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The Mathematics of Flow Similarity of the Velocity Boundary Layer

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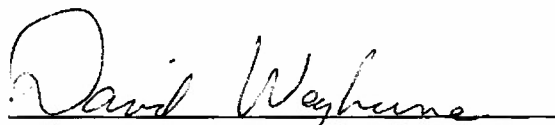
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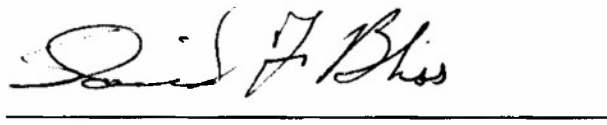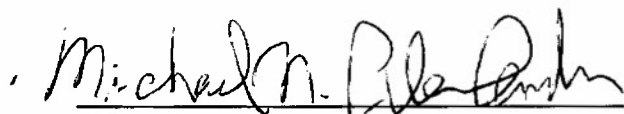Hanscom AFB MA 01731-2909

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14. ABSTRACT

New results for similarity scaling of the velocity profile of the 2-D wall-bounded flows are presented. The theoretical results are based on a simple concept; the area under similar scaled velocity profile curves must be equal. By taking certain integrals of the scaled velocity profiles and its first derivative, we obtain a number of similarity scaling requirements. For example, it is shown that if whole profile similarity exists, then: 1) the similar length scaling variable must be proportional to the displacement thickness, and 2) the velocity similarity scaling variable must be proportional to the free-stream velocity. Conventional thinking is that whole profile similarity is limited to laminar flow boundary layers. If true, this would drastically restrict the utility of the new scaling results. However, we show that to experimental accuracy, certain turbulent boundary layer flow datasets can display whole profile similarity and that as predicted the new scaling works well.   In addition, we develop new definitions for partial similarity for the inner, the outer, and the log law regions of a turbulent boundary layer. These definitions are most suitable for quantifying similarity of experimental profiles.

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1. Introduction

Similarity is one of the basic tenets of fluid flow theory, going back to the pioneering work of Reynolds[1] in the 1800's. For wall-bounded flows, the formal definition of similarity is that two velocity profiles are similar if they differ only by a scaling constant in y and u (where y is the normal direction to the wall and u is the velocity parallel to the wall). In the following, we discover some new properties of similar velocity profiles of 2-D wall-bounded boundary layer flows. The results are based on a new integral-based definition of similarity of the velocity profiles of 2-D wall-bounded flows. The new similarity scaling requirements are obtained without having to find solutions of the flow governing equations. Therefore, the new results apply to similar profiles whether they are laminar, transitional, or turbulent velocity profiles as long as the velocity is taken as the Reynolds averaged velocity profile.

The new similarity results are derived from simple integrals of the velocity profile (or its derivative) and are based on the requirement that the area under the scaled velocity profile curves must be equal. In what follows, we show that the mathematics requires that if similarity exists, then the displacement thickness must be a similarity length scaling variable for 2-D wall bounded boundary layer flows. We also show that the velocity scaling variable must be proportional to the free-stream velocity for this flow geometry. While other scaling variables are possible, we show that the length scale and the velocity scale variables are not independent. In fact, our scaling variable relationship is identical to the empirically derived scaling velocity successfully used by Zagarola and Smits [2] to scale turbulent boundary flows over wedges, in channels, and in pipes.

The new mathematical definition of similarity assumes the entire profile displays similarity behavior like one finds for the laminar boundary layer. However, the turbulent boundary layer is more complicated than the laminar boundary layer since the viscous region is largely confined to the near-wall region of the boundary layer. As a consequence, there has been considerable effort in the community to establish the possible existence of partial similarity by finding "scaling laws" for different regions of the velocity profile for turbulent boundary layer flows (see for example, review by Buschmann and Gad-el-Hak [3]). These include the log law region, the near-wall inner layer region and the outer layer region. In what follows, we use the equal area idea to develop new mathematical definitions for partial similarity for these various regions. The main advantage of these new definitions is that similarity can be discovered and verified by comparing numerical integral values of the velocity profile datasets being investigated. This removes the subjectivity of the commonly applied "Chi-by-eye" method of comparing velocity profile plots that is presently being used to claim similarity in experimental datasets.

Finally, we examine the case for similarity of the 2-D turbulent boundary layer on a wedge. In particular we are interested in what, if anything, we can say about the scaling variables for this flow situation. Based on a momentum balance argument, Castillo and George [4] have pointed out that for turbulent flows with a pressure gradient, the velocity scaling variable for the outer layer region must be the free-stream velocity. Recently, Weyburne [5] examined this flow configuration theoretically using the transformed x-momentum balance equation and the transformed Reynolds stress transport equation. The derived similarity criteria were used to find eleven experimental turbulent boundary layer datasets that show velocity profile similarity. It was found that in all cases the displacement thickness worked well as the outer layer similarity thickness scale and that the free-stream velocity works well as the outer layer velocity scaling variable. This is the same result we obtained theoretically herein for whole-profile similarity for this flow situation.

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Is this result merely a coincidence? Conventional thinking says that turbulent boundary layer flow over a wedge cannot display whole-profile similarity due to viscous effects (Townsend [6]). After careful study, we conclude that the result is not a coincidence. We show the reason the similarity scalings are the same for certain datasets is that the changes in the profile shape along the flow direction due to viscous effects are smaller than the experimental noise errors encountered in measuring the profile velocity. Therefore, to experimental accuracy, certain turbulent boundary layer data sets that show outer layer similarity actually show whole-profile similarity. If this result holds irrespective of Reynolds number, than this will have profound implications on modeling the turbulent boundary layer velocity profile. In what follows, we start with a review of the relevant mathematical theory of similarity and the equations that result.

2. The Mathematics of Similarity

2.1 Velocity Profile Scaling

In the analysis below, no assumptions are necessary as to the functional form of the velocity profile $u(y)$. The only requirements are that the boundary conditions for the velocity profile are known. The mathematical development presented herein is based on a simple concept; for similarity, the area under properly scaled velocity profile curves must be equal. Consider a 2-D flow along a body such that the y-direction is normal to the body's surface. The length and velocity scaling variables δ and u_s, as well as the other scaling variables used below, can vary with the flow direction but not in the y-direction. Starting with the formal definition of similarity, that is two velocity profiles are similar if they differ only by a scaling constant in y and $u(y)$, then it is self evident that for the profiles to be similar, the area under the properly scaled velocity profiles must be equal. The area under the scaled profiles, in mathematical terms, is given by

$$c_0 \;=\; \int_0^{h/\delta} d\left\{\frac{y}{\delta}\right\} \; \{u_e - u(y)\}/u_s \quad, \tag{1}$$

where c_0 is a nonzero numerical constant, $y = h$ is deep into the free stream, and u_e is the value of the stream-wise velocity $u(y)$ at the edge of the boundary layer. The integral is written using the velocity difference so that the integral value is not dependent on the numerical value of h as long as h is located deep in the free stream. Using a variable switch ($d\{y/\delta\} \Rightarrow (1/\delta)dy$) and simple algebra, Eq. 1 reduces to

$$c_0 \;=\; \frac{u_e \delta_1}{u_s \delta} \quad, \tag{2}$$

where the δ_1 is the displacement thickness given by

$$\delta_1 \;=\; \int_0^h dy \; \{1 - u(y)/u_e\} \quad. \tag{3}$$

Eq. 2 is an exact equation that applies whether the profiles are similar or not. A necessary condition for similarity is that the c_0 values for each profile of the set of profiles being tested are equal. For scaling purposes, one can take $c_0 = 1$ in Eq. 2, which then becomes the empirically

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derived velocity scale successfully used by Zagarola and Smits [2] to scale turbulent boundary flows over wedges, in channels, and in pipes. The importance of Eq. 2 in regards to similar profiles is that it means that the thickness scaling and the velocity scaling variables are not independent for 2-D wall-bounded similarity flows.

Having equal c_0 values is a necessary but not a sufficient condition for similarity of a set of profiles. If the scaled velocity profiles are similar, then it is self-evident that the scaled velocity profiles multiplied by the scaled y-coordinate raised to the nth power must also be similar. In mathematical terms, having equal area under the scaled velocity profiles multiplied by the scaled y-coordinate raised to the nth power is equivalent to

$$c_n = \int_0^{h/\delta} d\left\{\frac{y}{\delta}\right\} \left(\frac{y}{\delta}\right)^n \left\{u_e - u(y)\right\}/u_s , \quad (4)$$

where c_n are, in general, non-zero numerical constants. Mathematically, it is self-evident that a sufficient condition for similarity of a set of profiles is that the c_n values for $n=0,1,2,\ldots,\infty$ for each profile of the set are equal.

2.2 First Derivative Profile Scaling

By considering the area under the velocity profiles we were able to: 1) establish the inter-dependence of the length and velocity scaling variables and 2) provide an alternative definition of similarity. Using the same methodology from above, we can extend the mathematics of similarity to the first derivative profile and find additional similarity requirements for the velocity profile. If similarity is present in a set of velocity profiles then it is self-evident that the scaled first derivative profiles (derivative with respect to the scaled y-coordinate) must also be similar. It is also self-evident that the area under the scaled first derivative profiles must be equal for similarity. Furthermore, the area under the scaled first derivative profiles multiplied by the scaled y-coordinate to the power one must also be equal for similarity to exist.

In mathematical terms, equal area under the scaled first derivative profiles is expressed by

$$b = -\int_0^{h/\delta} d\left\{\frac{y}{\delta}\right\} \frac{d\left\{u_e - u(y)\right\}/u_s}{d\left\{\frac{y}{\delta}\right\}} , \quad (5)$$

where b is a non-zero numerical constant. Using the boundary conditions $u(0) = 0$ and $u(h) = u_e$ combined with a simple variable switch, Eq. 5 reduces to

$$b = \frac{u_e}{u_s} . \quad (6)$$

Therefore, for similarity of the velocity profiles in this geometry, the scaling velocity must be proportional to the free-stream velocity.

Next, having equal area under the scaled first derivative profiles times the scaled y-coordinate (to the power one) is equivalent to

$$d = \int_0^{h/\delta} d\left\{\frac{y}{\delta}\right\} \frac{y}{\delta} \frac{d\left\{u/u_e\right\}}{d\left\{\frac{y}{\delta}\right\}} , \quad (7)$$

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where d is a non-zero numerical constant. After a simple variable switch and integrating by parts, this equation reduces to

$$d = \delta_1 / \delta \quad . \tag{8}$$

Eq. 8 is important in that it states that if similarity exists in a set of profiles, then the displacement thickness must be a length scale that results in similarity for both the velocity profiles and the first derivative profiles. Therefore, using simple mathematics, we have determined that the displacement thickness δ_1 must be a similarity length scaling variable if similarity is found to exist in a set of velocity profiles. From Eq. 6, we also know that the free-stream velocity u_e must be a similarity velocity scaling variable for 2-D wall-bounded flows.

2.3 Partial Similarity Definitions

The above results apply to any 2-D wall-bounded flow for which similarity is found to be present in a set of velocity profiles. It is generally accepted by the fluid flow community that the 2-D wall-bounded turbulent boundary layer flow does not show whole-profile similarity (an assessment we do not agree with). Since closed solutions are not presently available, there has been considerable effort in the community to establish the possible existence of similarity solutions by finding "scaling laws" for different regions of the velocity profile for turbulent boundary layer flows (see for example, Buschmann and Gad-el-Hak [3]). These include the log law region, the near wall inner layer region, defined as the region for which the wall-induced viscous forces are important and the outer region, which is defined as the rest of the velocity profile that is not part of the inner region.

Consider first the case for partial similarity that deals with the so-called log law region that overlaps the inner and outer layers of a wall-bounded flow. There is some debate whether this region even exists at low Reynolds numbers and whether the profile should follow a logarithmic or exponential form [3]. Assuming the region exists, we can define a set of equations using the same technology from above to define similarity of the log law region. We start by defining a length scaling variable δ_o and a velocity scaling variable u_o for this overlap region of a turbulent boundary layer. A sufficient condition for proving similarity of the overlap region of a set of velocity profiles is that the integrals given by

$$p_n = \int_{h_i / \delta_o}^{h_o / \delta_o} d\left\{ \frac{y}{\delta_o} \right\} \left(\frac{y}{\delta_o} \right)^n \left\{ u_e - u(y) \right\} / u_o \quad , \tag{9}$$

are equal for $n = 1, 2, 3, \ldots, \infty$ for each profile in the set. It this set of equations $y = h_o$ is the location of the upper boundary of applicability and $y = h_i$ is the lower boundary of applicability for this overlap region.

In a similar fashion, we can define similarity of the inner region. For this case we define a length scale δ_i and a velocity scale u_i for the inner region of a turbulent boundary layer. A sufficient condition for similarity of a set of profiles in the inner region is that the integrals given by

$$g_n = \int_{0}^{h_i / \delta_i} d\left\{ \frac{y}{\delta_i} \right\} \left(\frac{y}{\delta_i} \right)^n \left\{ u_e - u(y) \right\} / u_i \quad , \tag{10}$$

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are equal for $n=1,2,3,\ldots,\infty$ for each profile in the set. In this set of integrals, $y=h_o$ is the upper limit of the inner region and the lower limit of the outer region.

Finally, to mathematically define similarity of the outer region, we start by defining a length scale δ_w and a velocity scale u_w for the outer region of a turbulent boundary layer. A sufficient requirement for similarity of a set of velocity profiles in the outer region is that the integrals given by

$$f_n = \int_{h_o/\delta_w}^{h/\delta_w} d\left\{\frac{y}{\delta_w}\right\} \left\{\frac{y}{\delta_w}\right\}^n \{u_e - u(y)\}/u_w \quad , \tag{11}$$

are equal for $n=0,1,2,\ldots,\infty$ for each profile in the set. In this set of integrals, $y=h$ is deep into the free stream and $y=h_o$ is the upper limit of the inner region. The usefulness for these partial similarity definitions will be discussed below.

3. Partial Similarity Scaling

For whole-profile similarity, we showed that the length and velocity scaling variables must be proportional to δ_1 and u_e. Unfortunately, the partial similarity case is not conducive to the same simple theoretical derivations that were used above to study scaling. In this section we will investigate what, if anything can be said about the length and velocity scaling for the partial similarity case. This section is separated from the previous case (whole-profile similarity) because the scaling results must be proven experimentally since the theoretical approach is not available.

Let us consider the particular case of partial similarity scaling of the outer region of a 2-D wall-bounded turbulent flow over a wedge. Based on a momentum balance argument, Castillo and George [4] have pointed out that for turbulent flows with a pressure gradient along the flow direction, the velocity scaling variable for this case must be u_e. Recently, Weyburne [5] investigated experimental datasets displaying outer region similarity of the turbulent flow over a wedge. The eleven experimental datasets included sets with an adverse, favorable, and zero pressure gradients. It was found that for the selected subset of profiles, the displacement thickness δ_1 worked well as the outer region similarity thickness scale and that the free-stream velocity at the boundary edge u_e worked well as the similarity velocity scale in all cases. This is the same result we obtained theoretically above for whole-profile similarity for this flow situation. This almost certainly is not a coincidence so the question becomes what underlying physical circumstances would lead to this result.

To investigate this situation, we start by assuming that a set of experimental profiles are measured and appear to result in similarity-like behavior in the outer region when plotted using the length scaling variable δ_1 and the velocity scaling variable u_e. This means that the f_0 values (Eq. 11, $n=0$) for each profile in the set must be equal to within experimental error. The mathematical difference between c_0 (Eq. 1) and f_0 using δ_1 and u_e as the scaling variables is

$$\Delta = \int_0^{h_o/\delta_1} d\left\{\frac{y}{\delta_1}\right\} \{u_e - u(y)\}/u_e \quad . \tag{12}$$

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If the value of Δ were equal for these same profiles that are similar in the outer region, then we would have the case that f_0 and Δ added together would result in equal c_0 values for each profile of the set. This means that the similarity results for the whole-profile discussed above would then apply. However, using a simple momentum balance approach, Townsend [6] has presented theoretical arguments that indicate that the viscous forces make it impossible to have exact similarity over the whole-profile for most 2D wall-bounded turbulent flows (with the exception of flow between a wedge, also called sink flow). This means that by Townsend's argument, Δ cannot be constant for 2-D boundary layer flow over a wedge as considered herein. However, consider the various adverse pressure gradient (APG), favorable pressure gradient (FPG) and zero pressure gradient (ZPG) experimental datasets that show similarity-like behavior in the outer region in Fig. 1 (the details of the experimental datasets used are given in Table 1). Conventional thinking says that the near-wall viscosity effects preclude whole-profile similarity. This is equivalent to saying that the **variation** of the value of Δ must be large for these cases. However, looking at plots of a number of turbulent boundary layer datasets in Fig. 1, it is obvious that the variation of the value of the area under the profiles from the wall ($y/\delta_1 = 0$) to
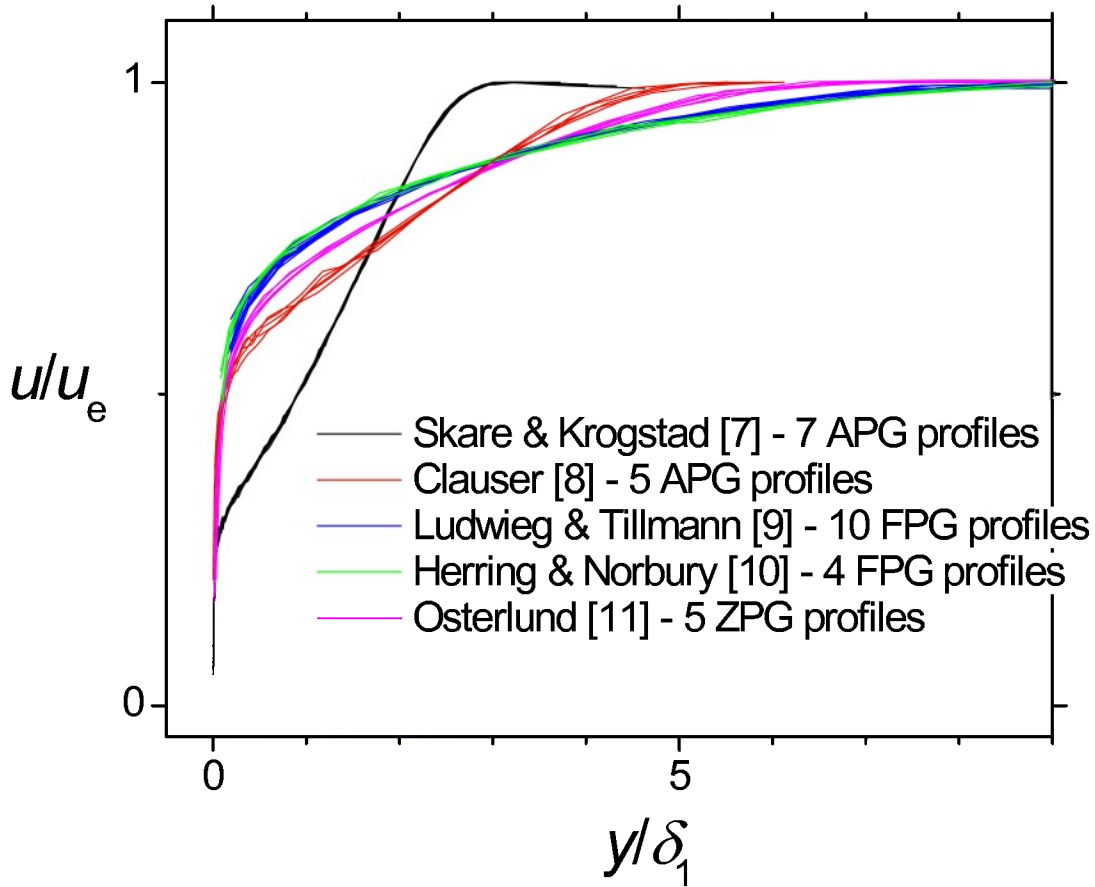

Fig. 1: APG, FPG, and ZPG velocity profiles from various sources.

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Author	Stations showing velocity profile similarity	Source of dataset
Skåre and Krogstad [7]	x=4.0, 4.2, 4.4, 4.6, 4.8,5.0, and 5.2	Author
Clauser [8]	x=18.58, 23.83, 26.92, 29.75, and 32.25	Coles and Hirst [12] Ident 2200
Ludwieg and Tillmann [9]	x=0.782, 1.282, 1.782, 2.282, 2.782, 3.132, 3.332, 3.532, 3.732, 3.932, 4.132, and 4.332	Coles and Hirst [12] Ident 1300
Herring and Norbury [10]	x= 2, 3, 4, and 5	Coles and Hirst [12] Ident 2700
Österlund [11]	x=1.5,2.5,3.5, 4.5 and 5.5 ($u_e \cong 10.3$ m/s) consisting of SW981129A, SW981128A, SW981127H, SW981126C, & SW981112A	Author

Table 1:   Summary of Datasets

the lower limit outer region ($\sim y / \delta_1$ = 0.2, see [18]) must be relatively small. The evolution of the viscous forces as one proceeds down the length of the wedge apparently are simply not large enough to drastically change the velocity profile shape. Hence the expected change in the area under the profile in this near-wall region must be small. If the variation of Δ is smaller than the experimental measurement error bars, then it can be said that to experimental accuracy, the Δ values would be equal. That would mean that to experimental accuracy, the set of velocity profiles would be similar. This, in turn, means that the scalings $\delta \propto \delta_1$ and $u_s \propto u_e$ must apply.

In order to prove that the variation of Δ is indeed small, consider a specific example, the outer layer data of Skåre and Krogstad [7] plotted in Fig. 2. In this figure, the black lines are seven outer layer experimentally measured velocity profiles taken with a spacing of 0.2 meters along a plate in an adverse pressure gradient. It is evident that the seven profiles show very good collapse using δ_1 and u_e scaling. An expanded view at different spots along the plot, not shown, shows that there is no noticeable Reynolds number-dependent behavior of the profile plots indicating outer region similarity. Now consider the intersection of the red and black lines in Fig. 2a. This is the location of the lower limit of the outer region (the identity of the red lines will be discussed below). According to established theory, if Skåre and Krogstad had measured the profiles all the way to the wall, then the area under the profiles would **not** be equal. However, looking at the data in expanded view in Fig. 2b, it appears that the near-wall differences are very small.

One way to establish that the viscous effects are truly small numerically would to use noise-free exact solutions obtained by solving the flow governing equations. However, this is not possible at this point in time for turbulent flows. An alternative approach is to use semi-empirical analytical models of the near-wall turbulent velocity profile, like the ones developed by Spadling [13], Musker [14], and Monkewitz, *et. al.* [15]. These semi-empirical models are
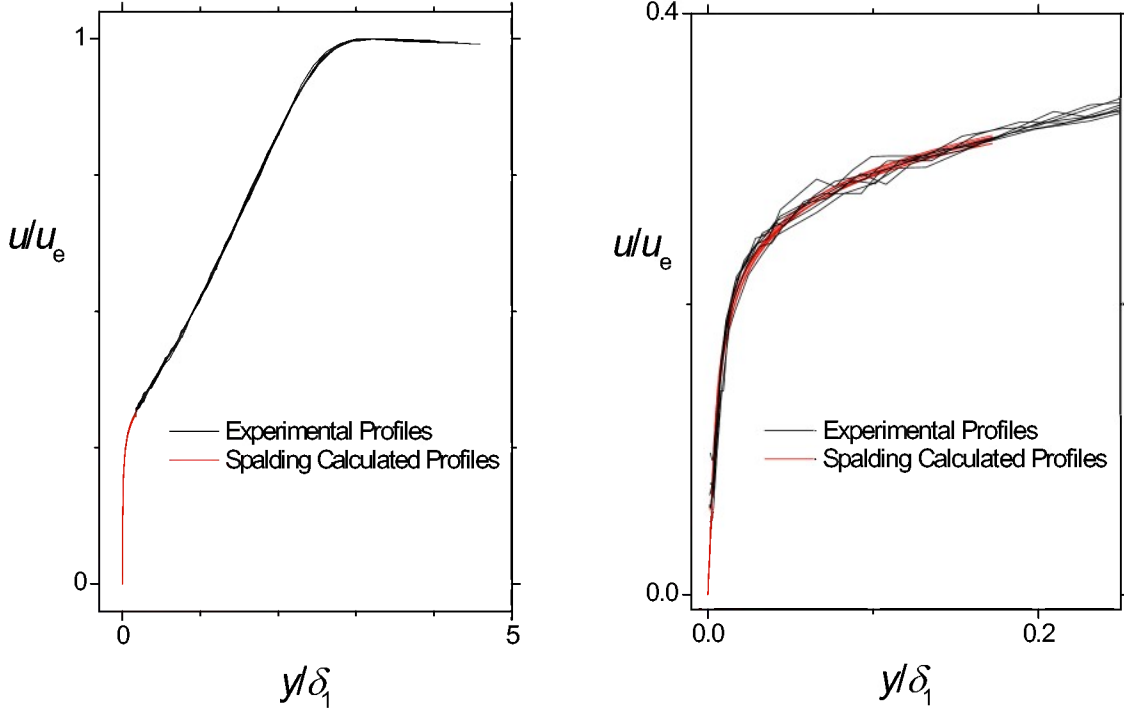
Figs. 2a and 2b: The seven velocity profiles (black lines) from Skåre and Krogstad [7] are depicted in *a*. The seven red lines are the semi-empirical Spadling [13] profiles. In *b* the *x*-*y* scale is expanded. In this figure all data points are included from Skåre and Krogstad [7] including the inner layer data.

considered to represent the inner viscous region of the turbulent boundary layer very well [15-17]. The calculated Spalding profiles are depicted in Fig. 2a by the red lines (using the Monkewitz, *et. al.* [15] log law constants, see Discussion below). The Musker and Monkewitz, *et. al.* plots, not shown, are essentially indistinguishable from the Spaulding plots of Fig. 2a. In all cases the analytical profiles were calculated using Skåre and Krogstad's tabulated data. Even at the expanded scale in Fig. 2b, the seven red lines appear to collapse to a single curve indicating similar-like profiles. In this plot, the seven Skåre and Krogstad profiles (includes all data points, not just the outer region points as in Fig. 2a) are plotted as black lines and the Spalding profiles as red lines. It is readily evident that the variation of the Spalding profiles (which model the actual profiles) is smaller than the experimental noise level. This means that the expected variation of the inner layer is smaller than what one can measure experimentally.

For confirmation, we now turn to the task of calculating the area under the scaled profiles. For the upper limit of integration for Δ (Eq. 12), we need to establish the value of h_o. Examining velocity plots from Skåre and Krogstad (their Fig. 3a) together with the log law curve indicates the inner region upper boundary is at about $y^+ \cong 300$. This corresponds to taking $h_o = 300\, u_\tau / v$. The value of $u(h_o)$ for each dataset was calculated using simple linear interpolation between adjacent experimental data points. To calculate δ_1 using the experimental data, we added the data point $u(0) = 0$ to each dataset. In Table 2, we summarize

Location	I_{sim}	Δ_{Musker}	$\Delta_{Spalding}$	Δ_{MCN}	f_0	$\Delta_{MCN} + f_0$
x=4.0	0.15603	0.12544	0.12676	0.12608	0.87254	0.99862
x=4.2	0.15620	0.12537	0.12669	0.12602	0.87311	0.99913
x=4.4	0.15486	0.12472	0.12605	0.12540	0.87375	0.99915
x=4.6	0.15521	0.12502	0.12633	0.12570	0.87471	1.00041
x=4.8	0.15575	0.12475	0.12607	0.12545	0.87401	0.99946
x=5.0	0.15531	0.12536	0.12666	0.12666	0.87267	0.99933
x=5.2	0.15467	0.12528	0.12657	0.12598	0.8755	1.00148
mean value =	0.15543	0.12513	0.12645	0.1259	0.87376	0.99965
s.d. =	5.8E-4	3.0E-4	3.0E-4	4.3E-4	0.0011	0.00152
Coef.Var.=	3.7E-3	2.4E-3	2.4E-3	3.4E-3	0.0013	0.00468

Table 2: Calculated integral values using $h_o = 300\, u_\tau / \nu$.

the numerical values for Δ (Eq. 12), designated Δ_{Musker}, $\Delta_{Spalding}$, and Δ_{MCN}, and f_0 (Eq. 11, n=0). The f_0 values were calculated using the experimental data points using the Trapezoidal rule and typically involved 35-38 points. The integrals Δ_{Musker}, $\Delta_{Spalding}$, and Δ_{MCN} are the values of the approximate Δ calculated using the semi-empirical analytical velocity profiles. These integrals were calculated using 1000 calculated data points and using the Trapezoidal rule integration formula.

The Coefficient of Variation (Coef.Var.), defined as the standard deviation (s.d.) divided by the mean, for the approximate Δ calculated in Table 2 only account for the variation due to viscosity induced changes to the profile shape. If it were possible to experimentally measure profiles all the way to the wall, the Δ calculation would also include an experimental noise component. In fact, looking at Fig. 2b, it is apparent that experimental noise on the profile will have a significant effect on the variation of Δ. To try to estimate this effect, we added the column designated I_{sim} to Table 2. The idea is to estimate the Coef.Var. of integrating an area of approximately equal size to the analytical Δ but in a section of the outer layer profile considered to be similar. Since the profiles in this region are similar, then the only contribution to the Coef.Var. will be from noise. Therefore we picked an area under the outer layer region of Fig. 2a that has approximately the same area as Δ and that is immediately adjacent to Δ. This area, I_{sim}, is calculated for each profile using a lower limit of $y/\delta_1 = 0.17$ ($\cong 300\, u_\tau / \nu$) and an upper limit of $y/\delta_1 = 0.41$. This area should give a reasonable estimate of the Coef.Var. of Δ due to just experimental noise.

There are two important points that can be made from the Table 2 data. First, notice that the Coef.Var. of I_{sim} is about the same as the Coef.Var. for Δ_{Musker}, $\Delta_{Spalding}$, or Δ_{MCN}. This means that even if one had a way to measure the velocity profile all the way to the wall with present day technology, the experimental noise would make it very difficult to see the variation of the velocity profile due to viscosity effects. We estimate that the variation of Δ due to experimental noise would need to be at least an order of magnitude smaller in order to see a Reynolds number-induced variation of Δ caused by just the viscosity effect. The second point is that the value of $\Delta_{MCN} + f_0$ is very close to a value of one for the Skåre and Krogstad data. The

same is true for $\Delta_{Musker} + f_0$ and $\Delta_{Spaulding} + f_0$ (not shown). For whole-profile similarity, Eq. 2 requires that $c_0 = \Delta + f_0 = 1$ for this choice of scaling variables. Therefore, to experimental accuracy, the seven Skåre and Krogstad velocity profiles exhibit whole-profile similarity behavior. This in turn means that for this dataset, we must have the scalings $\delta \propto \delta_1$ and $u_s \propto u_e$ according to the results in Section 2.

The Skåre and Krogstad dataset was used because the predicted inner layer variation was small. We found that not all profiles that showed outer layer similarity, as in Fig. 1 for example, will necessarily show small variations of the similarity plots predicted by the Spadling, Musker, or Monkewitz, *et. al.* semi-empirical profiles. However, we did find two other datasets in the literature that do show the same predicted small variation of the inner layer behavior as the Skåre and Krogstad dataset. The data of Clauser [8] and Herring and Norbury [10] datasets, plotted in Figs. 3a and 3b, do support the notion that the expected viscosity-induced variation of the profile shape is indeed small for certain datasets displaying outer layer similarity. Furthermore, the plots again indicate that the inner layer variation would be very difficult to measure even with present-day technology.
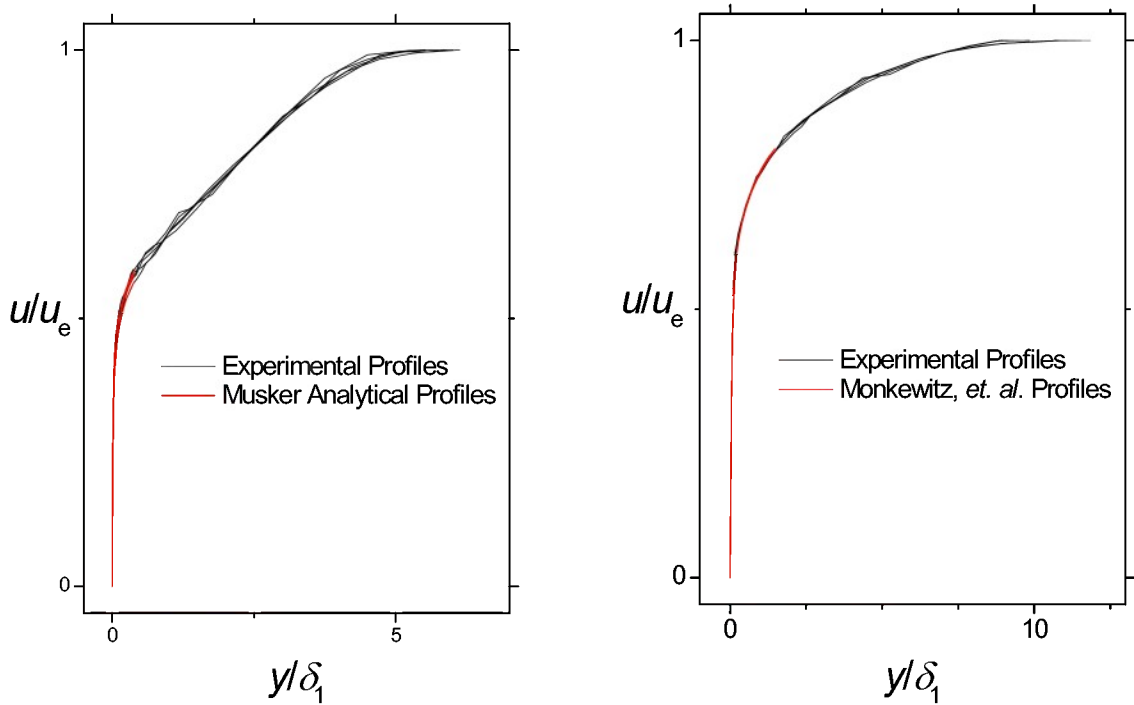
Figs. 3a and 3b: Five velocity profiles (black lines) from Clauser [8] are depicted in *a*. In *b* the four velocity profiles (black lines) are from the Herring and Norbury [10]. The red lines in both figures are the analytical profiles.

3. Discussion

The new integral-based theoretical approach to similarity was successful in showing that if similarity exists in a set of velocity profiles for 2-D wall bounded flows, then the similarity length scale must be proportional to the displacement thickness δ_1 and the similarity velocity scale must be proportional to the free-stream velocity u_e. This is the first time that certain length and velocity scales have been shown to be similarity length and velocity scales for a whole class of flows, in this case 2-D wall-bounded laminar, transitional, and turbulent flows. Other length and velocity scaling variables are possible besides δ_1 and u_e. However, the thickness scaling and the velocity scaling variables cannot be independent and must be related according to Eq. 2.

Another important practical contribution in this paper deals with the problem of discovering and quantifying partial similarity for the turbulent boundary layer. In the past, similarity of experimental curves has been determined using the "Chi-by-eye" method to judge whether plots of a set of profiles collapse to a single curve. With the new integral method, we can calculate and compare the values using Eqs. 9, 10, or 11 (depending on which area is of interest). Similar velocity profiles, that is, scaled velocity profiles that are superposable so as to be nearly coincident throughout the boundary layer region of interest, should have equal integral values within experimental error and show no Reynolds number trends. These numerical values can be tested for equivalence using standard statistical methods thereby removing the subjectivity inherent in the "Chi-by-eye" method.

The new similarity scaling results are based on the mathematical implications of similarity. That is, if one has a set of mathematical curves that collapse to a single new curve upon scaling, then one can derive information about the properties of the scaling without having to know how the curves were generated. This is a fundamentally different approach to similarity than the flow governing equation approach. In the flow governing equation approach, one tries to determine the mathematical implications of reducing the set of scaled partial differential equations that govern the flow to a set of ordinary differential equations. The advantage of the new approach over the flow governing equation approach is that it works for laminar, transitional, and turbulent flows. However, the new approach is only strictly valid for whole-profile similarity. Therefore, while interesting from a theoretical perspective, the whole-profile similarity restriction would seem to marginalize the new results to applications involving laminar flows since it has been universally accepted that whole-profile similarity is not possible for turbulent boundary layer flow. Since laminar flows are already known to be similar and its scaling variable behavior is known, then what is the utility of the new the scaling variable results?

The answer comes from the last Section concerning similarity scaling of the outer region of turbulent boundary layer flow. Based on theoretical arguments, conventional thinking says that the turbulent boundary layer cannot display whole-profile similarity due to viscosity effects. We showed that within experimental accuracy, whole-profile similarity does exist for certain turbulent boundary layer datasets. Thus the similarity results in Section 2 apply and the similarity scaling must be given by $\delta \propto \delta_1$ and $u_s \propto u_e$. Therefore, the new theoretical scaling results are supported by experimental results for both the laminar and turbulent boundary layer flows.

How can one rationalize this result with conventional thinking that says that the turbulent boundary layer cannot display whole-profile similarity. The conventional wisdom case is based on the argument that the x-dependent variable groupings appearing in the flow momentum equations must have the same functional dependence as the flow develops along the wedge.

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Equivalently, one can divide the momentum equations through by one of the variable groupings and check for constancy of the resulting parameters. Townsend [6] used this approach to show that if one includes the viscous force term parameter for turbulent flows, then these parameter ratios require that the similarity length scale δ must be linear in x and the similarity velocity scale u_s must go as $1/x$ (*i.e.*, wedge sink flow). These results are indisputable. However, the momentum equation argument does not account for the magnitude of the viscosity term as one proceeds down the wedge, or in this case the variation of the magnitude of the term as one proceeds down the wedge. Since the flow governing equations are partial differential equations, it is always necessary to neglect certain terms because their magnitude is small relative to the other terms in order to obtain similarity solutions. Consider, for example, the case of laminar flow on a flat plate first treated by Blasius [19]. Reduction of the x-momentum equation to an ordinary differential equation is only possible if one neglects the pressure term $\partial P/\partial x$ (among others). This term is small relative to the other terms but it is definitely nonzero and its value changes as one proceeds down the plate. The consequences of completely neglecting the $\partial P/\partial x$ term is that the Blasius solution predicts a reasonable parallel velocity profile solution but we are forced to accept the fact that the predicted normal velocity component reduces to a non-physically-realizable constant value at an infinite distance from the plate surface. It is our contention that for certain turbulent boundary layer flows, that while the magnitude of the viscous term in the momentum equation is not negligible as one proceeds down the length of the wedge, the variation of the magnitude of the term is smaller than what can be measured experimentally. In the section above, we presented semi-empirical based arguments that indicate that the viscous term variation as one proceeds down the length of a wedge for certain turbulent flows is indeed small. Therefore, the Townsend-based requirements are not always applicable and whole-profile similarity is possible for certain 2-D wall bounded turbulent boundary layer flows.

Additional support for the violation of the Towsend viscosity-based similarity argument comes from the success of the Prandtl scaling in producing similarity of the velocity profiles in the inner region of the turbulent boundary layer. How can one explain this near universality of the success in light of Towsend's viscosity argument? According to this argument, inner layer similarity should not be possible for the turbulent boundary layer except for sink flow. And yet there has been overwhelming experimental evidence that inner layer similarity using the Prandtl plus scaling is universally successful for all turbulent boundary layers. The only answer that makes sense is that, as we have already contended above; the expected differences in the velocity profiles due to viscosity must be smaller than what can be measured experimentally.

The experimental results for Clauser [8], Herring and Norbury [10], and Skåre and Krogstad [7] are some of the datasets that we found that showed small variations of the inner viscous region. What they all have in common is that the experimental setup for these three datasets were specifically designed to obtain equilibrium flows of the Clauser type. One of the keys in trying to experimentally develop Clauser equilibrium layers is to try to adjust the pressure gradient along the wedge to try to keep the skin friction coefficient small and constant. We believe that this is the reason for small viscosity induced changes to the velocity profile along the wedge length. Keeping the skin friction coefficient constant means keeping the velocity gradient at the wall constant. Note that this is the un-scaled velocity gradient. This in turn means the viscous induced changes along the wedge will be small since the un-scaled near-wall velocity profiles will be similar since the velocity gradient boundary conditions are similar.
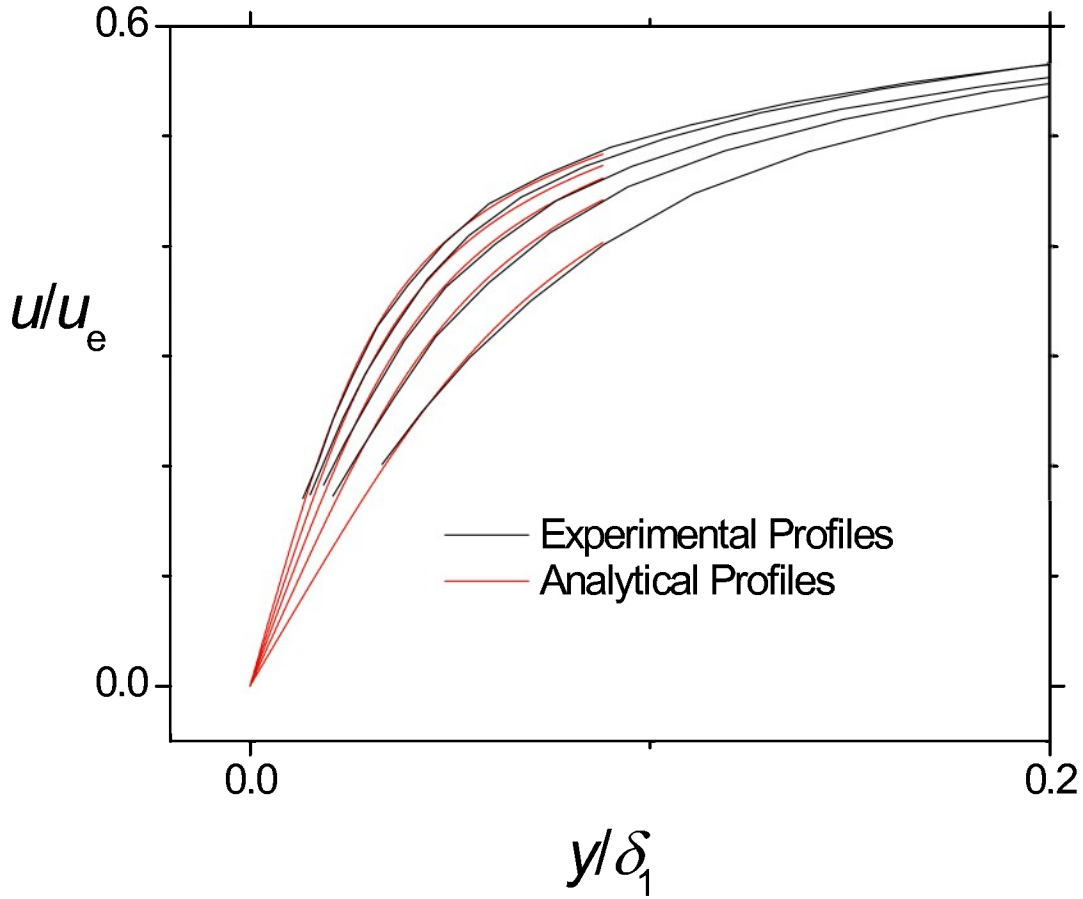
Fig. 4: The five velocity profiles (black lines) from Österlund [11] and the five red lines are the analytical Musker [14] profiles.

Of the other datasets that we looked at that show similarity-like behavior in the outer layer (see Fig. 1), most did not show the small variations in the velocity plots as those presented above. Instead, plots of the profiles are similar in appearance to Österlund's [11] data shown in Fig. 4. The data in Fig. 4 clearly indicates large Reynolds number dependent variations of the viscous forces resulting in a lack of velocity profile similarity in the near wall region. Thus not all turbulent boundary layer profiles showing outer layer similarity with Fig. 1 type scaling will necessarily show whole-profile similarity. This type of similarity, which we term similarity-like behavior, is obviously not true similarity. However, true similarity like that in Figs. 2-3, is relatively rare while the similarity-like behavior is more common. Thus, for engineering purposes, the similarity-like behavior scenario is still a useful concept for characterizing turbulent boundary layers.

More experimental work needs to be done to prove or disprove whole-profile similarity of the turbulent boundary layer. As we have already pointed out, the experimental verification of viscosity-induced profile variations will be very difficult with the measurement apparatus presently available. For example, DeGraaff and Eaton [20] indicate that using latest available technology, Laser Doppler Anemometry, the velocity can only be measured with $\pm 1.5\%$

uncertainty. This is simply not adequate to see Reynolds number-dependent velocity profile variations due to viscosity-induced effects for certain flows. Near-wall effects will only compound the measurement problem. While this may be the opening salvo on the debate over whole-profile similarity, it is clear that to experimental accuracy, certain existing datasets from the literature do display whole-profile similarity for turbulent boundary layer flows.

The ZPG data plotted in Fig. 4 may not show whole-profile similarity but it does support our contention that the semi-empirical analytical velocity profiles follow the experimental data rather well. Note that the semi-empirical velocity profiles from Spadling [13], Musker [14], and Monkewitz, *et. al.* [15] all subsume the log law for turbulent flows. It is known that pressure gradients can cause a shift in the velocity profile from the "universal" log law profile [21]. For the velocity profile datasets with pressure gradients used herein we tested various von Karmon log law constants from the literature including the traditional values, the newer Monkewitz, *et. al.* [15] ZPG values, and the APG values of Nagib, *et. al.* [21] by plotting the data along wth the

	Δ_{Musker}	$\Delta_{Spalding}$	Δ_{MCN}
Coef.Var. for $\kappa = 0.384$	2.4E-3	2.4E-3	3.4E-3
Coef.Var. for $\kappa = 0.359$	2.6E-3	2.6E-3	2.8E-3
Coef.Var. for $\kappa = 0.41$	2.3E-3	2.4E-3	2.3E-3

Table 3: Coefficient of Variation (Coef.Var.) using various Log Law constants.

various log law lines. It was found that the Monkewitz, *et. al.* [15] ZPG values worked well even for the velocity profiles with pressure gradients. Nevertheless, for assurance, we calculated the semi-empirical approximate Δ (Eq. 12) integrals using the different sets of the von Karmon log law constants. The results are summarized in Table 3. It is evident that the calculated Coef.Var. of the Δ_{Musker}, $\Delta_{Spalding}$, and Δ_{MCN} values are all similar. This means that even if the log law curves are shifted, the variation of the profiles will remain small. Therefore, we are confident that the arguments based on the use of the semi-empirical velocity profiles given above are valid. Furthermore, we must point out that the only way to discredit the semi-empirical based argument above is to discredit the log law for the turbulent boundary layers flows involved, that being equilibrium flows of the Clauser type.

Recall that Eq. 2 is identical to the velocity scaling first suggested empirically by Zagarola and Smits [2] for turbulent boundary layer flow including channel, wedge, and pipe flow (for scaling purposes, one can take any nonzero numerical value for c_0 including $c_0 = 1$). Zagarola and Smits [2] and others [4,22,23] (see also discussion and references in Buschmann and Gad-el-Hak [3]) have shown that Eq. 2 is successful at collapsing a number of experimentally obtained velocity profile data sets. The reason for the success has never been fully explained from a theoretical standpoint [3]. At first glance the success may seem reasonable but recall that Eq. 2 was derived herein based on whole-profile similarity. Therefore, one explanation for the success of the Zagarola and Smits scaling is that the data profiles actually exhibit whole-profile similarity (to experimental accuracy) or the outer region similarity-like behavior demonstrated

by Österlund's [11] datasets in Figs. 1 and 4. A simple first step to verify this finding is to replot the data exhibiting good profile collapse with Zagarola and Smits scaling using $\delta \propto \delta_1$ and $u_s \propto u_e$ scaling instead. Weyburne [5] has already had good success in collapsing many of the available experimental datasets using these scaling. This would add credibility to the whole-profile similarity scenario.

4. Conclusion

A fundamentally new way of deriving flow similarity criteria for velocity profiles was presented. It was shown that for whole-profile similarity of 2-D wall bounded velocity profiles; the displacement thickness must be a similarity length scaling variable and the velocity scaling variable must be the free-stream velocity at the boundary layer edge. It was shown that to experimental accuracy, certain turbulent boundary layers can exhibit whole-profile similarity and, in agreement with the new theoretical results, the displacement thickness and the free stream velocity at the boundary layer edge work well as scaling variables. In addition, new similarity definitions were developed that avoid the subjectivity inherent in the conventional method used to assert the existence of similarity in a set of experimental profiles.

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