# Formal Theories for Semantic Fusion

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***Abstract – A formal theory approach to semantic fusion is proposed and its philosophical, mathematical and computational development is illustrated through a formal theory for existence.***

**Keywords:** data fusion, higher-level fusion, semantic fusion, formal theory, ontology, situation assessment.

## 1 Introduction

Lambert ([1],[2],[3],[4]) contends that situation assessment is about assessing situations, with situations as fragments of the world represented by a set of assertions. Consequently the transition from object assessments to situation assessments requires a move from numeric to symbolic representations, and so, when contemplating machine based situation assessments, one confronts the question: "What symbols should be used and how do those symbols acquire meaning?" - termed "the Semantic Challenge" for Information Fusion by Lambert ([4]). The solution proposed in [4] was to engage "… formal theories that define the meaning of selected primitive symbols …" using a 5 tier inheritance structure of formal theories with suggested primitives, which is reproduced in Figure 1.

| | |
|---|---|
| Social: | group, ally, enemy, neutral, own, possess, invite, offer, accept, authorise, allow. |
| Intentional: | individual, routine, learnt, achieve, perform, succeed, fail, intend, desire, belief, expect, anticipate, sense, inform, effect, approve, disapprove, prefer. |
| Functional: | sense, move, strike, attach, inform, operational, disrupt, neutralise, destroy. |
| Physical: | land, sea, air, outer_space, incline, decline, number, temperature, weight, energy. |
| Metaphysical: | exist, fragment, identity, time, before, space, connect, distance, area, volume, angle. |

Figure 1. Hierarchy of semantic primitives.

Nowak and Lambert ([5]) subsequently used Model Theory ([6]) to: (a) explain how formal theories can define the meaning of formal language symbols; and (b) explain how ontologies as description logic formal theories can define the meaning of symbols, before demonstrating two implemented applications in which an ATTITUDE agent ([7]) engaged an ontology to semantically retrieve and reason with information.

The description logic approach restricts the expressivity of the formal language in order to guarantee the decidability of formal theories developed with that formal language. An alternative is to use logics of greater expressivity, but to consider the decidability of the formal theories developed with that formal language. In contrast to [5], this paper explores the latter approach.

The author suggests that in developing any computational semantic formal theory, three stages are required: (a) *philosophical*, in which the conceptualisation of the domain of discourse is specified; (b) *mathematical*, in which the formal structure of that philosophy is specified; and (c) *computational*, in which a computational implementation of that mathematical theory is specified. The remainder of this paper presents these three stages of development, choosing the concept of existence from the metaphysical tier in Figure 1 to illustrate the approach. Section 2 looks at the concept of existence; section 3 presents a formal theory of that conceptualisation; and section 4 delivers a computational implementation for that theory. Section 5 concludes by using category theory to show how the theory of existence could be combined within a hierarchy of formal theories to provide a computational metaphysics within the semantic framework of Figure 1.

## 2 Philosophy of Existence

The philosophy of existence outlined herein considers the nature of: metaphysics; nominalism; processes; and language.

### 2.1 Metaphysics

The term "metaphysics" derives from works by Aristotle from around 350 B.C. ([8]). Aristotle's understanding centred on understanding *things*, and he asserted that there were different *kinds of things*, each of which could be understood on the basis of the principles governing that kind. In seeking to uncover the underlying principles of each kind, the inquirer derives a science or systematic body of knowledge. As things of one kind may also be things of another kind, these sciences accommodate alternative perspectives toward the same things, and to that end, in his "Posterior Analytics" Aristotle defines an inheritance ordering over the sciences. Aristotle's ordering of the sciences and his aversion for the infinite were suggestive of a foundational science. Aristotle

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proffered the science of being, or metaphysics, as foundational, and of the different senses of being, nominated substance as fundamental. Substances are the individual things that exist. *Metaphysics was to provide a generic account of all individual existing things, irrespective of the classes to which they belonged*.

Within metaphysics, substance was studied through the roles it supported. One of the four roles concerns form and matter. For Aristotle, things are formed matter. Matter is the "stuff" of which a thing is composed, the characteristic that makes a statue *this* statue rather than that statue. Form is that which determines *what* a thing is, the characteristic that makes a statue, a *statue*. Form is the basis by which reality is individuated, while relying upon matter as a mechanism for relating forms. Matter supports a hierarchical structure of form but restricts access to immediate forms. Thus Aristotle cites earth as the matter of wood and wood as the matter of a casket, but earth cannot be the matter of a casket.

## 2.2 Nominalism

In contemplating a formal theory resembling Aristotle's hierarchical world of objects, one might consider set theory that, in its various forms, has come to represent our understanding of composition. Of these, Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC) (e.g. [9]) has, at least mathematically, been the most prominent, and ZFC's Regularity Axiom delivers the sort of compositional relationship noted by Aristotle between earth, wood and casket. The axioms of ZFC are listed below, expressed in terms of the first order formal language $\{\in, =\}$.

Extensionality Axiom

$\forall x \, \forall y \, (x = y \Leftrightarrow \forall z \, (z \in x \Leftrightarrow z \in y))$.

Empty Set Axiom

$\exists x \, \forall y \, (\neg \, (y \in x))$.

Unions Axiom

$\forall x \, \exists y \, \forall z \, (z \in y \Leftrightarrow \exists w \, (z \in w \, \& \, w \in x))$.

Power Set Axiom

$\forall x \, \exists y \, \forall z \, (z \in y \Leftrightarrow \forall w \, (w \in z \Rightarrow w \in x))$.

Infinity Axiom

$\exists x \, (\exists y \, (y \in x) \, \& \, \forall y \, (y \in x \Rightarrow \exists z \, (y \in z \, \& \, z \in x)))$.

Regularity Axiom

$\forall x \, (\exists y \, (y \in x) \Rightarrow$
$\exists y \, (y \in x \, \& \, \neg \, (\exists z \, (z \in y \, \& \, z \in x))))$.

Replacement Axioms

$\forall x \, (\forall u \, (u \in x \Rightarrow$
$\exists! v \, (\xi(u, v))) \Rightarrow \exists y \, \forall w \, (w \in y \Leftrightarrow \exists t \, (t \in x \, \& \, \xi(t, w))))$,

for any well formed formula $\xi(-, -)$ with two free variables.

Axiom of Choice

$\forall x \, (\forall y \, (y \in x \Rightarrow$
$(\exists z \, (z \in y) \, \& \forall u \, \forall v \, ((u \in x \, \& \, v \in x) \Rightarrow$
$\exists z \, ((z \in u \, \& \, z \in v) \Rightarrow u = v))))$
$\Rightarrow \exists w \, \forall y \, (y \in x \Rightarrow \exists z \, (z \in y \, \&$
$\forall t \, (t \in x \Rightarrow (t \in w \Leftrightarrow t = z)))))$.

The ordinals 0, 1, 2, …, $\omega$, $\omega^+$, …, $2^\omega$, … are used to generate the universe of the "intended model", as the ZFC axioms allow for: the existence of a zero ordinal $0 = \varnothing$; the existence of a successor ordinal $\alpha^+ = \alpha \cup \{\alpha\}$ for any ordinal $\alpha$ (e.g. $1 = 0^+ = \varnothing \cup \{\varnothing\} = \{\varnothing\}$); the existence of limit ordinals where $\lambda$ is a limit ordinal if it satisfies $\forall \alpha \, (\alpha \in \lambda \Rightarrow \alpha^+ \in \lambda)$ (e.g. $\omega = \{0, 1, 2, \ldots\}$). It is a relatively simple matter to demonstrate that every well ordered set supports a Transfinite Induction Principle and, as a consequence, it is possible to recursively define functions over these well ordered sets. Since ordinals are sets well ordered by $\in$, it follows that transfinite recursion can be performed over the ordinals. One such recursively defined function is the rank function **R** defined by $\mathbf{R}(0) = 0 = \varnothing$; $\mathbf{R}(\alpha^+) = \mathbf{P}(\mathbf{R}(\alpha))$ for powerset **P** and every ordinal $\alpha$; $\mathbf{R}(\lambda) = \bigcup_{\alpha \in \lambda} \mathbf{R}(\alpha)$ for every limit ordinal $\lambda > 0$. The Regularity Axiom is provably equivalent to the statement that for every set x there is an ordinal $\alpha$ such that $x \in \mathbf{R}(\alpha)$. In other words, *if something is a set then sooner or later it will be generated by the rank function*. The intended model of set theory then becomes the structure $<\text{Ran } \mathbf{R}; \in>$ where Ran $\mathbf{R} = \{\mathbf{R}(\alpha) \mid \alpha \text{ is an ordinal}\}$ and $\in$ is the "set" which, as a relation, well orders the ordinal sets, each of which is generated by the rank function.

The first problem with a metaphysics based on ZFC and its intended model is the choice of *foundation*. Sets in the intended model are *not* sets about times, missiles, governments and ships. Under the intended model, every set is an abstraction from the empty set $\mathbf{R}(0) = \varnothing$. The intended model could be extended with 'urelements' by expanding the foundation $\mathbf{R}(0)$ to include those elements of reality which one might wish to consider abstractions of. This approach then engenders something like an Aristotelian outlook. The set $\mathbf{R}(0)$ describes the fundamental conception of *matter*, termed "prime matter" by Aristotle, while the abstractions of $\mathbf{R}(1)$ from $\mathbf{R}(0)$ identify the *forms* the matter of $\mathbf{R}(0)$ might assume. The abstractions of $\mathbf{R}(2)$ from $\mathbf{R}(1)$ then detail the forms the matter of $\mathbf{R}(1)$ might assume. But what then is the prime matter foundation $\mathbf{R}(0)$? Aristotle opted for the earth, air, fire, and water of Empedocles.

A second problem with a metaphysics based on ZFC and its intended model is *overpopulation*. If HMAS_Adelaide is an object in the universe, then so is {HMAS_Adelaide}, and {{HMAS_Adelaide}}, and so on uncountably, with the Regularity Axiom ensuring that each of these sets is distinct. Through the Infinity Axiom and the Power Set Axiom in particular, ZFC presides over an abstraction explosion to the extent that we can intuitively formulate some abstractions which we fully intend to be sets, but which, upon analysis, turn out to be too large and too far removed from the empty set to ever appear in the stepwise proliferation of sets that **R** generates. This includes the two "sets" Ran **R** and $\in$ in the intended model. Moreover, Gödel's Second Incompleteness Theorem ([10]) in effect states that no theorem of ZFC can assert the consistency of ZFC. Since the Extended Completeness Theorem of first order logic holds that a theory is consistent if and only if it has a model, it follows that there can be no theorem of ZFC

asserting a model for ZFC. ZFC provably cannot produce a model for itself.

A third problem with a metaphysics based on ZFC and its intended model is *extensionality*. A representation is extensional if the truth value of an expression relating its compositional structure is unaltered by substituting components for other components having the same reference, and is intensional if otherwise. When sets are used to characterise formed matter, they become intensional representations of the world. For example, if objects are understood as sets of properties, then sets characterise matter through the properties it contains. The state (object) depicted in Figure 2 might therefore be conceptualised as

$S_1$ = {field, sky, foliage, helicopter, truck1, container1, truck2, container2, soccer_goals}.

The level of detail can be increased to obtain

$S_2$ = {field, sky, {trees, green}, {main_rotor, tail_rotor, fuselage}, truck1, container1, truck2, container2, soccer_goals}

with foliage = {trees, green} and helicopter = {main_rotor, tail_rotor, fuselage}, and by the Axiom of Extensionality, $S_1 = S_2$. But the *very same state* of the world might equally be conceptualised as,

$S_3$ = {field, sky, {trees, green}, main_rotor, tail_rotor, fuselage, truck1, container1, truck2, container2, soccer_goals}

and the Axiom of Extensionality ensures $S_1 \neq S_3$. So in this context, ZFC provides an intensional account of the world by referring to the same thing in non-identical ways.


Figure 2. Helicopter moving containers at location s at time t.

Aristotle grounded his understanding in individual objects. The properties or forms he observed therefore needed to be anchored to objects and so affected his concept of object identity. But the emergence of the idea of a relation over the last two hundred years, allows the properties (and relations) associated with an object to be linguistically separated from the identity of that object. Rather than conceive of the world through objects as set theoretic complexes like {tree, green}, we can entertain atomic formulae like green(tree), in which the object tree has the property green. Moreover, if we choose to embrace an extensional view of objects, then we might replace green(tree) with the molecular formula green($r_1$) & tree($r_1$), in which $r_1$ is a label that references an element of an extensional set theory. Stanislaw Leśniewski pioneered an *extensional set theory* in 1916 and in the 1930s, Goodman, Leonard, Quine and later Lewis, subsequently advocated Harvard extensional set theory under the title of nominalism.

## 2.3 Processes

Aristotle identified the world as a world of objects and tried to account for change in objects. Aristotle's solution to the problem of changing objects was to insist that such objects have at least two forms, one form that alters during the change, and another form that remains constant. The transient form is responsible for there being a change, while the permanent form is responsible for that change being a change in the one thing. But object change is a technically cumbersome mechanism, for it allows the same thing to differ, and this fosters ambiguity. The extensional Leibniz's law $(x = y) \Leftrightarrow (Q(x) \Leftrightarrow Q(y))$ in general fails for any property or quality Q if = is object identity, because the x object may have shed its accidental Q-ness before that same object was referred to as the y object. Abelson *et.al.* ([11] p. 178 and p. 290) notes similar difficulties with the assignment construct in Computer Science.

The alternative is to insist that changed things are different things, thereby safeguarding Leibniz's law. Heraclitus is famed for proposing this idea around 480 B.C., and drawing the conclusion that one can never cross the same river twice. The elements of this ontology are processes, as they are temporally individuated, while also admitting sums of temporally individuated processes as processes. Aristotle identifies the world as a world of objects and tries to account for change in objects. In contrast, the process philosopher (e.g. [12]) identifies the world as a world of processes and tries to locate persistence among the constant change. The metaphysical tier of Figure 1 is about being able to describe *what things are where, when*. This can be achieved through an extensional concept of processes, where *a process is any spatio-temporal fragment of the universe*.

## 2.4 Language

For any subject `s` and predicate `P`, Frege's 1892 ([13]) inclination was to represent the sentence `s is a P` by the formal expression `P(s)`. Under naïve realism, the content of `P(s)` is in turn the claim that the interpreted referent $\mathfrak{A}$(`s`) of subject `s` has the interpreted property $\mathfrak{A}$(`s`) of predicate `P`. This approach works satisfactorily for the sentence `HMAS Adelaide is a ship` as the content of the sentence `ship(HMAS_Adelaide)` is the claim that the object $\mathfrak{A}$(`HMAS_Adelaide`) referred to by `HMAS_Adelaide` has the property $\mathfrak{A}$(`ship`) referred to by `ship`. Expressed set theoretically, the predicate `ship` refers to the set of all things $\mathfrak{A}$(`ship`) that we label as ships, and for object $\mathfrak{A}$(`HMAS_Adelaide`) we have, $\mathfrak{A}$(`HMAS_Adelaide`) $\in$ $\mathfrak{A}$(`ship`).

The matter complicates, however, when we consider the sentence `HMAS Weirerstrass is a ship` as there is no object $\mathfrak{A}$(`HMAS_Weirerstrass`) to be referred to by `HMAS_Weirerstrass`. Russell's 1905 ([14]) solution was to suggest that the term `HMAS_Weirerstrass` is not a name, but a denoting phrase, and so the claim `HMAS Weirerstrass is a ship` is really asserting that: (a) there is something having the property of being HMAS Weirerstrass; and (b) that thing also has the property of being a ship. Consequently, on Russell's analysis, the correct formal representation is $\exists x$ `(HMAS_Weirerstrass(x) & ship(x))`. When confronted with any sentence of the form `s is a P`, one will not necessarily know whether or not the subject `s` refers to anything, and so as a precaution, one should always formally represent all such sentences by $\exists x$ `(s(x) & P(x))`.

Quine ([15]) used this style of analysis as a basis for ontological commitment. One could determine another's ontology by attending to the subjects they asserted in sentences. To exist is to be the value of an existentially quantified variable in a true sentence. This raises an interesting issue when the predicate in question is existence, however. The true sentences

`HMAS Adelaide exists (1)`
`HMAS Weirerstrass does not exist (2)`

would be formally represented by

$\exists x$ `(HMAS_Adelaide(x) & exist(x))`
$\exists x$ `(HMAS_Weirerstrass(x) &` $\neg$ `exist(x))`

with a corollary of the second formal expression being

$\exists x$ ($\neg$ `exist(x))`.

On the face of it, this is asserting the existence of something that doesn't exist! To prevent this, it has become customary to prohibit all usage of an `exist` predicate, and as a special case, to instead formally represent the two sentences (1) and (2) by

$\exists x$ `(HMAS_Adelaide(x))`
$\neg$ ($\exists x$ `(HMAS_Weirerstrass(x))`.

A presumption in the aforementioned progression from Frege to Russell to Quine, is that asserting `s is a P` entails asserting `s is`. In this paper, that presumption is challenged. Greater utility is to be had if we separate content from ontological commitment. If told `s is a P`, then I should be able to believe `s is a P` without also having to believe `s is`. The existence of `s` is a separate issue. On Quine's account the sentence `Superman wears a cape` (3) is false because it is formalised as

$\exists x$ `(Superman(x) &` $\exists y$ `(wears(x, y))`

and there is no `x` satisfying `Superman(x)`. On the author's account, sentence (3) is true because our meaning of `Superman` involves Superman wearing a cape, irrespective of whether Superman exists. Ignoring the existence of Superman allows us to truthfully state

`wears(Superman, cape),`

while considering the existence of Superman allows us to truthfully state

$\neg$ `exist(Superman) &`
`wears(Superman, cape).`

Separating content and ontological commitment, allows for reasoning about fictitious entities. *Practically, this is a necessary requirement, as it is sometimes necessary to reason about alleged entities whose existence has not been established.* For example, `HMAS_Adelaide @ 200606131400Z @ Celtic_Sea` may or may not have a referent, depending on where the ship actually is at the nominated time. Formally handling this requires use of an `exist` predicate whose meaning is not determined by direct reference in the way that the existential quantifier $\exists$ is defined. Section 3 defines the `exist` predicate by defining a mathematical (formal) theory of existence. The mathematical theory of existence, subsequently denoted $\mathbb{E}$, does not tell us what exists. It instead indicates what is meant by existence.

# 3 Mathematics of Existence

Section 2 promoted an existential process metaphysics defined through an `exist` predicate, where a process is any fragment of the spatio-temporal universe. To refine the concept of *fragment*, an initial formal theory of fragmentation is proposed. The formal theory is a weakening of the ZFC axioms noted in section 2.2. A first order framework is utilised with language $\{\equiv, \leq\}$ and with the domain of discourse restricted to processes. Fragmentation is expressed through $\leq$. The symbols $x \leq y$ means that process x is a fragment of process y. Identity is expressed by $\equiv$. $x \equiv y$ means that process x is identical to process y.

The Identity Axiom is an adaptation of the Extensionality Axiom from ZFC. It establishes process extensionality by making processes identical on account of their process fragments.

Identity Axiom

$\forall x \forall y (x \equiv y \Leftrightarrow \forall z (z \leq x \Leftrightarrow z \leq y))$.

In Figure 2 if helicopter_at_s_t is fully identified by the three relations main_rotor_at_s_t $\leq$ helicopter_at_s_t, tail_rotor_at_s_t $\leq$ helicopter_at_s_t, fuselage_at_s_t $\leq$ helicopter_at_s_t, and a label CH53_at_s_t is fully identified by the three relations main_rotor_at_s_t $\leq$ CH53_at_s_t, tail_rotor_at_s_t $\leq$ CH53_at_s_t, fuselage_at_s_t $\leq$ CH53_at_s_t, then helicopter_at_s_t and CH53_at_s_t are identical fragments of the world, *id est*, helicopter_at_s_t $\equiv$ CH53_at_s_t, because each fragment of helicopter_at_s_t is a fragment of CH53_at_s_t and *vice versa*.

Fragments are likewise defined on the basis of constituent fragments.

Fragmentation Axiom

$\forall x \forall y (x \leq y \Leftrightarrow \forall z (z \leq x \Rightarrow z \leq y))$.

So x is a fragment of y if and only if every fragment of x is also a fragment of y. Therefore main_rotor_at_s_t $\leq$ helicopter_at_s_t is the case because for each unlabelled z $\leq$ main_rotor_at_s_t in Figure 2, it is also the case that z $\leq$ helicopter_at_s_t. The Identity Axiom and Fragmentation Axiom are sufficient to establish $\leq$ as a partial ordering.

A process is any fragment of the spatio-temporal *universe*. The third axiom specifies a greatest process

with respect to fragmentation. That process is the universe.

Universe Axiom

$\exists x \forall y\ (y \leq x)$.

As the notion of process is intended to be *comprehensive*, the universe should itself also be a process. Under the Universe Axiom, it is. With the way the intuitive notion of process is framed, processes must be related to the universe through *fragmentation*. Once again, under the Universe Axiom, they are. The informal conception of process speaks of *the* universe, implying that the universe is *unique*. This is also provably the case.

Processes can be identified by both *uniting* and *separating* other processes. In the previous example, an analyst might take helicopter_at_s_t to be the unity of main_rotor_at_s_t, tail_rotor_at_s_t and fuselage_at_s_t. The Join Axiom facilitates this expressivity. For any two processes x and y, the Join Axiom identifies the join process z as the smallest fragment that includes both x and y.

Join Axiom

$\forall x \forall y \exists z\ (x \leq z \ \& \ y \leq z \ \&$
$\forall u\ ((x \leq u \ \& \ y \leq u) \Rightarrow z \leq u))$.

It is a simple matter to prove the uniqueness of join processes and so for any two processes x and y, to define $x + y$ as the join of x and y.

Definition (join)

$z \equiv x + y =_{df} (x \leq z \ \& \ y \leq z \ \&$
$\forall u\ ((x \leq u \ \& \ y \leq u) \Rightarrow z \leq u))$.

Equally, given any two processes x and y, x can be separated into two processes by using y as a separator. In this way x can separate y into two fragments, one consisting of those fragments that are shared by both x and y, and the other containing all those remaining fragments in x that are not in both x and y. The first is called the *meet process*. The second is called the *difference process*.

The Meet Axiom defines meet processes. For any two processes x and y, the Meet Axiom identifies the meet process z as the largest fragment contained by both x and y.

Meet Axiom

$\forall x \forall y \exists z\ (z \leq x \ \& \ z \leq y \ \&$
$\forall u\ ((u \leq x \ \& \ u \leq y) \Rightarrow u \leq z))$.

On the basis of uniqueness, for any two processes x and y, the meet of x and y is denoted by $x \bullet y$.

Definition

$z \equiv x \bullet y =_{df} (z \leq x \ \& \ z \leq y \ \&$
$\forall u\ ((u \leq x \ \& \ u \leq y) \Rightarrow u \leq z))$.

The Meet Axiom does not express how meets interact with joins, however. The following Distribution Axiom secures that outcome.

Distribution Axiom

$\forall x \forall y \forall z\ (x \bullet (y + z)) \leq (x \bullet y) + (x \bullet z))$.

The meet of x and y is the largest fragment contained by both x and y. The difference of x with y consists of the remainder of x when the meet is removed. The difference is therefore specified by two constraints. The first stipulates that the join of the two separation products returns the original process. This ensures the remainder of x is contained in the difference as required. However, with this sole constraint the difference may also contain additional fragments of the meet. So to prevent the difference from becoming too large, the join of any fragment of both separation products with any other process must yield that other process. Thus, if there were fragments common to both separation products, they would be inconsequential.

Difference Axiom

$\forall x \forall y \exists z\ (x \equiv z + (x \bullet y) \ \&$
$\forall u\ ((u \leq z \ \& \ u \leq (x \bullet y)) \Rightarrow \forall v\ (v \equiv u + v)))$.

The axiom ensures the uniqueness of differences and so gives rise to the definition of a difference process.

Definition (difference)

$z \equiv x - y =_{df} (x \equiv z + (x \bullet y) \ \&$
$\forall u\ ((u \leq z \ \& \ u \leq (x \bullet y)) \Rightarrow \forall v\ (v \equiv u + v)))$.

An important property emerging from the current axioms is the existence of complementary processes formed for any process x by $\mathbf{\Omega} - x$. As differences are unique, it follows that complementary processes are necessarily unique. The complement of x is denoted by $-x$.

Definition (complement)

$y \equiv -x =_{df} y \equiv \mathbf{\Omega} - x$.

A related consideration is the process $\mathbf{\Omega} - \mathbf{\Omega}$, or equivalently, $-\mathbf{\Omega}$. As differences are unique, $\mathbf{\Omega} - \mathbf{\Omega}$ is unique. This process is denoted by $\bot$. Since the universal process is casually called everything, it is natural to call the complement of everything, nothing. $\bot$ serves as the lower bound for processes.

Definition (nothing)

$x \equiv \bot =_{df} x \equiv \mathbf{\Omega} - \mathbf{\Omega}$.

An important theorem resulting from these axioms is that $<\Omega;\ +,\ \bullet,\ -,\ \bot,\ \mathbf{\Omega}>$ is a Boolean algebra. But the foundation for that algebra remains unresolved. Again the ancient Greeks come to the fore, with Democritus asserting that everything is composed of indivisible atoms, and Anaxagoras insisting that everything is infinitely divisible. The Democritus concept of atom survives in modern algebra. The following definition defines the concept of atom for the evolving theory.

Definition

$\mathbf{atomic}(x) =_{df} \neg(\bot \equiv x) \ \&$
$\forall y\ (y \leq x \Rightarrow (y \equiv x \vee y \equiv \bot))$.

Under this definition, a fragment of the world is an atomic fragment if it is not nothing and its only fragments are nothing and itself. The Foundation Axiom instead favours Anaxagoras by endorsing an *atomless* Boolean algebra.

Foundation Axiom (Anaxagoras)

$\neg\ (\exists x\ (\mathbf{atomic}(x)))$.

The axioms of the formal theory $\mathbb{E}$ have now been specified. The axioms have effectively been defined in terms of the fragmentation relation $\leq$, given the Identity Axiom. But as a theory of existence, it has presented no

commentary on existence *per se*. The following definition remedies that by defining an **exist** predicate.

Definition

**exist**(x) $=_{df} \neg (x \equiv \perp)$.

A resulting theorem $\forall x$ (**exist**(x) $\Leftrightarrow \neg (x \leq \perp)$) shows that the existence relation **exist** can be equivalently expressed in terms of the fragmentation relation $\leq$, and this reflects the way **exist** was defined. Conversely, another theorem $\forall x \forall y$ ($x \leq y \Leftrightarrow \neg$ **exist**($x \bullet -y$)) shows that the fragmentation relation $\leq$ can be equivalently expressed in terms of the existence predicate **exist**. *The mathematical theory of fragmentation is therefore equivalent to a mathematical theory of existence.* The mathematical theory of existence also fulfils the philosophical intent of section 2. Rather than define existence in the manner of $\exists$, through reference to things metatheoretically claimed to exist in the world, existence is instead defined here through a formal theory. The existential, foundationless formal theory does not identify what exists in the world. It instead defines *what is meant by existence*, so that appropriate conclusions can subsequently be drawn from beliefs about what exists in the world.

As the axioms and definitions of this section, $\mathbb{E}$ is a theory expressed in a first order metaphysics language $\mathbb{M}$, and so the conventional semantics of first order languages applies to $\mathbb{E}$ ([6]). Thus in the usual way one can define $\mathbb{E} \models \tau$ if and only if $\tau$ is a logical consequence of $\mathbb{E}$. Three important metatheorems of $\mathbb{E}$ are: $\mathbb{E}$ is satisfiable, *id est* there is no $\tau$ such that $\mathbb{E} \models \tau$ and $\mathbb{E} \models \neg \tau$; $\mathbb{E}$ is complete, *id est* for all $\tau$, $\mathbb{E} \models \tau$ or $\mathbb{E} \models \neg \tau$; and $\mathbb{E}$ is recursive, and so $\mathbb{E} \models \tau$ can be computed for all $\tau$ (given Church's Thesis [16]).

# 4 Computation of Existence

The final metatheorem of section 3 indicates that a computer can reason in accordance with the mathematical theory of existence $\mathbb{E}$ in language $\mathbb{M}$. The challenge of section 4 is to determine *how* to do that. The value of the theory of existence is that it *defines what is meant by existence*, so that appropriate conclusions can subsequently be drawn from beliefs about what exists in the world. In general there will be some domain theory $\mathbb{D}$ in the language $\mathbb{M}$ that expresses what is believed to exist in the world. By combining theory $\mathbb{D}$ with theory $\mathbb{E}$, meaning is attached to the expressions of $\mathbb{D}$ so that the intended semantic consequences of $\mathbb{D}$ follow. The intended semantic consequences of $\mathbb{D}$ are $\{\sigma \mid (\mathbb{D} \cup \mathbb{E}) \models \sigma\}$. Importantly, as $\mathbb{E}$ is provably satisfiable and complete, uncertainties in $(\mathbb{D} \cup \mathbb{E})$ must derive from uncertainties about the domain knowledge $\mathbb{D}$, not uncertainties about the meaning of the terms used in $\mathbb{E}$. The semantic consequence relation can be formally defined as $\models$.

Metadefinition (semantic consequence)

$\mathbb{D} \models \sigma =_{df} (\mathbb{D} \cup \mathbb{E}) \models \sigma$.

Two things are required for a machine to compute semantic consequences from domain knowledge.

- A computational language $\mathcal{C}$ in which to express *machine readable* domain knowledge $\mathcal{D}$. Semantically, $\mathcal{D}$ should correspond to some domain knowledge $\mathbb{D}$ when expressed in language $\mathbb{M}$.
- A deductive consequence relation $\vdash$ that can *compute* the deductive consequences $\{\delta \mid \mathcal{D} \vdash \delta\}$ of $\mathcal{D}$. Semantically, the deductive consequences should correspond to the semantic consequences $\{\sigma \mid \mathbb{D} \models \sigma\}$ of domain knowledge $\mathbb{D}$ corresponding to $\mathcal{D}$.

## 4.1 Computational Language

A computational language $\mathcal{C}$ has been implemented with a well defined translation function **Tr** : $\mathcal{C} \rightarrow \mathbb{M}$. It has nothing and universe to express the constant symbols for $\perp$ and $\Omega$ respectively; allows identifiers and variables (with restrictions) as terms; uses $(- \xi)$, $(\xi * \psi)$ and $(\xi + \psi)$ to express the terms $(- x)$, $(x \bullet y)$ and $(x + y)$ respectively, where $\xi$ and $\psi$ express x and y respectively; employs atomic formulae exists($\xi$), fragment($\xi$, $\psi$) and identical($\xi$, $\psi$) for **exist**(x), $x \leq y$ and $x \equiv y$ respectively, where $\xi$ and $\psi$ express x and y respectively; admits literals $\alpha$ and $(\sim \alpha)$ for atomic formulae $\alpha$, where $(\sim \alpha)$ denotes the negation of $\alpha$; allows conditionals $(\lambda$ iF $\beta)$ for literal $\lambda$ and either literal $\beta$ or literal conjunction $\beta = (\lambda_1 \,\&\, \ldots \,\&\, \lambda_m)$; and allows conjunctions $(\beta_1 \,\&\, \ldots \,\&\, \beta_k)$ for literals or conditionals $\beta_1, \ldots \beta_k$. A believe predicate is employed to enter expressions from language $\mathcal{C}$ into a knowledge base as domain knowledge $\mathcal{D}$.

To illustrate some domain knowledge $\mathcal{D}$, the processes of interest at region s and time t in Figure 2 can be segregated into distinct transitory and stationary fragments which at time t happen to be a fragment of region s. In formal language $\mathbb{M}$, this can be expressed by transitory $\bullet$ stationary $\equiv \perp$ & (transitory + stationary) $\bullet$ t $\leq$ s. This can be entered into the machine in language $\mathcal{C}$ as

believe((~ exists(transitory * stationary))
& fragment((transitory + stationary) * t, s)).

The transitory fragments comprise the mutually distinct helicopter, trucks and containers at time t, which is expressed to the machine through

believe(identical(transitory * t, (helicopter + trucks + containers) * t) & (~ exists(trucks * containers)) & (~ exists(helicopter * (trucks + containers)))).

There are two distinct trucks at time t, truck1 and truck2,

believe(identical(trucks * t, (truck1 + truck2) * t) & (~ exists(truck1 * truck2)) & exists(truck1 * t) & exists(truck2 * t)).

with the two distinct containers reported analogously. The helicopter has been conceptualised in terms of its distinct main rotor, tail rotor and fuselage fragments. This is expressed to the machine by

believe(identical(helicopter, main_rotor + tail_rotor + fuselage) & (~ exists(main_rotor * (tail_rotor + fuselage))) & (~ exists(tail_rotor * fuselage)) & exists(main_rotor * t) & exists(tail_rotor * t) & exists(fuselage * t)).

The four mutually distinct stationary fragments field, sky, foliage and soccer_goals are similarly expressed in $\mathcal{C}$.

## 4.2 Deductive Consequences

When domain knowledge $\mathcal{D}$ in the language of $\mathcal{C}$ is presented to the knowledge base, an existential normal form function $\exists\mathbf{NF} : \mathcal{C} \rightarrow \mathbf{P}(\mathcal{N})$ (not presented here) converts each $\delta \in \mathcal{D}$ to a set of normal form expressions $\exists\mathbf{NF}(\delta) \subset \mathcal{N}$, for existential normal form language $\mathcal{N} \subset \mathcal{C}$. and powerset $\mathbf{P}$. $\delta \in \mathcal{D}$ and $\exists\mathbf{NF}(\delta) \subset \mathcal{N}$ are semantically equivalent with respect to $\mathbb{E}$.

Metatheorem ($\exists\mathbf{NF}$ preserves the semantics of $\mathbb{E}$)

$\mathbb{E} \models (\wedge\mathbf{Tr}(\delta) \Leftrightarrow \wedge\{\wedge\mathbf{Tr}(\nu) \mid \nu \in \exists\mathbf{NF}(\delta)\}))$, where $\wedge\{ \varphi_1, \ldots, \varphi_k\} = (\varphi_1 \,\&\, \ldots \,\&\, \varphi_k)$, for translation function $\mathbf{Tr} : \mathcal{C} \rightarrow (\mathbb{M} \cup \mathbf{Name} \cup \mathbf{Variable})$

Entering believe($\delta$) for $\delta \in \mathcal{D}$ results in $\exists\mathbf{NF}(\delta)$ being stored in the knowledge base. For example, believing the identity statement for the helicopter in section 4.1 produces,

stored exists((fuselage * t)) into the KB
stored exists((main_rotor * t)) into the KB
stored exists((t * tail_rotor)) into the KB
stored ~exists((fuselage * -helicopter)) into the KB
stored ~exists((fuselage * main_rotor)) into the KB
stored ~exists((fuselage * tail_rotor)) into the KB
stored ~exists((helicopter * -fuselage * -main_rotor * -tail_rotor)) into the KB
stored ~exists((main_rotor * -helicopter)) into the KB
stored ~exists((main_rotor * tail_rotor)) into the KB
stored ~exists((tail_rotor * -helicopter)) into the KB

Deduction from $\mathcal{D}$ is defined in terms of deduction from the existential normal form translation $\exists\mathbf{NF}[\mathcal{D}] = \cup\{\exists\mathbf{NF}(\delta) \mid \delta \in \mathcal{D}\}$ of the expressions in $\mathcal{D}$.

Metadefinition (deductive consequence)

$\mathcal{D} \models \sigma =_{df} \exists\mathbf{NF}[\mathcal{D}] \models \nu$ for all $\nu \in \exists\mathbf{NF}(\sigma)$.

To ensure that the normal form deductive consequences $\{\nu \mid \exists\mathbf{NF}[\mathcal{D}] \models \nu\}$ correspond to the semantic consequences $\{\sigma \mid \mathbb{D} \models \sigma\}$ for $\mathbb{D} = \{\mathbf{Tr}(\nu) \mid \nu \in \exists\mathbf{NF}[\mathcal{D}]\}$, the normal form deductive consequence relation $\models$ is defined with reference to $\models$. Firstly, the two axioms for the proof theory of $\models$ are (~ exists(nothing)) and exists(universe) as $\mathcal{N}$ language counterparts to $\neg\,\mathbf{exist}(\perp)$ and $\mathbf{exist}(\boldsymbol{\Omega})$ (with the latter preventing the trivial model of $\mathbb{E}$ in which $\perp \equiv \boldsymbol{\Omega}$) in the language of $\mathbb{E}$. The proof theory couples these two axioms with 6 rules of inference. Four of these rules of inference correspond to the following four provable metatheorems of $\mathbb{E}$ which the author has named.

Generalised Existential Modus Ponens (G∃MP)

$\{\neg\,(\mathbf{exist}(\xi \bullet \psi)), \mathbf{exist}(\xi \bullet \eta)\} \models \mathbf{exist}(\text{-}\,\psi \bullet \eta)$.

Generalised Existential Modus Tollens (G∃MT)

$\{\neg\,(\mathbf{exist}(\xi \bullet \psi)), \neg\,(\mathbf{exist}(\text{-}\xi \bullet \eta))\} \models \neg\,(\mathbf{exist}(\psi \bullet \eta))$.

Generalised Existential Disjunctive Syllogism (G∃DS)

$\{\mathbf{exist}((\xi \bullet \psi) + \eta), \neg\,(\mathbf{exist}(\xi))\} \models \mathbf{exist}(\eta))$.

Generalised Existential DeMorgan's Law (G∃DL)

$\{\neg\,(\mathbf{exist}((\xi_1 + \ldots + \xi_n))\} \models \neg\,(\mathbf{exist}(\xi_i))$.

The other two rules of inference support inference with conditionals.

Conditional Assertion (CA)

$\{(\beta \text{ if true})\} \models \beta$.

Conditional Simple Constructive Dilemma (CSCD)

$\{(\beta_1 \text{ if } (\beta_2 \,\&\, \beta_3)), \beta_2\} \models (\beta_1 \text{ if } \beta_3)$.

Normal form deductive inference is then defined as follows.

Metadefinition (normal form deductive consequence)

$\exists\mathbf{NF}[\mathcal{D}] \models \nu =_{df}$ there exists a proof sequence $< \nu_1, \ldots, \nu_n>$ such that: $\nu_1 = (\sim \nu)$; $\nu_n$ = exists(nothing); and for all $\nu_k$ $(1 < k < n)$ either $\nu_k = (\sim$ exists(nothing)) or $\nu_k$ = exists(universe) or $\nu_k \in \exists\mathbf{NF}[\mathcal{D}]$ or there exists i, j < k and $\{ \varphi_i, \varphi_j\} \models \varphi_k$ by G∃MP, G∃MT, G∃DS, G∃DL, CA or CSCD, where $\mathbf{Tr}(\nu_m) = \varphi_m \in \mathbb{D}$ is the counterpart to $\nu_m \in \exists\mathbf{NF}[\mathcal{D}]$.

The author has written a theorem prover to compute $\mathcal{D} \models \sigma$, with the option of displaying a proof. To illustrate, Figure 3 shows the output generated by the theorem prover in response to the query ask_proof(fragment(main_rotor * t, s)) with the section 4.1 domain knowledge $\mathcal{D}$ of Figure 2.

To prove
    fragment(main_rotor*t,s)
it is necessary to prove the following normal form theorems
    ~exists((main_rotor * t * -s))

The number of theorems for the proof is 1.

Theorem: ~exists((main_rotor * t * -s))
Proof:
Hypothesise denial of the initial proposition
    ~exists((main_rotor * t * -s))
deduce exists((main_rotor * t * -s))
    by HYP
deduce ~exists((t * transitory * -s))
    by KB
deduce ~exists((helicopter * t * -transitory))
    by KB
deduce ~exists((main_rotor * -helicopter))
    by KB
deduce ~exists((main_rotor * t * -transitory))
    from ~exists((main_rotor * -helicopter))
    with ~exists((helicopter * t * -transitory))
    by GEMT
deduce ~exists((main_rotor * t * -s))
    from ~exists((t * transitory * -s))
    with ~exists((main_rotor * t * -transitory))
    by GEMT
deduce exists(nothing)
    from ~exists((main_rotor * t * -s))
    with exists((main_rotor * t * -s))
    by GEMP
As this is a contradiction, by reductio ad absurdum the theorem follows.
Q.E.D.

Figure 3. Sample generated proof.

# 5 Metaphysical Category

The theory of existence $\mathbb{E}$ is the primary metaphysical formal theory. To complete a computational metaphysics of processes as spatio-temporal fragments of the universe, space and time must augment $\mathbb{E}$. A metaphysical category ascending from $\mathbb{E}$ can be expressed in category theory

(e.g. [17]) by forming various specifications $S_F = <\sigma_F, F>$, each with a formal theory (set of axioms) F and its associated formal language signature $\sigma_F$ (composed of sorts and operators on them), and by defining morphisms between these specifications. Figure 4 illustrates the role of the limit specification $S_E$, which specifies processes. Time can be specified by colimit $S_T$ composed from a theory of temporal ordering T< (e.g. [18]) and a theory of temporal distance TD, as refinements of an ontological theory of time TO in which times conform to a sub-algebra TO of Boolean algebra E through the identification of times as equivalence classes of concurrent processes. Space can be specified by $S_S$ composed from a theory of spatial orientation SR; a theory of spatial distance SD; and a theory of spatial connection SC; as refinements of an ontological theory of space SO in which regions conform to a sub-algebra SO of Boolean algebra E through the identification of equivalence classes of co-located processes, (e.g. the region connection calculus can be expressed as a Boolean algebra SO and a collection of connection axioms SC ([19])). $S_T$ and $S_S$ then combine to form the metaphysical specification $S_M$, which in turn transitively underpins the specifications associated with the other tiers in Figure 1.

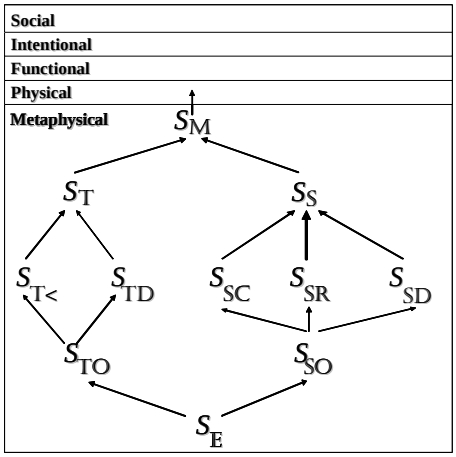

Figure 4. Hierarchy of semantic theories.

# 6 Conclusions

The choice of symbols and their meaning is a fundamental issue for higher-level data fusion. The paper has suggested that a hierarchy of formal theories is a promising approach provided the underlying philosophy, mathematics and computer science is carefully considered, and it has demonstrated that approach for a theory of existence.

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