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14. ABSTRACT
Radiation belt electrons and chorus waves are an outstanding instance of the important role cyclotron resonant wave-particle interactions play in the magnetosphere. Chorus waves are particularly complex, often occurring with large amplitude, narrowband but drifting frequency and fine structure. Nevertheless, modeling their effect on radiation belt electrons with bounce-averaged broadband quasi-linear theory seems to yield reasonable results. It is known that coherent interactions with monochromatic waves can cause particle diffusion, as well as radically different phase bunching and phase trapping behavior. Here the two formulations of diffusion, while conceptually different, are shown to give identical diffusion coefficients, in the narrowband limit of quasi-linear theory. It is further shown that suitably averaging the monochromatic diffusion coefficients over frequency and wave normal angle parameters reproduces the full broadband quasi-linear results. This may account for the rather surprising success of quasi-linear theory in modeling radiation belt electrons undergoing diffusion by chorus waves.

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Diffusion by one wave and by many waves

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[1] Radiation belt electrons and chorus waves are an outstanding instance of the important role cyclotron resonant wave-particle interactions play in the magnetosphere. Chorus waves are particularly complex, often occurring with large amplitude, narrowband but drifting frequency and fine structure. Nevertheless, modeling their effect on radiation belt electrons with bounce-averaged broadband quasi-linear theory seems to yield reasonable results. It is known that coherent interactions with monochromatic waves can cause particle diffusion, as well as radically different phase bunching and phase trapping behavior. Here the two formulations of diffusion, while conceptually different, are shown to give identical diffusion coefficients, in the narrowband limit of quasi-linear theory. It is further shown that suitably averaging the monochromatic diffusion coefficients over frequency and wave normal angle parameters reproduces the full broadband quasi-linear results. This may account for the rather surprising success of quasi-linear theory in modeling radiation belt electrons undergoing diffusion by chorus waves.

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1. Introduction

[2] Cyclotron resonant wave-particle interactions play a key role in both the acceleration and loss of radiation belt electrons. Chorus waves, in particular, are believed to be key to both the energization and loss of energetic electrons in the outer zone [*Chen et al.*, 2007; *Horne*, 2007; *Bortnik and Thorne*, 2007]. Chorus waves propagate in the whistler mode and are observed, with sufficient time resolution, to be coherent, with well-defined frequencies that drift during their growth to large amplitude [*Santolik et al.*, 2003; *Breneman et al.*, 2009]. The wave growth is intimately connected to the linear [*Li et al.*, 2008, 2009] and nonlinear [e.g., *Nunn*, 1974; *Katoh and Omura*, 2007] behavior of resonant electrons with energy in the keV range. MeV range electrons are also subject to nonlinear behavior induced by the developed waves, but their motion can be considered "parasitic," i.e., not feeding back to the development of the waves.

[3] Coherent cyclotron resonant interactions of test electrons with individual whistler mode waves has been treated by many authors, and yields three distinctly different kinds of particle behavior, namely diffusion, phase bunching, and phase trapping. Both phase bunching (without trapping) and phase trapping are favored by large amplitude waves and low inhomogeneity of the background magnetic field; a quantitative criterion has been developed by many authors [e.g., *Inan et al.*, 1978; *Albert*, 1993; *Omura et al.*, 2008]. The relevant regime also depends strongly on the particle energy and pitch angle, so all three types of behavior may occur under the same conditions. *Albert* [1993, 2000, hereafter Papers I and II, respectively] derived analytical expressions for the changes in pitch angle and energy for all three types of motion, using a Hamiltonian formulation, though frequency drift was neglected. Similar considerations also apply to large amplitude electromagnetic ion cyclotron waves [*Albert and Bortnik*, 2009]. In the diffusive regime, a key quantity is the effective interaction time, which is controlled by how long (or far) the particle has to move in the varying background field before the resonance condition is violated.

[4] The large-scale effects of chorus waves on the radiation belts have also been modeled using quasi-linear theory in one, two, and three dimensions (see *Albert* [2009] for a brief review). This framework assumes a continuum of uncorrelated, small amplitude waves, with wide distributions in frequency and wave normal angle, in a constant background magnetic field. Here, the diffusion can be considered limited by the relative parallel velocity of the particle and the group velocity of a nearly resonant wave packet [*Albert*, 2001]. The resulting local pitch angle and energy diffusion coefficients are computed locally and then bounce averaged, which finally introduces variation of the background magnetic field. Recently, the expressions for broadband quasi-linear diffusion coefficients were expressed in a relatively transparent form [*Albert*, 2005], which turned out to be convenient for isolating single waves within the broad frequency and wave normal angle distributions. Such single waves, suitably chosen, are perhaps surprisingly well able to represent the entire distributions, leading to accurate

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approximations [*Albert*, 2007, 2008]. These may be considered a generalization of the parallel propagation approximation [*Summers et al.*, 2007].

[5] Thus diffusion emerges from both quasi-linear and nonlinear treatments, but the underlying pictures are quite different. Since the quasi-linear diffusion approach seems to model the actual particle behavior fairly well [*Albert*, 2009], it is of great interest to relate the two sets of diffusion coefficients. This was done by *Albert* [2001], working with quasi-linear expressions for whistler mode waves in the high-density, low-frequency limit [*Lyons et al.*, 1972], which invoked considerable simplifications of both the whistler mode dispersion relation and the resonance condition. It was concluded that the narrowband limit of the quasi-linear pitch angle diffusion coefficient was approximately equal to the Hamiltonian-derived pitch angle diffusion coefficient for monochromatic waves. Here, the comparison of the two analytical frameworks is reconsidered in much greater generality, using the full description of stationary cold plasma waves. It is shown that the narrowband limit of bounce-averaged quasi-linear theory and the diffusive regime of the Hamiltonian analysis yield exactly the same pitch angle, energy, and cross diffusion coefficients. Furthermore, averaging the monochromatic results over distributions of frequency and wave normal angle, which statistically models a sequence of resonant interactions with individual waves, recovers exactly the full broadband quasi-linear diffusion coefficients. This seems a meaningful step toward reconciling the behavior expected from coherent dynamics, in the diffusion regime, with the apparent utility of bounce-averaged quasi-linear theory for modeling of radiation belt electrons.

[6] Any possible coupling between changes in α_0 and p with changes in L will be ignored. This is usually justified by the wide separation of time scales associated with the first two adiabatic invariants compared to that of the third, i.e., the drift period compared to the cyclotron and bounce periods. Such coupling, which would lead to cross diffusion terms involving $D_{\alpha_0 L}$ and D_{pL}, has only been considered occasionally, usually in the context of so-called drift shell splitting [*Roederer*, 1970; *Schulz and Lanzerotti*, 1974], although *Brizard and Chan* [2004] recently formulated the "full" matrix of diffusion coefficients generated by an arbitrary wave spectrum in axisymmetric geometry. The resulting diffusion equation could be solved numerically by an algorithm based on stochastic differential equations [*Tao et al.*, 2008] or the layer method described by *Tao et al.* [2009].

[7] Section 2 exhibits the local quasi-linear diffusion coefficients and their monochromatic limit, following *Albert* [2007], and carries out the bounce average following *Albert* [2001], leading to closed form expressions with no remaining integrals. Section 3 presents the diffusion coefficients of *Albert* [1993, 2000] resulting from coherent interactions with a single, monochromatic wave, which are found to be identical to the final results of section 2. Section 4 then considers the coherent diffusion coefficients suitably averaged over wave frequency and wave normal angle parameters, reproducing the full quasi-linear expressions. Section 5 presents some numerical examples of diffusion coefficients for a model of nightside chorus waves, calculated from each approach, and explicitly demonstrates their equivalence. This is followed by a brief discussion.

2. Quasi-Linear Diffusion Coefficients

[8] The condition for gyroresonance between a particle and a wave is

$$\omega - k_\parallel v_\parallel = \Omega_n, \quad \Omega_n \equiv sn\Omega_c/\gamma, \tag{1}$$

where n is an integer, $s = \pm 1$ is the sign of the charge of the particle, $\Omega_c = |q|\, B/mc$ is its local nonrelativistic gyrofrequency, and γ is its relativistic factor. The local pitch angle of the particle is α, the index of refraction is $\mu \equiv kc/\omega$, and the wave normal angle is θ. The underlying mechanism of quasi-linear diffusion can be thought of as involving continuous resonance: even as the particle diffuses in α and γ, it is always able to find an instantaneously resonant wave within the ω and θ distributions.

[9] *Albert* [2001] considered whistler waves, using expressions based on the approximations $\omega/\Omega_c \ll 1 \ll \omega_{pe}^2/\Omega_c^2$ [*Lyons et al.*, 1972; *Lyons*, 1974b], but here any cold plasma mode is considered, without any such approximations.

2.1. Local Expressions

[10] The local diffusion coefficients in a spectrum of waves were given by *Lyons* [1974a, 1974b], as derived from the Vlasov equation [*Kennel and Engelmann*, 1966; *Lerche*, 1968], although it can also be obtained by considering motion of a single particle acted on by single wave, for an interaction time related to the wave packet bandwidth [e.g., *Albert*, 2001]. In either case, the spatial variation of the background magnetic field and all other parameters is ignored for the local calculation, and accounted for later by bounce averaging.

[11] The derivation is fairly involved (see also the presentations by *Walker* [1993] and *Swanson* [1989]), but the results for pitch angle α and momentum p can be expressed as

$$D_{\alpha\alpha} = \sum_{n=-\infty}^{\infty} \frac{\pi}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta(\omega - k_\parallel v_\parallel - \Omega_n) \times q^2 \frac{|\mathbf{B_k}|^2}{V} \frac{|\Phi_n|^2}{\mu^2} \frac{\left(-\sin^2\alpha + \Omega_n/\omega\right)^2}{p^2\cos^2\alpha},$$

$$\frac{D^n_{\alpha p}}{D^n_{\alpha\alpha}} = \frac{p\sin\alpha\cos\alpha}{-\sin^2\alpha + \Omega_n/\omega}, \quad \frac{D^n_{pp}}{D^n_{\alpha\alpha}} = \left(\frac{D^n_{\alpha p}}{D^n_{\alpha\alpha}}\right)^2. \tag{2}$$

$D_{\alpha\alpha}$ has dimensions of $1/t$, because of the explicit division by p^2. $\mathbf{B_k}$ is the Fourier transform of the wave magnetic field taken over the plasma volume V (which is effectively infinite), and $|\Phi_n|^2$, as given by equation (9) of *Lyons* [1974b], is the result of resonance averaging the geometric details of the particle motion in the electromagnetic field of an oblique plane wave. The ratios of the diffusion coefficients were interpreted by *Kennel and Engelmann* [1966] in terms of single-wave characteristics of a quasi-linear diffusion operator, and were further discussed by *Lyons* [1974a] and *Summers et al.* [1998].

[12] The expressions get more involved after transforming the integration variables from $(k_\perp, k_\parallel)$ to (ω, θ), and mod-

eling $|\mathbf{B_k}|^2/V$ as a function of (ω, θ), which brings in normalization integrals. As expressed by *Albert* [2005], and similarly by *Glauert and Horne* [2005], the resulting form of the diffusion coefficients can be written as the sum over n of terms D^n given by

$$D^n_{\alpha\alpha} = \frac{\Omega_c}{\gamma^2}\frac{B^2_{\rm wave}}{B^2}\int_0^\pi \sin\theta d\theta\, \Delta_n G_1 G_2, \tag{3}$$

with

$$\begin{aligned} \Delta_n &= \frac{\pi}{2}\frac{\sec\theta}{|v_\parallel/c|^3}\Phi_n^2\frac{(-\sin^2\alpha+\Omega_n/\omega)^2}{|1-(\partial\omega/\partial k_\parallel)_\theta/v_\parallel|},\\ G_1 &= \frac{\Omega_c B^2(\omega)}{\int_{\omega_{LC}}^{\omega_{UC}} B^2(\omega')\, d\omega'},\\ G_2 &= \frac{g_\omega(\theta)}{\int_0^\pi d\theta' \sin\theta' g_\omega(\theta')\,\Gamma},\\ \Gamma &= \mu^2\left|\mu+\omega\frac{\partial\mu}{\partial\omega}\right|. \end{aligned} \tag{4}$$

[13] The refractive index μ is a known function of (ω, θ) for the given cold plasma wave mode [e.g., *Stix*, 1962]. $B^2(\omega)$ describes the frequency distribution of wave power, and is nonzero only between lower and upper cutoffs, $\omega_{LC} \le \omega \le \omega_{UC}$. Similarly, the distribution of wave power with wave normal angle θ is described by $g_\omega(\theta)$, which is nonzero only for $\theta_{\rm min} \le \theta \le \theta_{\rm max}$. Both $B^2(\omega)$ and $g_\omega(\tan\theta)$ are usually modeled as truncated Gaussians, peaked at ω_m and θ_m, respectively. The quantities G_1 and G_2 are explicitly normalized versions of $B^2(\omega)$ and $g_\omega(\theta)$, and are discussed further in Appendix A.

2.2. Narrowband Limit

[14] As shown by *Albert* [2007, 2008], the integral in equation (3) may be approximated as a weighted average, which becomes exact as $g_\omega(\theta)$ becomes narrowly peaked. In that limit,

$$D^n_{\alpha\alpha} = \frac{\Omega_c}{\gamma^2}\frac{B^2_{\rm wave}}{B^2}\frac{\Delta_n G_1}{\Gamma}, \tag{5}$$

evaluated at some resonant pair (ω, θ) within the specified distributions. For the purposes of *Albert* [2007, 2008], ω_{LC} and ω_{UC} were used to find θ ranges containing resonances, and $D^n_{\alpha\alpha}$ was approximated using representative values from within these ranges. In section 2.3, equation (5) is evaluated at θ_m, with ω taken to be the corresponding resonant value at each location.

2.3. Bounce Averaging

[15] The bounce-averaged diffusion coefficient for the equatorial pitch angle, α_0, is given by the sum over n of

$$D^n_{\alpha_0\alpha_0} = \frac{1}{\tau_b}\int\frac{dz}{v_\parallel}\left(\frac{\partial\alpha_0}{\partial\alpha}\right)^2 D^n_{\alpha\alpha}, \tag{6}$$

where z is distance along the magnetic field line (and is easily converted to latitude). In equation (3), $B^2(\omega)$ is evaluated at the resonant frequency, which depends on both θ and z. As the θ distribution is narrowed, $\omega_{\rm res}$ becomes a well-defined function of z. And as the ω distribution is narrowed, $B^2(\omega)$ approaches a δ function of ω. Assuming θ_m and ω_m are compatible with resonance at some location z_m, the bounce average and G_1 combine to give

$$\begin{aligned} \int\frac{dz}{v_\parallel}G_1(\omega_{\rm res}(z,\theta_m)) &\Rightarrow \int\frac{dz}{v_\parallel}\Omega_c\,\delta(\omega_{res}(z)-\omega_m)\\ &= \int\frac{dz}{v_\parallel}\Omega_c\frac{\delta(z-z_m)}{|d\omega/dz|}. \end{aligned} \tag{7}$$

The full wave intensity, $B_{\rm wave}$, is now considered to be concentrated at the single pair (θ_m, ω_m).

[16] The derivative of ω is evaluated using the resonance condition, and it is important to note that $k_\parallel$ is a function of both z and ω, as specified by the dispersion relation. Therefore implicit differentiation of the resonance condition gives

$$\frac{d\omega}{dz} = \left(1-v_\parallel\frac{\partial k_\parallel}{\partial\omega}\right)^{-1}\frac{\partial}{\partial z}(k_\parallel v_\parallel+\Omega_n). \tag{8}$$

[17] The factors of $(\Delta_n/\Gamma)/|d\omega/dz|$ containing partial derivatives combine and simplify:

$$\left|1-\frac{1}{v_\parallel}\frac{\partial\omega}{\partial k_\parallel}\right|^{-1}\left|1+\frac{\omega}{\mu}\frac{\partial\mu}{\partial\omega}\right|^{-1}\left|1-v_\parallel\frac{\partial k_\parallel}{\partial\omega}\right| = \left|\frac{k_\parallel v_\parallel}{\omega}\right|. \tag{9}$$

Putting everything together gives

$$\begin{aligned} D^n_{\alpha_0\alpha_0} = &\frac{\pi}{2\tau_b}\frac{B^2_{\rm wave}}{B^2}\frac{c^2\Omega^2}{|v_\parallel|v^2\gamma^2}\frac{\Phi_n^2}{\mu^2}\frac{B_{eq}}{B\cos^2\alpha_0}\\ &\times\left(-\sin^2\alpha+\frac{\Omega_n}{\omega}\right)^2\left|\frac{\partial}{\partial z}(k_\parallel v_\parallel+\Omega_n)\right|^{-1}, \end{aligned} \tag{10}$$

where B_{eq} and α_0 are equatorial values but all other quantities are evaluated at the resonance location. This is the monochromatic limit of the bounce-averaged, broadband quasilinear diffusion coefficient for each n.

[18] The bounce-averaged coefficients $D^n_{\alpha_0 p}$ and D^n_{pp} are derived similarly, and in the monochromatic limit are related to $D^n_{\alpha_0\alpha_0}$ by

$$\frac{D^n_{\alpha_0 p}}{D^n_{\alpha_0\alpha_0}} = \frac{p\sin\alpha_0\cos\alpha_0}{-\sin^2\alpha+\Omega_n/\omega}\frac{B}{B_{eq}},\quad \frac{D^n_{pp}}{D^n_{\alpha_0\alpha_0}} = \left(\frac{D^n_{\alpha_0 p}}{D^n_{\alpha_0\alpha_0}}\right)^2 \tag{11}$$

for each n. *Albert* [2004] discussed the role of these ratios in enforcing the condition $D_{\alpha_0\alpha_0}D_{pp} > D^2_{\alpha_0 p}$.

3. Coherent Interactions

[19] A quite different scenario is that of a particle interacting with a single wave in a spatially varying magnetic field, so that the resonance condition of equation (1) is only satisfied at discrete, isolated locations through which the particle passes. As mentioned, analytical estimates of the resulting particle motion were obtained in Papers I and II. For large amplitude waves and small background inhomogeneity, nonlinear behavior (phase bunching and phase

trapping) can occur, but here the opposite limit is considered, which leads to random walks, or diffusion.

[20] Papers I and II write out the full equations of motion in Hamiltonian form, transform to gyroresonance variables, expand to first order in B_{wave}/B, and appropriately average away nonresonant terms. For $n \neq 0$, this leads to two constants of motion which can be used to reduce the number of variables to a single action-angle pair, (I, ξ). To lowest order, I is proportional to the familiar first adiadatic invariant, and ξ is the usual wave-particle phase which is stationary at resonance. The evolution equations for I and ξ can be expressed in terms of a reduced Hamiltonian, $K = K_0(I, z) + K_1(I, z)\sin\xi$, with z playing the role of time. The adiabatic motion is described by K_0, while K_1 captures the effects of the resonant wave. For $n = 0$, a similar treatment yields a reduced Hamiltonian $M = M_0(\Upsilon, z) + M_1(\Upsilon, z)\sin\xi$, where $\Upsilon = \gamma^2$. The reduced Hamiltonians can be used to derive analytic approximations to the resonant changes in the adiabatic invariants I or Υ. An "inhomogeneity parameter" R, proportional to $(\partial B/\partial z)/B_{wave}$, delineates diffusion from the nonlinear regimes involving phase bunching and/or phase trapping. Here we only consider the case $|R| \gg 1$, which indicates diffusion.

3.1. Cyclotron Resonance

[21] At an isolated resonance $n \neq 0$, according to Papers I and II,

$$(\delta I)^2 = K_1^2 \frac{2\pi}{|\partial^2 K_0/\partial z \partial I|} \cos^2\left(\xi_{res} + \sigma_I \frac{\pi}{4}\right). \tag{12}$$

Again, z is distance along the field line, and σ_I is the sign of $\partial^2 K_0/\partial z \partial I$ at resonance. Averaging over ξ_{res}, which depends on the gyrophase and is randomized between bounces, yields 1/2. Papers I and II also give the perturbation Hamiltonian K_1 in terms of a_n, which describes the wave field components. The relation between K_1, a_n, and Φ_n noted by *Albert* [2001] holds for general cold plasma waves

$$K_1^2 = \frac{n^2}{4} \frac{a_n^2}{(p_\parallel/mc)^2} = \frac{n^2}{\mu^2} \frac{v_\perp^2}{v_\parallel^2} \frac{\Omega_c^2}{\omega^2} \frac{B_{wave}^2}{B^2} \Phi_n^2. \tag{13}$$

The Hamiltonian equation of motion for ξ yields

$$\frac{\partial^2 K_0}{\partial z \partial I} = \frac{\partial}{\partial z}\left(\frac{d\xi}{dz}\right) = \frac{c^2}{v_\parallel \omega^2} \frac{\partial}{\partial z}(k_\parallel v_\parallel + \Omega_n - \omega), \tag{14}$$

where ω is the constant frequency of the single wave and can be omitted in the z derivative.

[22] Diffusion coefficients are constructed from

$$\left\{D^n_{\alpha_0\alpha_0}, D^n_{\alpha_0 p}, D^n_{pp}\right\} = \frac{\langle(\delta I)^2\rangle}{2\tau_b}\left\{\left(\frac{\partial\alpha_0}{\partial I}\right)^2, \frac{\partial\alpha_0}{\partial I}\frac{\partial p}{\partial I}, \left(\frac{\partial p}{\partial I}\right)^2\right\}, \tag{15}$$

where $\langle\ \rangle$ denotes the average over ξ_{res}. From Paper II,

$$\frac{\partial\alpha_0}{\partial I} = \frac{-\sin^2\alpha + \Omega_n/\omega}{\sin\alpha_0\cos\alpha_0} \frac{B_{eq}}{B} \frac{m^2c^2}{p^2} \frac{\gamma}{sn}, \quad \frac{\partial p}{\partial I} = \frac{m^2c^2}{p} \frac{\gamma}{sn}. \tag{16}$$

The corresponding ratio $dp/d\alpha_0$ is closely related to the ratios in equation (11). Substituting equations (12)–(14) and (16) into equation (15) yields the first major result of this paper: the coherent interaction versions of $D^n_{\alpha_0\alpha_0}$, $D^n_{\alpha_0 p}$, and D^n_{pp} work out to be exactly the same as in equations (10) and (11) for the narrowband limit of the bounce-averaged quasi-linear expressions.

3.2. Landau Resonance

[23] For the special case $n = 0$, Paper II gives

$$(\delta\Upsilon)^2 = M_1^2 \frac{2\pi}{|\partial^2 M_0/\partial z \partial\Upsilon|} \cos^2\left(\xi_{res} + \sigma_\Upsilon \frac{\pi}{4}\right), \tag{17}$$

and σ_Υ is the sign of $\partial^2 M_0/\partial z \partial\Upsilon$ at resonance. Here a_0 is just a_n with $n = 0$, but now

$$M_1^2 = a_0^2\mu^2\cos^2\theta = 4\cos^2\theta \frac{v_\perp^2}{v_\parallel^2} \frac{\Omega_c^2}{\omega^2} \frac{B_{wave}^2}{B^2} \Phi_0^2. \tag{18}$$

The Hamiltonian equation of motion for ξ yields

$$\frac{\partial^2 M_0}{\partial z \partial\Upsilon} = \frac{\partial}{\partial z}\left(\frac{d\xi}{dz}\right) = \frac{c^2}{v_\parallel \omega^2} \frac{\partial}{\partial z}(k_\parallel v_\parallel - \omega), \tag{19}$$

where again ω can be omitted in the z derivative.

[24] The diffusion coefficients are now

$$D^{n=0}_{\alpha_0\alpha_0} = \frac{\langle(\delta\Upsilon)^2\rangle}{2\tau_b}\left(\frac{\partial\alpha_0}{\partial\Upsilon}\right)^2 \tag{20}$$

and so on. Using

$$\frac{\partial\alpha_0}{\partial\Upsilon} = -\tan\alpha_0 \frac{m^2c^2}{2p^2}, \quad \frac{\partial p}{\partial\Upsilon} = \frac{m^2c^2}{2p} \tag{21}$$

from Paper II, the resulting coherent interaction expressions for $D^{n=0}_{\alpha_0\alpha_0}$, $D^{n=0}_{\alpha_0 p}$, and $D^{n=0}_{pp}$ again agree exactly with equations (10) and (11) from the narrowband limit of bounce-averaged quasi-linear theory.

4. Average Over Wave Distributions

[25] It has just been shown that the monochromatic limit of bounce-averaged, broadband quasi-linear theory is well behaved, and reduces to the results of a Hamiltonian analysis of a resonant interaction with a single wave (in the diffusive regime). Conversely, pitch angle diffusion by a single, coherent wave can be expressed in terms of the quantities defined for quasi-linear diffusion

$$D^n_{\alpha_0\alpha_0} = \frac{\Omega_c}{\gamma^2} \frac{B_{wave}^2}{B^2} \frac{\Delta_n}{\Gamma}\left(\frac{\partial\alpha_0}{\partial\alpha}\right)^2 \frac{1}{\tau_b}\int\frac{dz}{v_\parallel}\Omega_c\delta(\omega_{res}(z) - \omega_m) \tag{22}$$

for either $n \neq 0$ or $n = 0$. We now consider the result of many coherent interactions with individual waves all with amplitude B_{wave} but with frequency and wavenormal angle statistically distributed according to $B^2(\omega)$ and $g_\omega(\theta)$.

[26] The appropriate average is

$$\langle D^n_{\alpha_0\alpha_0}\rangle = \frac{\int D_{\mathbf{k}}|\mathbf{B_k}|^2 d^3\mathbf{k}}{\int |\mathbf{B_k}|^2 d^3\mathbf{k}}, \tag{23}$$

where $D_{\mathbf{k}}$ refers to the single-wave equation (22). The denominator of (23) is just B_{wave}^2. Converting from $d^3\mathbf{k}$ to

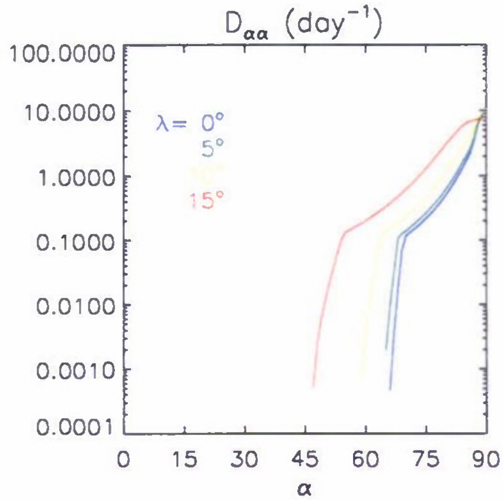

Figure 1. Local quasi-linear pitch angle diffusion coefficient for 1 MeV electrons interacting with a broadband spectrum of chorus waves at $L = 4.5$ at different latitudes. At the equator, the spectrum is peaked at $\theta_m = 0$, $\omega_m/\Omega_e = 0.35$. Only the lowest harmonic ($n = -1$) term is shown.

$d\omega d\theta$ in the numerator, using the results of Appendix A, gives

$$\langle D^n_{\alpha_0\alpha_0}\rangle = \int d\omega d\theta D_k \frac{B^2(\omega)}{\int B^2(\omega')d\omega'} \frac{\sin\theta g_\omega(\theta)\,\Gamma}{\int d\theta' \sin\theta' g_\omega(\theta')\,\Gamma}. \quad (24)$$

Then, schematically,

$$\begin{aligned} &\int d\omega \int d\theta \int dz\, \delta(\omega_{\rm res}(z) - \omega_m) \\ \Rightarrow &\int dz \int d\theta \int d\omega\, \delta(\omega - \omega_m) \Rightarrow \int dz \int d\theta, \end{aligned} \quad (25)$$

which yields

$$\langle D^n_{\alpha_0\alpha_0}\rangle = \frac{1}{\tau_b}\int \frac{dz}{v_\parallel}\left(\frac{\partial\alpha_0}{\partial\alpha}\right)^2 \frac{\Omega_c}{\gamma^2}\frac{B^2_{\rm wave}}{B^2}\int d\theta \sin\theta \Delta_n G_1 G_2. \quad (26)$$

This is the second major result of this paper: the coherent interaction diffusion coefficient, suitably averaged, is identical to the full quasi-linear result given by equation (6). The analogous relations hold for $\langle D^n_{\alpha_0 p}\rangle$ and $\langle D^n_{pp}\rangle$.

5. Numerical Example

[27] For illustration, we consider the model of *Li et al.* [2007] for nightside chorus during a magnetic storm main phase, at $L = 4.5$ with $\omega_{pe}/\Omega_e = 3.8$ at the equator. They computed quasi-linear diffusion coefficients for waves with $B_{\rm wave} = 50$ pT, with the equatorial frequency distribution specified by $\omega_m = 0.35\ \Omega_e$, $\delta\omega = 0.15\ \Omega_e$, $\omega_{LC} = 0.05\Omega_e$, and $\omega_{UC} = 0.65\ \Omega_e$. The waves are considered present only for latitude $\lambda \leq 15°$. In that work the waves were all taken to propagate with $\theta = 0$, but here, following *Horne et al.* [2005] and *Albert* [2008], the wave normal angle distribution is modeled with $\theta_m = 0$, $\delta\theta = 30°$, $\theta_{\min} = 0$, and $\theta_{\max} = 45°$.

[28] Figure 1 shows the local quasi-linear pitch angle diffusion coefficients for 1 MeV electrons for several values of latitude, calculated from equation (3). Only contributions by $n = -1$ are shown. For each wave normal angle in the distribution, the resonant frequency is found; if both lie within the model distributions, a contribution is made to the diffusion coefficient integrals. The 'usual' quasi-linear results [e.g., *Horne et al.*, 2005; *Albert*, 2005] consist of just such calculations, converted from α to α_0 and bounce averaged, as in equation (6), and summed over n.

[29] Figure 2 shows equatorial pitch angle diffusion coefficients for individual waves with $\theta = \theta_m = 0$ and various frequencies between ω_{LC} and ω_{UC}, calculated according to the coherent formulation, equations (12) and (15), with $n = -1$. Related calculations were previously presented by *Albert* [1993, 2000, 2002] and *Albert and Bortnik* [2009]. As mentioned in section 3, integration along the field line is inherent in the formulation. The curve for $\theta = \theta_m$, $\omega = \omega_m$, is emphasized by the dashed curve. Figure 3 is similar, but shows the results holding $\omega = \omega_m$ fixed and varying θ from $\theta_{\min}$ to $\theta_{\max}$.

[30] Figure 4 shows, as solid curves, the quasi-linear diffusion coefficients after carrying out the bounce averages of the local results illustrated in Figure 1. The sum of contributions from $n = -1$ and $n = +1$ are shown in the top row, and just $n = 0$ is shown in the bottom row. Also shown, as red squares, are the results of numerically averaging the diffusion coefficients for monochromatic waves, from Figures 2 and 3, weighted according to equation (24). It is apparent that, allowing for numerical accuracy, the com-

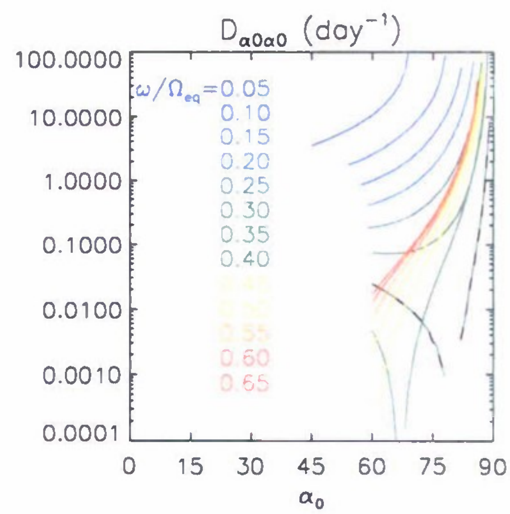

Figure 2. Equatorial pitch angle diffusion coefficient for 1 MeV electrons interacting with monochromatic chorus waves at $L = 4.5$, treated as a coherent interaction. Results are shown for a fixed value of wave normal angle and several fixed values of frequency; the dashed line indicates $\theta = \theta_m = 0$, $\omega = \omega_m = 0.35\Omega_e$ (at the equator). Only the lowest harmonic ($n = -1$) term is shown.

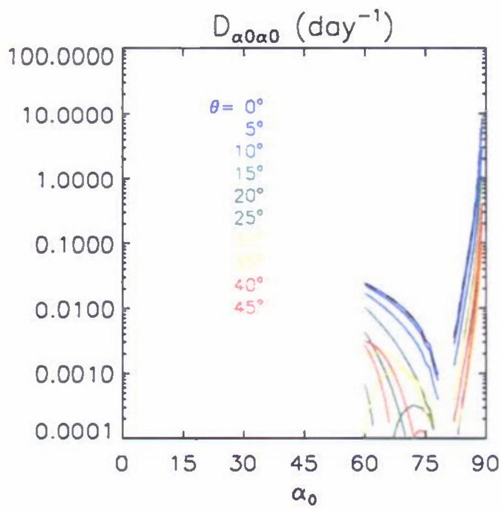

Figure 3. Same as Figure 2 but showing results for fixed frequency and several values of wave normal angle.

putational evaluations verify the analytical result that the two formulations are the same.

6. Summary and Discussion

[31] This paper has investigated the relationship between two seemingly different formulations of wave-particle interactions. Generalizing a previous study, it has been shown analytically that taking the narrowband limit of bounce-averaged, broadband quasi-linear diffusion coefficients agrees exactly with the diffusive limit of coherent interactions with a monochromatic wave. Moreover, considering the individual waves to be drawn from specified frequency and wavenormal angle distributions, and averaging diffusion coefficients accordingly, reproduces the full quasi-linear expressions.

[32] It has been a puzzle why global simulations using quasi-linear theory [*Li et al.*, 2007; *Albert*, 2009] are at least moderately successful in reproducing the observed effects of chorus waves, which upon close examination are discrete and coherent [*Santolik et al.*, 2003]. Parameters used to model chorus waves as a population which are based on wave measurements with coarse time resolution [*Meredith et al.*, 2003] should reflect the distribution of the underlying individual waves. As just shown, multiple interactions with this distribution of waves will be well described statistically by the quasi-linear approach, as long as the individual waves are not large enough to induce nonlinear particle behavior [*Cattell et al.*, 2008; *Cully et al.*, 2008].

[33] It should be noted that in all cases, the wave parameters (amplitude, frequency, wave normal angle) have been treated as constant during each individual wave-particle interaction. Although the quantities can vary significantly, indeed, frequency drift is a characteristic feature of chorus waves, the duration of an isolated interaction is brief in the diffusive regime. This would not apply to phase-trapped particles, which experience an extended resonant interaction

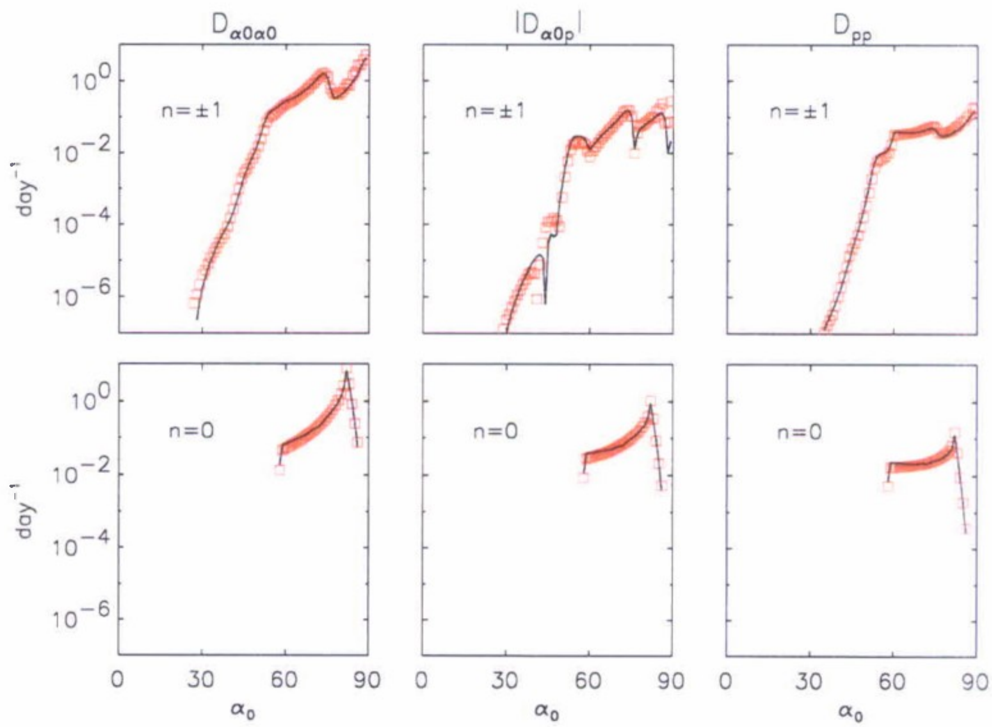

Figure 4. Bounce-averaged quasi-linear diffusion coefficients (solid curves) and diffusion coefficients for coherent interactions with monochromatic waves, averaged over the same frequency and wave normal angle distributions (red squares). (top) The contributions from $n = \pm 1$ and (bottom) the contributions of just $n = 0$ are shown. As predicted analytically, calculations using the two approaches agree.

time, and which are believed to be key for the self-consistent, nonlinear growth of chorus waves.

[34] For computing diffusion coefficients, there is no apparent major advantage to either viewpoint; the same number of integrals must be done either way. However, the coherent interaction approach has the large benefit of indicating when the diffusion approach becomes invalid, and nonlinear effects must be considered. Estimates of these effects have the form of velocity space advection, and may be included in a combined diffusion-advection equation [*Albert*, 1993, 2000, 2002]. The refinement of these estimates, and their use in global simulations, is the subject of ongoing work.

Appendix A: Parameterization of the Wave Distribution

[35] The Fourier transform of the squared wave magnetic field is

$$B^2_{\rm wave} = \int \frac{|\mathbf{B_k}|^2}{V}\frac{d^3k}{(2\pi)^3}, \tag{A1}$$

where

$$\begin{aligned}\int \frac{d^3\mathbf{k}}{(2\pi)^3} &= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi_k \int_{-\infty}^{\infty} dk_\parallel \int_0^\infty k_\perp dk_\perp \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_\parallel \int_0^\infty k_\perp dk_\perp \\ &= \frac{1}{(2\pi)^2}\int_0^\infty d\omega \int_0^\pi d\theta\, k_\perp J,\end{aligned} \tag{A2}$$

for any cold plasma mode,

$$J = \left|\frac{\partial(k_\perp, k_\parallel)}{\partial(\omega,\theta)}\right| = \frac{\omega^2}{c^3}\frac{\sin\theta}{k_\perp}\Gamma, \tag{A3}$$

and $\Gamma(\omega, \theta)$ is $\mu^2|\mu + \omega(\partial\mu/\partial\omega)|$ as in equation (4).

[36] *Lyons* [1974b] explicitly assumed that the wave distribution was independent of both ϕ_k and the sign of $k_\parallel$, so that the integrals could be restricted to $0 \le k_\parallel \le \infty$ or $0 \le \theta \le \pi/2$, with an additional factor of 2. However, to connect to single-wave results, it is more natural not to assume symmetry with respect to $\pm k_\parallel$, and to take θ integrals from 0 to π. Then

$$B^2_{\rm wave} = \frac{1}{(2\pi)^2}\int_0^\pi\int_0^\infty \frac{|\mathbf{B_k}|^2}{V} k_\perp J d\omega d\theta. \tag{A4}$$

Following *Lyons*, we now write

$$B^2_{\rm wave} = \int_0^\infty B^2(\omega)d\omega \tag{A5}$$

and also factor $B^2(\omega)$ out of $|\mathbf{B_k}|^2/V$. This leads to

$$\begin{aligned}\frac{|\mathbf{B_k}|^2}{V} &= \frac{4\pi^2c^3}{\omega^2}B^2_{\rm wave}\frac{B^2(\omega)}{\int B^2(\omega')d\omega'} \\ &\times \frac{g_\omega(\theta)}{\int g_\omega(\theta')\Gamma(\omega,\theta')\sin\theta' d\theta'},\end{aligned} \tag{A6}$$

which corresponds to equation A8 of Lyons, and which satisfies equation (A1) above for any choice of $B^2(\omega)$ and $g_\omega(\theta)$. In the notation of equation (4),

$$\frac{|\mathbf{B_k}|^2}{V} = \frac{4\pi^2c^3}{\omega^2\Omega_c}B^2_{\rm wave}G_1G_2, \tag{A7}$$

which is used in section 4.

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