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Discrete Optimization

The minimum size instance of a Pallet Loading Problem equivalence class

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Abstract

The Pallet Loading Problem (PLP) maximizes the number of identical rectangular boxes placed within a rectangular pallet. Boxes may be rotated 90° so long as they are packed with edges parallel to the pallet's edges, i.e., in an orthogonal packing. This paper defines the Minimum Size Instance (MSI) of an equivalence class of PLP, and shows that every class has one and only one MSI. We develop bounds on the dimensions of box and pallet for the MSI of any class. Applying our new bounds on MSI dimensions, we present an algorithm for MSI generation and use it to enumerate all 3,080,730 equivalence classes with an area ratio (pallet area divided by box area) smaller than 101 boxes. Previous work only provides bounds on the ratio of box dimensions and only considers a subset of all classes presented here.

Keywords: Packing; Pallet Loading Problem

1. Introduction

We identify each instance of a Pallet Loading Problem (PLP) by a quadruple (X, Y, a, b). We have a rectangular pallet with length X and width Y $(X \geqslant Y)$, and a rectangular box with length a and width b $(a \geqslant b)$. Boxes may be rotated 90° so long as they are placed with edges parallel to the pallet's edges, i.e., the packing must be orthogonal. We can assume, without loss of generality, that X, Y, a, b are positive integers (e.g., Bischoff and Dowsland, 1982). We also assume that at least one box can be packed in the pallet: $X \geqslant a$ and $Y \geqslant b$.

We encounter PLP when trying to maximize the number of identical boxes with dimensions a and b, placed on a pallet with dimensions X and Y where each box has a "this side up" restriction (e.g., Bischoff and Dowsland, 1982). Even without the "this side up" restriction, operational considerations may dictate the use of vertical layers with the same height. Issues of stability and safety of the boxes imply the use of orthogonal packing

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(e.g., Dowsland, 1987a; Nelissen, 1995; Young-Gun and Maing-Kyu, 2001). PLP also arises in some cutting stock and floor design settings.

Although PLP has been widely studied and it is known that instances can be divided into classes with the same optimal placement pattern (Dowsland, 1984), no procedure to generate all distinct classes for a given number of boxes has previously been reported. Dowsland (1987b) works with a subset of approximately 8000 equivalence classes and Scheithauer and Terno (1996) work with a randomly-generated subset of approximately 50,000 equivalence classes. A common set of restrictions on pallet and box dimensions first proposed by Dowsland (1984) has been used by other authors (e.g., Nelissen, 1993; Scheithauer and Terno, 1996; Morabito and Morales, 1998). These restrictions are on the aspect ratio of the pallet ($1 \leqslant X/Y \leqslant 2$), of the box ($1 \leqslant a/b \leqslant 4$), and the area ratio ($1 \leqslant (X * Y)/(a * b) < 51$). Nelissen (1995) and Naujoks (as reported by Nelissen (1995)) also investigate instances where $51 \leqslant (X * Y)/(a * b) < 101$ as do more recent papers (e.g., Alvarez-Valdes et al., 2005; Birgin et al., 2005; Lins et al., 2003) that apply exact algorithms and heuristics to a common set of about 50,000 instances. Recent work also includes detailed analysis of upper bounds for PLP (Letchford and Amaral, 2001).

This paper defines the Minimum Size Instance (MSI) of an equivalence class of PLP, and shows that every class has one and only one MSI. We develop bounds on the dimensions of box and pallet in the MSI of each class. Applying our newly-developed bounds on the MSI dimensions, we present an algorithm for MSI generation and use it to enumerate all 3,080,730 equivalence classes with an area ratio (pallet area divided by box area) smaller than 101 boxes. Previous work only provides bounds on the ratio of box dimensions and only considers a subset of all classes: this limits results. Martins (2003) finds all instances from 3,073,724 of these 3,080,730 classes can be solved easily. Given the small number of difficult instances, it is not surprising that many have been previously overlooked.

2. Efficient partitions and equivalence classes

Some PLP instances, with different dimensions, possess the same arrangement of boxes in an optimal solution. For example, the arrangement depicted in Fig. 1 is an optimal solution to the instance (22, 16, 5, 3) where the shaded regions indicate unused (wasted) areas of the pallet. The same arrangement is also optimal, for example, to instances (30, 22, 7, 4) and (50, 36, 11, 7).

Let (n, m) denote an ordered pair of non-negative integers satisfying

$$n * a + m * b \leqslant S \tag{1}$$

for a pallet dimension S, which could be X or Y. Such an ordered pair (n, m) is called a *feasible partition* of S. If n and m also satisfy

$$0 \leqslant S - n * a - m * b < b \tag{2}$$

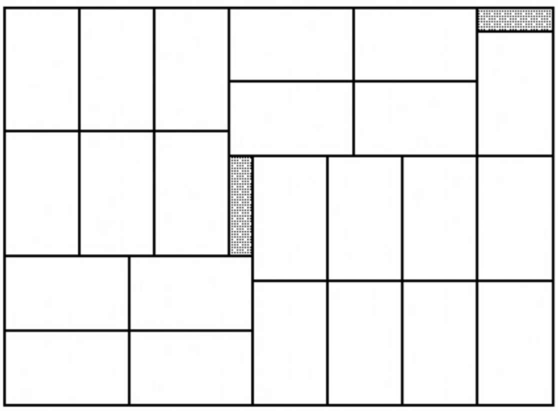

Fig. 1. Optimal solution arrangement for instances (22, 16, 5, 3), (30, 22, 7, 4), (50, 36, 11, 7), and all instances within the same equivalence class.

then (n,m) is called an *efficient partition* of S (Bischoff and Dowsland, 1982). For a pallet dimension S, the set of efficient partitions of S, denoted by $E(S,a,b)$, is defined to be the set of all feasible partitions (n,m) satisfying

$$n \in \{0,1,\ldots,\lfloor S/a \rfloor\} \quad \text{and} \quad m = \lfloor (S - n*a)/b \rfloor.$$

Dowsland (1984) shows that if two instances of PLP possess the same set of efficient partitions for both the pallet width and length, then both instances share the same set of optimal solutions. This defines a relation in the set of instances of PLP, which is reflexive, symmetric, and transitive. Therefore, the set of instances of PLP can be divided into equivalence classes, based on the set of efficient partitions. If a solution is known for a class representative, then this solution can be used on any other instance in the class. Because multiplying all dimensions by an integer produces a new instance in the same class, it is easy to see that each class contains infinitely many instances.

If, in addition, n and m satisfy

$$n*a + m*b = S,$$

then (n,m) is called a *perfect partition* of S (Dowsland, 1984). In general, each of these sets can be empty, but the instance of an equivalence class with minimal pallet dimensions contains at least one perfect partition for each dimension, X and Y (Dowsland, 1984). This is easily observed if we consider an arbitrary instance without a perfect partition for a given dimension. In this case, we can reduce the corresponding dimension of the pallet without altering the set of efficient partitions. This implies that the new instance, with a smaller pallet dimension, also belongs to the same class.

3. Representing equivalence classes

Because instances of PLP in the same equivalence class share the same set of optimal solutions, once one instance is solved, the solution can be stored in a database and retrieved whenever a solution to an instance of PLP of the same class is necessary (Dowsland, 1987a). Many PLP instances are easily solved so storage may only be necessary for difficult instances. The most straightforward way to identify an equivalence class in a database is to encode the set of efficient partitions defining the class. This way, given a new instance, it is possible to compute the set of efficient partitions and compare it with the entries in the database. One possible problem is that the cardinality of this set increases with the number of boxes packed.

Another approach is to select a unique class representative. This way, only four integers are necessary to represent the class, independent of the number of boxes in the optimum packing. One option for defining an equivalence-class representative is the instance that minimizes the area ratio, the *Minimum Area Ratio Instance* (*MARI*). But the minimization problem can have a solution at an open boundary, or at a non-integral interior point (Dowsland, 1984). In these cases the dimensions of the MARI can only be approximated, when using integers. Different approximations can generate different instances within the same class, complicating the identification process.

Another candidate for equivalence-class representative is the Minimum Size Instance (MSI), the instance that minimizes the dimensions of both the pallet and the box. We say $(\widetilde{X},\widetilde{Y},\tilde{a},\tilde{b})$ is the *Minimum Size Instance* of a class if for all instances (X,Y,a,b) in the same class, $\widetilde{X} \leqslant X$, $\widetilde{Y} \leqslant Y$, $\tilde{a} \leqslant a$, $\tilde{b} \leqslant b$.

4. Existence and uniqueness of the minimum size instance

When a dimension of the pallet is not a non-negative integer combination of the box's dimensions, Dowsland (1984) observes that the dimension of the pallet can be reduced. Let $G(S,a,b) = \max_{(i,j)\in E(S,a,b)} \{i*a + j*b\}$. We call $G(S,a,b)$ the *Perfect Partition Equivalent* function. Given an instance (X,Y,a,b) of PLP, the reduced dimensions of the pallet are given by $X^* = G(X,a,b)$, and $Y^* = G(Y,a,b)$. Therefore, if the dimensions of the box in the MSI, $\tilde{a}$ and $\tilde{b}$, are known, then the dimensions of the pallet are given by $\widetilde{X} = G(X,\tilde{a},\tilde{b})$ and $\widetilde{Y} = G(Y,\tilde{a},\tilde{b})$.

We show that the MSI is unique in a class and its dimensions can be easily bounded, simplifying the process of enumerating equivalence classes.

Theorem 1. *Every equivalence class of PLP has one and only one MSI.*

Proof. We initially show that there is no more than one MSI in each class. Then we show that every class has at least one MSI.

Suppose (X_1, Y_1, a_1, b_1) and (X_2, Y_2, a_2, b_2) are two MSIs in an equivalence class. By definition, both instances minimize all dimensions of the pallet and the box ($X_1 \leqslant X_2$, $Y_1 \leqslant Y_2$, $a_1 \leqslant a_2$, $a_1 \leqslant a_2$ and $X_2 \leqslant X_1$, $Y_2 \leqslant Y_1$, $a_2 \leqslant a_1$, $a_2 \leqslant a_1$), implying $X_1 = X_2$, $Y_1 = Y_2$, $a_1 = a_2$, $a_1 = a_2$. Therefore, if there is a MSI, it is unique.

Now consider an equivalence class. Because the dimensions of the pallet in the MSI are a function of the dimensions of the box in the MSI, the only way for a class not to have an MSI is if there exists one instance, say $(X_1, Y_1, \tilde{a}, b_1)$, with minimum length for the box (i.e., $\tilde{a} \leqslant a$ for all instances (X, Y, a, b) in the same class) and another instance, $(X_2, Y_2, a_2, \tilde{b})$, in which the box has minimum width (i.e., $\tilde{b} \leqslant b$ for all instances (X, Y, a, b) in the class). In this case, $a_2 > \tilde{a}$ and $b_1 > \tilde{b}$. The strict inequalities hold because otherwise at least one of the instances would have the box with both minimum dimensions. As both instances belong to the same class, $E(X_1, \tilde{a}, b_1) = E(X_2, a_2, \tilde{b})$ and $E(Y_1, \tilde{a}, b_1) = E(Y_2, a_2, \tilde{b})$. We show that the MSI can be identified from these two instances.

As Dowsland (1987a) shows, a scaled instance of PLP remains in the same equivalence class. After scaling, the dimensions of the pallet and box may no longer be integers. Normalizing the width of the box to 1 in the above instances, we obtain instances $(X_1/b_1, Y_1/b_1, \tilde{a}/b_1, 1)$ and $(X_2/\tilde{b}, Y_2/\tilde{b}, a_2/\tilde{b}, 1)$. Because $b_1 > \tilde{b}$ and $\tilde{b} > 0$, then $1/\tilde{b} > 1/b_1$, and this result together with $a_2 > \tilde{a}$ give us $a_2/\tilde{b} > \tilde{a}/\tilde{b} > \tilde{a}/b_1$. Because an equivalence class is a convex set (Nelissen, 1993), there is an instance $(X_0, Y_0, \tilde{a}/\tilde{b}, 1)$ in the class. If we multiply the dimensions by $\tilde{b}$ we obtain the instance $(X_0 * \tilde{b}, Y_0 * \tilde{b}, \tilde{a}, \tilde{b})$. We can apply the perfect partition equivalent function, obtaining $\widetilde{X} = G(X_0 * \tilde{b}, \tilde{a}, \tilde{b}) = \max_{(i,j)\in E(X_1,\tilde{a},b_1)}\{i * \tilde{a} + j * \tilde{b}\}$ because $E(X_0 * \tilde{b}, \tilde{a}, \tilde{b}) = E(X_1, \tilde{a}, b_1)$ and $\widetilde{Y} = G(Y_0 * \tilde{b}, \tilde{a}, \tilde{b}) = \max_{(i,j)\in E(Y_1,\tilde{a},b_1)}\{i * \tilde{a} + j * \tilde{b}\}$ because $E(Y_0 * \tilde{b}, \tilde{a}, \tilde{b}) = E(Y_1, \tilde{a}, b_1)$. The instance $(\widetilde{X}, \widetilde{Y}, \tilde{a}, \tilde{b})$ satisfies the requirements to be the MSI of the class. Therefore, the class has a MSI. □

5. Bounds on the dimensions of the MSI of an equivalence class

For instance (X, Y, a, b), let $A_x \equiv \lfloor X/a \rfloor$, $A_y \equiv \lfloor Y/a \rfloor$, $B_x \equiv \lfloor X/b \rfloor$, and $B_y \equiv \lfloor Y/b \rfloor$.

Dowsland (1987a) shows that $a \leqslant B_x + 1$ and $b \leqslant A_x + 1$ when considering the set of ratios a/b corresponding to equivalence classes. While these limits on a and b bound the ratio, they do not bound a and b in an equivalence class. For example, instance (104, 90, 15, 13), MSI of its class, where $A_x = 6$, $A_y = 6$, $B_x = 8$, $B_y = 6$, with $b = A_x + A_y + 1$ and $a = B_x + B_y + 1$. Theorem 2 shows these are upper bounds for any MSI.

Theorem 2. $\tilde{b} \leqslant A_x + A_y + 1$ *and* $\tilde{a} \leqslant B_x + B_y + 1$.

Proof. Given instance (X, Y, a, b), the optimal solution to the integer program below is the MSI for its equivalence class. Using the optimal solution to its linear programming relaxation, we show how to construct a PLP instance from the equivalence class that satisfies the bounds of Theorem 2.

Indices

i efficient partitions on length, $i = 0, \ldots, A_x$,
f efficient partitions on width, $f = 0, \ldots, A_y$.

Data

px_i number of boxes with their largest dimension oriented, vertically in partition i of the length,
py_f number of boxes with their largest dimension oriented horizontally in partition f of the width.

Variables

$(\breve{X}, \breve{Y}, \breve{a}, \breve{b})$ variables for (X, Y, a, b).

Formulation

$$\begin{aligned}
&\text{Minimize} \quad \breve{b}\\
&\text{subject to}\\
&\breve{X} - i * \breve{a} - px_i * \breve{b} \geqslant 0, \quad \forall i \in \{0,1,\ldots,A_x\}, && \text{(S1)}\\
&i * \breve{a} + (px_i + 1) * \breve{b} - \breve{X} \geqslant 1, \quad \forall i \in \{0,1,\ldots,A_x\}, && \text{(S2)}\\
&(A_x + 1) * \breve{a} - \breve{X} \geqslant 1, && \text{(S3)}\\
&\breve{Y} - f * \breve{a} - py_f * \breve{b} \geqslant 0, \quad \forall f \in \{0,1,\ldots,A_y\}, && \text{(S4)}\\
&f * \breve{a} + (py_f + 1) * \breve{b} - \breve{Y} \geqslant 1, \quad \forall f \in \{0,1,\ldots,A_y\}, && \text{(S5)}\\
&(A_y + 1) * \breve{a} - \breve{Y} \geqslant 1. && \text{(S6)}\\
&\breve{X}, \breve{Y}, \breve{a}, \breve{b} \ \text{integer}
\end{aligned}$$

We know from Theorem 1 that we can minimize $\breve{a}$ or $\breve{b}$ and obtain the MSI, here we minimize $\breve{b}$. The constraint sets (S1) and (S4) ensure feasible partitions, Eq. (1), and are called *fitting constraints*. The other constraint sets ensure efficient partitions, Eq. (2), and are called *efficiency constraints*. We call the linear programming relaxation (Primal).

Inspection of constraints (S1) and (S2) reveals that the addition of the constraints corresponding to the same value of i in each set bounds $\breve{b}$ below by 1, i.e., $(\breve{X} - i * \breve{a} - px_i * \breve{b}) + (i * \breve{a} + (px_i + 1) * \breve{b} - \breve{X}) = \breve{b} \geqslant 1$. Therefore, when (Primal) is feasible, it has an optimal solution $(\breve{X}^*, \breve{Y}^*, \breve{a}^*, \breve{b}^*)$, with objective function value at least 1. In this optimal solution, at least four constraints are binding because it is a four-dimensional linear program and the variables have no non-negativity constraints.

Because the MSI of a class has one perfect partition in each dimension, then, at least two fitting constraints, one in the length (S1) and one in the width (S4), are binding in an optimal solution. Also, at least one efficiency constraint is binding – from (S2), (S3), (S5) or (S6).

The dual of (Primal) has only four rows because the primal has only four variables. This makes it easier to work with the dual.

Let $\mathbf{s} = (s_{1,0}, \ldots, s_{1,A_x}, s_{2,0}, \ldots, s_{2,A_x}, s_3, s_{4,0}, \ldots, s_{4,A_y}, s_{5,0}, \ldots, s_{5,A_y}, s_6)$, be the vector of dual variables of (Primal). The vector shows six groups of variables corresponding to the first and last dual variable for constraint sets (S1)–(S6). The dual (Dual) of (Primal) is given by

$$\begin{aligned}
&\text{Max} \quad \sum_{i=0}^{A_x} s_{2i} + s_3 + \sum_{f=0}^{A_y} s_{5f} + s_6\\
&\text{subject to}\\
&\sum_{i=0}^{A_x} (s_{2i} - s_{1i}) * i + (A_x + 1) * s_3 + \sum_{f=0}^{A_y} (s_{5f} - s_{4f}) * f + (A_y + 1) * s_6 = 0,\\
&\sum_{i=0}^{A_x} ((px_i + 1) * s_{2i} - px_i * s_{1i}) + \sum_{f=0}^{A_y} ((py_f + 1) * s_{5f} - py_f * s_{4f}) = 1,\\
&\sum_{i=0}^{A_x} (s_{1i} - s_{2i}) - s_3 = 0,\\
&\sum_{f=0}^{A_y} (s_{4f} - s_{5f}) - s_6 = 0,\\
&\mathbf{s} \geqslant \mathbf{0}.
\end{aligned}$$

Let $\boldsymbol{P}$ be the matrix of technological coefficients of (Primal). Then $\boldsymbol{P}^{\mathrm{T}}$ has the following structure:

$$\begin{bmatrix} 0 & \dots & -A_x & 0 & \dots & A_x & A_x+1 & 0 & \dots & -A_y & 0 & \dots & A_y & A_y+1 \\ -px_0 & \dots & -px_{A_x} & px_0+1 & \dots & px_{A_x}+1 & 0 & -py_0 & \dots & -py_{A_y} & py_0 & \dots & py_{A_y}+1 & 0 \\ 1 & \dots & 1 & -1 & \dots & -1 & -1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 1 & -1 & \dots & -1 & -1 \end{bmatrix}.$$

In every optimal dual solution, there is an optimal basis that contains one column corresponding to a perfect X-partition, and another column corresponding to a perfect Y-partition. This follows from (Primal), in which there is always one binding X-partition row and one binding Y-partition row. Therefore, two of the columns in the basis look like

$$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also, the optimal basis contains at least one column corresponding to a binding efficiency constraint in the primal, or otherwise the dual objective function would have value zero. Therefore, the basis contains at least one of the following columns:

$$\begin{bmatrix} \cdot \\ \cdot \\ -1 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cdot \\ \cdot \\ 0 \\ -1 \end{bmatrix}.$$

Considering the conditions above, the optimal basis has one of the following layouts, not considering an exchange in the last two rows:

$$(1)\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (2)\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (3)\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \text{or} \quad (4)\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Let $\boldsymbol{B}$ be a basis, and let $\mathbf{s}_b = (s_p, s_q, s_r, s_s)$ be the corresponding vector of basic variables. The coefficients of the first two rows of the basis are identified as $b_{k,l}$, $k = 1, 2$, $l = 1, 2, 3, 4$, where k indicates the row and l the column. If the column corresponds to a fitting constraint in (Primal), its coefficients are identified by $\overline{b_{k,l}}$.

The cost coefficients for the (Dual) objective function take value 0 for fitting constraints, and 1 for efficiency constraints. Therefore, the vectors of cost coefficients corresponding to the cases above are respectively

(1) $(0, 0, 1, 1)$,
(2) $(0, 0, 1, 0)$,
(3) $(0, 0, 1, 0)$ or
(4) $(0, 0, 1, 1)$.

In all four cases, we can reduce the system of equations to a 2×2 system.

Case (1). The system is given by

$$\begin{aligned} &\overline{b_{1,1}} * s_p + \overline{b_{1,2}} * s_q + b_{1,3} * s_r + b_{1,4} * s_s = 0, \\ &\overline{b_{2,1}} * s_p + \overline{b_{2,2}} * s_q + b_{2,3} * s_r + b_{2,4} * s_s = 1, \\ &s_p - s_r - s_s = 0, \\ &s_q = 0. \end{aligned}$$

We use $s_q = 0$ and $s_p = s_r + s_s$, and obtain the following system:

$$(b_{1,3} + \overline{b_{1,1}}) * s_r + (b_{1,4} + \overline{b_{1,1}}) * s_s = 0,$$
$$(b_{2,3} + \overline{b_{2,1}}) * s_r + (b_{2,4} + \overline{b_{2,1}}) * s_s = 1.$$

Case (2). The system is given by

$$\overline{b_{1,1}} * s_p + \overline{b_{1,2}} * s_q + b_{1,3} * s_r + \overline{b_{1,4}} * s_s = 0,$$
$$\overline{b_{2,1}} * s_p + \overline{b_{2,2}} * s_q + b_{2,3} * s_r + \overline{b_{2,4}} * s_s = 1,$$
$$s_p - s_r + s_s = 0,$$
$$s_q = 0.$$

We use $s_q = 0$ and $s_r = s_p + s_s$, and obtain the following system:

$$(b_{1,3} + \overline{b_{1,1}}) * s_p + (b_{1,3} + \overline{b_{1,4}}) * s_s = 0,$$
$$(b_{2,3} + \overline{b_{2,1}}) * s_p + (b_{2,3} + \overline{b_{2,4}}) * s_s = 1.$$

Case (3). The system is given by

$$\overline{b_{1,1}} * s_p + \overline{b_{1,2}} * s_q + b_{1,3} * s_r + \overline{b_{1,4}} * s_s = 0,$$
$$\overline{b_{2,1}} * s_p + \overline{b_{2,2}} * s_q + b_{2,3} * s_r + \overline{b_{2,4}} * s_s = 1,$$
$$s_p - s_r = 0,$$
$$s_q + s_s = 0.$$

We use $s_p = s_r$ and $s_q = -s_s$, and obtain the following system:

$$(b_{1,3} + \overline{b_{1,1}}) * s_r + (\overline{b_{1,4}} - \overline{b_{1,2}}) * s_s = 0,$$
$$(b_{2,3} + \overline{b_{2,1}}) * s_r + (\overline{b_{2,4}} - \overline{b_{2,2}}) * s_s = 1.$$

Case (4). The system is given by

$$\overline{b_{1,1}} * s_p + \overline{b_{1,2}} * s_q + b_{1,3} * s_r + b_{1,4} * s_s = 0,$$
$$\overline{b_{2,1}} * s_p + \overline{b_{2,2}} * s_q + b_{2,3} * s_r + b_{2,4} * s_s = 1,$$
$$s_p - s_r = 0,$$
$$s_q - s_s = 0.$$

We use $s_p = s_r$ and $s_q = s_s$, and obtain the following system:

$$(b_{1,3} + \overline{b_{1,1}}) * s_r + (b_{1,4} + \overline{b_{1,2}}) * s_s = 0,$$
$$(b_{2,3} + \overline{b_{2,1}}) * s_r + (b_{2,4} + \overline{b_{2,2}}) * s_s = 1.$$

In each of the four cases, let d be the determinant of the 2×2 matrix. The matrix is a basis and all elements are integers, so $|d| \geqslant 1$. Therefore, the respective optimal objective function values are

(1) $\breve{b}^* = s_r + s_s = (-(\overline{b_{1,1}} + b_{1,4}) + (\overline{b_{1,1}} + b_{1,3}))/d \leqslant |(b_{1,3} - b_{1,4})/d| \leqslant (A_x + 1)/|d|$,
(2) $\breve{b}^* = s_r = (\overline{b_{1,1}} - \overline{b_{1,4}})/d \leqslant |(\overline{b_{1,1}} - \overline{b_{1,4}})/d| \leqslant A_x/|d|$,
(3) $\breve{b}^* = s_r = (\overline{b_{1,2}} - \overline{b_{1,4}})/d \leqslant |(\overline{b_{1,2}} - \overline{b_{1,4}})/d| \leqslant A_x/|d|$, and
(4) $\breve{b}^* = s_r + s_s = (-(b_{1,4} + \overline{b_{1,2}}) + (b_{1,3} + \overline{b_{1,1}}))/d \leqslant |((b_{1,3} + \overline{b_{1,1}}) - (b_{1,4} + \overline{b_{1,2}}))/d| \leqslant (A_x + A_y + 1)/|d|$.

Because all elements of the basis are integers, $\breve{b}^* |d|$ is an integer.

In case (1), both coefficients are non-negative, corresponding to efficiency constraints, and one can take value 0. Therefore, the maximum value of $\breve{b}^* |d|$ is $A_x + 1$.

In cases (2) and (3), all coefficients are non-positive, corresponding to fitting constraints, and one coefficient, in each case, can take value 0. Therefore, the maximum value of $\breve{b}^* |d|$ is A_x.

In case (4), if we consider the order of the rows as presented, the result follows because $(b_{1,3}+\overline{b_{1,1}})$ is computed over constraints in X, taking maximum value A_x+1, and $(b_{1,4}+\overline{b_{1,2}})$ is computed over constraints in Y, taking minimum value $-A_y$. If the order of the rows is exchanged, the values obtained are also exchanged, with $(b_{1,3}+\overline{b_{1,1}})$ taking maximum value A_y+1 and $(b_{1,4}+\overline{b_{1,2}})$ taking minimum value $-A_x$.

Considering all four cases, we verify that A_x+A_y+1 is an upper bound on $b^*|d|$. Scaling the optimal (Primal) solution by $|d|$, $|d|(\breve{X}^*,\breve{Y}^*,\breve{a}^*,\breve{b}^*)$ is an integer solution that is feasible to (Primal). We therefore have an instance in the equivalence class with $b=b^*|d|$ and, by definition, $\tilde{b}\leqslant b$ so

$$\tilde{b}\leqslant A_x+A_y+1. \tag{3}$$

If we minimize $\breve{a}$ in (Primal), then the upper bound for $\tilde{a}$ is

$$\tilde{a}\leqslant B_x+B_y+1. \qquad \square \tag{4}$$

6. Identifying the MSI

Given an instance of PLP, (X,Y,a,b) or the corresponding set of efficient partitions, we test values for $\tilde{b}$, starting at 1, and compute the other variables, until the MSI is found. Our algorithm operates with two main loops. The outer loop selects values for $\tilde{b}$, from 1 to $\min\{b,A_x+A_y+1\}$. For each value of $\tilde{b}$, we compute the range of $\tilde{a}$, $R_{\tilde{a}}$, given by $R_{\tilde{a}}=\{a\in Z^+:\lfloor B_x*\tilde{b}/(A_x+1)\rfloor<a\leqslant\lceil((B_x+1)*\tilde{b}-1)/A_x\rceil\}$. The second loop enumerates values for $\tilde{a}$ in $R_{\tilde{a}}$. With $\tilde{a}$ and $\tilde{b}$, we compute $\widetilde{X}=G(X,\tilde{a},\tilde{b})$. If the selected values for $\widetilde{X}$, $\tilde{a}$, and $\tilde{b}$ satisfy the efficiency constraints, we repeat the same computations for $\widetilde{Y}$, or otherwise try the next value for $\tilde{a}$. If the inequalities for $\widetilde{Y}$ are not satisfied, we continue with the procedure: otherwise, instance $(\widetilde{X},\widetilde{Y},\tilde{a},\tilde{b})$ is the MSI. This algorithm is simple to implement, with time complexity $O(B_x^4)$.

7. Generating equivalence classes

We enumerate the MSI of all equivalence classes with an area ratio smaller than 101 boxes per pallet. Also we use the MSI to uniquely identify each class and, therefore, record only one instance per class.

If N is the maximum number of boxes that can be packed on a pallet, we have

$$A_x+A_y\leqslant N+1 \tag{5}$$

and

$$B_x+B_y\leqslant 2N. \tag{6}$$

We recall some definitions to demonstrate these bounds. Any optimal must have at least A_x (A_y) boxes placed side by side across the length (width) of the pallet, so $A_x*A_y\leqslant N$. If $A_y=0$, then $A_x\leqslant N$ and $A_x+A_y\leqslant N$. If $A_y\geqslant 1$, then $A_x\leqslant N/A_y$ and $A_x+A_y\leqslant N/A_y+A_y\leqslant N+1$. Also, B_x (B_y) is the maximum number of boxes that can be placed side by side across the length (width) of the pallet. Therefore, $B_y\leqslant B_x\leqslant N$ and $B_y+B_x\leqslant 2N$.

Our *PLP Equivalence Class Generation Algorithm* (PLP-ECGA) has six main loops, and uses a list ordered lexicographically by b, a and Y to maintain the distinct generated classes for given values of a and b. The outermost loop determines the values for b, from 1 to $N+2$, because $\tilde{b}\leqslant A_x+A_y+1$ (3) and $A_x+A_y\leqslant N+1$ (5). The second loop selects values for a, from $b+1$ to $2N+1$, because $\tilde{a}\leqslant B_x+B_y+1$ (4) and $B_x+B_y\leqslant 2N$ (6), except when b equals 1, when a can also be equal to 1. If the greatest common divisor of a and b is greater than 1, then the second loop proceeds to the next value for a. Otherwise, the ordered list, with all generated classes, is emptied. The third and fourth loops select among all the possible perfect partitions of the width for candidate Y. The fifth and sixth loops select among the perfect partitions of the length for candidate X. If instance (X,Y,a,b) has an area ratio bound not exceeding N and has not been generated before then it is recorded in the list.

It is possible to verify through the algorithm that the number of equivalence classes is bounded by a polynomial in N, albeit a large polynomial. There are $O(N^2)$ ways of assigning values to a and b, corresponding to the number of pairs of relatively prime numbers less than or equal to $2N$. More precisely, the number is given

Table 1
Number of equivalence classes in groups based on the maximum number of boxes

Maximum number of boxes in each group	Number of classes in each group	Run time to generate all classes (seconds)
10	662	0
20	7309	3
50	216,095	295
100	3,080,730	14,667

Table 2
Distribution of values of b in the MSI in each class

Number of boxes	Number of classes	$b = 1$	$b \leqslant 2$	$b \leqslant 5$	$b \leqslant 10$	$b \leqslant 20$	$b \leqslant 50$
10	662	92	276	609	662	662	662
20	7309	520	1760	4873	6659	7309	7309
50	216,095	6362	23,270	71,686	119,298	182,870	216,095
100	3,080,730	46,300	174,177	544,004	964,673	1,710,574	2,822,767

by $\sum_{k=1}^{2N}\phi(k)$, where $\phi(k)$ is the Euler phi-function, which gives the number of integers less than k that are relatively prime to k (Gallian, 1998). The loops corresponding to the width are executed $O(N^2)$ times for each pair a and b. The same happens with the loops corresponding to the length. Therefore, the number of equivalence classes, with area ratio bound up to N, is bounded above by a sixth-degree polynomial in N.

The instances generated with the PLP-ECGA procedure are divided in groups of up to 10, 20, 50 and 100 boxes per pallet, as defined by the area ratio bound. Table 1 presents in the second column the number of equivalence classes of PLP in each group. The third column contains the time required, in seconds, to generate the equivalence classes within each group on a Pentium III 600 MHz personal computer.

Table 2 presents the distribution of classes in each group, where the MSI is defined with b smaller than or equal to 1, 2, 5, 10, 20, and 50. The first column defines the maximum number of boxes that can be packed in an instance in the group of classes covered, as given by the area ratio bound. The second column lists the number of distinct classes in each group. The following columns present the number of classes.

For PLP instances with area ratio smaller than 101 boxes, 91% of the equivalence classes have a MSI where the value of b is less than or equal to 50. The complete set of instances can be accessed at http://www.pallet-loading.org. More details can be found in (Martins, 2003).

8. Conclusions

In this paper, we define the Minimum Size Instance (MSI) of an equivalence class of PLP, and show that every class has one and only one MSI. This makes the MSI helpful in distinguishing equivalence classes. We also develop bounds on the dimensions of the box and pallet in the MSI of a class. Previous work only provides bounds on the ratio of box dimensions. Applying the newly developed bounds to the MSI, we enumerate the MSI of all equivalence classes with area ratio smaller than 101 and provide some statistics.

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