

Stable Restoration and Separation of Approximately Sparse Signals

Christoph Studer and Richard G. Baraniuk

Rice University, Houston, TX, USA

Abstract

This paper develops new theory and algorithms to recover signals that are approximately sparse in some general (i.e., basis, frame, over-complete, or in-complete) dictionary but corrupted by a combination of measurement noise and interference having a sparse representation in a second general dictionary. Particular applications covered by our framework include the restoration of signals impaired by impulse noise, narrowband interference, or saturation, as well as image in-painting, super-resolution, and signal separation. We develop efficient recovery algorithms and deterministic conditions that guarantee stable restoration and separation. Two application examples demonstrate the efficacy of our approach.

Keywords: Sparse signal recovery, signal restoration, signal separation, deterministic recovery guarantee, basis-pursuit denoising

1. Introduction

We investigate the problem of *recovery* of the coefficient vector $\mathbf{x} \in \mathbb{C}^{N_a}$ from the corrupted M -dimensional observations

$$\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{e} + \mathbf{n}, \quad (1)$$

where $\mathbf{A} \in \mathbb{C}^{M \times N_a}$ and $\mathbf{B} \in \mathbb{C}^{M \times N_b}$ are general (basis, frame, over-complete, or in-complete) deterministic dictionaries, i.e., matrices whose columns have unit Euclidean (or ℓ_2) norm. The vector $\mathbf{x}$ is assumed to be *approximately sparse*, i.e., its main energy (in terms of the sum of absolute values) is concentrated in only a few entries. The M -dimensional signal vector is defined as $\mathbf{y} = \mathbf{A}\mathbf{x}$. The vector $\mathbf{e} \in \mathbb{C}^{N_b}$ represents interference and is assumed to be

Report Documentation Page			Form Approved OMB No. 0704-0188		
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE 02 JUL 2011		2. REPORT TYPE		3. DATES COVERED 00-00-2011 to 00-00-2011	
4. TITLE AND SUBTITLE Stable Restoration and Separation of Approximately Sparse Signals			5a. CONTRACT NUMBER		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S)			5d. PROJECT NUMBER		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Rice University, Department Electrical and Computer Engineering, Houston, TX, 77005			8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)			10. SPONSOR/MONITOR'S ACRONYM(S)		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S)		
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT This paper develops new theory and algorithms to recover signals that are approximately sparse in some general (i.e., basis, frame, over-complete, or incomplete) dictionary but corrupted by a combination of measurement noise and interference having a sparse representation in a second general dictionary. Particular applications covered by our framework include the restoration of signals impaired by impulse noise, narrowband interference, or saturation as well as image in-painting, super-resolution, and signal separation. We develop efficient recovery algorithms and deterministic conditions that guarantee stable restoration and separation. Two application examples demonstrate the efficacy of our approach.					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT Same as Report (SAR)	18. NUMBER OF PAGES 46	19a. NAME OF RESPONSIBLE PERSON
a. REPORT unclassified	b. ABSTRACT unclassified	c. THIS PAGE unclassified			

perfectly sparse, i.e., only a few entries are nonzero, and $\mathbf{n} \in \mathbb{C}^M$ corresponds to measurement noise. Apart from the bound $\|\mathbf{n}\|_2 < \varepsilon$, the measurement noise is allowed to be arbitrary. The interference and noise components $\mathbf{e}$ and $\mathbf{n}$ are allowed to depend on the vector $\mathbf{x}$ and/or the dictionary $\mathbf{A}$.

The setting (1) also allows us to study *signal separation*, i.e., the separation of two distinct features $\mathbf{Ax}$ and $\mathbf{Be}$ from the noisy observation $\mathbf{z}$. Here, the vector $\mathbf{e}$ in (1) is also allowed to be approximately sparse and is used to represent a second desirable feature (rather than undesired interference). Signal separation amounts to simultaneously recovering the vectors $\mathbf{x}$ and $\mathbf{e}$ from the noisy measurement $\mathbf{z}$ followed by computation of the individual signal features $\mathbf{Ax}$ and $\mathbf{Be}$.

1.1. Applications for the model (1)

Both the recovery and separation problems outlined above feature prominently in numerous applications (see [1–17] and references therein), including:

- *Impulse noise*: The recovery of approximately sparse signals corrupted by impulse noise [13] corresponds to recovery of $\mathbf{x}$ from (1) by setting $\mathbf{B} = \mathbf{I}_M$ and associating the interference $\mathbf{e}$ with the impulse-noise vector. Practical examples include restoration of audio signals impaired by click/pop noise [1, 2] and reading out from unreliable memory [14].
- *Narrowband interference*: Audio, video, or communication signals are often corrupted by narrowband interference. A particular example is electric hum, which typically occurs in improperly designed audio equipment. Such impairments naturally exhibit a sparse representation in the frequency domain, which amounts to setting $\mathbf{B}$ to the Fourier matrix.
- *Saturation*: Non-linearities in amplifiers may result in signal saturation, e.g., [7, 16, 17]. Here, instead of the signal vector $\mathbf{y}$ of interest, one observes a saturated (or clipped) version $\mathbf{z} = \mathbf{y} + \mathbf{e} + \mathbf{n}$, where the nonzero entries of $\mathbf{e}$ correspond to the difference between the saturated signal and the original signal $\mathbf{y}$. The noise vector $\mathbf{n}$ can be used to model residual errors that are not captured by the interference component $\mathbf{Be}$.
- *Super-resolution and in-painting*: In super-resolution [3, 15] and in-painting [6, 8–10] applications, only a subset of the entries of the (full-resolution) signal vector $\mathbf{y} = \mathbf{Ax}$ is available. With (1), the interference

vector $\mathbf{e}$ accounts for the missing parts of the signal, i.e., the locations of the nonzero entries of $\mathbf{e}$ correspond to the missing entries in $\mathbf{y}$ and are set to some arbitrary value. The missing parts of $\mathbf{y}$ are then filled in by recovering $\mathbf{x}$ from $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{e} + \mathbf{n}$ followed by computation of the (full-resolution) signal vector $\mathbf{y} = \mathbf{A}\mathbf{x}$.

- *Signal separation:* The framework (1) can be used to model the decomposition of signals into two distinct features. Prominent application examples are the separation of texture from cartoon parts in images [4, 6] and separation of neuronal calcium transients from smooth signals caused by astrocytes in calcium imaging [5]. In both applications, $\mathbf{A}$ and $\mathbf{B}$ are chosen such that each feature can be represented by approximately sparse vectors in one dictionary. Signal separation then amounts to simultaneously extracting $\mathbf{x}$ and $\mathbf{e}$ from $\mathbf{z}$, where $\mathbf{A}\mathbf{x}$ and $\mathbf{B}\mathbf{e}$ represent the individual features.

In almost all of the applications outlined above, a predetermined (and possibly optimized) dictionary pair $\mathbf{A}$ and $\mathbf{B}$ is used. It is therefore of significant practical interest to identify the fundamental limits on the performance of restoration or separation from the model (1) for the deterministic setting, i.e., assuming no randomness in the dictionaries, the signal, interference, or the noise vector. Deterministic recovery guarantees for the special case of *perfectly* sparse vectors $\mathbf{x}$ and $\mathbf{e}$ and *no* measurement noise have been studied in [12, 18]. The results in [12, 18] rely on an uncertainty relation for pairs of general dictionaries and depend on the number of nonzero entries of $\mathbf{x}$ and $\mathbf{e}$, on the coherence parameters of the dictionaries $\mathbf{A}$ and $\mathbf{B}$, and on the amount of prior knowledge on the support of the signal and interference vector. However, the algorithms and proof techniques used in [12, 18] cannot be adapted for the general (and practically more relevant) setting formulated in (1), which features approximately sparse signals and additive measurement noise.

1.2. Contributions

In this paper, we generalize the recovery guarantees of [12, 18] to the framework (1) detailed above. In particular, we provide novel, computationally efficient restoration and separation algorithms, and derive corresponding recovery guarantees for the deterministic setting. Our guarantees depend in a natural way upon the number of dominant nonzero entries of $\mathbf{x}$ and $\mathbf{e}$, on

the coherence parameters of the dictionaries $\mathbf{A}$ and $\mathbf{B}$, and on the Euclidean norm of the measurement noise. Our results also depend on the amount of knowledge on the location of the dominant entries available prior to recovery. In particular, we investigate the following cases: 1) The locations of the dominant entries of the approximately sparse vector $\mathbf{x}$ and the support set of the perfectly sparse interference vector $\mathbf{e}$ are known (prior to recovery), 2) only the support set of the interference vector $\mathbf{e}$ is known, and 3) no support-set knowledge about $\mathbf{x}$ and $\mathbf{e}$ is available. Moreover, we present new coherence-based bounds on the restricted isometry constants (RICs) for the cases 2) and 3), which we then use to derive alternative recovery conditions using the restricted isometry property (RIP) framework. We provide a comparison to the recovery conditions for perfectly sparse signals and noiseless measurements presented in [12, 18]. Finally, we demonstrate the efficacy of our approach with two representative applications: Restoration of audio signals that are impaired by a mixture of impulse noise and Gaussian noise, and removal of scratches from old photographs.

1.3. Notation

Lowercase and uppercase boldface letters stand for column vectors and matrices, respectively. The transpose, conjugate transpose, and (Moore–Penrose) pseudo-inverse of the matrix $\mathbf{M}$ are denoted by $\mathbf{M}^T$, $\mathbf{M}^H$, and $\mathbf{M}^\dagger = (\mathbf{M}^H \mathbf{M})^{-1} \mathbf{M}^H$, respectively. The k th entry of the vector $\mathbf{m}$ is $[\mathbf{m}]_k$, and the k th column of $\mathbf{M}$ is $\mathbf{m}_k$ and the entry in the k th row and ℓ th column is designated by $[\mathbf{M}]_{k,\ell}$. The $M \times M$ identity matrix is denoted by $\mathbf{I}_M$ and the $M \times N$ all zeros matrix by $\mathbf{0}_{M \times N}$. The Euclidean (or ℓ_2) norm of the vector $\mathbf{x}$ is denoted by $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_1 = \sum_k |[\mathbf{x}]_k|$ stands for the ℓ_1 -norm of $\mathbf{x}$, and $\|\mathbf{x}\|_0$ designates the number of nonzero entries of $\mathbf{x}$. The spectral norm of the matrix $\mathbf{M}$ is $\|\mathbf{M}\|_2 = \sqrt{\lambda_{\max}(\mathbf{M}^H \mathbf{M})}$, where the minimum and maximum eigenvalue of a positive-semidefinite matrix $\mathbf{M}$ are denoted by $\lambda_{\min}(\mathbf{M})$ and $\lambda_{\max}(\mathbf{M})$, respectively. $\|\mathbf{M}\|_F = \sqrt{\sum_{k,\ell} |[\mathbf{M}]_{k,\ell}|^2}$ stands for the Frobenius matrix norm. Sets are designated by upper-case calligraphic letters. The cardinality of the set $\mathcal{T}$ is $|\mathcal{T}|$ and the complement of a set $\mathcal{S}$ in some superset $\mathcal{T}$ is denoted by $\mathcal{S}^c$. The support set of the vector $\mathbf{m}$, i.e., the index set corresponding to the nonzero entries of $\mathbf{m}$, is designated by $\text{supp}(\mathbf{m})$. We define the $M \times M$ diagonal (projection) matrix $\mathbf{P}_{\mathcal{S}}$ for the set

$\mathcal{S} \subseteq \{1, \dots, M\}$ as follows:

$$[\mathbf{P}_{\mathcal{S}}]_{k,\ell} = \begin{cases} 1, & k = \ell \text{ and } k \in \mathcal{S} \\ 0, & \text{otherwise,} \end{cases}$$

and $\mathbf{m}_{\mathcal{T}} = \mathbf{P}_{\mathcal{T}}\mathbf{m}$. The matrix $\mathbf{M}_{\mathcal{T}}$ is obtained from $\mathbf{M}$ by retaining the columns of $\mathbf{M}$ with indices in $\mathcal{T}$ and the $|\mathcal{T}|$ -dimensional vector $[\mathbf{m}]_{\mathcal{T}}$ is obtained analogously. For $x \in \mathbb{R}$, we set $[x]^+ = \max\{x, 0\}$.

1.4. Outline of the paper

The remainder of the paper is organized as follows. In Section 2, we briefly summarize the relevant prior art. Our new recovery algorithms and corresponding recovery guarantees are presented in Section 3. An alternative set of recovery guarantees obtained through the RIP framework and a comparison to existing recovery guarantees are provided in Section 4. The application examples are shown in Section 5, and we conclude in Section 6.

2. Relevant Prior Art

In this section, we review the relevant prior art in recovering sparse signals from noiseless and noisy measurements in the deterministic setting and summarize the existing guarantees for recovery of sparsely corrupted signals.

2.1. Recovery of perfectly sparse signals from noiseless measurements

Recovery of a vector $\mathbf{x} \in \mathbb{C}^{N_a}$ from the noiseless observations $\mathbf{y} = \mathbf{A}\mathbf{x}$ with $\mathbf{A}$ over-complete (i.e., $M < N_a$) corresponds to solving an underdetermined system of linear equations, which is well-known to be ill-posed. However, assuming that $\mathbf{x}$ is perfectly sparse (i.e., that only small number of its entries are nonzero) enables us to uniquely recover $\mathbf{x}$ by solving

$$(P0) \quad \text{minimize } \|\tilde{\mathbf{x}}\|_0 \quad \text{subject to } \mathbf{y} = \mathbf{A}\tilde{\mathbf{x}}.$$

Unfortunately, P0 has a prohibitive (combinatorial) computational complexity, even for small dimensions N_a . One of the most popular and computationally tractable alternative to solving P0 is basis pursuit (BP) [19–24], which corresponds to the convex program

$$(BP) \quad \text{minimize } \|\tilde{\mathbf{x}}\|_1 \quad \text{subject to } \mathbf{y} = \mathbf{A}\tilde{\mathbf{x}}.$$

Recovery guarantees for P0 and BP are usually expressed in terms of the sparsity level $n_x = \|\mathbf{x}\|_0$ and the coherence parameter of the dictionary $\mathbf{A}$, which is defined as

$$\mu_a = \max_{k, \ell, k \neq \ell} |\mathbf{a}_k^H \mathbf{a}_\ell|.$$

Specifically, a sufficient condition for $\mathbf{x}$ to be the unique solution of P0 and for BP to deliver this solution¹ is [21, 22, 24]

$$n_x < \frac{1}{2} \left(1 + \frac{1}{\mu_a} \right). \quad (2)$$

2.2. Recovery of approximately sparse signals from noisy measurements

For the case of bounded (otherwise arbitrary) measurement noise, i.e., $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{n}$ with $\|\mathbf{n}\|_2 \leq \varepsilon$, recovery guarantees based on the coherence parameter μ_a were developed in [27–31]. The corresponding recovery conditions mostly treat the case of perfectly-sparse signals, i.e., where only a small fraction of the entries $\mathbf{x}$ are nonzero. In almost all practical applications, however, only a few entries of $\mathbf{x}$ are actually zero. Nevertheless, many real-world signals exhibit the property that most of the signal’s energy (in terms of the sum of absolute values) is concentrated in only a few entries. We refer to this class of signals as *approximately sparse* in the remainder of the paper. For such signals, the support set associated to the best n_x -sparse approximation (in ℓ_1 -norm) corresponds to

$$\hat{\mathcal{X}} = \text{supp}_{n_x}(\mathbf{x}) = \arg \min_{\tilde{\mathcal{X}} \in \Sigma_{n_x}} \|\mathbf{x} - \mathbf{x}_{\tilde{\mathcal{X}}}\|_1,$$

where the set Σ_{n_x} contains all support sets of size n_x corresponding to perfectly n_x -sparse vectors having the same dimension as $\mathbf{x}$. A particular subclass of approximately sparse signals is the set of *compressible signals*, whose approximation error decreases according to a power law with exponent $s > 0$ as n_x increases [32].

The following theorem provides a sufficient condition for which a suitably modified version of BP, known as BP denoising (BPDN) [19], stably recovers an approximately sparse vector $\mathbf{x}$ from the noisy observation $\mathbf{z}$.

¹The condition (2) also ensures perfect recovery using orthogonal matching pursuit (OMP) [24–26], which is, however, not further investigated in this paper.

Theorem 1 (BP denoising [31, Thm. 2.1]). *Let $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{n}$, $\|\mathbf{n}\|_2 \leq \varepsilon$, and $\mathcal{X} = \text{supp}_{n_x}(\mathbf{x})$. If (2) is met, then the solution $\hat{\mathbf{x}}$ of the convex program*

$$(\text{BPDN}) \quad \text{minimize } \|\tilde{\mathbf{x}}\|_1 \quad \text{subject to } \|\mathbf{z} - \mathbf{A}\tilde{\mathbf{x}}\|_2 \leq \eta$$

with $\varepsilon \leq \eta$ satisfies

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_2 \leq C_0(\varepsilon + \eta) + C_1\|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1, \quad (3)$$

where both (non-negative) constants C_0 and C_1 depend on μ_a and n_x .

PROOF. The proof in [31, Thm. 2.1] is detailed for perfectly sparse vectors only. Since some of the proofs presented in the remainder of the paper are for approximately sparse signals and noisy measurements, we present the general case in Appendix A.

We emphasize that *perfect* recovery of $\mathbf{x}$ is, in general, impossible in the presence of bounded (but otherwise arbitrary) measurement noise $\mathbf{n}$. In the remainder of the paper, we consider *stable recovery* instead, i.e., in a sense that the ℓ_2 -norm of the difference between the estimate $\hat{\mathbf{x}}$ and the ground truth $\mathbf{x}$ is bounded from above by the ℓ_2 -norm of the noise $\|\mathbf{n}\|_2$ and the best n_x -sparse approximation in ℓ_1 -norm sense, i.e., as in (3). We finally note that Theorem 1 generalizes the results for noiseless measurements and perfectly sparse signals in [21, 22, 24] using BP (cf. Section 2.1). Specifically, for $\|\mathbf{n}\|_2 = 0$ and $\|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1 = 0$, BPDN with $\eta = 0$ corresponds to BP and (3) results in $\|\mathbf{x} - \hat{\mathbf{x}}\|_2 = 0$, which ensures perfect recovery of the vector $\mathbf{x}$ whenever (2) is met.

2.3. Recovery guarantees for perfectly sparse signals from sparsely corrupted and noiseless measurements

A large number of restoration and separation problems occurring in practice can be formulated as sparse signal recovery from sparsely corrupted signals using the input-output relation (1). Special cases of the general model (1) were studied in [7, 11–13, 18, 33–37].

Probabilistic recovery guarantees. Recovery guarantees for the probabilistic setting (i.e., recovery of $\mathbf{x}$ is guaranteed with high probability) for random Gaussian matrices, which are of particular interest for applications based on compressive sensing (CS), were reported in [7, 11, 35, 37]. Similar results

for randomly sub-sampled unitary matrices $\mathbf{A}$ were developed in [36]. The problem of sparse signal recovery from a particular nonlinear measurement process in the presence of impulse noise was considered in [13], and probabilistic results for signal detection based on ℓ_1 -norm minimization in the presence of impulse noise was investigated in [34]. In the remainder of the paper, however, we will consider the deterministic setting exclusively.

Deterministic recovery guarantees. Recovery guarantees in the deterministic setting for noiseless measurements and signals being perfectly sparse, i.e., the model $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{e}$, were studied in [12, 18, 33]. In [33], it was shown that when $\mathbf{A}$ is the Fourier matrix, $\mathbf{B} = \mathbf{I}_M$ and when the support set of the interference $\mathbf{e}$ is known, perfect recovery of $\mathbf{x}$ is possible if $2n_x n_e < M$, where $n_e = \|\mathbf{e}\|_0$. The case of $\mathbf{A}$ and $\mathbf{B}$ being arbitrary dictionaries (whereas $\mathbf{x}$ and $\mathbf{e}$ are assumed to be perfectly sparse and for noiseless measurements) was studied for different cases of support-set knowledge in [12, 18]. There, deterministic recovery guarantees were presented, which depended upon the number of nonzero entries n_x and n_e in $\mathbf{x}$ and $\mathbf{e}$, respectively, and on the coherence parameters μ_a and μ_b of $\mathbf{A}$ and $\mathbf{B}$, as well as on the mutual coherence between the dictionaries $\mathbf{A}$ and $\mathbf{B}$, which is defined as

$$\mu_m = \max_{k,\ell} |\mathbf{a}_k^H \mathbf{b}_\ell|.$$

A summary of the recovery guarantees presented in [12, 18] (along with the novel recovery guarantees presented in the next section) is given in Table 1, where, for the sake of simplicity of exposition, we define the function

$$f(u, v) = [1 - \mu_a(u - 1)]^+ [1 - \mu_b(v - 1)]^+.$$

We emphasize that the results presented in [12, 18] are for perfectly sparse and noiseless measurements only, and furthermore, the algorithms and proof techniques cannot be adapted for the more general setting proposed in (1). In order to gain insight into the practically more relevant case of approximately sparse signals and noisy measurements, we next develop new restoration and separation algorithms for several different cases of support-set knowledge and provide corresponding recovery guarantees. Our results complement those in [12, 18] (cf. Table 1).

3. Main Results

We now develop several computationally efficient methods for restoration or separation under the model (1) and derive corresponding recovery condi-

Table 1: Summary of deterministic recovery guarantees for perfectly/approximately sparse signals that are corrupted by interference in the absence/presence of measurement noise.

Support-set knowledge	Recovery condition	Perfectly sparse and no noise	Approx. sparse and noise
$\mathbf{x}$ and $\mathbf{e}$	$n_x n_e \mu_m^2 < f(n_x, n_e)$	[12, Thm. 3]	Theorem 2
$\mathbf{e}$ only	$2n_x n_e \mu_m^2 < f(2n_x, n_e)$	[12, Thms. 4 and 5]	Theorem 3
None	[18, Eq. 12] ^a	[18, Thm. 3]	—
	Eq. 13	—	Theorem 4

^aThe recovery condition is valid for BP and OMP; a less restrictive condition for P0 is given in [18, Thm. 2].

tions that guarantee their stability. Our recovery guarantees depend on the ℓ_2 -norm of the noise vector and on the amount of knowledge on the dominant nonzero entries of the signal and noise vectors. Specifically, we consider the following three cases: 1) *Direct restoration*: The locations of the entries corresponding to the best n_x -sparse approximation of $\mathbf{x}$ and the support set of the (perfectly sparse) interference vector $\mathbf{e}$ are known prior to recovery, 2) *BP restoration*: Only the support set of $\mathbf{e}$ is known, 3) *BP separation*: No knowledge about $\mathbf{x}$ and $\mathbf{e}$ is available, except for the fact that each vector exhibits an approximately an sparse representation in either $\mathbf{A}$ or $\mathbf{B}$.

3.1. Direct restoration: Support-set knowledge of $\mathbf{x}$ and $\mathbf{e}$

We start by addressing the case where the locations of the dominant entries (in terms of absolute value) of the approximately sparse vector $\mathbf{x}$ and the support set $\mathcal{E}$ associated with the perfectly sparse interference vector $\mathbf{e}$ are known prior to recovery. This scenario is relevant, for example, in the restoration of old phonograph records [1, 2], where one wants to recover a bandlimited signal that is impaired by impulse noise, e.g., clicks and pops. The occupied frequency band of phonograph recordings is typically known prior to recovery. In this case, one may assume $\mathbf{A}$ to be the M -dimensional discrete cosine transform (DCT) matrix and $\mathbf{B} = \mathbf{I}_M$. The locations of the clicks and pops, i.e., the support set $\mathcal{E} = \text{supp}(\mathbf{e})$, can be determined (prior to recovery) using the techniques described in [2], for example.

The restoration approach considered for this setup is as follows. Since $\mathcal{E}$ and $\mathcal{X} = \text{supp}_{n_x}(\mathbf{x})$ are both known prior to recovery, we start by projecting

the noisy observation vector $\mathbf{z}$ onto the orthogonal complement of the range space spanned by $\mathbf{B}_\mathcal{E}$, which eliminates the sparse corruptions caused by the interference $\mathbf{e}$. In particular, we consider

$$\mathbf{R}_\mathcal{E}\mathbf{z} = \mathbf{R}_\mathcal{E}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{e}_\mathcal{E} + \mathbf{n}) = \mathbf{R}_\mathcal{E}\mathbf{A}\mathbf{x} + \mathbf{R}_\mathcal{E}\mathbf{n}, \quad (4)$$

where $\mathbf{R}_\mathcal{E} = \mathbf{I}_M - \mathbf{B}_\mathcal{E}\mathbf{B}_\mathcal{E}^\dagger$ is the projector onto the orthogonal complement of the range space of $\mathbf{B}_\mathcal{E}$, and we used the fact that $\mathbf{R}_\mathcal{E}\mathbf{B}_\mathcal{E} = \mathbf{0}_{M \times 1}$. Next, we separate (4) by exploiting the fact that $\mathcal{X}$ is known

$$\begin{aligned} \mathbf{R}_\mathcal{E}\mathbf{z} &= \mathbf{R}_\mathcal{E}\mathbf{A}(\mathbf{x}_\mathcal{X} + \mathbf{x}_{\mathcal{X}^c}) + \mathbf{R}_\mathcal{E}\mathbf{n} \\ &= \mathbf{R}_\mathcal{E}\mathbf{A}_\mathcal{X}[\mathbf{x}]_\mathcal{X} + \mathbf{R}_\mathcal{E}\mathbf{A}\mathbf{x}_{\mathcal{X}^c} + \mathbf{R}_\mathcal{E}\mathbf{n} \end{aligned}$$

and isolate the dominant entries $[\mathbf{x}]_\mathcal{X}$ as follows:

$$(\mathbf{R}_\mathcal{E}\mathbf{A}_\mathcal{X})^\dagger \mathbf{R}_\mathcal{E}\mathbf{z} = [\mathbf{x}]_\mathcal{X} + (\mathbf{R}_\mathcal{E}\mathbf{A}_\mathcal{X})^\dagger \mathbf{R}_\mathcal{E}(\mathbf{A}\mathbf{x}_{\mathcal{X}^c} + \mathbf{n}). \quad (5)$$

In the case where both vectors $\mathbf{x}_{\mathcal{X}^c}$ and $\mathbf{n}$ are equal to zero, we obtain

$$(\mathbf{R}_\mathcal{E}\mathbf{A}_\mathcal{X})^\dagger \mathbf{R}_\mathcal{E}\mathbf{z} = [\mathbf{x}]_\mathcal{X}, \quad (6)$$

and therefore the dominant entries of $\mathbf{x}$ are recovered perfectly by this approach. Note that conjugate gradient methods (see, e.g., [38]) offer an efficient way of computing (6).

The following theorem provides a sufficient condition for $(\mathbf{R}_\mathcal{E}\mathbf{A}_\mathcal{X})^\dagger \mathbf{R}_\mathcal{E}$ to exist and for which the vector $\mathbf{x}$ can be restored stably from the noisy measurement $\mathbf{z}$ using the direct restoration (DR) procedure outlined above.

Theorem 2 (Direct restoration). *Let $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{e} + \mathbf{n}$ with $\|\mathbf{n}\|_2 \leq \varepsilon$, $\mathbf{e}$ perfectly n_e -sparse with support set $\mathcal{E}$, and $\mathcal{X} = \text{supp}_{n_x}(\mathbf{x})$. Furthermore, assume that $\mathcal{E}$ and $\mathcal{X}$ are known prior to recovery. If*

$$n_x n_e \mu_m^2 < f(n_x, n_e), \quad (7)$$

then the vector $\hat{\mathbf{x}}$ computed according to

$$(\text{DR}) \quad [\hat{\mathbf{x}}]_\mathcal{X} = (\mathbf{R}_\mathcal{E}\mathbf{A}_\mathcal{X})^\dagger \mathbf{R}_\mathcal{E}\mathbf{z}, \quad [\hat{\mathbf{x}}]_{\mathcal{X}^c} = \mathbf{0}_{|\mathcal{X}^c| \times 1}$$

with $\mathbf{R}_\mathcal{E} = \mathbf{I}_M - \mathbf{B}_\mathcal{E}\mathbf{B}_\mathcal{E}^\dagger$ satisfies

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_2 \leq C_3 \varepsilon + C_4 \|\mathbf{x} - \mathbf{x}_\mathcal{X}\|_1,$$

where the (non-negative) constants C_3 and C_4 depend on the coherence parameters μ_a , μ_b , and μ_m , and on the sparsity levels n_x and n_e .

PROOF. The proof is given in Appendix B.

Theorem 2 and in particular (7) provides a sufficient condition on the number n_x of dominant entries of $\mathbf{x}$ for which DR enables the stable recovery of $\mathbf{x}$ from $\mathbf{z}$, given the coherence parameters μ_a , μ_b , and μ_m , and the number of sparse corruptions n_e . Specifically, (7) states that for a given number of sparse corruptions n_e , the smaller the coherence parameters μ_a , μ_b , and μ_m , the more dominant entries of $\mathbf{x}$ can be recovered stably from $\mathbf{z}$. The case that guarantees the recovery of the largest number n_x of dominant entries in $\mathbf{x}$ is when $\mathbf{A}$ and $\mathbf{B}$ are orthonormal bases (ONBs) (i.e., $\mu_a = \mu_b = 0$) that are maximally incoherent (i.e., $\mu_m = 1/\sqrt{M}$); this is the case for the Fourier–identity pair, whence the recovery condition (7) corresponds to $n_x n_e < M$.

The recovery guarantee in Theorem 2 generalizes that in [12, Thm. 3] to approximately sparse signals and noisy measurements. In particular, for $\|\mathbf{n}\|_2 = 0$ and $\|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1 = 0$, we have $\varepsilon = 0$ and that DR perfectly recovers $\mathbf{x}$ if (7) is met. Since (7) is identical to the condition [12, Thm. 3] (cf. Table 1) we see that considering approximately sparse signals and (bounded) measurement noise does *not* result in a more restrictive recovery condition. We finally note that the recovery condition in (7) was shown in [12] to be tight for certain comb signals in the case where $\mathbf{A}$ is the Fourier matrix and $\mathbf{B} = \mathbf{I}_M$.

3.2. BP restoration: Support-set knowledge of $\mathbf{e}$ only

Next, we find conditions guaranteeing the stable recovery in the setting where the support set of the interference vector $\mathbf{e}$ is known prior to recovery. A prominent application for this setting is the restoration of saturated signals [7, 16]. Here, no knowledge on the locations of the dominant entries of $\mathbf{x}$ is required. The support set $\mathcal{E}$ of the sparse interference vector can, however, be easily identified by comparing the measured signal entries $[\mathbf{z}]_i$, $i = 1, \dots, M$, to a saturation threshold. Further application examples for this setting include the removal of impulse noise [1, 2, 14], in-painting, and super-resolution [3, 8, 15] of signals admitting an approximately sparse representation in some arbitrary dictionary $\mathbf{A}$.

The recovery procedure for this case is as follows. Since $\mathcal{E}$ is known prior to recovery, we may recover the vector $\mathbf{x}$ by projecting the noisy observation vector $\mathbf{z}$ onto the orthogonal complement of the range space spanned by $\mathbf{B}_{\mathcal{E}}$ (cf. Section 3.1). This projection eliminates the sparse noise and leaves us with a sparse signal recovery problem similar to that in Theorem 1. In

particular, we consider recovery from

$$\mathbf{R}_{\mathcal{E}}\mathbf{z} = \mathbf{R}_{\mathcal{E}}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{e}_{\mathcal{E}} + \mathbf{n}) = \mathbf{R}_{\mathcal{E}}\mathbf{A}\mathbf{x} + \mathbf{R}_{\mathcal{E}}\mathbf{n}, \quad (8)$$

where $\mathbf{R}_{\mathcal{E}} = \mathbf{I}_M - \mathbf{B}_{\mathcal{E}}\mathbf{B}_{\mathcal{E}}^{\dagger}$, and we used the fact that $\mathbf{R}_{\mathcal{E}}\mathbf{B}_{\mathcal{E}} = \mathbf{0}_{M \times 1}$. Note that the idea of projecting the observation vector onto the orthogonal complement of the space spanned by the active columns of $\mathbf{B}$ was put forward in [7], where the special case of $\mathbf{B}$ being an ONB was considered. The following theorem provides a sufficient condition that guarantees the stable restoration of the vector $\mathbf{x}$ from (8).

Theorem 3 (BP restoration). *Let $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{e} + \mathbf{n}$ with $\|\mathbf{n}\|_2 \leq \varepsilon$. Assume $\mathbf{e}$ to be perfectly n_e -sparse and $\mathcal{E} = \text{supp}(\mathbf{e})$ to be known prior to recovery. Furthermore, let $\mathcal{X} = \text{supp}_{n_x}(\mathbf{x})$. If*

$$2n_x n_e \mu_m^2 < f(2n_x, n_e), \quad (9)$$

then the result $\hat{\mathbf{x}}$ of BP restoration

$$(\text{BP-RES}) \quad \begin{cases} \text{minimize} & \|\tilde{\mathbf{x}}\|_1 \\ \text{subject to} & \|\mathbf{R}_{\mathcal{E}}(\mathbf{z} - \mathbf{A}\tilde{\mathbf{x}})\|_2 \leq \eta \end{cases}$$

with $\mathbf{R}_{\mathcal{E}} = \mathbf{I}_M - \mathbf{B}_{\mathcal{E}}\mathbf{B}_{\mathcal{E}}^{\dagger}$ and $\varepsilon \leq \eta$ satisfies

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_2 \leq C_5(\varepsilon + \eta) + C_6\|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1,$$

where the (non-negative) constants C_5 and C_6 depend on the coherence parameters μ_a , μ_b , and μ_m , and on the sparsity levels n_x and n_e .

PROOF. The proof is given in Appendix C.

The recovery condition (9) provides a sufficient condition on the number n_x of dominant entries of $\mathbf{x}$, for which BP-RES can stably recover $\mathbf{x}$ from $\mathbf{z}$. The condition depends on the coherence parameters μ_a , μ_b , and μ_m , and the number of sparse corruptions n_e . As for the case of DR, the situation that guarantees that the largest number n_x of dominant coefficients in $\mathbf{x}$ will be recovered stably using BP-RES, is when $\mathbf{A}$ and $\mathbf{B}$ are maximally incoherent ONBs. In this situation, the recovery condition (9) reduces to $2n_x n_e < M$, which is two times more restrictive than that for DR (see [12] for a detailed discussion on this factor-of-two penalty).

The following observations are immediate consequences of Theorem 3:

- If the vector $\mathbf{x}$ is perfectly n_x -sparse, i.e., $\|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1 = 0$, and no measurement noise is present, i.e., $\varepsilon = 0$, then BP-RES using $\eta = 0$ *perfectly* recovers $\mathbf{x}$ whenever (9) is satisfied. Note that two different restoration procedures were developed for this particular setting in [12, Thms. 4 and 5]. Both methods enable perfect recovery under exactly the same conditions (cf. Table 1). Hence, generalizing the recovery procedure to approximately sparse signals and measurement noise does not incur a penalty in terms of the recovery condition.
- The restoration method in [12, Thm. 5] requires a column-normalization procedure to guarantee perfect recovery under the condition (9). Since in this special case, BP-RES (with $\eta = 0$) corresponds to BP, Theorem 3 implies that this normalization procedure is not necessary for guaranteeing perfect recovery under (9). Note, however, that this observation does not apply to orthogonal matching pursuit (see [39] for more details).

We finally point out that the recovery condition in (9) was shown in [12] to be tight for particularly-chosen comb signals in the case where $\mathbf{A}$ is the Fourier matrix and $\mathbf{B} = \mathbf{I}_M$.

3.3. BP separation: No knowledge on the support sets

We finally consider the case where no knowledge about the support sets of the approximately sparse vectors $\mathbf{x}$ and $\mathbf{e}$ is available. A typical application scenario is signal separation [4, 6], e.g., the decomposition of audio, image, or video signals into two or more distinct features, i.e., in a part that exhibits an approximately sparse representation in the dictionary $\mathbf{A}$ and another part that exhibits an approximately sparse representation in $\mathbf{B}$. Decomposition then amounts to performing simultaneous recovery of $\mathbf{x}$ and $\mathbf{e}$ from $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{e} + \mathbf{n}$, followed by computation of the individual signal features according to $\mathbf{A}\mathbf{x}$ and $\mathbf{B}\mathbf{e}$. The main idea underlying this signal-separation approach is to rewrite (1) as

$$\mathbf{z} = \mathbf{D}\mathbf{w} + \mathbf{n} \tag{10}$$

where $\mathbf{D} = [\mathbf{A} \ \mathbf{B}]$ is the concatenated dictionary of $\mathbf{A}$ and $\mathbf{B}$ and the stacked vector $\mathbf{w} = [\mathbf{x}^T \ \mathbf{e}^T]^T$. Signal separation now amounts to performing BPDN on (10) for recovery of $\mathbf{w}$ from $\mathbf{z}$, which is also known as the synthesis separation problem for microlocal analysis (see [40, 41] and the references therein).

A straightforward way to arrive at a corresponding deterministic recovery guarantee for this problem is to consider $\mathbf{D}$ as the new dictionary with the *dictionary coherence* defined as

$$\mu_d = \max_{i,j,i \neq j} |\mathbf{d}_i^H \mathbf{d}_j| = \max \{\mu_a, \mu_b, \mu_m\}. \quad (11)$$

One can now use BPDN to recover $\mathbf{w}$ from (10) and invoke Theorem 1 with the recovery condition in (2), resulting in

$$w = n_x + n_e < \frac{1}{2} \left(1 + \frac{1}{\mu_d} \right). \quad (12)$$

However, it is important to realize that (12) ignores the structure underlying the dictionary $\mathbf{D}$, i.e., it does not take into account the fact that $\mathbf{D}$ is a concatenation of two dictionaries that are characterized by the coherence parameters μ_a , μ_b , and μ_m . Hence, the recovery guarantee (12) does not provide insight into which pairs of dictionaries $\mathbf{A}$ and $\mathbf{B}$ are most useful for signal separation. The following theorem takes into account the structure underlying $\mathbf{D}$, enabling us to gain insight into which pairs of dictionaries $\mathbf{A}$ and $\mathbf{B}$ support signal separation.

Theorem 4 (BP separation). *Let $\mathbf{z} = \mathbf{D}\mathbf{w} + \mathbf{n}$, with $\mathbf{D} = [\mathbf{A} \ \mathbf{B}]$, $\mathbf{w} = [\mathbf{a}^T \ \mathbf{e}^T]^T$, and $\|\mathbf{n}\|_2 \leq \varepsilon$. The dictionary $\mathbf{D}$ is characterized by the coherence parameters μ_a , μ_b , μ_m , and μ_d , and we assume $\mu_b \leq \mu_a$ without loss of generality. Furthermore, let $\mathcal{W} = \text{supp}_w(\mathbf{w})$. If*

$$w < \max \left\{ \frac{2(1 + \mu_a)}{\mu_a + 2\mu_d + \sqrt{\mu_a^2 + \mu_m^2}}, \frac{1 + \mu_d}{2\mu_d} \right\}, \quad (13)$$

then the solution $\hat{\mathbf{w}}$ of BP separation

$$(\text{BP-SEP}) \quad \begin{cases} \text{minimize} & \|\tilde{\mathbf{w}}\|_1 \\ \text{subject to} & \|\mathbf{z} - \mathbf{D}\tilde{\mathbf{w}}\|_2 \leq \eta \end{cases}$$

using $\varepsilon \leq \eta$ satisfies

$$\|\mathbf{w} - \hat{\mathbf{w}}\|_2 \leq C_7(\varepsilon + \eta) + C_8\|\mathbf{w} - \mathbf{w}_{\mathcal{W}}\|_1, \quad (14)$$

with the (non-negative) constants C_7 and C_8 .

PROOF. The proof is given in Appendix D.

The recovery condition (13) refines that in (12); in particular, considering the two-ONB setting for which $\mu_a = \mu_b = 0$ and $\mu_m = \mu_d$. In this case, the straightforward recovery condition (11) corresponds to $w < (1 + 1/\mu_d)/2$, whereas the one for BP separation (13) is

$$w < \frac{2}{3\mu_d}. \quad (15)$$

Hence, (13) guarantees the stable recovery for a larger number of dominant entries w in the stacked vector $\mathbf{w}$. Recovery guarantees for perfectly sparse signals and noiseless measurements in the two-ONB setting were developed in [23, 24, 42]. The corresponding recovery condition $w < (\sqrt{2}-0.5)/\mu_d$ turns out to be better (i.e., less restrictive) than the recovery condition for approximately sparse signals and measurement noise provided in (15). Whether this behavior is a fundamental result of considering approximately sparse signals and noisy measurements or is an artifact of the proof technique remains an open research problem.

4. Recovery Guarantees from the RIP-Framework and Comparison

In this section, we develop alternative recovery guarantees for BP-RES and BP-SEP using results from the restricted isometry property (RIP) framework. We furthermore provide a comparison with the (coherence-based) recovery guarantees obtained in the previous section and the guarantees for perfectly sparse signals and noiseless measurements presented in [12, 18] (recall Table 1).

4.1. RIP-based recovery guarantees

An alternative way of obtaining deterministic recovery guarantees for approximately sparse signals and measurement noise, i.e., for $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{n}$, has been developed under the RIP framework [34, 43–47]. There, the dictionary $\mathbf{A}$ is characterized by restricted isometry constants (RICs) instead of the coherence parameter μ_a .

Definition 1 ([34]). For each integer $n_x \geq 1$, the RIC δ_{n_x} of $\mathbf{A}$ is the smallest number such that

$$(1 - \delta_{n_x}) \|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\|_2^2 \leq (1 + \delta_{n_x}) \|\mathbf{x}\|_2^2 \quad (16)$$

holds for all perfectly n_x -sparse vectors $\mathbf{x}$.

The corresponding recovery conditions, which guarantee stable recovery of $\mathbf{x}$ with $\|\mathbf{n}\|_2 \leq \varepsilon$ using BDPN, are, e.g., of the form i) $\delta_{2n_x} < \sqrt{2} - 1$ [44] or ii) $\delta_{n_x} < 0.307$ [47]. The main issue with such recovery conditions is the fact that the RICs cannot be computed efficiently, in general. In order to arrive at recovery conditions that are explicit in the number of nonzero entries n_x , one may bound the RIC (16) from above using the coherence parameter μ_a as [31, 45]

$$\delta_{n_x} \leq \mu_a(n_x - 1) \quad (17)$$

which can then be used to arrive at the corresponding recovery conditions i) $n_x < (1 + (\sqrt{2} - 1)/\mu_a)/2$ or ii) $n_x < 1 + 0.307/\mu_a$. Both of these recovery conditions are more restrictive than that in (2). Nevertheless, recovery guarantees obtained through the RIP framework turn out to be useful for deriving probabilistic recovery conditions (guaranteeing recovery with high probability) in the field of compressive sensing (CS) [48, 49]. The coherence-based bound in (17) is useful also because it puts limits on the RIC that are explicit in n_x .

4.2. Recovery guarantees for sparsely corrupted signals

We next provide coherence-based bounds on the RIC constants for BP restoration and BP separation, and derive corresponding alternative recovery guarantees using results obtained in the RIP framework [44, 47].

Recovery guarantee for BP restoration. As a byproduct of the proof for BP-RES detailed in Appendix C, the following coherence-based upper bound on the RIC for the matrix $\tilde{\mathbf{A}} = \mathbf{R}_\varepsilon \mathbf{A}$ was obtained:

Lemma 5 (RIC bound for $\tilde{\mathbf{A}}$). *Let $\tilde{\mathbf{A}} = \mathbf{R}_\varepsilon \mathbf{A}$ with $\mathbf{R}_\varepsilon = \mathbf{I}_M - \mathbf{B}_\varepsilon \mathbf{B}_\varepsilon^\dagger$. For each integer $n_x \geq 1$, the smallest number δ_{n_x} such that*

$$(1 - \delta_{n_x}) \|\mathbf{x}\|_2^2 \leq \|\tilde{\mathbf{A}}\mathbf{x}\|_2^2 \leq (1 + \delta_{n_x}) \|\mathbf{x}\|_2^2$$

holds for all perfectly n_x -sparse vectors $\mathbf{x} \in \mathbb{C}^{N_a}$, is bounded from above by

$$\delta_{n_x} \leq \mu_a(n_x - 1) + \frac{n_x n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+}. \quad (18)$$

Combining the bound (18) with the condition $\delta_{n_x} < 0.307$ from [47] enables us to obtain the following recovery condition:

$$n_x < \frac{(0.307 + \mu_a)[1 - \mu_b(n_e - 1)]^+}{\mu_a[1 - \mu_b(n_e - 1)]^+ + n_e \mu_m^2} \quad (19)$$

that guarantees the stable restoration of $\mathbf{x}$ from $\mathbf{R}_\mathcal{E} \mathbf{z} = \tilde{\mathbf{A}} \mathbf{x} + \mathbf{R}_\mathcal{E} \mathbf{n}$ using BP-RES. We emphasize that for $\mu_a < 1 - 2 \cdot 0.307 \approx 0.386$ the condition (19) is more restrictive than the condition (9), which reads

$$n_x < \frac{1}{2} \frac{(1 + \mu_a)[1 - \mu_b(n_e - 1)]^+}{\mu_a[1 - \mu_b(n_e - 1)]^+ + n_e \mu_m^2}.$$

Hence, for most relevant values of the coherence parameter μ_a , the recovery condition in (9) is superior (in a sense of enabling the recovery of a larger number n_x of dominant coefficients in $\mathbf{x}$) to that obtained using the RIP framework.

Recovery guarantee for BP separation. During the proof for BP separation detailed in Appendix D, the following coherence-based upper bound on the RIC for the concatenated dictionary $\mathbf{D} = [\mathbf{A} \ \mathbf{B}]$ was obtained.

Lemma 6 (RIC bound for $\mathbf{D}$). *Let $\mathbf{D} = [\mathbf{A} \ \mathbf{B}]$ be characterized by μ_a , μ_b , μ_m , and μ_d , and assume $\mu_b \leq \mu_a$ without loss of generality. For each integer $w \geq 1$ the smallest number δ_w such that*

$$(1 - \delta_w) \|\mathbf{w}\|_2^2 \leq \|\mathbf{D} \mathbf{w}\|_2^2 \leq (1 + \delta_w) \|\mathbf{w}\|_2^2$$

holds for all perfectly w -sparse vectors $\mathbf{w} \in \mathbb{C}^{N_a + N_b}$, is bounded from above by

$$\delta_w \leq \min \left\{ \frac{1}{2} \left(\mu_a(w - 2) + w \sqrt{\mu_a^2 + \mu_m^2} \right), \mu_d(w - 1) \right\}. \quad (20)$$

As in Section 4.2, we use the right hand side of (20) in combination with the recovery condition $\delta_w < 0.307$ [47], to obtain a recovery guarantee for signal separation. In particular, BP-SEP guarantees stable recovery if

$$w < \max \left\{ \frac{2(0.307 + \mu_a)}{\mu_a + \sqrt{\mu_a^2 + \mu_m^2}}, 1 + \frac{0.307}{\mu_d} \right\}. \quad (21)$$

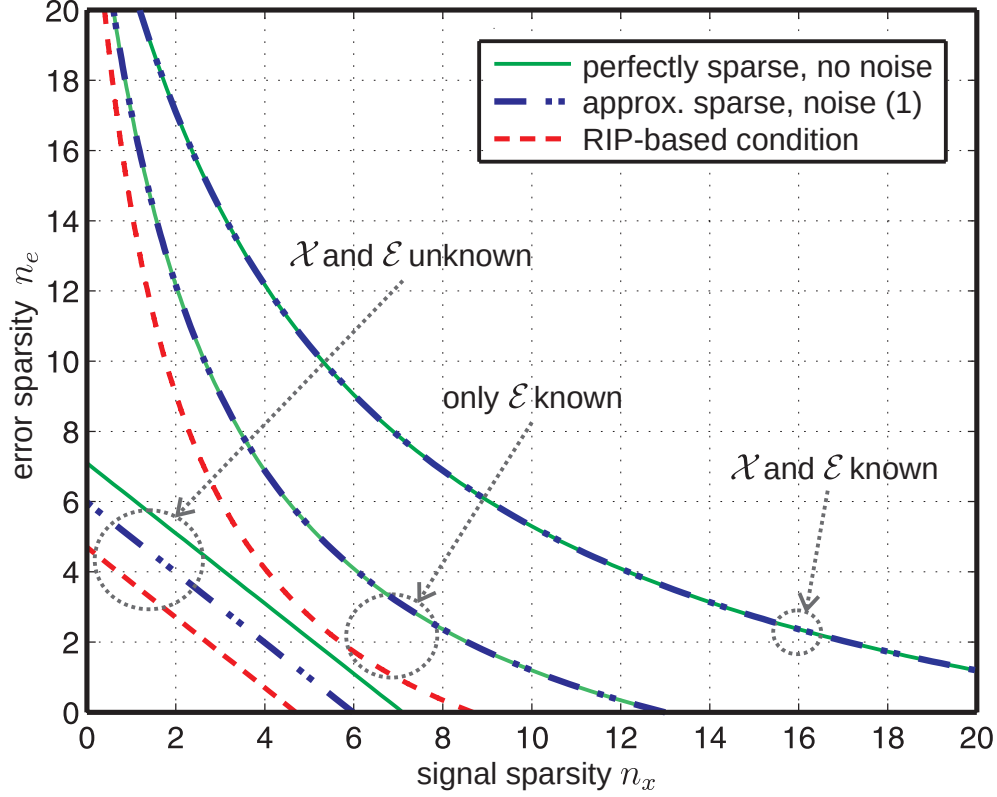


Figure 1: Comparison of the recovery conditions for $\mu_m = \mu_d = 0.1$ and $\mu_a = \mu_b = 0.04$.

As in the case of BP restoration, this condition turns out to be more restrictive than that in (13) for most relevant cases. However, the difference between the recovery condition in (21) and that in (13) is small, in general. To see this, consider the two-ONB case, i.e., $\mu_a = \mu_b = 0$ and $\mu_m = \mu_d$. Here, the recovery condition (21) corresponds to $w < 0.614/\mu_d$, which is only slightly more restrictive than the condition $w < 2/(3\mu_d)$ given in (15).

4.3. Comparison of the recovery guarantees

Fig. 1 compares the recovery conditions for the general model (1) to those obtained in [12, 18] for perfectly sparse signals and noiseless measurements (see also Table 1). We set $\mu_d = \mu_m = 0.1$ and $\mu_a = \mu_b = 0.04$, and compare the three different cases analyzed in Section 3. The following observations can be made:

- *Direct restoration:* For DR, no recovery guarantees are available through

the RIP framework. The recovery conditions for the general model (1) detailed in (7) and that in [12, Eq. 11] for perfectly sparse signals and noiseless measurements coincide. Hence, the generalization to approximately sparse signals and measurement noise does not entail a degradation in terms of the corresponding recovery condition.

- *BP restoration:* In this case, the recovery conditions for the general setup considered in this paper and the condition [12, Eq. 14] for perfectly sparse signals and noiseless measurements also coincide. Hence, generalizing the results does not incur a loss in terms of the recovery conditions. As mentioned in Section 4.2, the recovery condition obtained through the RIP framework turns out to be more restrictive.
- *BP separation:* In this case, we see that all of the recovery conditions differ. In particular, the condition [18, Eq. 13] for perfectly sparse signals and noiseless measurements is less restrictive than that for the general case (1). Furthermore, the recovery condition (21) obtained in the RIP framework turns out to be the most restrictive. However, the differences between the three recovery conditions are rather small.

In summary, we see that having more knowledge on the support sets prior to recovery yields less restrictive recovery conditions. This intuitive behavior can also be observed in practice and is illustrated in Section 5.

We finally emphasize that all of the recovery conditions derived above are deterministic in nature and therefore conservative in the sense that, in practice, recovery often succeeds for sparsity levels n_x and n_e much higher than the corresponding guarantees indicate. In particular, it is well-known that deterministic recovery guarantees are fundamentally limited by the so-called *square-root bottleneck*, e.g., [12, 18, 50], as they are valid for *all* dictionary pairs $\mathbf{A}$ and $\mathbf{B}$ with given coherence parameters, and all signal and interference realizations with given sparsity levels n_x and n_e . Nevertheless, we next show that our recovery conditions enable us to gain considerable insights into practical applications; i.e., they are useful for identification of appropriate dictionary pairs (e.g., being sufficiently incoherent) that should be used for signal restoration or separation.

5. Application Examples

We now develop two application examples to illustrate the main results of the paper. First, we show that direct restoration, BP restoration, and BP

separation can be used for simultaneous denoising and declipping of corrupted speech signals. Then, we illustrate the impact of support-set knowledge for a sparsity-based in-painting application.

5.1. Simultaneous denoising and declipping

In this example, we attempt the recovery of a speech signal that has been corrupted by a combination of additive Gaussian noise and impulse noise. To this end, we corrupt a 9.5s segment (44 100 kHz and 16 bit) from the speech signal in [51] by adding zero-mean i.i.d. Gaussian noise and impulse noise. The variance of the additive noise is chosen such that the mean-squared error (MSE) between the L -dimensional original audio signal $\mathbf{y}$ and the noisy version $\tilde{\mathbf{y}}$, defined as

$$\text{MSE} = 10 \log_{10} (\|\mathbf{y} - \tilde{\mathbf{y}}\|_2^2 / L) ,$$

is -30 decibel (dB). The impulse interference (used to model the clicking artifacts in the audio signal) is generated as follows: We corrupt 10% of the samples and chose the locations of the random clicks, which are modeled by the interference vector $\mathbf{e}$, uniformly at random. We then generate the clicks at these locations by adding i.i.d. zero-mean Gaussian random samples with variance 0.1 to the noisy signal (whose maximum amplitude was normalized to $+1$).

Recovery procedure. Recovery is performed with overlapping blocks of dimension $M = 1024$. The amount of overlap between adjacent blocks is 64 samples. We set $\mathbf{A}$ to the 1024×1024 -dimensional DCT matrix, $\mathbf{B} = \mathbf{I}_M$, and perform recovery based on $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{e} + \mathbf{n}$. The main reasons for using the DCT matrix in this example are i) the speech signal is approximately sparse in the DCT basis and ii) the mutual coherence between the identity and the DCT is small, i.e., $\mu_m = 1/\sqrt{512}$, which leads to less restrictive recovery conditions (7), (9), and (13). For all three recovery methods, we first compute an estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$ (and of $\mathbf{e}$ in the case of BP separation) and then compute an estimate of the denoised speech signal according to $\hat{\mathbf{y}} = \mathbf{A}\hat{\mathbf{x}}$. In order to reduce undesired artifacts occurring at the boundaries between two adjacent blocks, we add and overlap the recovered blocks using a raised-cosine window function when resynthesizing the entire speech signal.

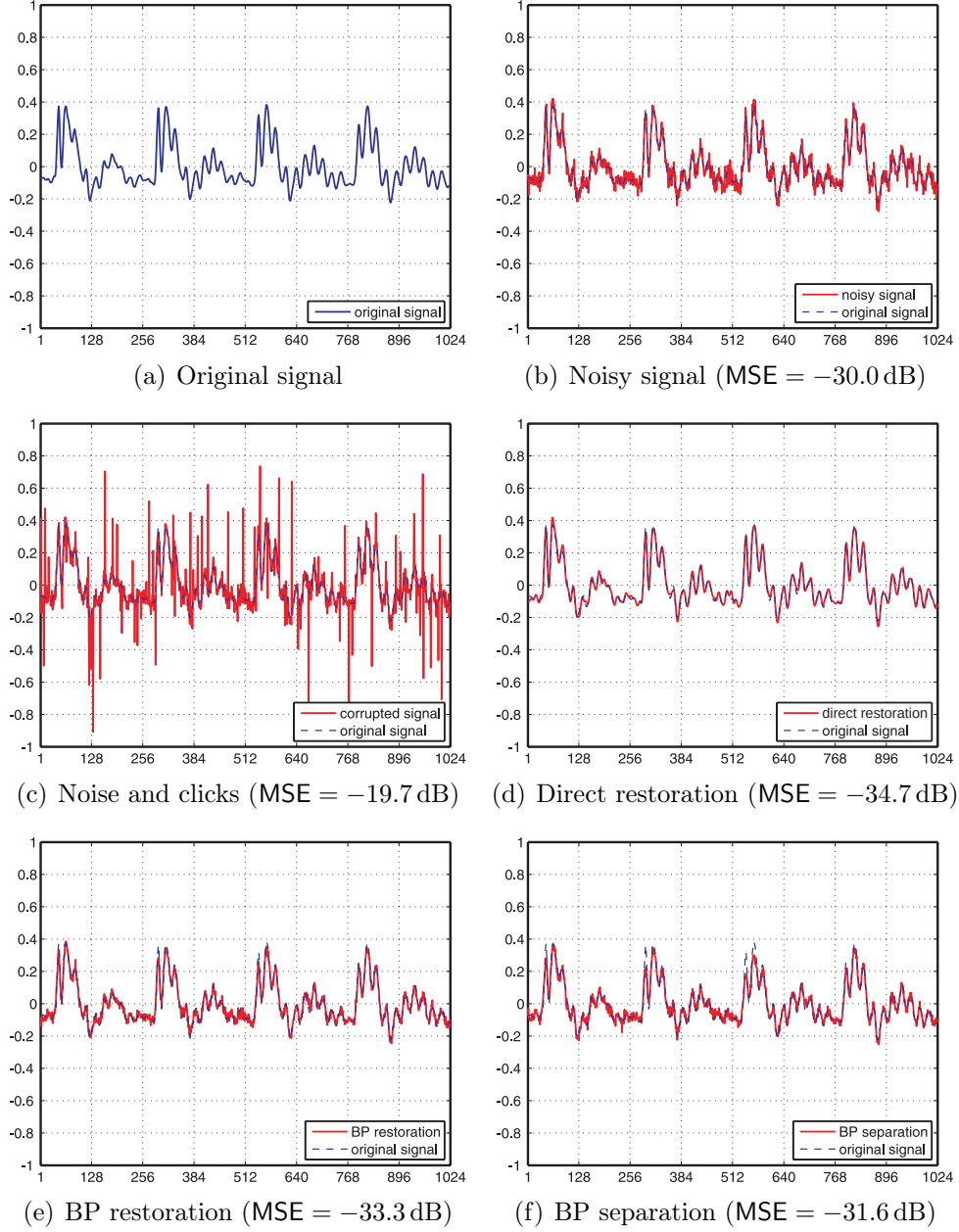


Figure 2: Mean squared error (MSE) results of simultaneous reduction of Gaussian and impulse interference in a corrupted speech signal (the x-axes correspond to sample indices, the y-axes to magnitudes).

Discussion of the results. Fig. 2 shows snapshots of the corruption and recovery procedure and the associated MSE results. The corrupted signal, which is impaired by Gaussian noise and impulse noise, has an MSE of -19.7 dB. The individual results of the three recovery procedures analyzed in this paper are as follows:

- *Direct restoration:* In this case (see Fig. 2(d)), the locations $\mathcal{E}$ of the impulse noise realizations are assumed to be known prior to recovery. We set $\mathcal{X} = \{1, \dots, 192\}$, assuming that the speech signal occupies only the lowest 192 frequencies. The MSE between the original signal and the one recovered through DR corresponds to -34.7 dB and, hence, DR is able to improve the MSE by 15 dB (compared to the signal that is impaired by Gaussian and impulse noise).
- *BP restoration:* Here, we assume that the locations of the impulse noise spikes $\mathcal{E}$ are known prior to recovery but nothing is known about $\mathbf{x}$ (except for the fact that it allows for an approximately sparse representation in $\mathbf{A}$). We perform BP restoration with $\eta = 0.6$, which results in an MSE of -33.3 dB (see Fig. 2(e)). Note that the parameter η determines the amount of denoising (for the Gaussian noise) and can be used to tune the resulting MSE.
- *BP separation:* In this case we assume that nothing is known about the support sets of either $\mathbf{x}$ or $\mathbf{e}$. We use BP separation with $\eta = 0.4$ and discard the recovered error component $\mathbf{e}$; the resulting MSE corresponds to -31.6 dB (see Fig. 2(f)). As it is the case for BP restoration, η determines the amount of denoising (of the Gaussian noise component). BP separation achieves surprisingly good recovery performance (compared to DR and BP-RES), while being completely *blind* to the locations of the sparse corruptions. Hence, BP separation offers an elegant way to mitigate impulse noise in speech signals, without requiring sophisticated algorithms that detect the locations of the sparse corruptions.

This application example shows that more knowledge on the support sets $\mathcal{X}$ and/or $\mathcal{E}$ leads to improved recovery results (i.e., smaller MSE). This intuitive behavior has been observed in [12] for perfectly sparse signals and noiseless measurements. We emphasize that DR, BP restoration, and BP separation are all able to *simultaneously* reduce Gaussian noise and impulse

interference as the resulting MSEs are all smaller than -30 dB (corresponding to the MSE of the noisy signal). The recovery procedure one should use in practice depends on the amount of support-set knowledge available prior to recovery.

We finally note that reduction of noise and clicks in impaired audio signals is a well-studied topic in the literature (see, e.g., [2] and references therein). However, most of the established methods rely on Bayesian estimation techniques, e.g., [2, 52, 53], which differ from the sparsity-based approach proposed here. Sparsity-based algorithms for restoration of impaired audio signals have been proposed recently in [16, 17]; however, no recovery guarantees are available for the proposed OMP-based restoration algorithm. We emphasize that virtually all proposed methods require knowledge of the locations of the sparse corruptions prior to recovery, whereas our results for BP separation show that sparse errors can effectively be removed *blindly* from speech signals. The main goal of this example is to illustrate the performance our algorithms and not to benchmark the performance relative to existing methods; a detailed performance and restoration-complexity comparison with existing methods for simultaneous denosing and declicking is left for future work.

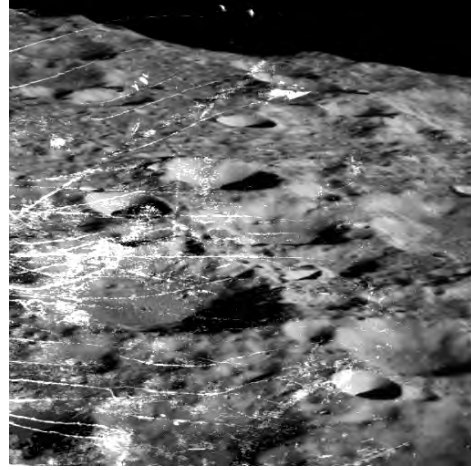
5.2. Removal of scratches in old photographs

We now consider a simple sparsity-based in-painting application. While a plethora of in-painting methods have been proposed in the literature (see, e.g., [6, 8–10] and the references therein), our goal here is not to benchmark our performance vs. theirs but rather to elucidate the differences between BP restoration and BP separation, i.e., to quantify the impact of support-set knowledge on the recovery performance. In this particular example, we intend to remove scratches from old photographs. To this end, we corrupt a 512×512 greyscale image by adding a mask containing artificially generated scratch patterns. This procedure corrupts 15% of the image. In order to demonstrate the recovery performance for approximately sparse signals, the image was *not* sparsified prior to adding the corruptions (which is in contrast to the in-painting example shown in [12]). The MSE between the original photo (shown in Fig. 3(a)) and the corrupted version (shown in Fig. 3(b)) corresponds to -16.7 dB.

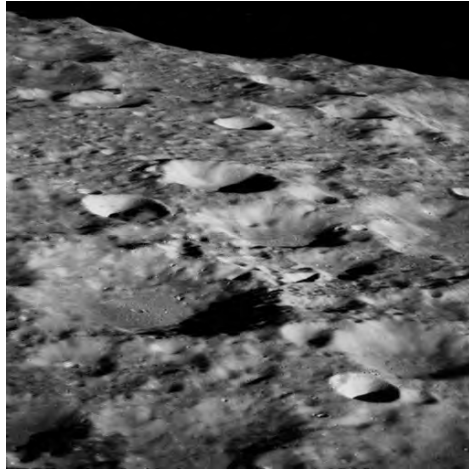
Restoration procedure. Scratch removal proceeds as follows. For BP restoration and BP separation, we assume that the image admits an approximately



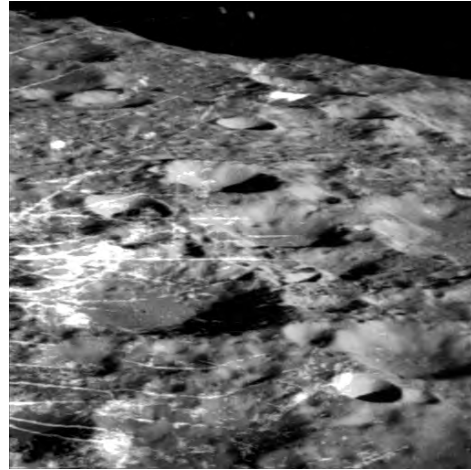
(a) Original (courtesy of NASA [54])



(b) Corrupted photo ($\text{MSE} = -16.7 \text{ dB}$)



(c) Result of BP-RES ($\text{MSE} = -29.2 \text{ dB}$)



(d) Result of BP-SEP ($\text{MSE} = -19.4 \text{ dB}$)

Figure 3: Example of using BP restoration (BP-RES) and BP separation (BP-SEP) for removal of scratches from old photographs.

sparse representation in the two-dimensional DCT basis $\mathbf{A}$, whereas the interference is assumed to be sparse in the identity basis $\mathbf{B} = \mathbf{I}_M$. Recovery is performed on the basis of the full 512×512 pixel image, i.e., we have $M = 512^2$ corrupted measurements. For BP restoration, we assume that the locations of the scratches are known prior to recovery, whereas no such knowledge is required for BP separation. For BP restoration we recover $\hat{\mathbf{x}}$ (for BP separation we additionally recover $\hat{\mathbf{e}}$) and then compute an estimate of the image as $\hat{\mathbf{y}} = \mathbf{A}\hat{\mathbf{x}}$. Since we consider noiseless measurements, we set $\eta = 0$ for both recovery procedures. Note that DR is not considered in this example as information on the location of the dominant entries of $\mathbf{x}$ is difficult to acquire in practice.

Discussion of the results. Fig. 3 shows results of the corruption and recovery procedure along with the associated MSE values. For BP restoration, we see that the recovered image has an MSE of -29.2 dB and well approximates the ground truth. For BP separation, the MSE improves over the corrupted image, but in parts where large areas of the image are corrupted, blind removal of scratches fails. Hence, knowing the locations of the sparse corruptions leads to a significant advantage in terms of MSE and is therefore highly desirable for sparsity-based in-painting methods.

We finally note that the recovery conditions (7), (9), and (13) turn out to be useful in practice as they show that the dictionary $\mathbf{A}$ must both i) *sparsify* the signal to be recovered and ii) be *incoherent* with the interference dictionary $\mathbf{B}$. Note that the second requirement is satisfied for the DCT–Identity pair used here, whereas other transform bases typically used to sparsify images (i.e., to satisfy the first requirement), such as wavelet bases, exhibit high mutual coherence with the identity basis. Hence, our recovery guarantees help to identify good dictionary pairs for a variety of restoration and separation problems.

6. Conclusions

In this paper, we have generalized the results presented in [12, 18] for the recovery of perfectly sparse signals that are corrupted by perfectly sparse interference to the much more practical case of approximately sparse signals and noisy measurements. We proposed novel restoration and separation algorithms for three different cases of knowledge on the location of the dominant entries (in terms of absolute value) in the vector $\mathbf{x}$, namely 1) direct

restoration, 2) BP restoration, and 3) BP separation. Moreover, we developed deterministic recovery guarantees for all three cases. The application examples demonstrated that our recovery guarantees explain which dictionary pairs $\mathbf{A}$ and $\mathbf{B}$ are most suited for sparsity-based signal restoration or separation. Our comparison of the presented deterministic guarantees with similar ones obtained using the restricted isometry property (RIP) framework and to those provided in [12, 18] reveals that, for BP restoration and BP separation, considering the general model does not result in more restrictive recovery conditions. For BP separation, however, the recovery conditions for the general model considered here turn out to be slightly more restrictive as it is for perfectly sparse signals and noiseless measurements.

There are many avenues for follow-on work. The derivation of probabilistic recovery guarantees (with randomness in the signals rather than in the dictionaries) leading to recovery conditions guaranteeing restoration and separation with high probability is an interesting open research topic. Furthermore, a detailed exploration of more real-world applications using the restoration and separation techniques analyzed in this paper is left for future work.

Appendix A. Proof of Theorem 1

The proof follows closely that given in [31, Thm. 2.1] and relies on techniques developed earlier in [20, 31, 43].

Appendix A.1. Prerequisites

We start with the following definitions. Let $\mathbf{h} = \hat{\mathbf{x}} - \mathbf{x}$, where $\hat{\mathbf{x}}$ denotes the solution of BPDN and $\mathbf{x}$ is the vector to be recovered. Furthermore, define $\mathbf{h}_0 = \mathbf{P}_{\mathcal{X}}\mathbf{h}$ with the set $\mathcal{X} = \text{supp}_{n_x}(\mathbf{x})$. The proof relies on the following facts.

Cone constraint. Let $e_0 = 2\|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1$ with $\mathbf{x}_{\mathcal{X}} = \mathbf{P}_{\mathcal{X}}\mathbf{x}$; then [20, 43, 46]

$$\|\mathbf{h} - \mathbf{h}_0\|_1 \leq \|\mathbf{h}_0\|_1 + e_0 \quad (\text{A.1})$$

which follows from the fact that BPDN delivers a feasible solution $\hat{\mathbf{x}}$ satisfying $\|\mathbf{x}\|_1 \geq \|\hat{\mathbf{x}}\|_1$ and from

$$\begin{aligned} \|\mathbf{x}\|_1 &\geq \|\hat{\mathbf{x}}\|_1 = \|\hat{\mathbf{x}}_{\mathcal{X}}\|_1 + \|\hat{\mathbf{x}}_{\mathcal{X}^c}\|_1 = \|\mathbf{x}_{\mathcal{X}} + \mathbf{h}_0\|_1 + \|\mathbf{h} - \mathbf{h}_0 + \mathbf{h}_{\mathcal{X}^c}\|_1 \\ &\geq \|\mathbf{x}_{\mathcal{X}}\|_1 - \|\mathbf{h}_0\|_1 + \|\mathbf{h} - \mathbf{h}_0\|_1 - \|\mathbf{x}_{\mathcal{X}^c}\|_1. \end{aligned}$$

Application of the reverse triangle inequality to the left-hand side term of (A.1) yields the following useful bound:

$$\|\mathbf{h}\|_1 \leq 2\|\mathbf{h}_0\|_1 + e_0. \quad (\text{A.2})$$

Tube constraint. We furthermore have [43, 46]

$$\begin{aligned} \|\mathbf{A}\mathbf{h}\|_2 &= \|\mathbf{A}\hat{\mathbf{x}} - \mathbf{y} - (\mathbf{A}\mathbf{x} - \mathbf{y})\|_2 \\ &\leq \|\mathbf{A}\hat{\mathbf{x}} - \mathbf{y}\|_2 + \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2 \leq \eta + \varepsilon. \end{aligned} \quad (\text{A.3})$$

Coherence-based restricted isometry property (RIP). Since $\mathbf{h}_0$ is perfectly n_x -sparse, Geršgorin's disc theorem [55, Thm. 6.1.1] applied to $\|\mathbf{A}\mathbf{h}_0\|_2^2$ yields

$$(1 - \mu_a(n_x - 1))\|\mathbf{h}_0\|_2^2 \leq \|\mathbf{A}\mathbf{h}_0\|_2^2 \leq (1 + \mu_a(n_x - 1))\|\mathbf{h}_0\|_2^2. \quad (\text{A.4})$$

Appendix A.2. Bounding the error $\|\mathbf{h}_0\|_2$ on the signal support

The goal of the following steps is to bound the recovery error $\|\mathbf{h}_0\|_2$ on the support set $\mathcal{X}$. We follow the steps in [31] to arrive at the following chain of inequalities:

$$\begin{aligned} |\mathbf{h}^H \mathbf{A}^H \mathbf{A} \mathbf{h}_0| &\geq |\mathbf{h}_0^H \mathbf{A}^H \mathbf{A} \mathbf{h}_0| - |(\mathbf{h} - \mathbf{h}_0)^H \mathbf{A}^H \mathbf{A} \mathbf{h}_0| \\ &\geq (1 - \mu_a(n_x - 1))\|\mathbf{h}_0\|_2^2 - \left| \sum_{k \in \mathcal{X}} \sum_{\ell \in \mathcal{X}^c} [\mathbf{h}_0^H]_k \mathbf{a}_k^H \mathbf{a}_\ell [\mathbf{h}]_\ell \right| \end{aligned} \quad (\text{A.5})$$

$$\geq (1 - \mu_a(n_x - 1))\|\mathbf{h}_0\|_2^2 - \mu_a \|\mathbf{h}_0\|_1 \|\mathbf{h} - \mathbf{h}_0\|_1 \quad (\text{A.6})$$

$$\geq (1 - \mu_a(n_x - 1))\|\mathbf{h}_0\|_2^2 - \mu_a \|\mathbf{h}_0\|_1 (\|\mathbf{h}_0\|_1 + e_0) \quad (\text{A.7})$$

$$\geq (1 - \mu_a(n_x - 1))\|\mathbf{h}_0\|_2^2 - \mu_a n_x \|\mathbf{h}_0\|_2^2 - \mu_a \sqrt{n_x} \|\mathbf{h}_0\|_2 e_0 \quad (\text{A.8})$$

$$= (1 - \mu_a(2n_x - 1))\|\mathbf{h}_0\|_2^2 - \mu_a \sqrt{n_x} \|\mathbf{h}_0\|_2 e_0, \quad (\text{A.9})$$

where (A.5) follows from (A.4), (A.6) is a consequence of $|\mathbf{a}_k^H \mathbf{a}_\ell| \leq \mu_a, \forall k \neq \ell$, (A.7) results from the cone constraint (A.1), and (A.8) from the Cauchy-Schwarz inequality. We emphasize that (A.9) is crucial, since it determines the recovery condition for BPDN. In particular, if the first RHS term in (A.9) satisfies $(1 - \mu_a(2n_x - 1)) > 0$ and $\mathbf{h}_0 \neq \mathbf{0}_{N_a \times 1}$, then the error $\|\mathbf{h}_0\|_2$ is bounded

from above as follows:

$$\|\mathbf{h}_0\|_2 \leq \frac{|\mathbf{h}^H \mathbf{A}^H \mathbf{A} \mathbf{h}_0| + \mu_a \sqrt{n_x} \|\mathbf{h}_0\|_2 e_0}{(1 - \mu_a(2n_x - 1)) \|\mathbf{h}_0\|_2} \quad (\text{A.10})$$

$$\leq \frac{\|\mathbf{A} \mathbf{h}\|_2 \|\mathbf{A} \mathbf{h}_0\|_2 + \mu_a \sqrt{n_x} \|\mathbf{h}_0\|_2 e_0}{(1 - \mu_a(2n_x - 1)) \|\mathbf{h}_0\|_2} \quad (\text{A.11})$$

$$\leq \frac{(\varepsilon + \eta) \sqrt{1 + \mu_a(n_x - 1)} \|\mathbf{h}_0\|_2 + \mu_a \sqrt{n_x} \|\mathbf{h}_0\|_2 e_0}{(1 - \mu_a(2n_x - 1)) \|\mathbf{h}_0\|_2} \quad (\text{A.12})$$

$$= \frac{(\varepsilon + \eta) \sqrt{1 + \mu_a(n_x - 1)} + \mu_a \sqrt{n_x} e_0}{1 - \mu_a(2n_x - 1)}. \quad (\text{A.13})$$

Here, (A.10) is a consequence of (A.9), (A.11) follows from the Cauchy-Schwarz inequality, and (A.12) results from the tube constraint (A.3) and the RIP (A.4). The case $\mathbf{h}_0 = \mathbf{0}_{N_a \times 1}$ is trivial as it implies $\|\mathbf{h}_0\|_2 = 0$.

Appendix A.3. Bounding the recovery error $\|\mathbf{h}\|_2$

We are now ready to derive an upper bound on the recovery error $\|\mathbf{h}\|_2$. To this end, we first bound $\|\mathbf{A} \mathbf{h}\|_2^2$ from below as in [31]

$$\begin{aligned} \|\mathbf{A} \mathbf{h}\|_2^2 &= \mathbf{h}^H \mathbf{A}^H \mathbf{A} \mathbf{h} = \sum_{k, \ell} [\mathbf{h}^H]_k \mathbf{a}_k^H \mathbf{a}_\ell [\mathbf{h}]_\ell \\ &= \sum_k \|\mathbf{a}_k\|_2^2 |[\mathbf{h}]_k|^2 + \sum_{k, \ell, k \neq \ell} [\mathbf{h}^H]_k \mathbf{a}_k^H \mathbf{a}_\ell [\mathbf{h}]_\ell \\ &\geq \|\mathbf{h}\|_2^2 - \mu_a \sum_{k, \ell, k \neq \ell} |[\mathbf{h}^H]_k [\mathbf{h}]_\ell| \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} &= \|\mathbf{h}\|_2^2 + \mu_a \sum_k |[\mathbf{h}]_k|^2 - \mu_a \sum_{k, \ell} |[\mathbf{h}^H]_k [\mathbf{h}]_\ell| \\ &= (1 + \mu_a) \|\mathbf{h}\|_2^2 - \mu_a \|\mathbf{h}\|_1^2, \end{aligned} \quad (\text{A.15})$$

where (A.14) follows from $\|\mathbf{a}_k\|_2 = 1$, $\forall k$, and $|\mathbf{a}_k^H \mathbf{a}_\ell| \leq \mu_a$, $\forall k \neq \ell$. With (A.15), the recovery error can be bounded as

$$\begin{aligned} \|\mathbf{h}\|_2^2 &\leq \frac{\|\mathbf{A} \mathbf{h}\|_2^2 + \mu_a \|\mathbf{h}\|_1^2}{1 + \mu_a} \\ &\leq \frac{(\varepsilon + \eta)^2 + \mu_a (2\|\mathbf{h}_0\|_1 + e_0)^2}{1 + \mu_a}, \end{aligned} \quad (\text{A.16})$$

where (A.2) is used to arrive at (A.16). By taking the square root of (A.16) and applying the Cauchy-Schwarz inequality, we arrive at the following bound:

$$\begin{aligned}\|\mathbf{h}\|_2 &\leq \frac{\sqrt{(\varepsilon + \eta)^2 + \mu_a (2\|\mathbf{h}_0\|_1 + e_0)^2}}{\sqrt{1 + \mu_a}} \\ &\leq \frac{(\varepsilon + \eta) + \sqrt{\mu_a} (2\|\mathbf{h}_0\|_1 + e_0)}{\sqrt{1 + \mu_a}}.\end{aligned}\tag{A.17}$$

Finally, using $\|\mathbf{h}_0\|_1 \leq \sqrt{n_x}\|\mathbf{h}_0\|_2$ with the bound in (A.13) followed by algebraic simplifications yields

$$\begin{aligned}\|\mathbf{h}\|_2 &\leq \frac{(\varepsilon + \eta) + \sqrt{\mu_a} (2\sqrt{n_x}\|\mathbf{h}_0\|_2 + e_0)}{\sqrt{1 + \mu_a}} \\ &\leq (\varepsilon + \eta) \frac{1 - \mu_a(2n_x - 1) + 2\sqrt{\mu_a n_x} \sqrt{1 + \mu_a}(n_x - 1)}{\sqrt{1 + \mu_a} (1 - \mu_a(2n_x - 1))} \\ &\quad + e_0 \frac{\sqrt{\mu_a + \mu_a^2}}{(1 - \mu_a(2n_x - 1))} \\ &= C_0(\eta + \varepsilon) + C_1\|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1,\end{aligned}$$

which concludes the proof. We note that by imposing a more restrictive condition than $n_x < (1 + 1/\mu_a)/2$ in (2), one may arrive at smaller constants C_0 and C_1 (see [46] for the details).

Appendix B. Proof of Theorem 2

The proof is accomplished by deriving an upper bound on the residual errors resulting from direct restoration. Furthermore, we show that the recovery condition (7) guarantees the existence of $\mathbf{R}_{\mathcal{E}} = \mathbf{I}_M - \mathbf{B}_{\mathcal{E}}\mathbf{B}_{\mathcal{E}}^\dagger$ and $(\mathbf{R}_{\mathcal{E}}\mathbf{A}_{\mathcal{X}})^\dagger$.

Appendix B.1. Bounding the recovery error

We start by bounding the recovery error of DR as

$$\begin{aligned}\|\mathbf{x} - \hat{\mathbf{x}}\|_2 &\leq \|\mathbf{x}_{\mathcal{X}} - \hat{\mathbf{x}}_{\mathcal{X}}\|_2 + \|\mathbf{x}_{\mathcal{X}^c} - \hat{\mathbf{x}}_{\mathcal{X}^c}\|_2 \\ &\leq \|\mathbf{x}_{\mathcal{X}} - \hat{\mathbf{x}}_{\mathcal{X}}\|_2 + \|\mathbf{x}_{\mathcal{X}^c}\|_1.\end{aligned}\tag{B.1}$$

The only term in (B.1) that needs further investigation is $\|\mathbf{x}_{\mathcal{X}} - \hat{\mathbf{x}}_{\mathcal{X}}\|_2$. As shown in (5), we have

$$[\hat{\mathbf{x}}]_{\mathcal{X}} = (\mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^{\dagger} \mathbf{R}_{\mathcal{E}} \mathbf{z} = \mathbf{x}_{\mathcal{X}} + (\mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^{\dagger} \mathbf{R}_{\mathcal{E}} (\mathbf{A} \mathbf{x}_{\mathcal{X}^c} + \mathbf{n})$$

and hence, it follows that

$$\|\mathbf{x}_{\mathcal{X}} - \hat{\mathbf{x}}_{\mathcal{X}}\|_2 \leq \|(\mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^{\dagger} \mathbf{R}_{\mathcal{E}} \mathbf{v}\|_2, \quad (\text{B.2})$$

where $\mathbf{v} = \mathbf{A} \mathbf{x}_{\mathcal{X}^c} + \mathbf{n}$ represents the residual error term. The remainder of the proof amounts to deriving an upper bound on the RHS in (B.2). We start with the definition of the pseudo-inverse

$$\|(\mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^{\dagger} \mathbf{R}_{\mathcal{E}} \mathbf{v}\|_2 = \|(\mathbf{A}_{\mathcal{X}}^H \mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^{-1} \mathbf{A}_{\mathcal{X}}^H \mathbf{R}_{\mathcal{E}} \mathbf{v}\|_2 \quad (\text{B.3})$$

$$\leq \|(\mathbf{A}_{\mathcal{X}}^H \mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^{-1}\|_2 \|\mathbf{A}_{\mathcal{X}}^H \mathbf{R}_{\mathcal{E}} \mathbf{v}\|_2, \quad (\text{B.4})$$

where (B.3) is a consequence of $\mathbf{R}_{\mathcal{E}} = \mathbf{R}_{\mathcal{E}}^H \mathbf{R}_{\mathcal{E}}$, and (B.4) follows from the Rayleigh-Ritz theorem [55, Thm. 4.2.2]. We next individually bound the RHS terms in (B.4) from above.

Appendix B.2. Bounding the ℓ_2 -norm of the inverse

The bound on the norm of the inverse in (B.4) is based upon an idea developed in [24]. Specifically, we use the Neumann series $(\mathbf{I}_{|\mathcal{X}|} - \mathbf{K})^{-1} = \mathbf{I}_{|\mathcal{X}|} + \sum_{k=1}^{\infty} \mathbf{K}^k$ [56, Lem. 2.3.3] to obtain

$$\begin{aligned} \|(\mathbf{A}_{\mathcal{X}}^H \mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^{-1}\|_2 &= \|(\mathbf{A}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{X}} - \mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^{\dagger} \mathbf{A}_{\mathcal{X}})^{-1}\|_2 \\ &= \|(\mathbf{I}_{|\mathcal{X}|} - \mathbf{K})^{-1}\|_2 = \left\| \mathbf{I}_{|\mathcal{X}|} + \sum_{k=1}^{\infty} \mathbf{K}^k \right\|_2 \\ &\leq 1 + \left\| \sum_{k=1}^{\infty} \mathbf{K}^k \right\|_2 \leq 1 + \sum_{k=1}^{\infty} \|\mathbf{K}\|_2^k \\ &= \frac{1}{1 - \|\mathbf{K}\|_2}, \end{aligned} \quad (\text{B.5})$$

which is guaranteed to exist whenever $\|\mathbf{K}\|_2 < 1$ with

$$\mathbf{K} = \mathbf{I}_{|\mathcal{X}|} - \mathbf{A}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{X}} + \mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^{\dagger} \mathbf{A}_{\mathcal{X}}.$$

We next derive a sufficient condition for which $\|\mathbf{K}\|_2 < 1$ and, hence, the matrix $\mathbf{A}_{\mathcal{X}}^H \mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}}$ is invertible. We bound $\|\mathbf{K}\|_2$ from above as

$$\|\mathbf{K}\|_2 \leq \|\mathbf{I}_{|\mathcal{X}|} - \mathbf{A}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{X}}\|_2 + \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^\dagger \mathbf{A}_{\mathcal{X}}\|_2 \quad (\text{B.6})$$

$$\leq \mu_a(n_x - 1) + \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}} (\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}})^{-1} \mathbf{B}_{\mathcal{E}}^H \mathbf{A}_{\mathcal{X}}\|_2, \quad (\text{B.7})$$

where (B.6) results from the triangle inequality and (B.7) is a consequence of Geršgorin's disc theorem [55, Thm. 6.1.1] applied to the ℓ_2 -norm of the hollow matrix $\mathbf{I}_{|\mathcal{X}|} - \mathbf{A}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{X}}$. We next bound the RHS term in (B.7) as

$$\|\mathbf{A}_{\mathcal{X}} \mathbf{B}_{\mathcal{E}} (\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}})^{-1} \mathbf{B}_{\mathcal{E}}^H \mathbf{A}_{\mathcal{X}}\|_2 \leq \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_2 \|(\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}})^{-1}\|_2 \|\mathbf{B}_{\mathcal{E}}^H \mathbf{A}_{\mathcal{X}}\|_2 \quad (\text{B.8})$$

$$\leq \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_F^2 \|(\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}})^{-1}\|_2 \quad (\text{B.9})$$

$$\leq n_x n_e \mu_m^2 \|(\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}})^{-1}\|_2 \quad (\text{B.10})$$

$$\leq \frac{n_x n_e \mu_m^2}{\lambda_{\min}(\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}})} \leq \frac{n_x n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+}, \quad (\text{B.11})$$

where (B.8) follows from the ℓ_2 -matrix-norm bound, (B.9) from $\|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_2 \leq \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_F$ and $\|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_F = \|\mathbf{B}_{\mathcal{E}}^H \mathbf{A}_{\mathcal{X}}\|_F$, and (B.10) from

$$\|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_F^2 = \sum_{k \in \mathcal{X}} \sum_{\ell \in \mathcal{E}} |\mathbf{a}_k^H \mathbf{b}_\ell|^2 \leq \sum_{k \in \mathcal{X}} \sum_{\ell \in \mathcal{E}} \mu_m^2 = n_x n_e \mu_m^2.$$

Note that (B.11) requires $n_e < 1 + 1/\mu_b$, which provides a sufficient condition for which the pseudo-inverse $\mathbf{B}_{\mathcal{E}}^\dagger$ exists.

Combining (B.5), (B.7), and (B.11) yields the upper bound

$$\|(\mathbf{A}_{\mathcal{X}}^H \mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^{-1}\|_2 \leq \frac{1}{1 - \mu_a(n_x - 1) - \frac{n_x n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+}}, \quad (\text{B.12})$$

which requires

$$1 - \mu_a(n_x - 1) - \frac{n_x n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+} > 0 \quad (\text{B.13})$$

for the matrix $(\mathbf{A}_{\mathcal{X}}^H \mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^{-1}$ to exist. We emphasize that the condition (B.13) determines the recovery condition for DR (7). In particular, if

$$(1 - \mu_a(n_x - 1)) [1 - \mu_b(n_e - 1)]^+ > n_x n_e \mu_m^2$$

then (B.13) and $n_e < 1 + 1/\mu_b$ are both satisfied and, hence, the recovery matrix $(\mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^\dagger \mathbf{R}_{\mathcal{E}}$ required for DR exists.

Appendix B.3. Bounding the residual error term

We now derive an upper bound on the RHS residual error term in (B.4) according to

$$\begin{aligned}\|\mathbf{A}_{\mathcal{X}}^H \mathbf{R}_{\mathcal{E}} \mathbf{v}\|_2 &\leq \|\mathbf{A}_{\mathcal{X}}^H \mathbf{v}\|_2 + \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^\dagger \mathbf{v}\|_2 \\ &\leq \sqrt{n_x} \|\mathbf{v}\|_2 + \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^\dagger \mathbf{v}\|_2,\end{aligned}\quad (\text{B.14})$$

where (B.14) is a result of

$$\|\mathbf{A}_{\mathcal{X}}^H \mathbf{v}\|_2 = \sqrt{\sum_{k \in \mathcal{X}} |\mathbf{a}_k^H \mathbf{v}|^2} \leq \sqrt{\sum_{k \in \mathcal{X}} \|\mathbf{a}_k\|_2^2 \|\mathbf{v}\|_2^2} \leq \sqrt{n_x} \|\mathbf{v}\|_2. \quad (\text{B.15})$$

The bound on the second RHS term in (B.14) is obtained by carrying out similar steps used to arrive at (B.11), i.e.,

$$\begin{aligned}\|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^\dagger \mathbf{v}\|_2 &\leq \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_2 \|(\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}})^{-1}\|_2 \|\mathbf{B}_{\mathcal{E}}^H \mathbf{v}\|_2 \\ &\leq \frac{\sqrt{n_x n_e \mu_m^2}}{\lambda_{\min}(\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}})} \|\mathbf{B}_{\mathcal{E}}^H \mathbf{v}\|_2 \\ &\leq \frac{n_e \sqrt{n_x \mu_m^2}}{[1 - \mu_b(n_e - 1)]^+} \|\mathbf{v}\|_2.\end{aligned}\quad (\text{B.16})$$

Finally, we bound the ℓ_2 -norm of the residual error term according to

$$\|\mathbf{v}\|_2 = \|\mathbf{A} \mathbf{x}_{\mathcal{X}^c} + \mathbf{n}\|_2 \leq \|\mathbf{A} \mathbf{x}_{\mathcal{X}^c}\|_2 + \|\mathbf{n}\|_2 \leq \|\mathbf{x}_{\mathcal{X}^c}\|_1 + \|\mathbf{n}\|_2 \quad (\text{B.17})$$

since

$$\|\mathbf{A} \mathbf{x}_{\mathcal{X}^c}\|_2 = \left\| \sum_{k \in \mathcal{X}^c} \mathbf{a}_k [\mathbf{x}]_k \right\|_2 \leq \sum_{k \in \mathcal{X}^c} \|\mathbf{a}_k [\mathbf{x}]_k\|_2 = \|\mathbf{x}_{\mathcal{X}^c}\|_1.$$

Appendix B.4. Putting the pieces together

In order to bound the recovery error on the support set $\mathcal{X}$, we combine (B.12) with (B.14) and (B.16) to arrive at

$$\begin{aligned}\|(\mathbf{R}_{\mathcal{E}} \mathbf{A}_{\mathcal{X}})^\dagger \mathbf{R}_{\mathcal{E}} \mathbf{v}\|_2 &\leq \frac{([1 - \mu_b(n_e - 1)]^+ + n_e \mu_m) \sqrt{n_x}}{(1 - \mu_a(n_x - 1)) [1 - \mu_b(n_e - 1)]^+ - n_x n_e \mu_m^2} \|\mathbf{v}\|_2 \\ &= c \|\mathbf{v}\|_2.\end{aligned}\quad (\text{B.18})$$

Finally, using (B.17) in combination with (B.1), (B.2), and (B.18) leads to

$$\begin{aligned}\|\mathbf{x} - \hat{\mathbf{x}}\|_2 &\leq c\varepsilon + (c+1)\|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1 \\ &= C_3\varepsilon + C_4\|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1,\end{aligned}$$

which concludes the proof.

Appendix C. Proof of Theorem 3

We first derive a set of key properties of the matrix $\tilde{\mathbf{A}} = \mathbf{R}_{\mathcal{E}}\mathbf{A}$, which are then used to prove the main result following the steps in Appendix A.

Appendix C.1. Properties of the matrix $\tilde{\mathbf{A}}$

It is important to realize that BP restoration operates on the input-output relation

$$\mathbf{R}_{\mathcal{E}}\mathbf{z} = \mathbf{R}_{\mathcal{E}}(\mathbf{A}\mathbf{x} + \mathbf{B}_{\mathcal{E}}\mathbf{e} + \mathbf{n}) = \tilde{\mathbf{A}}\mathbf{x} + \mathbf{R}_{\mathcal{E}}\mathbf{n} \quad (\text{C.1})$$

with $\mathbf{R}_{\mathcal{E}} = \mathbf{I}_M - \mathbf{B}_{\mathcal{E}}\mathbf{B}_{\mathcal{E}}^{\dagger}$ and $\tilde{\mathbf{A}} = \mathbf{R}_{\mathcal{E}}\mathbf{A}$. The recovery condition for BP restoration (9), which will be derived next, also ensures that $\mathbf{R}_{\mathcal{E}}$ exists; this is due to fact that the recovery condition for DR (7) ensures that $\mathbf{R}_{\mathcal{E}}$ exists and the condition for BP restoration (9) is met whenever (7) is satisfied.

In order to adapt the proof in Appendix A for the projected input-output relation in (C.1), the following properties of $\tilde{\mathbf{A}}$ are required.

Tube constraint. Analogously to Appendix A, we obtain

$$\begin{aligned}\|\tilde{\mathbf{A}}\mathbf{h}\|_2 &\leq \|\mathbf{R}_{\mathcal{E}}(\mathbf{A}\hat{\mathbf{x}} - \mathbf{z})\|_2 + \|\mathbf{R}_{\mathcal{E}}(\mathbf{A}\mathbf{x} - \mathbf{z})\|_2 \\ &\leq \eta + \|\mathbf{R}_{\mathcal{E}}\mathbf{n}\|_2 \leq \eta + \varepsilon,\end{aligned}$$

where the last inequality follows from the fact that $\mathbf{R}_{\mathcal{E}}$ is a projection matrix and, hence, $\|\mathbf{R}_{\mathcal{E}}\mathbf{n}\|_2 \leq \|\mathbf{n}\|_2 \leq \varepsilon$.

Coherence-based bound on the RIC. We next compute a coherence-based bound on the RIC for the matrix $\tilde{\mathbf{A}}$. To this end, let $\mathbf{h}_0$ be perfectly n_x -sparse and

$$\|\tilde{\mathbf{A}}\mathbf{h}_0\|_2^2 = \left| \mathbf{h}_0^H \mathbf{A}^H \mathbf{A} \mathbf{h}_0 - \mathbf{h}_0^H \mathbf{A}^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^{\dagger} \mathbf{A} \mathbf{h}_0 \right| \quad (\text{C.2})$$

$$\leq (1 + \mu_a(n_x - 1))\|\mathbf{h}_0\|_2^2 + \left| \mathbf{h}_0^H \mathbf{A}^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^{\dagger} \mathbf{A} \mathbf{h}_0 \right|, \quad (\text{C.3})$$

where (C.2) follows from $\mathbf{R}_\varepsilon^H \mathbf{R}_\varepsilon = \mathbf{R}_\varepsilon$ and (C.3) from Geršgorin's disc theorem [55, Thm. 6.1.1]. Next, we bound the second RHS term in (C.3) as follows:

$$\begin{aligned} \left| \mathbf{h}_0^H \mathbf{A}^H \mathbf{B}_\varepsilon \mathbf{B}_\varepsilon^\dagger \mathbf{A} \mathbf{h}_0 \right| &= \left| \mathbf{h}_0^H \mathbf{A}^H \mathbf{B}_\varepsilon (\mathbf{B}_\varepsilon^H \mathbf{B}_\varepsilon)^{-1} \mathbf{B}_\varepsilon^H \mathbf{A} \mathbf{h}_0 \right| \\ &\leq \lambda_{\min}^{-1}(\mathbf{B}_\varepsilon^H \mathbf{B}_\varepsilon) \|\mathbf{B}_\varepsilon^H \mathbf{A} \mathbf{h}_0\|_2^2 \end{aligned} \quad (\text{C.4})$$

$$\leq \lambda_{\min}^{-1}(\mathbf{B}_\varepsilon^H \mathbf{B}_\varepsilon) \|\mathbf{B}_\varepsilon^H \mathbf{A}_\mathcal{X}\|_2^2 \|\mathbf{h}_0\|_2^2 \quad (\text{C.5})$$

$$\leq \frac{n_x n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+} \|\mathbf{h}_0\|_2^2 \quad (\text{C.6})$$

where (C.4) follows from [55, Thm. 4.2.2], (C.5) from the ℓ_2 -norm inequality. The inequality (C.6) results from

$$\|\mathbf{B}_\varepsilon^H \mathbf{A}_\mathcal{X}\|_2^2 \leq \|\mathbf{B}_\varepsilon^H \mathbf{A}_\mathcal{X}\|_F^2 = \sum_{\ell \in \mathcal{E}} \sum_{k \in \mathcal{X}} |\mathbf{b}_\ell^H \mathbf{a}_k|^2 \leq n_x n_e \mu_m^2.$$

Note that (C.6) requires $n_e < 1 + 1/\mu_b$, which is a sufficient condition for $(\mathbf{B}_\varepsilon^H \mathbf{B}_\varepsilon)^{-1}$ to exist. Note that $n_e < 1 + 1/\mu_b$ holds whenever the recovery condition for BP-RES in (7) is satisfied. Combining (C.3) with (C.6) results in

$$\begin{aligned} \|\mathbf{R}_\varepsilon \mathbf{A} \mathbf{h}_0\|_2^2 &\leq \left(1 + \mu_a(n_x - 1) + \frac{n_x n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+} \right) \|\mathbf{h}_0\|_2^2 \\ &= (1 + \hat{\delta}) \|\mathbf{h}_0\|_2^2. \end{aligned} \quad (\text{C.7})$$

We next compute the lower bound as

$$\begin{aligned} \left\| \tilde{\mathbf{A}} \mathbf{h}_0 \right\|_2^2 &= \left| \mathbf{h}_0^H \mathbf{A}^H \mathbf{A} \mathbf{h}_0 - \mathbf{h}_0^H \mathbf{A}^H \mathbf{B}_\varepsilon \mathbf{B}_\varepsilon^\dagger \mathbf{A} \mathbf{h}_0 \right| \\ &\geq (1 - \mu_a(n_x - 1)) \|\mathbf{h}_0\|_2^2 - \left| \mathbf{h}_0^H \mathbf{A}^H \mathbf{B}_\varepsilon \mathbf{B}_\varepsilon^\dagger \mathbf{A} \mathbf{h}_0 \right| \end{aligned} \quad (\text{C.8})$$

$$\geq (1 - \mu_a(n_x - 1)) \|\mathbf{h}_0\|_2^2 - \frac{n_x n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+} \|\mathbf{h}_0\|_2^2 \quad (\text{C.9})$$

$$\begin{aligned} &\geq \left(1 - \mu_a(n_x - 1) - \frac{n_x n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+} \right) \|\mathbf{h}_0\|_2^2 \\ &= (1 - \hat{\delta}) \|\mathbf{h}_0\|_2^2 \end{aligned} \quad (\text{C.10})$$

where (C.8) follows from [55, Thm. 6.1.1] and (C.9) is obtained by carrying out similar steps used to arrive at (C.6). Note that (C.7) and (C.10) provide a coherence-based upper bound $\hat{\delta}$ on the RIC of the projected matrix $\tilde{\mathbf{A}} = \mathbf{R}_\varepsilon \mathbf{A}$.

Upper bound on the inner products. The proof detailed in Appendix A requires an upper bound on the inner products of columns of the matrix $\tilde{\mathbf{A}}$. For $i \neq j$, we obtain

$$\begin{aligned} |\tilde{\mathbf{a}}_i^H \tilde{\mathbf{a}}_j| &= |\mathbf{a}_i^H \mathbf{R}_{\mathcal{E}} \mathbf{a}_j| \leq |\mathbf{a}_i^H \mathbf{a}_j| + |\mathbf{a}_i^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^\dagger \mathbf{a}_j| \\ &\leq \mu_a + |\mathbf{a}_i^H \mathbf{B}_{\mathcal{E}} (\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}})^{-1} \mathbf{B}_{\mathcal{E}}^H \mathbf{a}_j| \end{aligned} \quad (\text{C.11})$$

$$\leq \mu_a + \frac{|\mathbf{a}_i^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^H \mathbf{a}_j|}{[1 - \mu_b(n_e - 1)]^+} \quad (\text{C.12})$$

$$\leq \mu_a + \frac{\|\mathbf{B}_{\mathcal{E}}^H \mathbf{a}_i\|_2 \|\mathbf{B}_{\mathcal{E}}^H \mathbf{a}_j\|_2}{[1 - \mu_b(n_e - 1)]^+}, \quad (\text{C.13})$$

where (C.11) follows from the definition of the coherence parameter μ_a , (C.12) is a consequence of Geršgorin's disc theorem, and (C.13) from the Cauchy-Schwarz inequality. Since

$$\|\mathbf{B}_{\mathcal{E}}^H \mathbf{a}_i\|_2 = \sqrt{\sum_{k \in \mathcal{E}} |\mathbf{b}_k^H \mathbf{a}_i|^2} \leq \sqrt{n_e \mu_m^2}$$

for all $i = 1, \dots, N_a$, the inner products with $i \neq j$ satisfy

$$|\tilde{\mathbf{a}}_i^H \tilde{\mathbf{a}}_j| \leq \mu_a + \frac{n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+} \triangleq a. \quad (\text{C.14})$$

Lower bound on the column norm. The last prerequisite for the proof is a lower bound on the column-norms of $\tilde{\mathbf{A}}$. Application of the reverse triangle inequality, using the fact that $\|\mathbf{a}_i\|_2 = 1$, $\forall i$, and carrying out the similar steps used to arrive at (C.14) results in

$$\begin{aligned} \|\tilde{\mathbf{a}}_i\|_2^2 &= \|\mathbf{R}_{\mathcal{E}} \mathbf{a}_i\|_2^2 \geq |\mathbf{a}_i^H \mathbf{a}_i| - |\mathbf{a}_i^H \mathbf{B}_{\mathcal{E}} \mathbf{B}_{\mathcal{E}}^\dagger \mathbf{a}_i| \\ &\geq 1 - \frac{n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+} \triangleq b. \end{aligned}$$

Appendix C.2. Recovery guarantee

We now derive the recovery condition and bound the corresponding error $\|\mathbf{h}\|_2$. The proof follows that of Appendix A. For the sake of simplicity of exposition, we make use of the previously defined quantities $\hat{\delta}$, a , and b .

Bounding the error on the signal support. We start by bounding the error $\|\mathbf{h}_0\|_2$ as follows:

$$\begin{aligned} \left| \mathbf{h}^H \tilde{\mathbf{A}}^H \tilde{\mathbf{A}} \mathbf{h}_0 \right| &\geq \left| \mathbf{h}_0^H \tilde{\mathbf{A}}^H \tilde{\mathbf{A}} \mathbf{h}_0 \right| - \left| (\mathbf{h} - \mathbf{h}_0)^H \tilde{\mathbf{A}}^H \tilde{\mathbf{A}} \mathbf{h}_0 \right| \\ &\geq (1 - \hat{\delta}) \|\mathbf{h}_0\|_2^2 - a n_x \|\mathbf{h}_0\|_2^2 - a \sqrt{n_x} \|\mathbf{h}_0\|_2 e_0 \\ &= c \|\mathbf{h}_0\|_2^2 - a \sqrt{n_x} \|\mathbf{h}_0\|_2 e_0 \end{aligned} \quad (\text{C.15})$$

with

$$c \triangleq 1 - \hat{\delta} - a n_x = 1 - \mu_a(2n_x - 1) - \frac{2n_x n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+}.$$

Note that the parameter c is crucial, since it determines the recovery condition for BP-RES (9). In particular, $c > 0$ is equivalent to (9)

$$[1 - \mu_a(2n_x - 1)]^+ [1 - \mu_b(n_e - 1)]^+ > 2n_x n_e \mu_m^2.$$

If this condition is satisfied, then we can bound $\|\mathbf{h}_0\|_2$ from above as follows:

$$\|\mathbf{h}_0\|_2 \leq \frac{(\varepsilon + \eta) \sqrt{1 + \hat{\delta}} + a \sqrt{n_x} e_0}{c}.$$

Bounding the recovery error. We next compute an upper bound on $\|\mathbf{h}\|_2$. To this end, we start with a lower bound on $\|\tilde{\mathbf{A}} \mathbf{h}\|_2^2$ as

$$\left\| \tilde{\mathbf{A}} \mathbf{h} \right\|_2^2 \geq (b + a) \|\mathbf{h}\|_2^2 - a \|\mathbf{h}\|_1^2 = (1 + \mu_a) \|\mathbf{h}\|_2^2 - a \|\mathbf{h}\|_1^2,$$

since $b + a = 1 + \mu_a$. Finally, we bound $\|\mathbf{h}\|_2$ as follows:

$$\begin{aligned} \|\mathbf{h}\|_2 &\leq (\varepsilon + \eta) \frac{c + 2\sqrt{a n_x} \sqrt{1 + \hat{\delta}}}{\sqrt{1 + \mu_a} c} + e_0 \frac{\sqrt{a} \sqrt{1 + \mu_a}}{c} \\ &= C_5(\eta + \varepsilon) + C_6 \|\mathbf{x} - \mathbf{x}_{\mathcal{X}}\|_1, \end{aligned}$$

where the constants C_5 and C_6 depend on μ_a , μ_b , n_x , and n_e , which concludes the proof.

Appendix D. Proof of Theorem 4

In order to prove the recovery guarantee in Theorem 4, we start by deriving a coherence-based bound on the RIC of the concatenated matrix $\mathbf{D} = [\mathbf{A} \ \mathbf{B}]$ which is then used to prove the main result.

Appendix D.1. Coherence-based RIC for $\mathbf{D} = [\mathbf{A} \ \mathbf{B}]$

In this section, we obtain an equivalent bound to that in Appendix A.1 for the dictionary $\mathbf{D}$ that depends only on the coherence parameters μ_a , μ_b , μ_m , and μ_d , and the total number of nonzero entries denoted by $w = n_x + n_e$.

Bounds that are explicit in n_x and n_e . Let $\mathbf{h}_0 = [\mathbf{h}_x^T \ \mathbf{h}_e^T]^T$ where $\mathbf{h}_x = \mathbf{P}_{\mathcal{X}}(\hat{\mathbf{x}} - \mathbf{x})$ and $\mathbf{h}_e = \mathbf{P}_{\mathcal{E}}(\hat{\mathbf{e}} - \mathbf{e})$ are perfectly n_x and n_e sparse, respectively. We start by the lower bound on the squared ℓ_2 -norm according to

$$\begin{aligned} \|\mathbf{D}\mathbf{h}_0\|_2^2 &= \begin{bmatrix} \mathbf{h}_x^H & \mathbf{h}_e^H \end{bmatrix} \begin{bmatrix} \mathbf{A}^H \mathbf{A} & \mathbf{A}^H \mathbf{B} \\ \mathbf{B}^H \mathbf{A} & \mathbf{B}^H \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{h}_x \\ \mathbf{h}_e \end{bmatrix} \\ &= \mathbf{h}_0^H \begin{bmatrix} \mathbf{I}_{N_a} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{N_b} \end{bmatrix} \mathbf{h}_0 + \mathbf{h}_0^H \begin{bmatrix} \mathbf{A}^H \mathbf{A} - \mathbf{I}_{N_a} & \mathbf{A}^H \mathbf{B} \\ \mathbf{B}^H \mathbf{A} & \mathbf{B}^H \mathbf{B} - \mathbf{I}_{N_b} \end{bmatrix} \mathbf{h}_0 \\ &\geq \|\mathbf{h}_0\|_2^2 - \left\| \begin{bmatrix} \mathbf{A}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{X}} - \mathbf{I}_{|\mathcal{X}|} & \mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}} \\ \mathbf{B}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{E}} & \mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}} - \mathbf{I}_{|\mathcal{E}|} \end{bmatrix} \right\|_2 \|\mathbf{h}_0\|_2^2, \quad (\text{D.1}) \end{aligned}$$

where (D.1) follows from the reverse triangle inequality and elementary properties of the ℓ_2 matrix norm. We next compute an upper bound on the matrix norm in (D.1) as follows:

$$\begin{aligned} &\left\| \begin{bmatrix} \mathbf{A}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{X}} - \mathbf{I}_{|\mathcal{X}|} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}} - \mathbf{I}_{|\mathcal{E}|} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}} \\ \mathbf{B}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{E}} & \mathbf{0} \end{bmatrix} \right\|_2 \\ &\leq \max \{ \|\mathbf{A}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{X}} - \mathbf{I}_{|\mathcal{X}|}\|_2, \|\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}} - \mathbf{I}_{|\mathcal{E}|}\|_2 \} + \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_2, \quad (\text{D.2}) \end{aligned}$$

where (D.2) is a result of the triangle inequality for matrix norms and the facts that the spectral norm of both a block-diagonal matrix and an anti-block-diagonal matrix is given by the largest among the spectral norms of the individual nonzero blocks. The application of Geršgorin's disc theorem to the $\max\{\cdot\}$ -term in (D.2) and

$$\|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_2 \leq \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_F \leq \sqrt{\sum_{k \in \mathcal{X}} \sum_{\ell \in \mathcal{E}} |\mathbf{a}_k^H \mathbf{b}_{\ell}|^2} \leq \sqrt{n_x n_e \mu_m^2}$$

leads to

$$\begin{aligned} &\max \{ \|\mathbf{A}_{\mathcal{X}}^H \mathbf{A}_{\mathcal{X}} - \mathbf{I}_{|\mathcal{X}|}\|_2, \|\mathbf{B}_{\mathcal{E}}^H \mathbf{B}_{\mathcal{E}} - \mathbf{I}_{|\mathcal{E}|}\|_2 \} + \|\mathbf{A}_{\mathcal{X}}^H \mathbf{B}_{\mathcal{E}}\|_2 \\ &\leq \max \{ \mu_a(n_x - 1), \mu_b(n_e - 1) \} + \sqrt{n_x n_e \mu_m^2}. \end{aligned}$$

Hence, we arrive at the following lower bound

$$\|\mathbf{D}\mathbf{h}_0\|_2^2 \geq \|\mathbf{h}_0\|_2^2 \left(1 - \max\{\mu_a(n_x - 1), \mu_b(n_e - 1)\} - \sqrt{n_x n_e \mu_m^2}\right). \quad (\text{D.3})$$

By performing similar steps used to arrive at (D.3) we obtain the upper bound

$$\|\mathbf{D}\mathbf{h}_0\|_2^2 \leq \|\mathbf{h}_0\|_2^2 \left(1 + \max\{\mu_a(n_x - 1), \mu_b(n_e - 1)\} + \sqrt{n_x n_e \mu_m^2}\right). \quad (\text{D.4})$$

Bounds depending on $w = n_x + n_e$. Both bounds in (D.3) and (D.4) are explicit in n_x and n_e . Since the individual sparsity levels n_x and n_e are unknown prior to recovery, a coherence-based RIP bound, which depends solely on the total number $w = n_x + n_e$ of nonzero entries of $\mathbf{h}_0$ rather than on n_x and n_e , is required. To this end, we define the function

$$g(n_x, n_e) = \max\{\mu_a(n_x - 1), \mu_b(n_e - 1)\} + \sqrt{n_x n_e \mu_m^2}$$

and find the maximum

$$\hat{g}(w) = \max_{0 \leq n_x \leq w} g(n_x, w - n_x). \quad (\text{D.5})$$

Since $\hat{g}(w)$ only depends on $w = n_x + n_e$ and $g(n_x, n_e) \leq \hat{g}(w)$, we can replace $g(n_x, n_e)$ by $\hat{g}(w)$ in both bounds (D.3) and (D.4).

We start by computing the maximum in (D.5). Assume $\mu_a(n_x - 1) \geq \mu_b(n_e - 1)$ and consider the function

$$g_a(n_x, w - n_x) = \mu_a(n_x - 1) + \sqrt{n_x(w - n_x) \mu_m^2}. \quad (\text{D.6})$$

It can easily be shown that $g_a(n_x, w - n_x)$ is strictly concave in n_x for all $0 \leq n_x \leq w$ and $0 \leq w < \infty$ and, therefore, the maximum is either achieved at a stationary point or a boundary point. Standard arithmetic manipulations show that the (global) maximum of the function in (D.6) corresponds to

$$\hat{g}_a(w) = \frac{1}{2} \left(\mu_a(w - 2) + w \sqrt{\mu_a^2 + \mu_m^2} \right). \quad (\text{D.7})$$

For the case where $\mu_a(n_x - 1) < \mu_b(n_e - 1)$, we carry out similar steps used to arrive at (D.6) and exploit the symmetry of (D.5) to arrive at

$$\hat{g}_b(w) = \frac{1}{2} \left(\mu_b(w - 2) + w \sqrt{\mu_b^2 + \mu_m^2} \right).$$

Hence, by assuming that $\mu_b \leq \mu_a$, we obtain upper and lower bounds on (D.3) and (D.4) in terms of $w = n_x + n_e$ with the aid of (D.7) as follows:

$$(1 - \hat{g}_a(w))\|\mathbf{h}_0\|_2^2 \leq \|\mathbf{D}\mathbf{h}_0\|_2^2 \leq (1 + \hat{g}_a(w))\|\mathbf{h}_0\|_2^2. \quad (\text{D.8})$$

It is important to realize that for some values of μ_a , μ_m , and w , the bounds in (D.8) are inferior to those obtained when ignoring the structure of the concatenated dictionary $\mathbf{D}$, i.e.,

$$(1 - \mu_d(w - 1))\|\mathbf{h}_0\|_2^2 \leq \|\mathbf{D}\mathbf{h}_0\|_2^2 \leq (1 + \mu_d(w - 1))\|\mathbf{h}_0\|_2^2. \quad (\text{D.9})$$

with $\mu_d = \max\{\mu_a, \mu_b, \mu_m\}$. However, for $w \geq 2$, $\mu_m = \mu_d$, and

$$\mu_a < \mu_m + \frac{\mu_m w}{2} \left(\sqrt{\frac{w-2}{w-1}} - 1 \right),$$

the RIP considering the structure of $\mathbf{D}$ in (D.8) turns out to be more tight than (D.9). For other values of w and/or μ_a , (D.8) turns out to be less tight than (D.9). In order to tighten the RIP in both cases, we consider

$$(1 - \hat{\delta}_w)\|\mathbf{h}_0\|_2^2 \leq \|\mathbf{D}\mathbf{h}_0\|_2^2 \leq (1 + \hat{\delta}_w)\|\mathbf{h}_0\|_2^2,$$

where the coherence-based upper bound on the RIC of the concatenated dictionary $\mathbf{D} = [\mathbf{A} \ \mathbf{B}]$ corresponds to

$$\hat{\delta}_w = \min \left\{ \frac{1}{2} \left(\mu_a(w - 2) + w \sqrt{\mu_a^2 + \mu_m^2} \right), \mu_d(w - 1) \right\}.$$

Appendix D.2. Recovery guarantee

We now bound the error $\|\mathbf{h}\|_2$ and derive the recovery guarantee by following the proof in Appendix A. In the following, we only show the case where

$$\frac{1}{2} \left(\mu_a(w - 2) + w \sqrt{\mu_a^2 + \mu_m^2} \right) \leq \mu_d(w - 1).$$

The other case, i.e., where the standard RIP (D.9) is tighter than (D.8), readily follows from the proof in Appendix A, by replacing $\mathbf{A}$ by $\mathbf{D}$, μ_a by μ_d , and n_x by w .

Bounding the error on the signal support. We start by bounding the error $\|\mathbf{h}_0\|_2$. Since $\mu_m \leq \mu_d$, we arrive at

$$\begin{aligned} |\mathbf{h}^H \mathbf{D}^H \mathbf{D} \mathbf{h}_0| &\geq |\mathbf{h}_0^H \mathbf{D}^H \mathbf{D} \mathbf{h}_0| - |(\mathbf{h} - \mathbf{h}_0)^H \mathbf{D}^H \mathbf{D} \mathbf{h}_0| \\ &\geq (1 - \hat{\delta}_w) \|\mathbf{h}_0\|_2^2 - \mu_d w \|\mathbf{h}_0\|_2^2 - \mu_d \sqrt{w} \|\mathbf{h}_0\|_2 e_0 \\ &= d \|\mathbf{h}_0\|_2^2 - \mu_d \sqrt{w} \|\mathbf{h}_0\|_2 e_0 \end{aligned}$$

with

$$d \triangleq 1 - \delta_w - \mu_d w = 1 - \frac{w}{2} \left(\mu_a + 2\mu_d + \sqrt{\mu_a^2 + \mu_m^2} \right) + \mu_a.$$

It is important to note that d is crucial for the recovery guarantee as it determines the condition for which BP-SEP in (13) enables stable separation. Specifically, if $d > 0$ or, equivalently, if

$$w < \frac{2(1 + \mu_a)}{\mu_a + 2\mu_d + \sqrt{\mu_a^2 + \mu_m^2}}$$

then the error on the signal support $\|\mathbf{h}_0\|_2$ is bounded from above as

$$\|\mathbf{h}_0\|_2 \leq \frac{(\varepsilon + \eta) \sqrt{1 + \hat{\delta}_w} + \mu_d \sqrt{w} e_0}{d}.$$

where $e_0 = 2\|\mathbf{w} - \mathbf{w}_{\mathcal{W}}\|_1$ with $\mathcal{W} = \text{supp}_w(\mathbf{w})$.

Bounding the recovery error. Analogously to the derivation in Appendix A.3, we now compute an upper bound on $\|\mathbf{h}\|_2$, i.e.,

$$\|\mathbf{D}\mathbf{h}\|_2^2 \geq (1 + \mu_d) \|\mathbf{h}\|_2^2 - \mu_d \|\mathbf{h}\|_1^2. \quad (\text{D.10})$$

Finally, bounding $\|\mathbf{h}\|_2$ similarly to Appendix A.3 results in

$$\begin{aligned} \|\mathbf{h}\|_2 &\leq (\varepsilon + \eta) \frac{d + 2\sqrt{\mu_d w} \sqrt{1 + \hat{\delta}_w}}{\sqrt{1 + \mu_d d}} + e_0 \frac{\sqrt{\mu_d}(d + 2\mu_d w)}{\sqrt{1 + \mu_d d}} \\ &= C_7(\eta + \varepsilon) + C_8 \|\mathbf{w} - \mathbf{w}_{\mathcal{W}}\|_1. \end{aligned}$$

where the constants C_7 and C_8 depend on the parameters μ_a , μ_b , μ_m , μ_d , and $w = n_x + n_e$, which concludes the proof.

Acknowledgments

The authors would like to thank C. Aubel, H. Bölcskei, P. Kuppinger, and G. Pope for inspiring discussions. This work was supported by the Swiss National Science Foundation (SNSF) under Grant PA00P2-134155 and by the Grants NSF CCF-0431150, CCF-0728867, CCF-0926127, DARPA/ONR N66001-08-1-2065, N66001-11-1-4090, N66001-11-C-4092, ONR N00014-08-1-1112, N00014-10-1-0989, AFOSR FA9550-09-1-0432, ARO MURIs W911NF-07-1-0185 and W911NF-09-1-0383, and by the Texas Instruments Leadership University Program.

References

- [1] S. V. Vaseghi, R. Frayling-Cork, Restoration of old gramophone recordings, *J. Audio Eng. Soc* 40 (10) (1992) 791–801.
- [2] S. J. Godsill, P. J. W. Rayner, *Digital Audio Restoration – A Statistical Model-Based Approach*, Springer-Verlag London, 1998.
- [3] M. Elad, Y. Hel-Or, Fast super-resolution reconstruction algorithm for pure translational motion and common space-invariant blur, *IEEE Trans. Image Proc.* 10 (8) (2001) 1187–1193.
- [4] M. Elad, J.-L. Starck, P. Querre, D. L. Donoho, Simultaneous cartoon and texture image inpainting using morphological component analysis (MCA), *Appl. Comput. Harmon. Anal.* 19 (2005) 340–358.
- [5] W. Göbel, F. Helmchen, In vivo calcium imaging of neural network function, *Physiology* 22 (6) (2007) 358–365.
- [6] J.-F. Cai, S. Osher, Z. Shen, Split Bregman methods and frame based image restoration, *Multiscale Model. Simul.* 8 (2) (2009) 337–369.
- [7] J. N. Laska, P. Boufounos, M. A. Davenport, R. G. Baraniuk, Democracy in action: Quantization, saturation, and compressive sensing, *Appl. Comput. Harmon. Anal.*, to appear (2011).
- [8] M. Bertalmio, G. Sapiro, V. Caselles, C. Ballester, Image inpainting, *Proc. of 27th Ann. Conf. Comp. Graph. Int. Tech.* (2000) 417–424.

- [9] A. Wong, J. Orchard, A nonlocal-means approach to exemplar-based inpainting, in: Proc. of IEEE ICIP, San Diego, CA, USA, 2008, pp. 2600–2603.
- [10] J.-F. Cai, R. H. Chan, L. Shen, Z. Shen, Simultaneously inpainting in image and transformed domains, *Num. Mathematik* 112 (4) (2009) 509–533.
- [11] J. N. Laska, M. A. Davenport, R. G. Baraniuk, Exact signal recovery from sparsely corrupted measurements through the pursuit of justice, Proc. of 43rd Asilomar Conf. on Signals, Systems, and Comput., Pacific Grove, CA, USA (2009) 1556–1560.
- [12] C. Studer, P. Kuppinger, G. Pope, H. Bölcskei, Recovery of sparsely corrupted signals, *IEEE Trans. Inf. Theory*, submitted to (2011).
- [13] R. E. Carrillo, K. E. Barner, T. C. Aysal, Robust sampling and reconstruction methods for sparse signals in the presence of impulsive noise, *IEEE J. Sel. Topics Sig. Proc.* 4 (2) (2010) 392–408.
- [14] C. Novak, C. Studer, A. Burg, G. Matz, The effect of unreliable LLR storage on the performance of MIMO-BICM, Proc. of 44th Asilomar Conf. on Signals, Systems, and Comput., Pacific Grove, CA, USA.
- [15] S. G. Mallat, G. Yu, Super-resolution with sparse mixing estimators, *IEEE Trans. Image Proc.* 19 (11) (2010) 2889–2900.
- [16] A. Adler, V. Emiya, M. G. Jafari, M. Elad, R. Gribonval, M. D. Plumbley, A constrained matching pursuit approach to audio declipping, in: Proc. of IEEE Int. Conf. Acoustics, Speech, and Sig. Proc. (ICASSP), Prague, Czech Republic, 2011.
- [17] A. Adler, V. Emiya, M. G. Jafari, M. Elad, R. Gribonval, M. D. Plumbley, Audio inpainting, *IEEE Trans. on Audio, Speech, and Language Processing*, to appear (2011).
- [18] P. Kuppinger, G. Durisi, H. Bölcskei, Uncertainty relations and sparse signal recovery for pairs of general signal sets, *IEEE Trans. Inf. Theory*, submitted to (2011).

- [19] S. S. Chen, D. L. Donoho, M. A. Saunders, Atomic decomposition by basis pursuit, *SIAM J. Sci. Comput.* 20 (1) (1998) 33–61.
- [20] D. L. Donoho, X. Huo, Uncertainty principles and ideal atomic decomposition, *IEEE Trans. Inf. Theory* 47 (7) (2001) 2845–2862.
- [21] D. L. Donoho, M. Elad, Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ_1 minimization, *Proc. Natl. Acad. Sci. USA* 100 (5) (2003) 2197–2202.
- [22] R. Gribonval, M. Nielsen, Sparse representations in unions of bases, *IEEE Trans. Inf. Theory* 49 (12) (2003) 3320–3325.
- [23] M. Elad, A. M. Bruckstein, A generalized uncertainty principle and sparse representation in pairs of bases, *IEEE Trans. Inf. Theory* 48 (9) (2002) 2558–2567.
- [24] J. A. Tropp, Greed is good: Algorithmic results for sparse approximation, *IEEE Trans. Inf. Theory* 50 (10) (2004) 2231–2242.
- [25] Y. C. Pati, R. Rezaifar, P. S. Krishnaprasad, Orthogonal matching pursuit: Recursive function approximation with applications to wavelet decomposition, *Proc. of 27th Asilomar Conf. on Signals, Systems, and Comput.*, Pacific Grove, CA, USA (1993) 40–44.
- [26] G. Davis, S. G. Mallat, Z. Zhang, Adaptive time-frequency decompositions, *Opt. Eng.* 33 (7) (1994) 2183–2191.
- [27] D. L. Donoho, M. Elad, V. N. Temlyakov, Stable recovery of sparse overcomplete representations in the presence of noise, *IEEE Trans. Inf. Theory* 52 (1) (2006) 6–18.
- [28] J. J. Fuchs, Recovery of exact sparse representations in the presence of bounded noise, *IEEE Trans. Inf. Theory* 51 (10) (2005) 3601–2608.
- [29] J. A. Tropp, Just relax: Convex programming methods for identifying sparse signals in noise, *IEEE Trans. Inf. Theory* 52 (3) (2006) 1030–1051.
- [30] Z. Ben-Haim, Y. C. Eldar, M. Elad, Coherence-based performance guarantees for estimating a sparse vector under random noise, *IEEE Trans. Sig. Proc.* 58 (10) (2010) 5030–5043.

- [31] T. T. Cai, L. Wang, G. Xu, Stable recovery of sparse signals and an oracle inequality, *IEEE Trans. Inf. Theory* 56 (7) (2010) 3516–3522.
- [32] R. G. Baraniuk, V. Cevher, M. F. Duarte, C. Hedge, Model-based compressive sensing, *IEEE Trans. Inf. Theory* 56 (4) (2010) 1982–2001.
- [33] D. L. Donoho, P. B. Stark, Uncertainty principles and signal recovery, *SIAM J. Appl. Math* 49 (3) (1989) 906–931.
- [34] E. J. Candès, T. Tao, Decoding by linear programming, *IEEE Trans. Inf. Theory* 51 (12) (2005) 4203–4215.
- [35] J. Wright, Y. Ma, Dense error correction via ℓ_1 -minimization, *IEEE Trans. Inf. Theory* 56 (7) (2010) 3540–3560.
- [36] N. H. Nguyen, T. D. Tran, Exact recoverability from dense corrupted observations via ℓ_1 minimization, *arXiv:1102.1227v1*.
URL <http://arxiv.org/abs/1102.1227v1>
- [37] X. Li, Compressed sensing and matrix completion with constant proportion of corruptions, *arXiv:1104.1041v1*.
URL <http://arxiv.org/abs/1104.1041v1>
- [38] M. Hestenes, E. Stiefel, Methods of conjugate gradients for solving linear systems, *J. Res. Nat. Bur. Stand.* 49 (6) (1952) 409–436.
- [39] G. Pope, H. Bölcskei, A unified theory for sparse signal recovery in Hilbert spaces: Deterministic thresholds, in preparation (2011).
- [40] D. L. Donoho, G. Kutyniok, Microlocal analysis of the geometric separation problem, *arXiv:1004.3006v1*.
URL <http://arxiv.org/abs/1004.3006v1>
- [41] G. Kutyniok, Data separation by sparse representations, *arXiv:1102.4527v1*.
URL <http://arxiv.org/abs/1102.4527v1>
- [42] A. Feuer, A. Nemirovski, On sparse representations in pairs of bases, *IEEE Trans. Inf. Theory* 49 (6) (2003) 1579–1581.

- [43] E. J. Candès, J. Romberg, T. Tao, Stable signal recovery from incomplete and inaccurate measurements, *Comm. Pure Appl. Math.* 59 (2006) 1207–1223.
- [44] E. J. Candès, The restricted isometry property and its implications for compressed sensing, *C. R. Acad. Sci. Paris, Ser. I* 346 (2008) 589–592.
- [45] T. T. Cai, G. Xu, J. Zhang, On recovery of sparse signals via ℓ_1 minimization, in: *IEEE Trans. Inf. Theory*, Vol. 55, 2009, pp. 3388–3397.
- [46] T. T. Cai, L. Wang, G. Xu, Shifting inequality and recovery of sparse signals, *IEEE Trans. Inf. Theory* 58 (3) (2010) 1300–1308.
- [47] T. T. Cai, L. Wang, G. Xu, New bounds for restricted isometry constants, *IEEE Trans. Inf. Theory* 56 (9) (2010) 4388–4394.
- [48] D. L. Donoho, Compressed sensing, *IEEE Trans. Inf. Theory* 52 (4) (2006) 1289–1306.
- [49] E. J. Candès, M. B. Wakin, An introduction to compressive sampling, *IEEE Sig. Proc. Mag.* 25 (2) (2008) 21–30.
- [50] J. A. Tropp, On the conditioning of random subdictionaries, *Appl. Comput. Harmon. Anal.* 25 (2008) 1–24.
- [51] R. G. Baraniuk, Goodbye, textbooks; hello, open-source learning, [online accessed on July 1, 2011] URL <http://www.ted.com/talks> (Feb. 2006).
- [52] C. Févotte, S. J. Godsill, A bayesian approach for blind separation of sparse sources, *IEEE Trans. on Audio, Speech, and Language Processing* 14 (6) (2006) 2174–2188.
- [53] S. J. Godsill, A. Cemgil, C. Févotte, P. J. Wolfe, Bayesian computational methods for sparse audio and music processing, in: *Proc. EUSIPCO*, Poznan, Poland, 2007, pp. 345–349.
- [54] National Aeronautics and Space Administration, Apollo 11: Orbital view, west of daedalus crater, [online accessed on Jun 6, 2011:] URL http://www.apolloarchive.com/apollo_gallery.html (July 1969).
- [55] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge Press, New York, NY, 1985.

- [56] G. H. Golub, C. F. van Loan, Matrix Computations, 3rd Ed., The Johns Hopkins Univ. Press, 1969.