **complex projective five space** $\mathbb{CP}^5$, the space of one-dimensional subspaces in $\mathbb{C}^6$. Since these conics are tangent to the line at a point with complex coordinates, we are also forced to allow the X, Y, and Z coordinates to take complex values. Thus our solutions are conic curves that live naturally in the **complex projective plane** $\mathbb{CP}^2$.

When we first introduced the parameter space $\mathbb{RP}^5$ we noted that its points are in one-to-one correspondence with the **equations** of the plane conics (up to scalar multiple). At that time we hid one complication: there can be several equations that define the same set of points; for example $x^2 + y^2 + 1 = 0$ and $x^2 + y^2 + 3 = 0$ both define the empty set. But in $\mathbb{CP}^2$ these equations become $X^2 + Y^2 + Z^2 = 0$ and $X^2 + Y^2 + 3Z^2 = 0$ and they define different complex curves. Points in $\mathbb{CP}^5$ are in one-to-one correspondence with **conic curves** in $\mathbb{CP}^2$. This fact follows from the observation that if we have a plane conic in $\mathbb{CP}^2$ defined by a degree two equation, then any other degree two equation for the conic must be a scalar multiple of the first. This can be proved using Hilbert's Nullstellensatz, one of the foundational theorems in algebraic geometry [**5**, Sec. 4.1].

From here on, we'll restrict our attention to points, lines, and conics in the complex projective plane. We'll drop the $\mathbb{C}$ from our notation and just use $\mathbb{P}^n$ to denote complex n-dimensional projective space. A general conic in $\mathbb{P}^2$ will have the form

$$aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0,$$

where the coefficients are allowed to be complex numbers.

To summarize, the solutions to our enumerative problems may include degenerate conics and points of tangency may occur at complex points or at infinity. In order to accommodate these issues, we work with the moduli space $\mathbb{CP}^5$ of complex conics in the complex projective plane $\mathbb{CP}^2$.

3.3. Bézout's Theorem. Using the complex projective plane allows us to discuss intersections of curves consistently. For example, in the projective plane, two lines *always* intersect in one point—parallel lines meet at infinity, just like in a perspective drawing. As another example, consider a circle and a line in the plane. Algebraically, plugging the line equation into the circle equation gives a degree two polynomial. If this polynomial has distinct real roots, then the circle and line will intersect in two points. If the polynomial has a pair of complex conjugate roots, the circle and line will look as though they miss each other as in Figure 6. But if we allow points in the plane to have complex coefficients, then they will intersect at these two complex points. So in general, the number of intersection points between the circle and the line is two.

This consistent counting in complex projective space is described by **Bézout's theorem**:

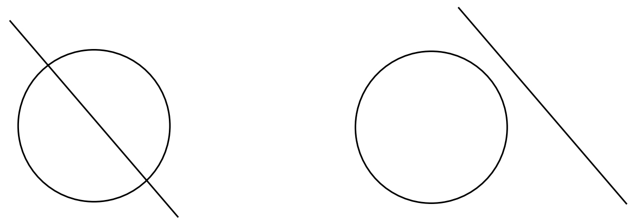

Figure 6. Two pictures of a line meeting a circle in two complex points.

Theorem 2 ([24]). *If n hypersurfaces of degrees $d_1, d_2, \dots, d_n$ in $\mathbb{P}^n$ intersect transversally, then the intersection consists of $d_1 \cdot d_2 \cdots d_n$ points.*

The hypersurfaces $X_1, \dots, X_n$ **intersect transversally at a point** $P \in X_1 \cap \cdots \cap X_n$ when their tangent spaces at P just intersect in the point P alone. The intersection $X_1 \cap \cdots \cap X_n$ is **transverse** if it is transverse at each of its points. As an example, consider again the line and circle. When the line meets the circle in two real points, then at each point the tangent line to the circle just meets the given line in one point; so the line and the circle meet transversally. This is also the case when they meet at two complex points, although it is harder to draw the picture! However, when the line is tangent to the circle, the two tangent spaces coincide and the intersection is not transverse. In this case, instead of two intersection points, we only get one. See Figure 7 for more examples.

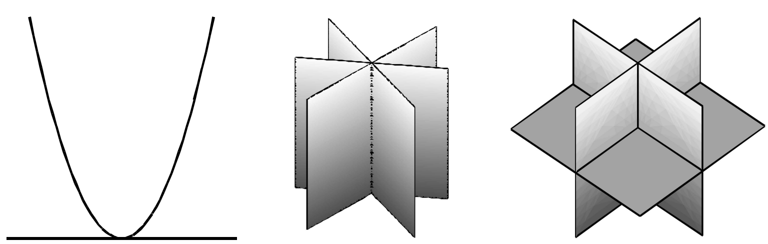

Figure 7. Nontransverse intersections—left and center—and a transverse intersection, right.

We've developed our intuition about Bézout's theorem in the projective plane $\mathbb{P}^2$, but now we want to use it in $\mathbb{P}^5$ to answer our enumerative questions. First let's revisit the problem of counting the conics that pass through 5 points. We saw that these conics are in bijection with the points in the intersection of five hypersurfaces H_p of degree 1. Since these H_p are hyperplanes, they intersect transversally when the hyperplanes impose linearly independent conditions. We saw that this was true if no four of the points are collinear. In this case, the five hyperplanes intersect in $1^5 = 1$ point by Bézout's theorem.

Let us return to the question, "How many conics pass through four given points and are tangent to a given line?" Now we are intersecting four degree one hyperplanes H_p and one degree two hypersurface H_ℓ. If these intersect transversally, Bézout's theorem says that they will intersect in $1^4 \cdot 2 = 2$ points, so there will be two such conics. In this case, it turns out that the intersection is transverse unless three of the points are collinear or one of the points lies on the given line.

It is possible to generalize Bézout's theorem to cases where the hypersurfaces do not meet transversally. If the hypersurfaces $X_1, \dots, X_n$ have degrees $d_1, \dots, d_n$ and the intersection $X_1 \cap \cdots \cap X_n$ consists of finitely many points, then we always get $d_1 d_2 \cdots d_n$ points in the intersection, provided we count the points with their proper multiplicities. For instance, if a line is tangent to a circle, then the only point of intersection must count with multiplicity $1 \cdot 2 = 2$. There are two ways to explain this result. If we plug the equation for the line into the equation for the circle, we get a quadratic equation $Q = 0$, where Q factors as a perfect square. The point of intersection corresponds to the double root of this quadratic.[2] There is also a dynamic way to compute the multiplicity of the tangent point. As we slide the line across the circle, we can keep track of the points of intersection. Since two points of intersection collapse to one point when the line is tangent to the circle, the point of tangency counts for two points. If we count with multiplicity, any circle meets any line in two (not necessarily distinct!) points.

In what follows, we will only need two facts about multiplicity: the multiplicity of the intersection $X_1 \cap \cdots \cap X_n$ at one of its points P is always a positive integer, and the number is equal to 1 precisely when the X_i meet transversally at P. However, if you know a little commutative algebra, then there is a nice way to assign multiplicity at a point of intersection of hypersurfaces X_i: the multiplicity is just the vector-space dimension of the quotient of the polynomial ring, localized at the maximal ideal corresponding to the point, modulo the polynomials defining the hypersurfaces.

3.4. Three points and two lines. We can use Bézout's theorem to count the number of conics passing through three points and tangent to two lines. The answer we expect is $1^3 \cdot 2^2 = 4$. In this case, the intersection will again be transverse unless the three points are collinear or one of the points lies on one of the lines. So far we've managed to fill in a few columns in Table 1.

Table 1. Number of conics through p points and tangent to ℓ lines.

Lines ℓ	0	1	2	3	4	5
Points p	5	4	3	2	1	0
Conic solutions	1	2	4	?	?	?

4. EXCESS INTERSECTION AND THE DUALITY OF $\mathbb{P}^2$

4.1. Excess intersection and general position. It is tempting to guess that the unknown entries in Table 1 continue as powers of 2. After all, if the five hypersurfaces involved in each problem intersect transversally, then this result would follow from Bézout's theorem. However, it turns out that for the last three point and line problems the corresponding hypersurfaces cannot intersect transversally, no matter how we choose the points and lines!

The reason that these hypersurfaces do not intersect transversally involves the double line conics. The defining polynomial of a double line factors as a perfect square. We refer to a conic whose defining polynomial is not a perfect square as a **reduced conic**. The reduced conics include nondegenerate conics, like circles and

[2]This double point is an example of what algebraic geometers call a **scheme**. Schemes occur naturally as limiting objects whenever geometric objects change under deformation; for example, if a pair of crossed lines pivot about their intersection point the limiting object is a double line.

parabolas, as well as degenerate conics consisting of a pair of distinct lines. The only **nonreduced** conics are the double line conics.

To see how double lines are connected to the transverse intersection property, let's consider conics that pass through one point and are tangent to four lines. Recall that in the projective plane, any two lines will intersect; in particular, any double line through the given point will intersect each of the four given lines. These double lines will be tangent to each of the given lines in the algebraic sense that the equation for the intersection has a double root. Thus we get infinitely many double line conics as solutions! Because we don't have a finite number of solutions, the five hypersurfaces in $\mathbb{P}^5$ must not intersect transversally. Here we see that the geometry of conics in $\mathbb{P}^2$ sheds light on the geometry of $\mathbb{P}^5$. This phenomenon, in which we expect our intersection to consist of finitely many points but in fact the intersection has higher dimension, is called **excess intersection**.

However, there are also a finite number of reduced conics that pass through the point and are tangent to the four given lines. We would like to ignore the double line solutions and just count the number of reduced conics that pass through p given points and are tangent to $5 - p$ given lines. In general there is a finite solution to this problem. Moreover, in most cases the reduced conics solving the problem are all nondegenerate. **In each of the remaining enumerative questions we will ask for the number of reduced conics satisfying the geometric constraints.**

In the problems we've already solved, none of the solutions can be double lines because of the constraints we put on the given points and lines—in each case, we did not allow all of the points in the problem to be collinear. We placed these constraints to ensure transverse intersection of the corresponding hypersurfaces, which then guaranteed that the number of conics was correctly calculated by the formula in Bézout's theorem. The constraints ensured that we do not get infinitely many solutions to our problem, nor do we get solutions with multiplicity. When we have a finite number of reduced solutions, each appearing with multiplicity one, then we say that the given points and lines are in **general position**. In each enumerative problem, the conditions that constitute general position (for example, that no three points lie on a line) will be slightly different, but almost all configurations of points and lines will satisfy the conditions.

4.2. Duality in $\mathbb{P}^2$. The tool that we will use to remove the double lines from our count is the duality of $\mathbb{P}^2$. Duality allows us to exchange points and lines, and at the same time it transforms conics into conics. The operation of duality respects inclusion and tangency. This will allow us to replace the remaining three point-line problems with the three problems we have already solved.

Consider the set of all lines in $\mathbb{P}^2$. We use a linear equation $L:\ AX + BY + CZ = 0$ to describe such a line. Of course, the line doesn't change if we multiply each of the coefficients by a nonzero constant, so the line can be represented by a point $[A : B : C]$ in a projective plane. Thus the set of lines in $\mathbb{P}^2$ is called the **dual projective plane**. The dual projective plane is just another copy of $\mathbb{P}^2$, but to distinguish the two spaces we will denote the dual projective plane by $\check{\mathbb{P}}^2$ and use the symbols A, B, and C for its coordinates.

By definition, a line L in $\mathbb{P}^2$ corresponds to a point in $\check{\mathbb{P}}^2$, which we call $\check{L}$ ("L dual"). We can also define the dual of a point p in $\mathbb{P}^2$ as the collection of lines that pass through p. If $p = [X_0 : Y_0 : Z_0]$ then this collection is $\{[A : B : C] : X_0A + Y_0B + Z_0C = 0\}$. This is a linear equation in the variables A, B, and C, so we see that the point p naturally corresponds to a line in $\check{\mathbb{P}}^2$, which we call $\check{p}$. Geometrically duality associates lines in $\check{\mathbb{P}}^2$ to points in $\mathbb{P}^2$ and vice versa. It is a good exercise to check algebraically that duality respects inclusion: if p is a point of $\mathbb{P}^2$ lying on the line L in $\mathbb{P}^2$, then $\check{p}$ is a line in $\check{\mathbb{P}}^2$ containing the point $\check{L}$. This is illustrated in Figure 8.

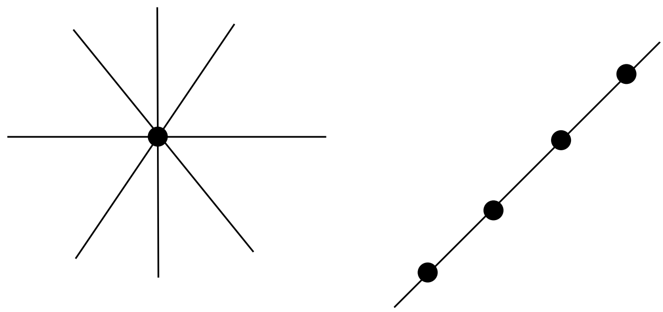

Figure 8. Four lines through one point (left) and their duals (right).

Let's see what duality does to conics. If Q is a conic curve in $\mathbb{P}^2$, then we define the dual curve $\check{Q}$ in $\check{\mathbb{P}}^2$ to be the collection of all lines tangent to Q. Note that this means that $\check{Q}$ will contain a point $\check{L}$ if and only if the corresponding line L in $\mathbb{P}^2$ was tangent to Q.

As an example, consider the conic Q given by the equation $X^2 - YZ = 0$. A general line $AX + BY + CZ = 0$ meets the curve in two points. If $A \neq 0$, we can find these two points by noting that $(AX)^2 - A^2YZ = 0$ on the curve and $AX = -(BY + CZ)$ on the line, so the points of intersection satisfy $(BY + CZ)^2 - A^2YZ = 0$. Rearranging gives a homogeneous quadratic equation in the variables Y and Z,

$$B^2Y^2 + (2BC - A^2)YZ + C^2Z^2 = 0. \tag{2}$$

As long as the **discriminant** of this equation is nonzero, its solution consists of two distinct points $[X : Y : Z] \in \mathbb{P}^2$ and the line $AX + BY + CZ = 0$ is the secant line to the curve Q joining the two points. When the discriminant $A^2(A^2 - 4BC)$ of equation (2) is zero, the line is tangent to the curve. Since we assumed that $A \neq 0$, we see that the line is tangent to the curve when $A^2 - 4BC = 0$. A similar analysis gives the same equation when B or C is nonzero. So in this example, the equation for the dual curve $\check{Q}$ is $A^2 - 4BC = 0$. Note that $\check{Q}$ is a conic in $\check{\mathbb{P}}^2$.

What is the equation of the dual for a more general conic? If the conic has equation $aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0$, then a general line $AX + BY + CZ = 0$ with $A \neq 0$ meets the conic in two points where

$$\begin{aligned} a(BY + CZ)^2 - bAY(BY + CZ) + cA^2Y^2 \\ - d(BY + CZ)AZ + eA^2YZ + fA^2Z^2 = 0. \end{aligned}$$

This is a homogeneous quadratic equation in Y and Z. Just as above, the line is tangent to the conic when the discriminant of this quadratic equation vanishes. Writing out the discriminant and using that $A \neq 0$, we see that

$$\begin{aligned} (e^2 - 4cf)A^2 + (4bf - 2de)AB + (d^2 - 4af)B^2 + (4cd - 2be)AC \\ + (4ae - 2bd)BC + (b^2 - 4ac)C^2 = 0. \end{aligned} \tag{3}$$

The same equation would result if we assumed that B or C is nonzero. So this equation characterizes the lines that are tangent to the conic Q; it is the equation for the dual $\check{Q}$. As in the example, the equation for $\check{Q}$ is degree two, so it is again a conic in $\check{\mathbb{P}}^2$. It can be checked that when Q is nondegenerate, then $\check{Q}$ is also nondegenerate.

We leave it to the reader to check that: (1) the dual to a degenerate conic consisting of a pair of crossed lines $L_1 \cup L_2$ is the double line conic through the two points $\check{L}_1$ and $\check{L}_2$; (2) the dual to a double line conic is just $\check{\mathbb{P}}^2$ itself.

Viewing equation (3) in a slightly different way gives the equation for the hypersurface H_ℓ of conics tangent to a line ℓ. Let $AX + BY + CZ = 0$ represent the line ℓ. Fixing A, B, and C, equation (3) gives constraints on the coefficients of the conics tangent to ℓ. This gives an explicit equation for H_ℓ as a degree 2 hypersurface.

In order to use duality to help us count conics tangent to lines, we need to understand how tangency transforms under duality. If a line L_1 is tangent to a nondegenerate conic Q then by definition $\check{L}_1$ is a point on $\check{Q}$. If p is a point on Q, is the line $\check{p}$ tangent to the dual conic $\check{Q}$? To start, let L_1 be the tangent line to Q at p. Then $\check{L}_1$ is a point lying on the line $\check{p}$. The line $\check{p}$ meets the dual conic in two points, $\check{L}_1$ and $\check{L}_2$, as depicted in Figure 9. We aim to show that these two points coincide, so that $\check{p}$ is a tangent line to $\check{Q}$. Since $\check{L}_2 \in \check{Q}$, L_2 must be a line that is tangent to Q. But $\check{L}_2 \in \check{p}$ so L_2 must also pass through p. So L_2 is a line through p that is tangent to the conic Q. Since each tangent line can meet the conic in at most one place, L_2 must be tangent to Q at p and so $L_2 = L_1$. It follows that $\check{L}_2 = \check{L}_1$, so the line $\check{p}$ meets the dual conic in a unique point; the line $\check{p}$ must be tangent to $\check{Q}$ at this point.

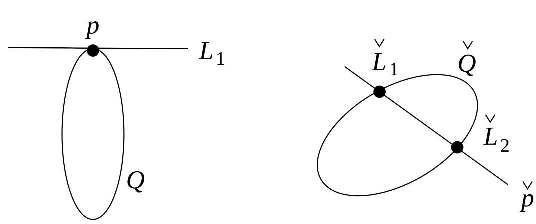

Figure 9. Tangency under duality (we show that $\check{L}_2 = \check{L}_1$).

4.3. The rest of the point-line problems. If we have five lines in the plane, then any nondegenerate conic tangent to them will have a dual conic that passes through the five dual points. There is only one such dual conic, so there can only be one conic tangent to all five lines. Any nondegenerate conic tangent to four given lines and passing trough a given point must have a dual conic that passes through the four dual points and is tangent to the dual line. If the dual points and dual lines are in general position, there are 2 such dual conics, so there are 2 conics tangent to four lines and passing through a point in general position. Finally, the reader should check that there are 4 conics tangent to three lines and passing though a pair of points in general position. We summarize the solutions to the point-line problems in Table 2.

5. STEINER'S PROBLEM

5.1. Blowing up the Veronese. Now that we have solved all of our point-line problems, we turn to Steiner's problem, "How many conics are tangent to five given conics?" We first try to intersect the five hypersurfaces H_Q corresponding to the five given

Table 2. Number of conics through p points and tangent to ℓ lines.

Lines ℓ	0	1	2	3	4	5
Points p	5	4	3	2	1	0
Conic solutions	1	2	4	4	2	1

conics. Bézout's formula would suggest that the intersection consists of $6^5 = 7776$ points, which was the answer that Steiner gave. However, Bézout's theorem does not apply because the five hypersurfaces H_Q cannot intersect transversally. Indeed, every double line conic is tangent to each of the five given conics, so the intersection of the five hypersurfaces H_Q in $\mathbb{P}^5$ contains the set of double lines. Unfortunately, we cannot use duality directly to filter out the double line conics from our answer. For this we introduce another tool, the blowup. To start, let's look more closely at the set of double lines that is causing us so much difficulty.

The set of all points in $\mathbb{P}^5$ corresponding to double lines is called the **Veronese surface**, V. Recall that a double line is a conic whose defining polynomial is the square of the defining polynomial of a line:

$$\begin{aligned}(AX + BY + CZ)^2 = A^2X^2 &+ 2ABXY + B^2Y^2 \\ &+ 2ACXZ + 2BCYZ + C^2Z^2 = 0.\end{aligned}$$

Lines in $\mathbb{P}^2$ are parameterized by $\check{\mathbb{P}}^2$, so the Veronese is indeed a surface (two-dimensional). It is the image of the injective map from $\check{\mathbb{P}}^2$ into $\mathbb{P}^5$,

$$[A : B : C] \mapsto [A^2 : 2AB : B^2 : 2AC : 2BC : C^2].$$

This map is defined by polynomials, and its domain is all of $\check{\mathbb{P}}^2$, so it is an example of what algebraic geometers call a **morphism**. The image of this morphism, which is the Veronese surface, is defined by the zeros of some homogeneous polynomials in the six coordinates on $\mathbb{P}^5$. Because the Veronese is two-dimensional, and it is in the five-dimensional $\mathbb{P}^5$, we expect that it can be described by three equations. This is true locally, but if we want to describe the full Veronese we in fact need six equations. The Veronese surface is an example of a variety which is not a **complete intersection**. A little algebra finds us the six equations:

$$\begin{aligned} b^2 - 4ac &= 0 \qquad & 4bf - 2de &= 0 \\ d^2 - 4af &= 0 \qquad & 4cd - 2be &= 0 \\ e^2 - 4cf &= 0 \qquad & 4ae - 2bd &= 0. \end{aligned} \tag{4}$$

If we consider an open set where one of these six variables is nonzero, we can use three of the above equations to solve for three variables and then derive the other three equations. But this only works on this open set; if we were to choose a different variable to be nonzero we would need three different equations to derive the others. So to describe the whole Veronese we need all six equations.

Next we will look at a map which is not a morphism, and try to turn it into one. Consider the map from $\mathbb{P}^5$ to $\mathbb{P}^5$ which sends a conic to its dual. Using equation (3) for the dual conic we found in section 4.2, this map is defined as

$$\begin{aligned} \delta : \mathbb{P}^5 &\to \mathbb{P}^5 \\ [a:b:c:d:e:f] &\mapsto [e^2-4cf : 4bf-2de : d^2-4af : \\ &\qquad 4cd-2be : 4ae-2bd : b^2-4ac]. \end{aligned} \tag{5}$$

The six polynomials that define δ in (5) are precisely the left sides of the six equations in (4), and therefore δ is undefined at a point if and only if that point is on the Veronese—that is, if and only if the point corresponds to a double line. The map is not a morphism on all of $\mathbb{P}^5$; however, it is defined by polynomials, so it is what algebraic geometers call a **rational map**. A rational map can always be extended to a morphism by expanding the domain space. In our case, we will take the Veronese surface out of $\mathbb{P}^5$ and replace it with a four-dimensional variety. This is done by looking at the graph of δ in $\mathbb{P}^5 \times \mathbb{P}^5$ and closing it up.

Definition. The **blowup of $\mathbb{P}^5$ along the Veronese surface**, $Bl_V\mathbb{P}^5$, is the closure of the graph of δ in $\mathbb{P}^5 \times \mathbb{P}^5$. The **blowing down morphism** $\pi : Bl_V\mathbb{P}^5 \to \mathbb{P}^5$ is given by the projection onto the first factor.

Let's try to write some equations for $Bl_V\mathbb{P}^5$. First of all, the graph is a set of points $([a:b:c:d:e:f],[r:s:t:u:v:w])$ in $\mathbb{P}^5 \times \mathbb{P}^5$ which must satisfy

$$\begin{aligned} \lambda r &= e^2-4cf & \qquad \lambda u &= 4cd-2be \\ \lambda s &= 4bf-2de & \lambda v &= 4ae-2bd \\ \lambda t &= d^2-4af & \lambda w &= b^2-4ac. \end{aligned}$$

Eliminating λ and cross multiplying, we get fifteen equations:

$$\begin{aligned}
r(4bf-2de) &= s(e^2-4cf) & s(d^2-4af) &= t(4bf-2de) & t(4ae-2bd) &= v(d^2-4af) \\
r(d^2-4af) &= t(e^2-4cf) & s(4cd-2be) &= u(4bf-2de) & t(b^2-4ac) &= w(d^2-4af) \\
r(4cd-2be) &= u(e^2-4cf) & s(4ae-2bd) &= v(4bf-2de) & u(4ae-2bd) &= v(4cd-2be) \\
r(4ae-2bd) &= v(e^2-4cf) & s(b^2-4ac) &= w(4bf-2de) & u(b^2-4ac) &= w(4cd-2be) \\
r(b^2-4ac) &= w(e^2-4cf) & t(4cd-2be) &= u(d^2-4af) & v(b^2-4ac) &= w(4ae-2bd).
\end{aligned}$$

In addition, because the original six equations for the Veronese were not algebraically independent, the blowup must satisfy eight more equations:

$$\begin{aligned} bu+2ew+2cv &= 0 & \qquad bv+2dw+2au &= 0 \\ eu+2br+2cs &= 0 & dv+2bt+2as &= 0 \\ ds+2et+2fv &= 0 & 4ar-4ct+du-ev &= 0 \\ es+2dr+2fu &= 0 & 4ct-4fw+bs-du &= 0. \end{aligned}$$

These eight equations are syzygies—linear relations (with polynomial coefficients) among the six equations (4) defining the Veronese surface. We obtained them using the `syz` command in Macaulay2. Syzygies play an important role in understanding the behavior of systems of equations and their solution sets (see [**7**] for more details).

Now, let us compare the blowup to $\mathbb{P}^5$. Suppose that a point $[a:b:c:d:e:f]$ on $\mathbb{P}^5$ is not on the Veronese surface, so it does not represent a double line conic. Then the duality map is well-defined at this point. In fact we see that the first fifteen equations completely determine $[r : s : t : u : v : w]$, so $\pi^{-1}([a : b : c : d : e : f])$ is a point. In fact, away from the Veronese surface, $\mathbb{P}^5$ and $Bl_V\mathbb{P}^5$ are isomorphic.

Now suppose that the point $[a : b : c : d : e : f]$ is on the Veronese surface. In this case, the first fifteen equations all reduce to $0 = 0$. Instead, the last eight equations tell us what the corresponding points are in the blowup. For example, let us consider the double line conic $X^2 = 0$. The corresponding point on the Veronese surface in $\mathbb{P}^5$ is $[1 : 0 : 0 : 0 : 0 : 0]$. If we let $b = c = d = e = f = 0$ in the equations on the blowup, we see that the last eight equations reduce to

$$2au = 0 \qquad 2as = 0 \qquad 4ar = 0.$$

Since $a \neq 0$, this tells us that r, s, and u are forced to be zero, but that t, v, and w are free. Thus the points in the blowup $Bl_V\mathbb{P}^5$ that are mapped by the blowing down morphism to this point are of the form

$$([1 : 0 : 0 : 0 : 0 : 0],\ [0 : 0 : t : 0 : v : w])\,.$$

These points define a $\mathbb{P}^2$ within the blowup, so $\pi^{-1}([a : b : c : d : e : f]) \cong \mathbb{P}^2$. The same will be true for any double line that we pick: the corresponding point on the Veronese in $\mathbb{P}^5$ has been replaced by an entire $\mathbb{P}^2$ in $Bl_V\mathbb{P}^5$.

In essence, in constructing the blowup we have ripped the Veronese out of $\mathbb{P}^5$ and replaced it with something two dimensions larger, a four-dimensional hypersurface. This hypersurface is called the **exceptional divisor** of the blowup, and we will call it E for short. The name "blowing up the Veronese" should make us think not of explosives, but of inserting a soda straw into $\mathbb{P}^5$ right at the Veronese, and blowing in a bubble of air to stretch it out into a four-dimensional object.

This act of stretching out the Veronese is exactly what we need to pull apart the excess intersection in Steiner's problem. Consider a hypersurface Y in $\mathbb{P}^5$ that contains the Veronese. Its preimage $\pi^{-1}(Y)$ in the blowup will then contain the exceptional divisor. On the other hand, away from V and E, $\mathbb{P}^5$ and the blowup are identical. If we remove V from Y and consider the inverse image $\pi^{-1}(Y \setminus V)$, this will be isomorphic to $Y \setminus V$. If we take the closure $\overline{\pi^{-1}(Y \setminus V)}$, we get a hypersurface in the blowup that intersects E, but does not contain it (see Figure 10). We call this new hypersurface the **strict transform** of Y and denote it by $\tilde{Y}$.

To solve Steiner's problem, we will intersect the strict transforms of the hypersurfaces in $\mathbb{P}^5$. Because the process of constructing the strict transform eliminates the E components, this will eliminate the excess intersection along the Veronese. We will show in section 7 that the proper transforms actually do intersect transversally, which will verify that blowing up eliminates the excess intersection.

5.2. The Chow ring. In $\mathbb{P}^5$ we were able to count the number of points in the intersection of five hypersurfaces using Bézout's theorem, but Bézout's theorem doesn't hold on the blowup. This is because on the blowup, the "degree" of a hypersurface is more complicated—it is not just one number! The extra information is encoded in the **Chow ring** of the blowup. In general, for an algebraic variety, the Chow ring is a ring that describes how its subvarieties intersect. Elements of the Chow ring are classes of subvarieties that have the same intersection properties. Bézout's theorem describes the Chow ring of projective space.

In the case of $\mathbb{P}^5$, Bézout's theorem says that the degree of a hypersurface is enough to determine its intersection properties. In particular, all hyperplanes will be in the

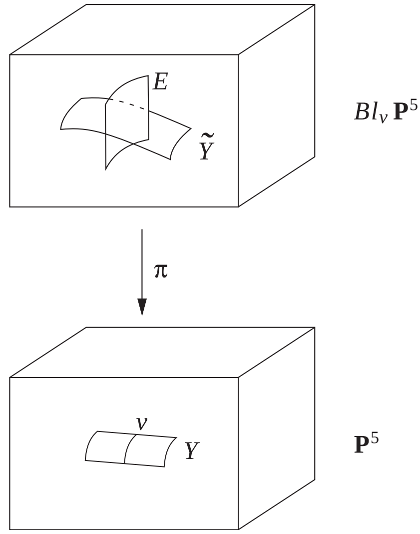

Figure 10. A picture of the blowup.

same class. Let us call that class $[H]$; it will be one element of the Chow ring of $\mathbb{P}^5$. The addition operation in the Chow ring roughly corresponds to the union of two varieties, so $[H]+[H]$ will be the class representing the union of two hyperplanes. But the union of two hyperplanes is a special case of a degree two hypersurface, and all degree two hypersurfaces have the same intersection properties, so any degree two hypersurface is in the class $2[H]$. Similarly, any degree d hypersurface in $\mathbb{P}^5$ will be in the class $d[H]$.

Multiplication in the Chow ring corresponds to intersection. If two varieties Y_1 and Y_2 intersect transversally, then the Chow ring product $[Y_1]\cdot[Y_2]$ is defined to be the class $[Y_1 \cap Y_2]$. In $\mathbb{P}^5$, the intersection of five general hyperplanes is just one point, so $[H]^5$ is the class of one point. Intersecting five hypersurfaces of general degree corresponds in the Chow ring to the multiplication

$$d_1[H]\cdot d_2[H]\cdot d_3[H]\cdot d_4[H]\cdot d_5[H] = d_1\cdot d_2\cdot d_3\cdot d_4\cdot d_5[H]^5,$$

which represents $d_1\cdot d_2\cdot d_3\cdot d_4\cdot d_5$ points.

Now let us describe the Chow ring of the blowup $Bl_V\mathbb{P}^5$ of $\mathbb{P}^5$ along the Veronese. First, consider a hyperplane H in $\mathbb{P}^5$ that does not contain the Veronese. Since H does not contain the Veronese, its strict transform $\tilde{H}$ is equal to its inverse image $\pi^{-1}(H)$. We will take the class of $\tilde{H}$ to be one generator of the Chow ring of $Bl_V\mathbb{P}^5$. If Y is a hypersurface of degree d in $\mathbb{P}^5$, then $[\pi^{-1}(Y)] = d[\tilde{H}]$.

The exceptional divisor does not behave like $d[\tilde{H}]$ for any d, so it represents a new class $[E]$ in the Chow ring. Since $Bl_V\mathbb{P}^5$ and $\mathbb{P}^5$ are isomorphic away from the Veronese, this is the only new generator of the Chow ring. Thus, any hypersurface in $Bl_V\mathbb{P}^5$ will be represented by a class $m[\tilde{H}]+n[E]$.

Now let Y be a general degree d hypersurface in $\mathbb{P}^5$ containing the Veronese. Then $[\pi^{-1}(Y)] = [\tilde{Y}] + n[E]$ for some n, so we see that

$$[\tilde{Y}] = d[\tilde{H}] - n[E]. \tag{6}$$

Next we would like to compute the integer n in equation (6). This integer will represent "how much" $\pi^{-1}(Y)$ contains E.

We say a function F **vanishes to order n along a variety** Z if F and all its partial derivatives of order $< n$ vanish everywhere on Z. For instance, $F = (x - 3y)^2$ vanishes to order 2 along the line $L : x - 3y = 0$ because F, $F_x = 2(x - 3y)$, and $F_y = -6(x - 3y)$ all vanish along L, while $F_{xx} = 2$ does not vanish on L.

Since Y is a hypersurface in $\mathbb{P}^5$, it is defined by one polynomial equation $P_Y = 0$. Its inverse image $\pi^{-1}(Y)$ is defined by $P_Y \circ \pi = 0$. The integer n appearing in equation (6) is the order of vanishing of the polynomial $P_Y \circ \pi$ along E. Fortunately it turns out that this is equal to the order of vanishing of P_Y along $V = \pi(E)$ (see [**7**, p. 106] for a proof). So the integer n is the largest integer such that P_Y and all its partial derivatives of order $< n$ lie in $\mathbb{I}(V)$ (here $\mathbb{I}(V)$ is the ideal of functions vanishing on the Veronese surface V; it is generated by the six equations (4)).

Let us find the strict transforms we need to solve our enumerative problems. Given a point p, there are many double line conics that do not pass through that point, so the hyperplane H_p of conics through p does not contain the Veronese. Thus $[\tilde{H}_p] = [\tilde{H}]$. The hypersurface H_ℓ of conics tangent to the line ℓ has degree 2. It is easy to see that the defining equation (3) for H_ℓ vanishes along V; a computer algebra system will verify that its first partial derivatives do not vanish along V. So by (6), $[\tilde{H}_\ell] = 2[\tilde{H}] - [E]$. The hypersurface H_Q of conics tangent to the conic Q has degree 6. The defining equation for H_Q vanishes along V; its first partial derivatives also vanish along V but its second derivatives do not. So the strict transform can be written as $[\tilde{H}_Q] = 6[\tilde{H}] - 2[E]$.

5.3. Counting conics. Now we use an intersection computation in the Chow ring of the blowup to compute the answer to Steiner's problem! By intersecting the strict transforms in the blowup, we are throwing away all of the extra double line solutions that caused us trouble before.

In the last section we showed that

$$\begin{aligned} [\tilde{H}_p] &= [\tilde{H}], \\ [\tilde{H}_\ell] &= 2[\tilde{H}] - [E], \\ \text{and } [\tilde{H}_Q] &= 6[\tilde{H}] - 2[E]. \end{aligned} \tag{7}$$

Now that we have everything in terms of $[\tilde{H}]$ and $[E]$, we could figure out how to intersect combinations of those and finish our calculations. But an easier way is to notice that we can instead write everything in terms of $[\tilde{H}_p]$ and $[\tilde{H}_\ell]$. We computed the intersections of these in sections 3 and 4.3 when we answered questions like "How many conics pass through 3 given points and are tangent to 2 given lines?" From these earlier calculations, we learned that in the Chow ring of the blowup,

$$\begin{aligned} [\tilde{H}_p]^5 &= [\tilde{H}_\ell]^5 = 1, \\ [\tilde{H}_p]^4[\tilde{H}_\ell] &= [\tilde{H}_p][\tilde{H}_\ell]^4 = 2, \\ [\tilde{H}_p]^3[\tilde{H}_\ell]^2 &= [\tilde{H}_p]^2[\tilde{H}_p]^3 = 4. \end{aligned}$$

(Note that on the right-hand side we are abusing notation: 1, 2, and 4 represent multiples of the Chow ring class of a point.) From (7) we see that

$$[\tilde{H}_Q] = 6[\tilde{H}] - 2[E] = 2[\tilde{H}_p] + 2[\tilde{H}_\ell]. \tag{8}$$

The answer to Steiner's original problem, "How many conics are tangent to five given conics?", is

$$
\begin{aligned}
[\tilde{H}_Q]^5 &= (2[\tilde{H}_p] + 2[\tilde{H}_\ell])^5 \\
&= 32([\tilde{H}_p]^5 + 5[\tilde{H}_p]^4[\tilde{H}_\ell] + 10[\tilde{H}_p]^3[\tilde{H}_\ell]^2 \\
&\quad + 10[\tilde{H}_p]^2[\tilde{H}_\ell]^3 + 5[\tilde{H}_p][\tilde{H}_\ell]^4 + [\tilde{H}_\ell]^5) \\
&= 32(1 + 5(2) + 10(4) + 10(4) + 5(2) + 1) \\
&= 3264.
\end{aligned}
$$

In a similar way, for each choice of c, ℓ, and p satisfying $c + \ell + p = 5$, we obtain the answer to the question "How many conics pass through p points and are tangent to ℓ lines and c conics in general position?" Table 3 lists these answers.

Table 3. The number of conics through p points, tangent to ℓ lines, and tangent to $5 - p - \ell$ conics in general position.

lines \ points	0	1	2	3	4	5
0	3264	816	184	36	6	1
1	816	224	56	12	2	
2	184	56	16	4		
3	36	12	4			
4	6	2				
5	1					

6. VISUALIZING THE 3264 CONICS. In order to supplement our understanding of Steiner's problem, we'll give a geometric construction that produces the 3264 conics tangent to five special conics. This construction is due to Fulton (see [**26**] and also [**22**]).

Consider five lines $L_1, \ldots, L_5$ in general position, each containing a marked point $P_i \in L_i$. Instead of looking at a single enumerative problem, we consider simultaneously all of the point-line enumerative problems involving some of the lines and some of the marked points. For example, we can ask for the conics through points P_1 and P_4 that are tangent to lines L_2, L_3, and L_5. We know from section 2 that there are four such conics, as in Figure 11.

More generally, for any two of the points we pick, there are four conics through those points that are tangent to the other three lines. There are $\binom{5}{2} = 10$ ways to choose the two points. If we choose a different number of points, say n, there are

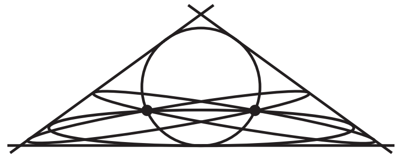

Figure 11. Four conics through two points and tangent to three lines.

$\binom{5}{n}[\tilde{H}_p]^n[\tilde{H}_\ell]^{5-n}$ conics passing through n of our five points and tangent to the other $5-n$ lines. Summing over all n, we get

$$[\tilde{H}_p]^5 + 5[\tilde{H}_p]^4[\tilde{H}_\ell] + 10[\tilde{H}_p]^3[\tilde{H}_\ell]^2 + 10[\tilde{H}_p]^2[\tilde{H}_\ell]^3 + 5[\tilde{H}_p][\tilde{H}_\ell]^4 + [\tilde{H}_\ell]^5$$
$$= ([\tilde{H}_p] + [\tilde{H}_\ell])^5 = 102$$

for the total number of conics that satisfy any 5 of the 10 conditions imposed by the points and lines.

Now we make each of the five lines into a conic, first by thinking of it as a double line (with a double marked point) and then by deforming the double line into a hyperbola, as illustrated in Figure 12.

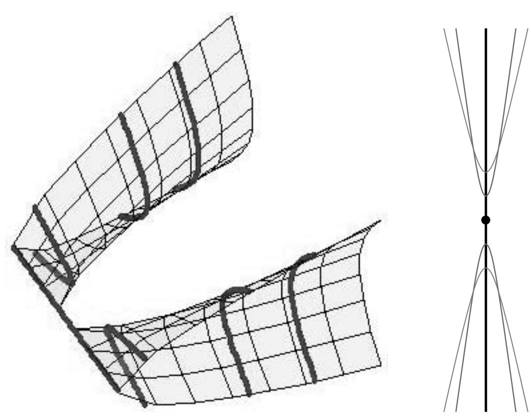

Figure 12. Left: Holding y constant gives cross sections of the surface $x^2y = y^2 + z^2$ that deform to a double line. Right: Superimposed snapshots of the deformation in the plane.

Let's look for conics tangent to these five hyperbolas. For each conic that was tangent to L_i, there are two conics tangent to the hyperbola. And for each conic that passed through the point P_i, there are two conics tangent to the hyperbola. In a sense, this is the geometric meaning of equation (8), $[\tilde{H}_Q] = 2[\tilde{H}_p] + 2[\tilde{H}_\ell]$. The hyperbolas are pictured in Figure 13.

Thus, deforming each line into a hyperbola doubles the number of conics. Since there are five lines, the total number of conics when all five are deformed is

$$2^5([\tilde{H}_p] + [\tilde{H}_\ell])^5 = 32(102) = 3264.$$

7. THE STRICT TRANSFORMS MEET TRANSVERSALLY. How can we be sure that our Chow ring computations give the correct number of reduced conics, when a similar technique (a naive application of Bézout's theorem) failed on $\mathbb{P}^5$? In section

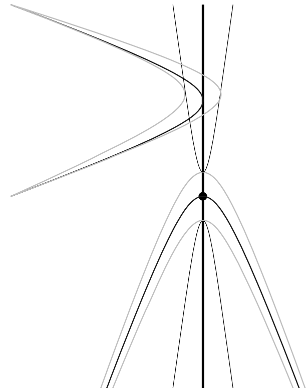

Figure 13. Deforming the line L_i gives rise to twice as many tangent conics.

5.3 we showed that the intersection of the five hypersurfaces $\tilde{H}_{Q_i}$ is in the class representing 3264 points. We would like to prove that for most sets of five conics, this intersection is in fact 3264 points of multiplicity one, each corresponding to a reduced conic. As we did before, we will say that a set of five conics with this property is in general position. The criteria for five conics to be in general position are quite complicated (including such restrictions as "No three of the conics have a common tangent line"); a list of such criteria can be found in [**3**]. However, without going into details about what constitutes general position for the five conics, we can prove that almost all sets of five conics are in general position.

One way to formalize the notion of "almost all sets of five conics" is to use the **Zariski topology**. Just as small open balls play a key role in analysis, the Zariski topology is a crucial tool in algebraic geometry. In the Zariski topology, a set is closed if it is the set of common solutions to a collection of polynomial equations. If you imagine a hypersurface, you see immediately that Zariski closed sets are very thin. A Zariski open set is just the complement of a Zariski closed set, so saying that a property holds on a nonempty Zariski open set means that it holds in almost all situations.[3]

Theorem 3. *The set of all* $(Q_1, Q_2, Q_3, Q_4, Q_5) \in (\mathbb{P}^5)^5$ *such that the* $\tilde{H}_{Q_i}$ *intersect transversally in points corresponding to reduced conics is open in the Zariski topology.*

Proof. We will prove that the complement is closed in the Zariski topology.

Suppose we have five conics $Q_1, \dots, Q_5$ and let

$$a_i x^2 + b_i xy + c_i y^2 + d_i xz + e_i yz + f_i z^2 = 0$$

be the defining equation for the conic Q_i. Let P be a point in the intersection of the $\tilde{H}_{Q_i} \subset Bl_V\mathbb{P}^5$. The blowup is a five-dimensional manifold embedded in the ten-dimensional space $\mathbb{P}^5 \times \mathbb{P}^5$ so at P the blowup has a five-dimensional tangent space $\mathbb{T}_P(Bl_V\mathbb{P}^5)$ that is a subspace of $\mathbb{T}_P(\mathbb{P}^5 \times \mathbb{P}^5)$ (we can think of P as lying at the origin of this space). Each of the five hypersurfaces $\tilde{H}_{Q_i}$ has a tangent space that is a hyperplane in $\mathbb{T}_P(Bl_V\mathbb{P}^5)$. To describe these hyperplanes, let's consider $\tilde{H}_{Q_i}$ in a chart near P. To be precise, we have $P \in Bl_V\mathbb{P}^5 \subset \mathbb{P}^5 \times \mathbb{P}^5$ and we can pick charts on both

[3]Warning: though the Zariski topology is fundamental for algebraic geometry, it takes some getting used to. For instance, since the nonempty open sets are so large, it is not Hausdorff. It is a good exercise for the reader to check that the Zariski open sets form a topology.

factors and intersect with the blowup to get a chart on the blowup. The point P was described by 12 homogeneous coordinates; after we pass to a chart, it is described by 10 affine coordinates $x_0, \dots, x_9$. The equation for $\tilde{H}_{Q_i}$ on this chart is the restriction of a polynomial $g_i(x_0, \dots, x_9)$ to the blowup; the coefficients of this polynomial g_i are themselves polynomials in the six homogeneous coordinates $a_i, \dots, f_i$ describing $Q_i \in \mathbb{P}^5$. The gradient $\nabla g_i(P)$ is a vector with ten entries, each of them a polynomial in 16 variables: $x_0, \dots, x_9$ and $a_i, \dots, f_i$. This vector consists of the coefficients of a linear function that, when restricted to $\mathbb{T}_P(Bl_V\mathbb{P}^5)$, vanishes precisely on the tangent space to $\tilde{H}_{Q_i}$ at P. This defines a codimension 1 subspace in $\mathbb{T}_P(Bl_V\mathbb{P}^5)$.

Now we ask: "What conditions do we need to put on our conics Q_i so that these five codimension 1 subspaces intersect in just a point?" We just need the five linear conditions to be independent. We can check this by forming the Jacobian matrix,

$$J = \left(\frac{\partial g_i}{\partial x_j}\right) = \begin{pmatrix} \nabla g_1(P) \\ \nabla g_2(P) \\ \nabla g_3(P) \\ \nabla g_4(P) \\ \nabla g_5(P) \end{pmatrix},$$

and seeing whether it has full rank. The matrix will *fail* to have full rank precisely when all its 5 by 5 minors vanish. Each of these minors is the determinant of a 5 by 5 submatrix, so they are polynomials in the entries of the $\nabla g_i(P)$. Thus, these are polynomial conditions involving the variables describing the Q_i and the variables describing P; that is, these are polynomial conditions on the product of $(\mathbb{P}^5)^5$ with our chart in $Bl_V\mathbb{P}^5$. Since $Bl_V\mathbb{P}^5$ is covered by charts, we obtain polynomials that cut out the subset $S \subset (\mathbb{P}^5)^5 \times Bl_V\mathbb{P}^5$ consisting of tuples

$$\{(Q_1, Q_2, Q_3, Q_4, Q_5, P) : \; P \in \cap \tilde{H}_{Q_i} \text{ and the intersection is not transverse at } P\}.$$

Because this set is defined by the vanishing of polynomial equations, it is closed in the Zariski topology.

Now we turn our attention to the points in the intersection that correspond to the double line conics. The exceptional divisor $E = \pi^{-1}(V) \subset Bl_V\mathbb{P}^5$ is a closed set because it is the inverse image of the closed set V under the continuous map π. Indeed, if the Veronese V is obtained as the set of common zeros of polynomials G_i then the exceptional divisor $E = \pi^{-1}(V)$ is the set of common zeros of the polynomials $G_i \circ \pi$ on $Bl_V\mathbb{P}^5$. It follows that the product $(\mathbb{P}^5)^5 \times E$ is closed in $(\mathbb{P}^5)^5 \times Bl_V\mathbb{P}^5$. As the Zariski closed sets form a topology, the union $S' = S \cup [(\mathbb{P}^5)^5 \times E]$ is also closed.

Now one great fact about projective varieties is that if we have a projection from one projective variety to another, then the image of a Zariski closed set is closed.[4] So the projection $\pi_1 : (\mathbb{P}^5)^5 \times Bl_V\mathbb{P}^5 \to (\mathbb{P}^5)^5$ that drops the last factor takes the closed set S' to a closed set. This says that the set of configurations $(Q_1, Q_2, Q_3, Q_4, Q_5) \in (\mathbb{P}^5)^5$ such that either the $\tilde{H}_{Q_i}$ fail to intersect transversally or their intersection includes points on the exceptional divisor is closed in the Zariski topology. Therefore its complement, the set of configurations where the $\tilde{H}_{Q_i}$ intersect transversally in points corresponding to reduced conics, is open in the Zariski topology. ■

This theorem is not quite enough to guarantee that for most configurations of five plane conics the intersection of the $\tilde{H}_{Q_i}$ is transverse and consists of isolated points

[4]This is not true for affine spaces though. For example, the image of the parabola $xy = 1$ projected onto the x-axis consists of the entire line except for the origin.

corresponding to reduced conics. We've shown that the set parameterizing such configurations is Zariski open, but we have not shown that it is nonempty. This is a common difficulty in algebraic geometry: it is easy to show that a property holds on an open set and hard to show that this open set is not empty! To do this, we just need to produce **one** example of five conics so that the intersection of the $\tilde{H}_{Q_i}$ is transverse and consists of points corresponding to reduced conics. We can avoid checking the transverse condition by just producing an instance of the problem where the solution consists of precisely 3264 reduced conics; these will all be isolated and count with multiplicity one since the intersection must be rationally equivalent to 3264 points. Fortunately, we've already produced such an example in section 6.[5] So the Zariski open set in Theorem 3 is not empty and for most configurations of five conics there are 3264 reduced conics tangent to all five. A precise characterization of the sets of five conics for which we do not get 3264 reduced conics tangent to all five can be found in [**3**].

In principle we'd need to repeat this work for each of the other problems stemming from combinations of five points, lines, and conics. This can be done by appealing to arguments similar to those given here, or by relying on a general transversality result due to Kleiman [**15**, p. 273]. The upshot is that for all our enumerative questions, almost all configurations give rise to a finite number of reduced solutions and this number is given in Table 3.

8. CODA. We've answered several enumerative questions involving conics; however, the true value of these problems lies in their connection to interesting mathematics. We hope that this article whets the reader's appetite for more algebraic geometry, and with this in mind, we make a few suggestions for further reading. As well, we've always felt that we understand a subject better after working a few exercises. We include some fun problems that further develop some of the material we've discussed.

Suggestions for further reading. In recent years, enumerative geometry has been heavily influenced by an influx of ideas from string theory. The major breakthrough that caused mathematicians to sit up and take notice was a prediction in 1991, using mirror symmetry, of the number of degree d rational curves on a degree 5 hypersurface in $\mathbb{P}^4$ [**4**]. This physics computation was not mathematically rigorous, but at the time these numbers were known to algebraic geometers only for very small d, so it was amazing to have predictions for all of the numbers at once. The development of the field of Gromov-Witten theory has put this computation on solid mathematical footing, as well as leading to many other interesting results. A recent book by Sheldon Katz [**17**] provides an introduction to this aspect of enumerative geometry, and we recommend it very highly.

Duality played a key role in our solutions to enumerative problems involving lines and points. The theory of duality (and the discriminants that define the dual varieties) is given extensive treatment in [**14**], where it is related to toric varieties and systems of hypergeometric differential equations. Blowing up also played a key role in our solution to Steiner's problem. The blowup is commonly used to resolve singularities in algebraic geometry. Indeed, Hironaka was awarded the Fields medal for showing that every variety can be desingularized by a sequence of blowups. A brief account of this theorem can be found in [**24**, Chap. 7] and an expository proof in [**16**].

[5] Fulton and MacPherson [**12**] use a dimension count to give a different proof that there must be such an example. They show that the collection of quintuples of conics for which we get solutions of multiplicity higher than one is of lower dimension than the set of quintuples of conics itself. So in particular, there is some configuration of five conics all of whose solutions have multiplicity one (all of the $\tilde{H}_{Q_i}$ intersect transversally).

We used techniques from computational algebra and computer algebra systems throughout the paper. The Macaulay2 book [**9**] explains how to use a computer algebra system to solve many problems in algebraic geometry. In particular, the reader is referred to Sottile's paper [**27**], which deals directly with enumerative questions.

There are several other ways to solve Steiner's problem, but all revolve around removing the double line conics from our count. One way is to generalize Bézout's theorem by assigning an intersection multiplicity to each **component** in the intersection of our hypersurfaces. Serre [**23**] showed that the correct intersection multiplicity for these and more general intersections can be computed using the Tor functor from commutative algebra. This can be done in low-dimensional examples on a computer [**8**].

Another approach to assigning intersection multiplicities is easier to visualize. To start, if we have n hypersurfaces lying in general position and having degrees $d_1, \dots, d_n$, then by Bézout's theorem their intersection consists of $d_1 \cdots d_n$ points. The reason that we have entire components in the intersection is that our hypersurfaces are not in general position. However, we can deform our hypersurfaces (changing each of the coefficients of our hypersurface from a constant to a function of a variable t, which is equal to our given coefficients when $t = 0$) so that they are in general position for $t \neq 0$. For nonzero t, we get $d = d_1 \cdots d_n$ points $p_1(t), \dots, p_d(t)$, each a function of the parameter t. If our family deforms nicely (the technical condition is that the family is **flat**) then we would expect these d points to approach d limiting points as $t \to 0$. The number of points that land on each component is the intersection multiplicity of the component. With this definition, we can determine the contribution to the Bézout number from each of the higher dimensional components and by subtraction compute the number of isolated points (corresponding to conics that are not counted with multiplicity) in our solution. Katz [**17**, Chap. 8] gives a nice example of this process in action.

A final approach to intersection multiplicities involves computations with Chern classes of vector bundles, leading to the so-called characteristic numbers. See [**11**, Sec. 10.4] for an application of these techniques to Steiner's problem and Fulton's lecture notes [**10**] for a broad overview of intersection theory. A very accessible discussion of characteristic numbers, together with a history of Steiner's problem, can be found in Kleiman's article [**19**]. This theory is sufficient to enumerate the conics tangent to five given plane curves in general position. As in the case of conics, each curve C gives rise to a class $\deg(C)[\tilde{H}_p] + \deg(\check{C})[\tilde{H}_\ell]$ and the answer comes from finding the product of the five classes in $Bl_V\mathbb{P}^5$.

There are plenty of other enumerative problems with connections to algebraic geometry. Schubert calculus deals with enumerative problems involving linear spaces, rather than conics; for example, "How many lines in $\mathbb{P}^3$ meet four other lines in general position?" Kleiman and Laksov [**21**] give a nice introduction to Schubert calculus.

Problem 1. How many lines are simultaneously tangent to two conics in general position? [*Hint:* Think of the dual picture.]

Problem 2. To each conic $Q: \; ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$ we associate the matrix

$$M_Q = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix},$$

so that if $\mathbf{x}^T = \begin{bmatrix} x & y & z \end{bmatrix}$ then the equation for the conic is given by $\mathbf{x}^T M_Q \mathbf{x} = 0$.

(a) Show that if $\tilde{\mathbf{x}} = L\mathbf{x}$ is a linear change of coordinates, then when the quadratic Q is written in the new variables $\tilde{\mathbf{x}}$, it corresponds to the matrix $(L^{-1})^T M_Q L^{-1}$.

(b) Show that given any nondegenerate conic Q, there is a linear change of coordinates transforming it to $yz = x^2$.

(c) Show that Q is degenerate if and only if $\det(M_Q) = 0$, and Q is a double line if and only if $\operatorname{rank}(M_Q) = 1$.

(d) Show that if Q is nonsingular, then its dual curve $\check{Q}$ corresponds to the matrix $M_{\check{Q}} = M_Q^{-1}$. In this sense the duality map is a generalization of the inverse operation for matrices. This topic is explored in great detail (for multidimensional matrices!) in [**14**].

Problem 3. Verify that if $[x_i : y_i : z_i]$ $(1 \leq i \leq 5)$ are five points in general position, then the unique conic $ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$ passing through them is given by the vanishing of the determinant of the matrix

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1z_1 & y_1z_1 & z_1^2 \\ x_2^2 & x_2y_2 & y_2^2 & x_2z_2 & y_2z_2 & z_2^2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3z_3 & y_3z_3 & z_3^2 \\ x_4^2 & x_4y_4 & y_4^2 & x_4z_4 & y_4z_4 & z_4^2 \\ x_5^2 & x_5y_5 & y_5^2 & x_5z_5 & y_5z_5 & z_5^2 \\ x^2 & xy & y^2 & xz & yz & z^2 \end{bmatrix}.$$

Problem 4. (a) Show that the point in $\mathbb{P}^5$ corresponding to a conic Q lies on the hypersurface H_Q of all conics tangent to Q. [*Hint:* Consider the case $Q : x^2 - yz = 0$.]

(b) Show that if P is a point on H_Q then the entire line joining P and the point in $\mathbb{P}^5$ corresponding to Q is on H_Q too. This shows that H_Q is a **cone over** Q.

Problem 5. (a) A **circle** is a conic passing through the two points $[1 : i : 0]$ and $[1 : -i : 0]$. Show that when we homogenize the curve defined by $x^2 + y^2 = r^2$ we get a circle in $\mathbb{P}^2$.

(b) Show that the parameter space for circles in $\mathbb{P}^2$ is a 3-dimensional projective space.

(c) One approach to count the circles tangent to three general circles is to compute $[\tilde{H}_Q]^3[\tilde{H}_p]^2 = 184$ in $Bl_V(\mathbb{P}^5)$. Why is this incorrect? Find the correct count.

[*Remark:* In fact, something very precise can be said if the 3 given circles are mutually tangent. If r_1, r_2, r_3, and r_4 are the radii of the circles, then

$$2(r_1^2 + r_2^2 + r_3^2 + r_4^2) = (r_1 + r_2 + r_3 + r_4)^2.$$

This is a famous theorem, known to Descartes and extended and immortalized in Sir Frederick Soddy's poem, "The Kiss Precise" [**25**].]

Problem 6. Let $\gamma : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ be the map that sends $([A : B : C], [D : E : F])$ to the point representing the conic $(Ax + By + Cz)(Dx + Ey + Fz) = 0$. We call the image of γ the degenerate variety because it parameterizes the degenerate conics.

(a) The map $\nu_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ given by $([A : B : C], [D : E : F]) \mapsto [AD : BD : CD : AE : BE : CE : AF : BF : CF]$ is called the Segre map and its image the Segre variety. Show that the Segre map is an injection and show that the map γ factors through the Segre map in the sense that there is a map $\pi : \mathbb{P}^8 \to \mathbb{P}^5$ such that $\gamma = \pi \circ \nu_{2,2}$.

(b) Show that the map γ is generically 2 to 1. Where does this map ramify? That is, where does this map fail to be 2 to 1?

(c) Show that the degenerate variety is a degree 3 hypersurface in $\mathbb{P}^5$ by a geometric argument.

(d) Now find the defining equation of the degenerate variety. Hint: use part (c) of Problem 2.

(e) Show that the exceptional divisor $E \subset Bl_V\mathbb{P}^5$ is isomorphic to the degenerate variety. In particular, if $\pi_2 : \mathbb{P}^5 \times \mathbb{P}^5 \to \mathbb{P}^5$ is the projection onto the second factor, then $\pi_2|_E$ is an isomorphism onto its image and $\pi_2(E)$ is the degenerate variety.

(f) Thus E consists of pairs $(Q, L_1 \cup L_2)$ where Q is a double line conic and $L_1 \cup L_2$ is a crossed line conic. Show that if $(Q, L_1 \cup L_2) \in E$ then the line corresponding to Q is the dual of the point of intersection of L_1 and L_2.

(g) The two points $\check{L}_1$ and $\check{L}_2$ lie on the double line Q, so we can think of Q as being a double line with two marked points. While double lines do not have well-defined duals, if we dualize a double line with two marked points, we get the pair of crossed lines corresponding to the duals of the marked points. This interpretation gives a way to define duality on the blowup. Show that with this interpretation of duality the points on the blowup all have the form $(Q, \check{Q})$ and the duality map just swaps the two coordinates. Because the duality map is well-defined on the blowup, $Bl_V\mathbb{P}^5$ is sometimes called the **space of complete conics**.

(h) Now suppose that Q is a double line with two marked points, and the two points come together. What happens in the dual picture?

(i) Using the duality map δ from part (g) show that we get $\delta(\tilde{H}_Q) = \tilde{H}_{\check{Q}}$, $[\delta(\tilde{H}_p)] = [\tilde{H}_\ell]$, and $[\delta(\tilde{H}_\ell)] = [\tilde{H}_p]$. Since Q and $\check{Q}$ are both conics, $[\tilde{H}_Q] = [\tilde{H}_{\check{Q}}] = a[\tilde{H}_p] + b[\tilde{H}_\ell]$, say. Applying δ, conclude that $a = b$. Then show that $a = b = 2$ only using the degrees of H_p, H_ℓ, and H_Q. This gives another proof of formula (8), originally due to Clebsch and Lindemann [**19**, p. 121].

Problem 7. Instead of using the moduli space of conics themselves, one can do enumerative calculations with the moduli space of stable maps. This is the point of view taken in Gromov-Witten theory.

(a) Consider the projective line $\mathbb{P}^1$ with homogeneous coordinates $[S : T]$. Find a degree 2 morphism from $\mathbb{P}^1$ to $\mathbb{P}^2$ whose image is the curve $yz = x^2$. (In this case "degree" means the degree of the polynomial functions defining the morphism.)

(b) By composing the (inverse of the) change of coordinates in part (b) of Problem 2 with the map in part (a), we see that every nondegenerate conic is the image of $\mathbb{P}^1$ under some degree 2 map. Show that any degree 2 map from $\mathbb{P}^1$ to $\mathbb{P}^2$ is either a one-to-one map onto a nondegenerate conic, or a two-to-one map onto a line with two branch points.

(c) Show that composing a map from $\mathbb{P}^1$ to $\mathbb{P}^2$ with a linear change of coordinates on $\mathbb{P}^1$, $[S : T] \mapsto [aS + bT : cS + dT]$, with $ad - bc \neq 0$, does not change the image. Two maps that differ by such an automorphism of $\mathbb{P}^1$ are considered isomorphic.

(d) In general, the moduli space of stable genus 0 maps to $\mathbb{P}^2$ consists of (isomorphism classes of) maps from a **tree** of $\mathbb{P}^1$'s to $\mathbb{P}^2$. In the degree 2 case, the only possible trees have either one or two "branches." Thus a stable degree 2 map is either a degree 2 map from $\mathbb{P}^1$ to $\mathbb{P}^2$, such as the maps in part (b), or a map from a pair of intersecting $\mathbb{P}^1$'s to $\mathbb{P}^2$, where the map has degree 1 on each "branch." Explain why the moduli space of stable degree 2 genus 0 maps to $\mathbb{P}^2$ is essentially the same as the blowup $Bl_V\mathbb{P}^5$, the space of complete conics.

ACKNOWLEDGMENTS. We first learned about this topic from Frank Sottile, whose expository lectures and writing guided our early investigations. Mike Roth provided help at a crucial time; his suggestions form the basis for our proof of Theorem 3. Steve Kleiman gave us extensive comments on a first draft of the paper and rescued us from several errors. Helene Speer's gracious help improved our figures. This paper grew out of Midshipman Andrew Bashelor's Trident project [**3**], a full-year research project carried out at the U.S. Naval Academy. We are grateful to the Trident Scholar Committee for their oversight and guidance. In particular we gratefully acknowledge the Office of Naval Research for partial support of this work in funding document N0001405WR20153.

REFERENCES

1. M. Artin, *Algebra*, Prentice Hall, Englewood Cliffs, NJ, 1991.
2. D. Ayala and R. Cavalieri, Counting bitangents with stable maps, *Expositiones Mathematicae* **24** (2006) 307–336; also available at `http://arxiv.org/abs/math.AG/0505139`.
3. A. Bashelor, *Enumerative Algebraic Geometry: Counting Conics*, Trident Scholar Project Report, no. 330, United States Naval Academy, Annapolis, MD, 2005.
4. P. Candelas, X. de la Ossa, P. Green, and L. Parkes, A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory, *Nuclear Phys. B* **359** (1991) 21–74.
5. D. Cox, J. Little, and D. O'Shea, *Using Algebraic Geometry*, Graduate Texts in Mathematics, vol. 185, Springer-Verlag, New York, 1998.
6. H. Derksen and G. Kemper, *Computational Invariant Theory*, Encyclopaedia of Mathematical Sciences, vol. 130, Springer-Verlag, Berlin, 2002.
7. D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.
8. ———, Projective geometry and homological algebra, in *Computations in Algebraic Geometry with Macaulay 2*, D. Eisenbud, D. R. Grayson, and M. Stillman, eds., Algorithms and Computation in Mathematics, vol. 8, Springer-Verlag, Berlin, 2002, 17–40.
9. D. Eisenbud, D. R. Grayson, and M. Stillman, eds., *Computations in Algebraic Geometry with Macaulay 2*, Algorithms and Computation in Mathematics, vol. 8, Springer-Verlag, Berlin, 2002.
10. W. Fulton, *Introduction to Intersection Theory in Algebraic Geometry*, CBMS Regional Conference Series in Mathematics, vol. 54, Conference Board of the Mathematical Sciences, Washington, DC, 1984.
11. ———, *Intersection Theory*, 2nd ed., Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.
12. W. Fulton and R. MacPherson, Defining algebraic intersections, in *Algebraic Geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977)*, Lecture Notes in Mathematics, vol. 687, Springer-Verlag, Berlin, 1978, 1–30.
13. W. Fulton and R. Pandharipande, Notes on stable maps and quantum cohomology, in *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math., vol. 62, American Mathematical Society, Providence, RI, 1997, 45–96.
14. I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Mathematics: Theory & Applications, Birkhäuser, Boston, MA, 1994.
15. R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, Berlin, 1977.
16. H. Hauser, The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand), *Bull. Amer. Math. Soc. (N.S.)* **40** (2003) 323–403.
17. S. Katz, *Enumerative Geometry and String Theory*, American Mathematical Society, Providence, RI, 2006.

18. S. Kleiman, Problem 15: Rigorous foundation of Schubert's enumerative calculus, in *Mathematical Developments Arising from Hilbert Problems*, Proc. Symp. Pure Math., vol. 28, American Mathematical Society, Providence, RI, 1976, 445-482.
19. ———, Chasles's enumerative theory of conics: A historical introduction, in *Studies in Algebraic Geometry*, Mathematical Association of America Studies in Mathematics, vol. 20, Mathematical Association of America, Washington, DC, 1980, 117–138.
20. ———, Intersection theory and enumerative geometry: A decade in review, With the collaboration of Anders Thorup on section 3, in *Algebraic Geometry–Bowdoin 1985*, Proc. Sympos. Pure Math., vol. 46, Part 2, American Mathematical Society, Providence, RI, 1987, 321–370.
21. S. L. Kleiman and D. Laksov, Schubert calculus, this MONTHLY **79** (1972) 1061–1082.
22. F. Ronga, A. Tognoli, and T. Vust, The number of conics tangent to 5 given conics: The real case, *Rev. Mat. Univ. Complut. Madrid* **10** (1997) 391–421.
23. J.-P. Serre, *Algèbre Locale. Multiplicités*, 2nd ed., Lecture Notes in Mathematics, vol. 11, Springer-Verlag, Berlin, 1965.
24. K. E. Smith, L. Kahanpää, P. Kekäläinen, and W. Traves, *An Invitation to Algebraic Geometry*, Universitext, Springer-Verlag, New York, 2000.
25. F. Soddy, The kiss precise, *Nature* **137** (1936) 1021.
26. F. Sottile, Enumerative geometry for real varieties, in *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math., vol. 62, American Mathematical Society, Providence, RI, 1997, 435–447.
27. ———, From enumerative geometry to solving systems of polynomials equations, in *Computations in Algebraic Geometry with Macaulay 2*, D. Eisenbud, D. R. Grayson, and M. Stillman, eds., Algorithms and Computation in Mathematics, vol. 8, Springer-Verlag, Berlin, 2002, 101–129.

ANDREW BASHELOR received his B.Sc. from the United States Naval Academy in 2005 and an M.Sc. in Operations Analysis from the Naval Postgraduate School in 2006. He is currently serving as an officer in the U.S. Navy, stationed in Hawaii.
andrew.bashelor@mac.com

AMY KSIR grew up in Laramie, Wyoming. She received her B.A. from Rice University and her Ph.D. from the University of Pennsylvania, and currently teaches at the U.S. Naval Academy. Her research interests are in applications of algebraic geometry to string theory and coding theory; she also enjoys playing the accordion.
Department of Mathematics, U.S. Naval Academy, Annapolis, MD, 21402
ksir@usna.edu

WILL TRAVES received his B.Sc.H. from Queen's University and his Ph.D. from the University of Toronto. He teaches at the U.S. Naval Academy, where he helps coach the women's rugby team.
Department of Mathematics, U.S. Naval Academy, Annapolis, MD, 21402
traves@usna.edu