

**EXISTENCE, UNIQUENESS AND STABILITY OF SOLUTIONS OF
GENERALIZED TIKHONOV-PHILLIPS FUNCTIONALS**

By

G.L. Mazzieri

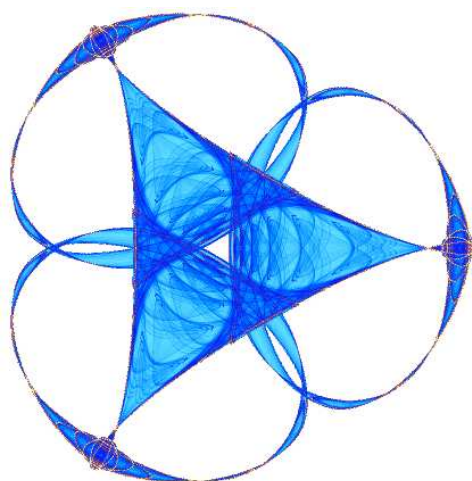
R.D. Spies

and

K.G. Temperini

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455-0436

Phone: 612/624-6066 Fax: 612/626-7370

URL: <http://www.ima.umn.edu>

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Existence, uniqueness and stability of solutions of generalized Tikhonov-Phillips functionals

G. L. Mazziere^{1,2}, R. D. Spies^{1,3} and K. G. Temperini^{1,4}

(1) Instituto de Matemática Aplicada del Litoral, IMAL, CONICET-UNL, Güemes 3450, S3000GLN, Santa Fe, Argentina.

(2) Departamento de Matemática, Facultad de Bioquímica y Ciencias Biológicas, Universidad Nacional del Litoral, Santa Fe, Argentina.

(3) Departamento de Matemática, Facultad de Ingeniería Química, Universidad Nacional del Litoral, Santa Fe, Argentina.

(4) Departamento de Matemática, Facultad de Humanidades y Ciencias, Universidad Nacional del Litoral, Santa Fe, Argentina.

E-mail: gmazzieri@hotmail.com, rspies@santafe-conicet.gov.ar, ktemperini@santafe-conicet.gov.ar

Abstract. The Tikhonov-Phillips method is widely used for regularizing ill-posed inverse problems mainly due to the simplicity of its formulation as an optimization problem. The use of different penalizers in the functionals associated to the corresponding optimization problems has originated a variety other methods which can be considered as “variants” of the traditional Tikhonov-Phillips method of order zero. Such is the case for instance of the Tikhonov-Phillips method of order one, the total variation regularization method, etc. In this article we find sufficient conditions on the penalizers in generalized Tikhonov-Phillips functionals which guarantee existence and uniqueness and stability of the minimizers. The particular cases in which the penalizers are given by the bounded variation norm, by powers of seminorms and by linear combinations of powers of seminorms associated to closed operators, are studied. Several examples are presented and a few results on image restoration are shown.

Keywords: Inverse problem, Ill-Posed, Regularization, Tikhonov-Phillips.

AMS Subject classifications: 47A52, 65J20

1. Introduction

In a quite general framework an inverse problem can be formulated as the need for determining x in an equation of the form

$$Tx = y, \quad (1)$$

where T is a linear bounded operator between two infinite dimensional Hilbert spaces X and Y (in general these will be function spaces), the range of T , $\mathcal{R}(T)$, is non-closed and y is the data, supposed to be known, perhaps with a certain degree of error. It is well known that under these hypotheses, problem (1) is ill-posed in the sense of Hadamard ([11]). In this case the ill-posedness is a result of the unboundedness of $T^\dagger$, the Moore-Penrose generalized inverse of T . The Moore-Penrose generalized inverse is a fundamental tool in the treatment of inverse ill-posed problems and their regularized solutions, mainly due to the fact that this operator is strongly related to the least-squares solutions of problem (1). In fact, the least-squares solution of minimum norm of problem (1), also known as the best approximate solution, is $x^\dagger \doteq T^\dagger y$, which exists if and only if $y \in \mathcal{D}(T^\dagger) = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$. Moreover, for $y \in \mathcal{D}(T^\dagger)$, the set of all least-squares solutions of problem (1) is given by $x^\dagger + \mathcal{N}(T)$, where $\mathcal{N}(T)$ denotes the null space of the operator T .

The unboundedness of $T^\dagger$ has as undesired consequence the fact that small errors or noise in the data y can result in arbitrarily large errors in the corresponding approximated solutions (see [18], [17]), turning unstable all standard numerical approximation methods, making them unsuitable for most applications and inappropriate from any practical point of view. The so called “regularization methods” are mathematical tools designed to restore stability to the inversion process and consist essentially of parametric families of continuous linear operators approximating $T^\dagger$. The mathematical theory of regularization methods is very wide (a comprehensive treatise on the subject can be found in the book by Engl, Hanke and Neubauer, [9]) and it is of great interest in a broad variety of applications in many areas such as Medicine, Physics, Geology, Geophysics, Biology, image restaurarion and processing, etc.

There exist numerous ways of regularizing an ill-posed inverse problem. Among the most standard and traditional methods we mention the Tikhonov-Phillips method ([15], [19], [20]), truncated singular value decomposition (TSVD), Showalter’s method, total variation regularization ([1]), etc. Among all regularization methods, probably the best known and most commonly and widely used is the Tikhonov-Phillips regularization method, which was originally proposed by Tikhonov and Phillips in 1962 and 1963 (see [15], [19], [20]). Although this method can be formalized within a very general framework by means of spectral theory ([9], [7]), the widespread of its use is undoubtedly due to the fact that it can also be formulated in a very simple way as an optimization problem. In fact, the regularized solution of problem (1) obtained by applying Tikhonov-Phillips method is the minimizer x_α of the functional

$$J_\alpha(x) \doteq \|Tx - y\|^2 + \alpha \|x\|^2, \quad (2)$$

where α is a positive constant known as the regularization parameter.

The penalizing term $\alpha \|x\|^2$ in (2) not only induces stability but it also determines certain regularity properties of the approximating regularized solutions x_α and of the corresponding least-squares solution which they approximate as $\alpha \rightarrow 0^+$. Thus, for instance, it is well known that minimizers of (2) are always “smooth” and, for $\alpha \rightarrow 0^+$, they approximate the least-squares solution of minimum norm of (1), that is $\lim_{\alpha \rightarrow 0^+} x_\alpha = T^\dagger y$. This method is more precisely known as the Tikhonov-Phillips method of order zero. Choosing other penalizing terms gives rise to different approximations with different properties, approximating different least-squares solutions of (1). Thus, for instance, the use of $\|\nabla x\|^2$ as penalizer instead of $\|x\|^2$ in (2) originates the so called Tikhonov-Phillips method of order one, the penalizer $\|x\|_{\text{BV}}$ (where $\|\cdot\|_{\text{BV}}$ denotes the bounded variation norm) gives rise to the so called bounded variation regularization method introduced by Acar and Vogel in 1994 ([1]), etc. In particular, in the latter case, the approximating solutions are only forced to be of bounded variation rather than smooth and they approximate, for $\alpha \rightarrow 0^+$, the least-squares solution of problem (1) of minimum $\|\cdot\|_{\text{BV}}$ -norm (see [1]). This method has been proved to be a good choice, for instance, in certain image restoration problems in which it is highly desirable to detect and preserve sharp edges and discontinuities of the original image.

Hence, the penalizing term in (2) is used not only to stabilize the inversion of the ill-posed problem but also to enforce certain characteristics on the approximating solutions and on the particular limiting least-squares solution that they approximate. As a consequence, it is reasonable to assume that an adequate choice of the penalizing term, based on *a-priori* knowledge about certain characteristics of the exact solution of problem (1), will lead to approximated “regularized” solutions which will appropriately reflect those characteristics.

With the above considerations in mind, we shall consider functionals of the form

$$J_{W,\alpha}(x) \doteq \|Tx - y\|^2 + \alpha W(x) \quad x \in \mathcal{D}, \quad (3)$$

where $W(\cdot)$ is an arbitrary functional with domain $\mathcal{D} \subset \mathcal{X}$ and α is a positive constant.

The purpose of this article is to find sufficient conditions on the penalizers in generalized Tikhonov-Phillips functionals of the form (3) which guarantee existence and uniqueness and stability of the minimizers. The particular cases in which the penalizers are given by the bounded variation norm, by powers of seminorms and by linear combinations of powers of seminorms associated to closed operators, are studied. Several examples are presented and a few results on image restoration are shown.

2. Existence and uniqueness for general penalizing terms

In this section we shall consider the problem of finding conditions on the penalizer $W(\cdot)$ which guarantee existence and uniqueness of global minimizers of (3). Previously we will need to introduce a few definitions.

Definition 2.1. Let $\mathcal{X}$ be a vector space, W a functional defined over a set $\mathcal{D} \subset \mathcal{X}$ and A a subset of $\mathcal{D}$. We say that A is W -bounded if there exists a constant $k < \infty$ such that $|W(a)| \leq k$ for every $a \in A$.

Definition 2.2. (W -coercivity) Let $\mathcal{X}$ be a vector space and W, F two functionals defined on a set $\mathcal{D} \subset \mathcal{X}$. We say that the functional F is W -coercive if $\lim_{n \rightarrow \infty} F(x_n) = +\infty$ for every sequence $\{x_n\} \subset \mathcal{D}$ for which $\lim_{n \rightarrow \infty} W(x_n) = +\infty$.

Remark 2.3. Note that if the functional F is W -coercive and W is bounded from below, then all lower level sets for F , i.e. all sets of the form $\{x \in \mathcal{D} : F(x) \leq a\}$ with $a \in \mathbb{R}$ are W -bounded sets.

Definition 2.4. Let $\mathcal{X}$ be a normed vector space, W, F two functionals with $\text{Dom}(F) \subset \text{Dom}(W) \subset \mathcal{X}$. We say that F is W -subsequentially (weakly) lower semicontinuous if for every W -bounded sequence $\{x_n\} \subset \text{Dom}(F)$ such that $x_n \xrightarrow{(w)} x \in \text{Dom}(F)$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $F(x) \leq \liminf_{j \rightarrow \infty} F(x_{n_j})$. If F is W -subsequentially lower semicontinuous we will simply say that F is W -sls. Similarly, if F is W -subsequentially weakly lower semicontinuous we will say that F is W -swls.

In the following theorem, sufficient conditions on the operator T and on the functional W guaranteeing the existence and uniqueness of the minimizer of the functional (3) are established.

Theorem 2.5. (Existence and uniqueness) Let $\mathcal{X}, \mathcal{Y}$ be normed vector spaces, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $y \in \mathcal{Y}$, $\mathcal{D} \subset \mathcal{X}$ a convex set and $W : \mathcal{D} \rightarrow \mathbb{R}$ a functional bounded from below, W -subsequentially weakly lower semicontinuous, and such that W -bounded sets are relatively weakly compact in $\mathcal{X}$. More precisely, suppose that W satisfies the following hypotheses:

- (H1): $\exists \gamma \geq 0$ such that $W(x) \geq -\gamma \quad \forall x \in \mathcal{D}$.
- (H2): for every W -bounded sequence $\{x_n\} \subset \mathcal{D}$ such that $x_n \xrightarrow{w} x \in \mathcal{D}$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $W(x) \leq \liminf_{j \rightarrow \infty} W(x_{n_j})$.
- (H3): for every W -bounded sequence $\{x_n\} \subset \mathcal{D}$ there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and $x \in \mathcal{D}$ such that $x_{n_j} \xrightarrow{w} x$.

Then the functional $J_{W,\alpha}(\cdot)$ in (3) has a global minimizer. If moreover W is convex and T is injective or W is strictly convex, then such a minimizer is unique.

Proof. First we note that for every sequence $\{z_n\} \subset \mathcal{D}$ we have that

$$z_n \xrightarrow{w} z \implies \|Tz - y\|^2 \leq \liminf_{n \rightarrow \infty} \|Tz_n - y\|^2. \quad (4)$$

This follows immediately from the continuity of T and the weak lower semicontinuity of the norm.

Let now $\{x_n\} \subset \mathcal{D}$ be such that

$$J_{W,\alpha}(x_n) \rightarrow \inf_{x \in \mathcal{D}} J_{W,\alpha}(x) \doteq J_{\min}. \quad (5)$$

Hypothesis (H1) guarantees that $-\infty < J_{\min} < +\infty$. From the definition of $J_{W,\alpha}(\cdot)$ and since $\alpha > 0$ it follows that $J_{W,\alpha}(\cdot)$ is W -coercive. Suppose now that the sequence $\{x_n\}$ is not W -bounded. Then, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $W(x_{n_j}) \rightarrow \infty$, from which, by virtue of the W -coercivity of $J_{W,\alpha}(\cdot)$ it follows that $J_{W,\alpha}(x_{n_j}) \rightarrow \infty$. This contradicts (5). Thus the sequence $\{x_n\}$ is W -bounded. It then follows by hypothesis (H3) that there must exist a sequence $\{x_{n_j}\} \subset \{x_n\}$ and $\bar{x} \in \mathcal{D}$ such that $x_{n_j} \xrightarrow{w} \bar{x}$ and since W satisfies (H2) there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ such that

$$W(\bar{x}) \leq \liminf_{k \rightarrow \infty} W(x_{n_{j_k}}). \quad (6)$$

Then

$$\begin{aligned} J_{W,\alpha}(\bar{x}) &= \|T\bar{x} - y\|^2 + \alpha W(\bar{x}) \\ &\leq \liminf_{k \rightarrow \infty} \|Tx_{n_{j_k}} - y\|^2 + \alpha \liminf_{k \rightarrow \infty} W(x_{n_{j_k}}) \quad (\text{by (4) and (6)}) \\ &\leq \liminf_{k \rightarrow \infty} \left(\|Tx_{n_{j_k}} - y\|^2 + \alpha W(x_{n_{j_k}}) \right) \quad (\text{by prop. of liminf}) \\ &= \liminf_{k \rightarrow \infty} J_{W,\alpha}(x_{n_{j_k}}) \quad (\text{by def. of } J_{W,\alpha}) \\ &= \lim_{n \rightarrow \infty} J_{W,\alpha}(x_n) \quad (\text{by (5) and since } \{x_{n_{j_k}}\} \text{ is a subseq. of } \{x_n\}) \\ &= J_{\min}. \end{aligned}$$

It then follows that $J_{W,\alpha}(\bar{x}) = J_{\min}$. This proves the existence of a global minimizer of (3). For the uniqueness, note that under the hypothesis that W be convex and T be injective or W be strictly convex, one has that the functional $J_{W,\alpha}(\cdot)$ is strictly convex and therefore the global minimizer is unique. $\blacksquare$

Remark 2.6. Note that in the previous theorem the convexity of $\mathcal{D}$ is not needed for the existence. Note also that if we replace hypotheses (H2) and (H3) on the functional W by the assumptions that W be W -sls and that W -bounded sets be relatively compact in $\mathcal{X}$, i.e. by the following hypotheses:

- (H2'): for every W -bounded sequence $\{x_n\} \subset \mathcal{D}$ such that $x_n \rightarrow x \in \mathcal{D}$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $W(x) \leq \liminf_{j \rightarrow \infty} W(x_{n_j})$;
- (H3'): for every W -bounded sequence $\{x_n\} \subset \mathcal{D}$ there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and $x \in \mathcal{D}$ such that $x_{n_j} \rightarrow x$,

then both existence and uniqueness remain valid.

Remark 2.7. Note that hypothesis (H3') is stronger than (H3) which, in turn, is stronger than the hypothesis that every W -bounded set be weakly precompact. Also, (H2') is weaker than (H2) which in turn is weaker than the hypothesis that W be weakly lower semicontinuous.

Remark 2.8. If $\mathcal{X}$ is a reflexive Banach space and $W(\cdot)$ is a norm defined on a subspace $\mathcal{D}$ of $\mathcal{X}$, which is on $\mathcal{D}$ equivalent or stronger than the norm of $\mathcal{X}$, then it follows that W satisfies hypothesis (H1), (H2), (H3) and therefore the functional (3) has a global minimizer on $\mathcal{D}$. If moreover T is injective or the normed space $(\mathcal{D}, W(\cdot))$ is complete and separable or Hilbert, then such a minimizer is unique.

Observe that hypothesis (H1), (H2) and (H3) as well as (H2') and (H3') impose conditions only on the penalizer $W(\cdot)$ and not on T , so that the corresponding existence and uniqueness results hold for any bounded linear operator T . It is therefore not surprising that those conditions can be relaxed if some information on T in connection to $W(\cdot)$ is provided. The next theorem shows a result in this direction.

Theorem 2.9. Let $\mathcal{X}, \mathcal{Y}$ be normed spaces, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $\mathcal{D} \subset \mathcal{X}$ a convex set and W a real functional on $\mathcal{D}$. Consider the following standing hypotheses:

- (I2): W is T - W -sls, i.e. for every sequence $\{x_n\} \subset \mathcal{D}$ such that $\{\|Tx_n\| + W(x_n)\}$ is bounded in $\mathbb{R}$ (in the sequel we shall refer to such a sequence as a “ T - W bounded sequence”) and $x_n \xrightarrow{w} x \in \mathcal{D}$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $W(x) \leq \liminf_{j \rightarrow \infty} W(x_{n_j})$.
- (I3): T - W -bounded sets are relatively weakly compact in $\mathcal{X}$, i.e., for every T - W -bounded sequence $\{x_n\} \subset \mathcal{D}$ there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and $x \in \mathcal{D}$ such that $x_{n_j} \xrightarrow{w} x$.

If T and $W(\cdot)$ satisfy the hypotheses (H1), (I2) and (I3), then the functional $J_{W,\alpha}(\cdot)$ in (3) has a global minimizer. If moreover W is convex and T is injective or W is strictly convex, then such a minimizer is unique.

Proof. Let $\{x_n\}$ be a minimizing sequence of $J_{W,\alpha}(\cdot)$. From the definition of $J_{W,\alpha}(\cdot)$ it follows that $\{x_n\}$ is T - W -bounded. Then by (I3) there must exist $\{x_{n_j}\} \subset \{x_n\}$ and $\bar{x} \in \mathcal{D}$ such that $x_{n_j} \xrightarrow{w} \bar{x}$. Now by virtue of (I2) there exists $\{x_{n_{j_k}}\} \subset \{x_{n_j}\}$ such that $W(\bar{x}) \leq \liminf_{k \rightarrow \infty} W(x_{n_{j_k}})$. Following now the same steps as in Theorem 2.5 we obtain that

$$J_{W,\alpha}(\bar{x}) = \min_{x \in \mathcal{D}} J_{W,\alpha}(x).$$

If W is convex and T is injective or $W(\cdot)$ is strictly convex, uniqueness follows from the strict convexity of $J_{W,\alpha}(\cdot)$ on $\mathcal{D}$. ■

Remark 2.10. Note that hypotheses (I2) and (I3) are weaker than (H2) and (H3), respectively. Also note that both (I2) and (I3) hold, for instance if $\mathcal{X}$ is reflexive, $W(\cdot)$ is subsequentially weakly lower semicontinuous and T and W are complemented, i.e. there exists a positive constant c such that $\|Tx\|^2 + W(x) \geq c\|x\|^2 \forall x \in \mathcal{D}$.

Remark 2.11. Just like in Theorem 2.5, in Theorem 2.9, the convexity of $\mathcal{D}$ is not needed for the existence. Also note that if hypothesis (I2) and (I3) are replaced by the assumption that W be T - W -sls and that T - W -bounded sets be relatively compact in $\mathcal{X}$, i.e. by the hypotheses:

- (I2'): for every T - W -bounded sequence $\{x_n\} \subset \mathcal{D}$ such that $x_n \rightarrow x \in \mathcal{D}$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $W(x) \leq \liminf_{j \rightarrow \infty} W(x_{n_j})$;
- (I3'): for every T - W -bounded sequence $\{x_n\} \subset \mathcal{D}$ there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and $x \in \mathcal{D}$ such that $x_{n_j} \rightarrow x$,

then the results of Theorem 2.9 remain valid.

3. Stability

As it was previously mentioned, inverse ill-posed problems appear in a wide variety of applications in diverse areas. Solving these problems usually involves several steps starting from modeling, through measurements and data acquisition for the experiment under study, to the discretization of the mathematical model and the derivation of numerical approximations for the regularized solutions. All these steps entail intrinsic errors, many of which are unavoidable. For this reason, in the context of the study of inverse ill-posed problems from the optic of Tikhonov-Phillips methods with general penalizing terms, it is of particular interest to analyze the stability of the minimizers of the functional (3) under different types of perturbations. To proceed with some results in this direction we shall need the following definitions.

Definition 3.1. (*W-coercivity*) Let $\mathcal{X}$ be a vector space, $W, F_n, n = 1, 2, \dots$, functionals defined on a set $\mathcal{D} \subset \mathcal{X}$. We will say that the sequence $\{F_n\}$ is W -coercive if $\lim_{n \rightarrow \infty} F_n(x_n) = +\infty$ for every sequence $\{x_n\} \subset \mathcal{D}$ for which $\lim_{n \rightarrow \infty} W(x_n) = +\infty$.

Definition 3.2. (*consistency*) Let $\mathcal{X}$ be a vector space and $W, F, F_n, n = 1, 2, \dots$, functionals defined on a set $\mathcal{D} \subset \mathcal{X}$. We will say that the sequence $\{F_n\}$ is consistent for F if $F_n x \rightarrow Fx$ for every $x \in \mathcal{D}$. We will say that the sequence $\{F_n\}$ is W -uniformly consistent for F if $F_n x \rightarrow Fx$ uniformly on every W -bounded set, that is if for any given $c > 0$ and $\epsilon > 0$, there exists $N = N(c, \epsilon)$ such that $|F_n(x) - F(x)| < \epsilon$ for every $n \geq N$ and every $x \in \mathcal{D}$ such that $|W(x)| \leq c$.

In the following theorem we present a weak stability result for the minimizers of a general functional on a normed space.

Theorem 3.3. Let $\mathcal{X}$ be a normed vector space, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \rightarrow \mathbb{R}$ a functional satisfying the hypotheses (H1) and (H3) of Theorem 2.5 (i.e. there exists $\gamma > 0$ such that $W(x) \geq -\gamma$ for every $x \in \mathcal{D}$ and every W -bounded sequence contains a weakly convergent subsequence with limit in $\mathcal{D}$), $J, J_n, n = 1, 2, \dots$, functionals on $\mathcal{D}$ such that J is W -swls and $\{J_n\}$ is W -coercive and W -uniformly consistent for J . Assume further that there exists a unique global minimizer $\bar{x} \in \mathcal{D}$ of J and that each functional J_n also possesses on $\mathcal{D}$ a global minimizer x_n (not necessarily unique). Then $x_n \xrightarrow{w} \bar{x}$.

Proof. Since for each $n \in \mathbb{N}$, x_n minimizes the functional J_n we have that $J_n(x_n) \leq J_n(\bar{x})$. Then

$$\limsup_{n \rightarrow \infty} J_n(x_n) \leq \limsup_{n \rightarrow \infty} J_n(\bar{x}) = J(\bar{x}) < \infty, \quad (7)$$

where the equality follows from the hypothesis that the sequence $\{J_n\}$ is W -uniformly consistent for J . From (7), the hypothesis (H1) on W and the hypothesis of W -coercitivity of $\{J_n\}$ it then follows that the sequence $\{x_n\}$ is W -bounded.

Suppose now that the sequence $\{x_n\}$ does not converge weakly to $\bar{x}$. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that no subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ converges weakly to $\bar{x}$. On the other hand, since the sequence $\{x_{n_j}\}$ is W -bounded (since the original sequence is) hypothesis (H3) on W implies that there exist $x^* \in \mathcal{D}$ and a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_k}} \xrightarrow{w} x^*$. It then follows that $x^* \neq \bar{x}$.

On the other hand, since the sequence $\{x_{n_{j_k}}\}$ is W -bounded, $x_{n_{j_k}} \xrightarrow{w} x^*$ and J is W -swls, it follows that there exists a subsequence $\{x_{n_{j_{k_\ell}}}\} \subset \{x_{n_{j_k}}\}$ such that

$$J(x^*) \leq \liminf_{\ell \rightarrow \infty} J(x_{n_{j_{k_\ell}}}). \quad (8)$$

Also, since the sequence $\{x_{n_{j_{k_\ell}}}\}$ is W -bounded and $\{J_n\}$ is W -uniformly consistent for J , it follows that

$$\lim_{\ell \rightarrow \infty} \left(J(x_{n_{j_{k_\ell}}}) - J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) = 0. \quad (9)$$

Hence

$$\begin{aligned} J(x^*) &\leq \liminf_{\ell \rightarrow \infty} J(x_{n_{j_{k_\ell}}}) && \text{(by (8))} \\ &\leq \limsup_{\ell \rightarrow \infty} J(x_{n_{j_{k_\ell}}}) \\ &= \limsup_{\ell \rightarrow \infty} \left[\left(J(x_{n_{j_{k_\ell}}}) - J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) + J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right] \\ &\leq \limsup_{\ell \rightarrow \infty} \left(J(x_{n_{j_{k_\ell}}}) - J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) + \limsup_{\ell \rightarrow \infty} J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \\ &= \lim_{\ell \rightarrow \infty} \left(J(x_{n_{j_{k_\ell}}}) - J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) + \limsup_{\ell \rightarrow \infty} J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) && \text{(by (9))} \\ &= \limsup_{\ell \rightarrow \infty} J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) && \text{(by (9))} \\ &\leq J(\bar{x}). && \text{(by (7), since } \{x_{n_{j_{k_\ell}}}\} \subset \{x_n\}) \end{aligned}$$

Since $\bar{x}$ is the unique minimizer of J it follows that $x^* = \bar{x}$, contradicting our previous result that $x^* \neq \bar{x}$. This contradiction came from assuming that the sequence $\{x_n\}$ did not converge weakly to $\bar{x}$. Hence $x_n \xrightarrow{w} \bar{x}$ as we wanted to show. $\blacksquare$

Note that by virtue of Remark 2.7, the hypothesis that J be W -swls in the previous theorem can be replaced by the hypothesis that J be weakly lower semicontinuous on $\mathcal{D}$.

In the particular case in which the functionals J and J_n are of Tikhonov-Phillips type, under certain general conditions on the penalizer $W(\cdot)$, the previous theorem yields a weak stability result for the minimizers of the functional (3). In fact we have the following corollary.

Corollary 3.4. *Let $\mathcal{X}$ be a normed vector space, $\mathcal{Y}$ an inner product space, $T, T_n \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $n = 1, 2, \dots$, $y \in \mathcal{Y}$, $\alpha > 0$, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \rightarrow \mathbb{R}$ a functional satisfying hypotheses (H1), (H2) and (H3) of Theorem 2.5, $J, J_n, n = 1, 2, \dots$, functionals on $\mathcal{D}$ defined as follows:*

$$J(x) \doteq \|Tx - y\|^2 + \alpha W(x), \quad (10)$$

$$J_n(x) \doteq \|T_n x - y_n\|^2 + \alpha_n W(x), \quad (11)$$

such that as $n \rightarrow \infty$, $\alpha_n \rightarrow \alpha$, $y_n \rightarrow y$ and $T_n x \rightarrow Tx$ uniformly for x in W -bounded sets (i.e. $\{T_n\}$ is W -uniformly consistent for T). Suppose further that J has a unique global minimizer $\bar{x}$. If x_n is a global minimizer of J_n then $x_n \xrightarrow{w} \bar{x}$.

Proof. To prove this corollary it suffices to verify that the functionals J and J_n satisfy the hypotheses of Theorem 3.3, that is, verify that J is W -swls and that the sequence $\{J_n\}$ is W -coercive and W -uniformly consistent for J .

To prove that J is W -swls, let $\{x_n\} \subset \mathcal{D}$ be a W -bounded sequence such that $x_n \xrightarrow{w} x \in \mathcal{D}$. From the continuity of T and the weak lower semicontinuity of every norm, it follow immediately that

$$\|Tx - y\|^2 \leq \liminf_{n \rightarrow \infty} \|Tx_n - y\|^2. \quad (12)$$

On the other hand, by (H2) it follows that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$W(x) \leq \liminf_{j \rightarrow \infty} W(x_{n_j}). \quad (13)$$

Then,

$$\begin{aligned} J(x) &= \|Tx - y\|^2 + \alpha W(x) \\ &\leq \liminf_{j \rightarrow \infty} \|Tx_{n_j} - y\|^2 + \liminf_{j \rightarrow \infty} \alpha W(x_{n_j}) \text{ (by (12) y (13))} \\ &\leq \liminf_{j \rightarrow \infty} \{\|Tx_{n_j} - y\|^2 + \alpha W(x_{n_j})\} \quad \text{(by property of lim inf)} \\ &= \liminf_{j \rightarrow \infty} J(x_{n_j}). \end{aligned}$$

Hence J is W -swls.

Now we will prove that the sequence $\{J_n\}$ is W -coercive. For that let $\{x_n\} \subset \mathcal{D}$ such that $W(x_n) \rightarrow +\infty$. Observe that

$$J_n(x_n) = \|T_n x_n - y_n\|^2 + \alpha_n W(x_n) \geq \alpha_n W(x_n). \quad (14)$$

Since W satisfies (H1) and $\alpha_n \rightarrow \alpha > 0$, it follows immediately from (14) that $J_n(x_n) \rightarrow +\infty$. Hence $\{J_n\}$ is uniformly W -coercive.

Finally we will show that $\{J_n\}$ is W -uniformly consistent for J . For that let $M \subset \mathcal{D}$ be a W -bounded set. Since $\{T_n\}$ is W -uniformly consistent for T we have that $T_n x \rightarrow Tx$ uniformly on M and since $y_n \rightarrow y$, it follows that $\|T_n x - y_n\|^2 \rightarrow \|Tx - y\|^2$ uniformly on M . Finally, since

$$\begin{aligned} |J_n(x) - J(x)| &= | \|T_n x - y_n\|^2 + \alpha_n W(x) - \|Tx - y\|^2 - \alpha W(x) | \\ &\leq | \|T_n x - y_n\|^2 - \|Tx - y\|^2 | + |(\alpha_n - \alpha)|W(x)|, \end{aligned} \quad (15)$$

it follows that $J_n(x) \rightarrow J(x)$ uniformly for $x \in M$. Thus $\{J_n\}$ is W -uniformly consistent for J .

Since J and $\{J_n\}$ satisfy the hypotheses of Theorem 3.3, the corollary then follows. ■

Remark 3.5. Note that by virtue of Remark 2.8, the weak stability result of Corollary 3.4 holds if i) $\mathcal{X}$ is a reflexive Banach space, ii) the penalizer $W(\cdot)$ in (10) is a norm defined on a subspace $\mathcal{D}$ of $\mathcal{X}$ which is on $\mathcal{D}$ equivalent or stronger than the original norm in $\mathcal{X}$ and iii) T is injective or the space $(\mathcal{D}, W(\cdot))$ is a separable Banach space or a Hilbert space.

Hypotheses on Theorem 3.3 and Corollary 3.4 can be weakened if adequate information on the operator T is available. Before we proceed to the statements of the corresponding results, we shall need the following definitions.

Definition 3.6. (*T - W -coercivity*) Let $\mathcal{X}, \mathcal{Y}$ be vector spaces, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $W, F_n, n = 1, 2, \dots$, functionals defined on a set $\mathcal{D} \subset \mathcal{X}$. We will say that the sequence $\{F_n\}$ is T - W -coercive if $\lim_{n \rightarrow \infty} F_n(x_n) = +\infty$ for every sequence $\{x_n\} \subset \mathcal{D}$ for which $\lim_{n \rightarrow \infty} \|Tx_n\| + W(x_n) = +\infty$.

Definition 3.7. (*T - W -uniform consistency*) Let $\mathcal{X}, \mathcal{Y}$ be vector spaces, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $W, F, F_n, n = 1, 2, \dots$, functionals defined on a set $\mathcal{D} \subset \mathcal{X}$. We will say that the sequence $\{F_n\}$ is T - W -uniformly consistent for F if $F_n \rightarrow F$ uniformly on every T - W -bounded set, that is if for any given $c > 0$ and $\epsilon > 0$, there exists $N = N(c, \epsilon)$ such that $|F_n(x) - F(x)| < \epsilon$ for every $n \geq N$ and every $x \in \mathcal{D}$ such that $\|Tx\| + |W(x)| \leq c$.

Theorem 3.8. Let $\mathcal{X}, \mathcal{Y}$ be normed vector spaces, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \rightarrow \mathbb{R}$ a functional satisfying hypotheses (H1) of Theorem 2.5 and (I3) of Theorem 2.9 (i.e. there exists $\gamma > 0$ such that $W(x) \geq -\gamma$ for every $x \in \mathcal{D}$ and every T - W -bounded sequence contains a weakly convergent subsequence with limit in $\mathcal{D}$), $J, J_n, n = 1, 2, \dots$, functionals on $\mathcal{D}$ such that J is T - W -swls and $\{J_n\}$ is T - W -coercive and T - W -uniformly consistent for J . Assume further that there exists a unique global minimizer $\bar{x} \in \mathcal{D}$ of J and that each functional J_n also possesses on $\mathcal{D}$ a global minimizer x_n (not necessarily unique). Then $x_n \xrightarrow{w} \bar{x}$.

Proof. The proof follows like in Theorem 3.3 with the obvious modifications. ■

Corollary 3.9. *Let $\mathcal{X}$ be normed vector space, $\mathcal{Y}$ an inner product space, $T, T_n \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $n = 1, 2, \dots$, $y \in \mathcal{Y}$, $\alpha > 0$, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \rightarrow \mathbb{R}$ a functional satisfying hypotheses (H1) of Theorem 2.5 and (I2) and (I3) of Theorem 2.9, $J, J_n, n = 1, 2, \dots$, functionals on $\mathcal{D}$ defined as follows:*

$$\begin{aligned} J(x) &\doteq \|Tx - y\|^2 + \alpha W(x), \\ J_n(x) &\doteq \|T_n x - y_n\|^2 + \alpha_n W(x), \end{aligned}$$

such that as $n \rightarrow \infty$, $\alpha_n \rightarrow \alpha$, $y_n \rightarrow y$ and $T_n x \rightarrow Tx$ uniformly for x in W -bounded sets (i.e. $\{T_n\}$ is W -uniformly consistent for T). Suppose further that J has a unique global minimizer $\bar{x}$. If x_n is a global minimizer of J_n then $x_n \xrightarrow{w} \bar{x}$.

Proof. We will show that J and $\{J_n\}$ satisfy the hypotheses of Theorem 3.8. For that it suffices to show that J is T - W -swls and that $\{J_n\}$ is T - W coercive and T - W -uniformly consistent for J . That J is T - W -swls follows immediately from (I2) and the weak lower semicontinuity of every norm. The T - W -uniform consistency of $\{J_n\}$ for J follows exactly as in the proof of Corollary 3.4 by noting that $|J_n(x) - J(x)| \leq |||T_n x - y_n|||^2 - |||Tx - y|||^2| + |(\alpha_n - \alpha)| |W(x)|$ and using the fact that T - W -bounded sets are also W -bounded. Finally, the T - W -coercivity of $\{J_n\}$ follows easily from the W -uniform consistency of $\{T_n\}$ for T . $\blacksquare$

Next we present a strong stability result for the minimizers of general functionals on a normed space.

Theorem 3.10. *Let $\mathcal{X}$ be a normed vector space, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \rightarrow \mathbb{R}$ a functional satisfying hypotheses (H1) of Theorem 2.5 and (H3') of Remark 2.6 (i.e., there exists $\gamma > 0$ such that $W(x) \geq -\gamma$ for every $x \in \mathcal{D}$ and every W -bounded sequence contains a convergent subsequence with limit in $\mathcal{D}$), $J, J_n, n = 1, 2, \dots$, functionals on $\mathcal{D}$ such that J is W -subsequentially lower semicontinuous (W -sls) and $\{J_n\}$ is W -coercive and W -uniformly consistent for J . Suppose further that J has a unique global minimizer $\bar{x} \in \mathcal{D}$ and that each functional J_n also possesses on $\mathcal{D}$ a global minimizer x_n (not necessarily unique). Then $x_n \rightarrow \bar{x}$.*

Proof. For each $n \in \mathbb{N}$, let x_n be a global minimizer of J_n . Following the same steps as those in the proof of Theorem 3.3 it follows that the sequence $\{x_n\}$ is W -bounded.

Suppose now that $\{x_n\}$ does not converge to $\bar{x}$. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that no subsequence of $\{x_{n_j}\}$ converges to $\bar{x}$. On the other hand, since the sequence $\{x_{n_j}\}$ is W -bounded (since the original sequence is), hypothesis (H3') on the functional W implies that there exist a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ and $x^* \in \mathcal{D}$ such that $x_{n_{j_k}} \rightarrow x^*$. From this it follows that $x^* \neq \bar{x}$ and since J is W -sls, there exists a subsequence $\{x_{n_{j_k\ell}}\} \subset \{x_{n_{j_k}}\}$ such that

$$J(x^*) \leq \liminf_{\ell \rightarrow \infty} J(x_{n_{j_k\ell}}). \quad (16)$$

Then

$$\begin{aligned}
J(x^*) &\leq \liminf_{\ell \rightarrow \infty} J(x_{n_{j_{k_\ell}}}) && \text{(by (16))} \\
&\leq \limsup_{\ell \rightarrow \infty} \left[\left(J(x_{n_{j_{k_\ell}}}) - J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) + J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right] \\
&\leq \limsup_{\ell \rightarrow \infty} \left(J(x_{n_{j_{k_\ell}}}) - J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) + \limsup_{\ell \rightarrow \infty} J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \\
&= \limsup_{\ell \rightarrow \infty} J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) && \text{(since } \{J_n\} \text{ is } W\text{-unif. consistent for } J) \\
&\leq \limsup_{n \rightarrow \infty} J_n(x_n) && \text{(since } \{x_{n_{j_{k_\ell}}}\} \subset \{x_n\}) \\
&\leq \limsup_{n \rightarrow \infty} J_n(\bar{x}) && \text{(since } x_n \text{ minimizes } J_n) \\
&= J(\bar{x}). && \text{(since } \{J_n\} \text{ is } W\text{-unif. consistent for } J)
\end{aligned}$$

Hence $J(x^*) \leq J(\bar{x})$ which contradicts the fact that $\bar{x} \neq x^*$ and x^* is the unique minimizer of J . This contradiction came from assuming that the sequence $\{x_n\}$ does not converge to $\bar{x}$. Hence $x_n \rightarrow \bar{x}$. $\blacksquare$

The previous theorem yields a strong stability result for minimizers of the functional (3) in the particular case in which J and J_n are of Tikhonov-Phillips type. More precisely we have the following corollary.

Corollary 3.11. *Let $\mathcal{X}$ be a normed vector space, $\mathcal{Y}$ an inner product space, $T, T_n \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $n = 1, 2, \dots$, $y \in \mathcal{Y}$, $\alpha > 0$, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \rightarrow \mathbb{R}$ a functional satisfying hypotheses (H1) of Theorem 2.5 and (H2') and (H3') of Remark 2.6, $J, J_n, n = 1, 2, \dots$, functionals on $\mathcal{D}$ defined as follows:*

$$J(x) \doteq \|Tx - y\|^2 + \alpha W(x), \quad (17)$$

$$J_n(x) \doteq \|T_n x - y_n\|^2 + \alpha_n W(x), \quad (18)$$

such that as $n \rightarrow \infty$, $\alpha_n \rightarrow \alpha$, $y_n \rightarrow y$ and $T_n x \rightarrow Tx$ uniformly on W -bounded sets (i.e. $\{T_n\}$ is W -uniformly consistent for T). Suppose further that J has a unique global minimizer $\bar{x}$. If x_n is a global minimizer of J_n then $x_n \rightarrow \bar{x}$.

Proof. Since the proof is immediately obtained from Theorem 3.10 following the same steps as in Corollary 3.4, we do not give details here. $\blacksquare$

Here again, the strong stability results of Theorem 3.10 and Corollary 3.11 remain valid under weaker hypotheses involving both the model operator T and the penalizer W .

Theorem 3.12. *Let $\mathcal{X}$ be a normed vector space, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \rightarrow \mathbb{R}$ a functional satisfying hypotheses (H1) of Theorem 2.5 and (I3') of Remark 2.11 (i.e., there exists $\gamma > 0$ such that $W(x) \geq -\gamma$ for every $x \in \mathcal{D}$ and every T - W -bounded sequence contains a convergent subsequence with limit in $\mathcal{D}$), $J, J_n, n = 1, 2, \dots$, functionals on $\mathcal{D}$ such that J is T - W -subsequentially lower semicontinuous (T - W -sls) and $\{J_n\}$ is T - W -coercive and T - W -uniformly consistent for J . Suppose further that J*

has a unique global minimizer $\bar{x} \in \mathcal{D}$ and that each functional J_n also possesses on $\mathcal{D}$ a global minimizer x_n (not necessarily unique). Then $x_n \rightarrow \bar{x}$.

Proof. The proof of this theorem proceeds exactly as the one of Theorem 3.10, by changing the W -boundedness, W -sls, W -uniform consistency and $(H3')$ hypotheses by T - W -boundedness, T - W -sls, T - W -uniform consistency and $(I3')$, respectively. ■

Here again, the previous strong stability theorem yields a corresponding stability result for minimizers of the functional (3) in the particular case in which J and J_n are of Tikhonov-Phillips type. This result is given in the following corollary.

Corollary 3.13. *Let $\mathcal{X}$ be a normed vector space, $\mathcal{Y}$ an inner product space, $T, T_n \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $n = 1, 2, \dots$, $y \in \mathcal{Y}$, $\alpha > 0$, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \rightarrow \mathbb{R}$ a functional satisfying hypotheses $(H1)$ of Theorem 2.5 and $(I2')$ and $(I3')$ of Remark 2.11, $J, J_n, n = 1, 2, \dots$, functionals on $\mathcal{D}$ defined as follows:*

$$J(x) \doteq \|Tx - y\|^2 + \alpha W(x), \quad (19)$$

$$J_n(x) \doteq \|T_n x - y_n\|^2 + \alpha_n W(x), \quad (20)$$

such that as $n \rightarrow \infty$, $\alpha_n \rightarrow \alpha$, $y_n \rightarrow y$ and $T_n x \rightarrow Tx$ uniformly on W -bounded sets (i.e. $\{T_n\}$ is W -uniformly consistent for T). Suppose further that J has a unique global minimizer $\bar{x}$. If x_n is a global minimizer of J_n then $x_n \rightarrow \bar{x}$.

Proof. We will show that J and $\{J_n\}$ satisfy the hypotheses of Theorem 3.12. For that it suffices to show that J is T - W -sls and that $\{J_n\}$ is T - W coercive and T - W -uniformly consistent for J . The fact that J is T - W -sls follows immediately from $(I2')$, the boundedness of T and the continuity of the norm in $\mathcal{X}$. The T - W -uniform consistency of $\{J_n\}$ for J follows exactly as in the proof of Corollary 3.4 by noting that $|J_n(x) - J(x)| \leq ||T_n x - y_n\|^2 - \|Tx - y\|^2| + |(\alpha_n - \alpha)|W(x)|$ and using the fact that T - W -bounded sets are also W -bounded and the hypothesis of the W -uniform consistency of $\{T_n\}$ for T . Finally, also the T - W -coercivity of $\{J_n\}$ follows easily from the W -uniform consistency of $\{T_n\}$ for T . ■

4. Particular cases

In this section we present several examples of penalizers $W(\cdot)$ for which some of the results obtained in the previous section are valid and therefore, existence, uniqueness and/or stability for the minimizers of the corresponding generalized Tikhonov-Phillips functional $J_{W,\alpha}(\cdot)$ in (3) are obtained.

4.1. Total variation penalization

Bounded variation penalty methods have been studied by Rudin, Osher and Fatemi in 1992 ([16]) and Acar and Vogel in 1994 ([1]), among others. These methods have been proved highly successful in certain image denoising problems where edge preserving

is an important issue ([4], [5], [6], [8]). Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ a convex, bounded set with Lipschitz continuous boundary, $1 \leq p \leq \frac{d}{d-1}$, $\mathcal{X} \doteq L^p(\Omega)$, $\mathcal{D} \doteq BV(\Omega)$, where $BV(\Omega)$ denotes the space of functions of bounded variations on Ω . Recall that $BV(\Omega) = \{u \in L^1(\Omega) : J_0(u) < \infty\}$, where $J_0(u) \doteq \sup_{v \in \mathcal{V}} \int_{\Omega} (-u \operatorname{div} v) dx$ and $\mathcal{V} \doteq \{v \in C_0^1(\Omega; \mathbb{R}^d) : |v(x)| \leq 1 \forall x \in \Omega\}$ (for $u \in C^1(\Omega)$ one has that $J_0(u) = \int_{\Omega} |\nabla u| dx$) and for $u \in BV(\Omega)$ the BV norm of u is defined by $\|u\|_{BV(\Omega)} \doteq \|u\|_{L^1(\Omega)} + J_0(u)$. Let W be the functional defined on $\mathcal{D}$ by $W(u) \doteq \|u\|_{BV(\Omega)}$.

We will show that $W(\cdot)$ satisfies the hypotheses (H1), (H2) and (H3) of Theorem 2.5. Clearly $W(\cdot)$ satisfies hypothesis (H1) with $\gamma = 0$. Hypothesis (H3) follows immediately from the compact imbedding of $BV(\Omega)$ into $L^p(\Omega)$ for $1 \leq p < \frac{d}{d-1}$ and from the weak compact imbedding for $p = \frac{d}{d-1}$. These results are extensions of the Rellich-Kondrachov Theorem and can be found for example in [2] and [3]. It only remains prove that $W(\cdot)$ satisfies hypothesis (H2). For that, let $\{u_n\} \subset \mathcal{D}$ be a W -bounded sequence such that $u_n \xrightarrow{w-L^p} u \in \mathcal{D}$. Then, $u_n \xrightarrow{w-L^1} u$ (since $p \geq 1$). From the weak lower semicontinuity of the $\|\cdot\|_{L^1(\Omega)}$ norm and of the functional $J_0(\cdot)$ in $L^1(\Omega)$ (see [1]), it follows that

$$\|u\|_{L^1(\Omega)} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^1(\Omega)} \quad \text{and} \quad J_0(u) \leq \liminf_{n \rightarrow \infty} J_0(u_n). \quad (21)$$

Then,

$$\begin{aligned} W(u) &= \|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + J_0(u) \\ &\leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^1(\Omega)} + \liminf_{n \rightarrow \infty} J_0(u_n) \quad (\text{by (21)}) \\ &\leq \liminf_{n \rightarrow \infty} \left(\|u_n\|_{L^1(\Omega)} + J_0(u_n) \right) \\ &= \liminf_{n \rightarrow \infty} \|u_n\|_{BV(\Omega)} \\ &= \liminf_{n \rightarrow \infty} W(u_n), \end{aligned}$$

which proves (H2). Hence $W(\cdot)$ satisfies the hypotheses of Theorem 2.5 and therefore for any $\alpha > 0$, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, ($\mathcal{Y}$ a normed space) the functional

$$J_{\|\cdot\|_{BV}, \alpha}(u) \doteq \|Tu - v\|^2 + \alpha \|u\|_{BV(\Omega)} \quad (22)$$

has a global minimizer on $BV(\Omega)$. If T is injective then such a global minimizer is unique. If T is not injective uniqueness cannot be guaranteed since the $\|\cdot\|_{BV}$ -norm is not strictly convex. Also, if $p < \frac{d}{d-1}$ and $J_{\|\cdot\|_{BV}, \alpha}(\cdot)$ has a unique global minimizer, then the problem of finding such a minimizer is strongly stable under perturbations in the model (T), in the data (y) and in the regularization parameter (α). This follows immediately from the fact that (H2) is stronger than (H2'), the relative compactness of BV-bounded sets in $L^p(\Omega)$ for $p < \frac{d}{d-1}$ (see [10]) and Corollary 3.11. For $p = \frac{d}{d-1}$ and $d \geq 2$ the problem is weakly stable, by virtue of Corollary 3.4.

4.2. Penalization with powers of semi-norms associated to closed operators

Theorem 4.1. *Let $\mathcal{X}$, $\mathcal{Z}$ be reflexive Banach spaces, $\mathcal{Y}$ a normed space, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $L : \mathcal{D}(L) \subset \mathcal{X} \rightarrow \mathcal{Z}$ a closed linear operator such that the range of L , $\mathcal{R}(L)$,*

is weakly closed. Assume further that T and L are complemented, i.e. there exists a constant $k > 0$ such that $\|Tx\|^2 + \|Lx\|^2 \geq k\|x\|^2$, $\forall x \in \mathcal{D}(L)$. Then, for any $q > 1$, $\alpha > 0$ and $y \in \mathcal{Y}$ the functional

$$J_{L,q,\alpha}(x) \doteq \|Tx - y\|^2 + \alpha\|Lx\|^q, \quad x \in \mathcal{D}(L), \quad (23)$$

has a unique global minimizer.

Proof. Let $q > 1$, $\mathcal{D} \doteq \mathcal{D}(L)$ and $W_{L,q} : \mathcal{D} \rightarrow \mathbb{R}_0^+$ defined by $W_{L,q}(x) \doteq \|Lx\|^q$. We will show that T and $W_{L,q}$ satisfy the hypotheses (H1), (H2) and (I3). Hypothesis (H1) is trivially satisfied since $W_{L,q}(x) \geq 0 \forall x \in \mathcal{D}$. To prove that (H2) holds, let $\{x_n\} \subset \mathcal{D}$ be a $W_{L,q}$ -bounded sequence such that $x_n \xrightarrow{w} x \in \mathcal{D}$. Then there exists a constant $c < \infty$ such that $\|Lx_n\| \leq c \forall n \in \mathbb{N}$. Since the Banach space $\mathcal{Z}$ is reflexive, there exist $z \in \mathcal{Z}$ and $\{x_{n_j}\} \subset \{x_n\}$ such that $Lx_{n_j} \xrightarrow{w} z$. Since $\mathcal{R}(L)$ is weakly closed $z \in \mathcal{R}(L)$. Now, the operator $L^\dagger$, the Moore-Penrose generalized inverse of L , is continuous (since $\mathcal{R}(L)$ is closed), and therefore $P_{\mathcal{N}(L)^\perp}x_{n_j} = L^\dagger Lx_{n_j} \xrightarrow{w} L^\dagger z$ (where $P_{\mathcal{N}(L)^\perp}$ is the orthogonal projection of $\mathcal{X}$ onto $\mathcal{N}(L)^\perp$). Since $x_{n_j} = P_{\mathcal{N}(L)^\perp}x_{n_j} + P_{\mathcal{N}(L)}x_{n_j}$ it follows that $P_{\mathcal{N}(L)}x_{n_j} \xrightarrow{w} x - L^\dagger z$ and therefore $x - L^\dagger z \in \mathcal{N}(L)$ (since $\mathcal{N}(L)$ is weakly closed, L being closed). Hence $0 = L(x - L^\dagger z) = Lx - LL^\dagger z = Lx - P_{\overline{\mathcal{R}(L)}}z = Lx - z$. Thus $z = Lx$ and $W_{L,q}(x) = \|Lx\|^q = \|z\|^q \leq \liminf_{j \rightarrow \infty} \|Lx_{n_j}\|^q = \liminf_{j \rightarrow \infty} W_{L,q}(x_{n_j})$, where the inequality follows from the fact that $Lx_{n_j} \xrightarrow{w} z$ and the weak lower semicontinuity of the norm in $\mathcal{Z}$. This proves (H2).

To prove that (I3) holds, let $\{x_n\} \subset \mathcal{D}$ be a T - $W_{L,q}$ -bounded sequence. By the complementation condition it follows that $\{x_n\}$ is bounded in $\mathcal{X}$ and by the reflexivity of $\mathcal{X}$ there must exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and $x \in \mathcal{X}$ such that $x_{n_j} \xrightarrow{w} x$. It only remains to be proved that $x \in \mathcal{D} = \mathcal{D}(L)$. For that observe that since $\{x_{n_j}\}$ is a $W_{L,q}$ -bounded sequence such that $x_{n_j} \xrightarrow{w} x$, following the same steps as in the proof of (H2) above, we obtain that there exists $z \in \mathcal{R}(L)$ such that $x - L^\dagger z \in \mathcal{N}(L)$. Since $L^\dagger z \in \mathcal{N}(L)^\perp \subset \mathcal{D}(L)$ it then follows that $x \in \mathcal{D}(L)$. This finally proves that (I3) holds.

Now, since hypothesis (H2) implies hypothesis (I2) (see Remark 2.10), Theorem 2.9 now implies that for any $\alpha > 0$, $y \in \mathcal{Y}$, the functional $J_{L,q,\alpha}(x)$ defined by (23), has a global minimizer on $\mathcal{D}(L)$. Since $q > 1$, from the complementation condition it follows easily that $J_{L,q,\alpha}$ is strictly convex and therefore such a global minimizer is unique. ■

It is appropriate point out here that the above hypotheses on L are satisfied by most differential operators and that the complementation condition holds, for instance, whenever $\dim \mathcal{N}(L) < \infty$ and $\mathcal{N}(T) \cap \mathcal{N}(L) = \{0\}$. Also, the previous theorem provides existence for any $q > 0$. However uniqueness can only be guaranteed for $q > 1$ and, if T is injective, also for $q = 1$.

The next Lemma shows that the problem of finding the global minimum of (23) is weakly stable under perturbations on y , α and T .

Lemma 4.2. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, T, L, \mathcal{D}$ as in Theorem 4.1, $q > 1$, $y, y_n \in \mathcal{Y}$, $\alpha, \alpha_n \geq 0$, $T_n \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $n = 1, 2, \dots$, and $J_{L,q,\alpha}, J_n, n = 1, 2, \dots$, functionals on $\mathcal{D}$ defined by*

$$J_{L,q,\alpha}(x) \doteq \|Tx - y\|^2 + \alpha \|Lx\|^q, \quad (24)$$

$$J_n(x) \doteq \|T_n x - y_n\|^2 + \alpha_n \|Lx\|^q. \quad (25)$$

Assume that $\alpha_n \rightarrow \alpha$, $y_n \rightarrow y$ as $n \rightarrow \infty$ and that $T_n x \rightarrow Tx$ uniformly for x in L -bounded sets (i.e. $\{T_n\}$ is L -uniformly consistent for T). Let $\bar{x}$ be the unique minimizer of $J_{L,q,\alpha}$ and x_n a global minimizer of J_n . Then $x_n \xrightarrow{w} \bar{x}$.

Proof. Let $W_{L,q} : \mathcal{D} \rightarrow \mathbb{R}_0^+$ defined by $W_{L,q}(x) \doteq \|Lx\|^q$. In Theorem 4.1 we proved that T and $W_{L,q}$ satisfy hypotheses (H1), (I2) and (I3). Since by hypothesis $\alpha_n \rightarrow \alpha$, $y_n \rightarrow y$ and $\{T_n\}$ is $W_{L,q}$ -uniformly consistent for T , the Lemma follows immediately from Corollary 3.9. ■

From the point of view of applications of the Tikhonov-Phillips methods, the weak stability result established by the previous Lemma, although important, could render insufficient. A strong stability result, at least on the data y is highly desired. In the next Lemma we show that such a result can be obtain by imposing an additional hypothesis to the operator L .

Lemma 4.3. *Let $\mathcal{X}, \mathcal{Z}, \mathcal{Y}, T, T_n L, \mathcal{D}, q, W_{L,q}, y, y_n, \alpha, \alpha_n, J_{L,q,\alpha}, \bar{x}, x_n$ and $J_n, n = 1, 2, \dots$ as in Lemma 4.2. Assume further that T - L -bounded sets are compact in $\mathcal{X}$. Then $x_n \rightarrow \bar{x}$.*

Proof. In Theorem 4.1 we proved that T and $W_{L,q}$ satisfy hypotheses (H1) and (I2). Since hypothesis (I2) implies hypothesis (I2') and the compactness of T - L -bounded sets implies (I3)', the lemma then follows from Corollary 3.13. ■

Remark 4.4. *If $q = 2$, under the same hypotheses of Lemma 4.2 one can get continuity of the solutions of (24) with respect to α and y . This can be easily verified from the fact that the unique global minimizer of (24) is given by $\bar{x} = (\alpha L^* L + T^* T)^{-1} T^* y$. Thus, if x_n is the minimizer of (25) with $T_n = T \forall n$, then one has that*

$$\bar{x} - x_n = (\alpha - \alpha_n) (\alpha L^* L + T^* T)^{-1} L^* L x_n + (\alpha L^* L + T^* T)^{-1} T^* (y - y_n). \quad (26)$$

Suppose now that $\alpha_n \rightarrow \alpha$ and $y_n \rightarrow y$. Then by Lemma 4.2 $x_n \xrightarrow{w} \bar{x}$ and therefore $\{x_n\}$ is bounded. Also, since $\|Tx\|^2 + \|Lx\|^2 \geq k \|x\|^2$ it follows that the operators $(\alpha L^ L + T^* T)^{-1} L^* L$ and $(\alpha L^* L + T^* T)^{-1} T^*$ are both bounded. In fact $(\alpha L^* L + T^* T)^{-1} L^* L \leq \alpha^{-1} I$ and $(\alpha L^* L + T^* T)^{-1} \leq \frac{1}{k \min(\alpha, 1)}$. Hence, it follows from (26) that $x_n \rightarrow \bar{x}$.*

4.3. Penalization by linear combination of powers of semi-norms associated to closed operators

We study here the case of generalized Tikhonov-Phillips regularization methods for which the functional $W(\cdot)$ in (3) is of the form $W(x) \doteq \sum_{i=1}^N \alpha_i \|L_i x\|^{q_i}$, where the L_i 's are closed operators. We start with the main existence and uniqueness result.

Theorem 4.5. *Let $\mathcal{X}$, $\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_N$ be reflexive Banach spaces, $\mathcal{Y}$ a normed space, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $\mathcal{D}$ a subspace of $\mathcal{X}$, $L_i : \mathcal{D} \rightarrow \mathcal{Z}_i$, $i = 1, 2, \dots, N$, closed linear operators with $\mathcal{R}(L_i)$ weakly closed for every $1 \leq i \leq N$ and such that $T, L_1, L_2, \dots, L_N$ are complemented, i.e. there exists a constant $k > 0$ such that $\|Tx\|^2 + \sum_{i=1}^N \|L_i x\|^2 \geq k\|x\|^2$, $\forall x \in \mathcal{D}$. Then, for any $y \in \mathcal{Y}$, $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{R}^+$ and $q_1, q_2, \dots, q_N \in \mathbb{R}$, $q_i > 1 \forall i = 1, 2, \dots, N$, the functional*

$$J(x) \doteq \|Tx - y\|^2 + \sum_{i=1}^N \alpha_i \|L_i x\|^{q_i}, \quad (27)$$

has a unique global minimizer.

Proof. Let $y \in \mathcal{Y}$, $\alpha_i > 0$, $q_i > 1$, $i = 1, 2, \dots, N$ and define $\vec{\alpha} \doteq (\alpha_1, \alpha_2, \dots, \alpha_N)^T$, $\vec{q} \doteq (q_1, q_2, \dots, q_N)^T$, the normed space $\mathcal{Z} \doteq \bigotimes_{i=1}^N \mathcal{Z}_i$, $\vec{L} : \mathcal{X} \rightarrow \mathcal{Z}$ as $\vec{L}x \doteq (L_1 x, L_2 x, \dots, L_N x)^T$, and the functional $W_{\vec{L}, \vec{q}, \vec{\alpha}} : \mathcal{D} \rightarrow \mathbb{R}_0^+$ by $W_{\vec{L}, \vec{q}, \vec{\alpha}}(x) = \sum_{i=1}^N \alpha_i \|L_i x\|^{q_i}$, so that $J(x) = \|Tx - y\|^2 + W_{\vec{L}, \vec{q}, \vec{\alpha}}(x)$. We will prove that T and $W_{\vec{L}, \vec{q}, \vec{\alpha}}$ satisfy the hypotheses (H1), (H2) and (I3). In fact, (H1) is trivial and for (H2), let $\{x_n\} \subset \mathcal{D}$ be a $W_{\vec{L}, \vec{q}, \vec{\alpha}}$ -bounded sequence such that $x_n \xrightarrow{w} x \in \mathcal{D}$. Then for every $i = 1, 2, \dots, N$, the sequence $\{L_i x_n\}_{n=1}^\infty$ is bounded in $\mathcal{Z}_i$ and since $\mathcal{Z}_i$ is reflexive there exist a subsequence $\{x_{n_k}\}$ and $z_i \in \mathcal{Z}_i$ such that $L_i x_{n_k} \xrightarrow{w} z_i$ as $k \rightarrow \infty$. Since $\mathcal{R}(L_i)$ is weakly closed, $z_i \in \mathcal{R}(L_i)$. By taken subsequences, we may assume that such a subsequence is the same for all i , i.e. $L_i x_{n_k} \xrightarrow{w} z_i$ as $k \rightarrow \infty$ for every $i = 1, 2, \dots, N$.

Now, since $\mathcal{R}(L_i)$ is closed, $L_i^\dagger$ is bounded and therefore $L_i^\dagger L_i x_{n_k} \xrightarrow{w} L_i^\dagger z_i$, as $k \rightarrow \infty$, for all $i = 1, 2, \dots, N$. Since $L_i^\dagger L_i = P_{\mathcal{N}(L_i)^\perp}$ is the orthogonal projection of $\mathcal{X}$ onto $\mathcal{N}(L_i)^\perp$, writing $x_{n_k} = P_{\mathcal{N}(L_i)^\perp} x_{n_k} + P_{\mathcal{N}(L_i)} x_{n_k}$, it follows that $P_{\mathcal{N}(L_i)} x_{n_k} \xrightarrow{w} x - L_i^\dagger z_i$ as $k \rightarrow \infty$ and therefore $x - L_i^\dagger z_i \in \mathcal{N}(L_i)$ (being $\mathcal{N}(L_i)$ closed, since L_i is closed). Hence for all $i = 1, 2, \dots, N$, it follows that $0 = L_i(x - L_i^\dagger z_i) = L_i x - P_{\overline{\mathcal{R}(L_i)}} z_i = L_i x - z_i$ (where the last equality follows since $z_i \in \mathcal{R}(L_i)$). Thus, $z_i = L_i x \forall i = 1, 2, \dots, N$. Then

$$\|L_i x\|^{q_i} = \|z_i\|^{q_i} \leq \liminf_{k \rightarrow \infty} \|L_i x_{n_k}\|^{q_i},$$

(where the last inequality follows from the fact that $L_i x_{n_k} \xrightarrow{w} z_i$ as $k \rightarrow \infty$ and the weak lower semicontinuity of the norm in $\mathcal{Z}_i$), and therefore

$$\begin{aligned} W_{\vec{L}, \vec{q}, \vec{\alpha}}(x) &= \sum_{i=1}^N \alpha_i \|L_i x\|^{q_i} \leq \sum_{i=1}^N \alpha_i \liminf_{k \rightarrow \infty} \|L_i x_{n_k}\|^{q_i} \\ &\leq \liminf_{k \rightarrow \infty} \sum_{i=1}^N \alpha_i \|L_i x_{n_k}\|^{q_i} = \liminf_{k \rightarrow \infty} W_{\vec{L}, \vec{q}, \vec{\alpha}}(x_{n_k}). \end{aligned}$$

Thus (H2) holds. That (I3) also holds follows from the complementation condition and the reflexivity of $\mathcal{X}$, following the same steps as in Theorem 4.1. Since (H2) implies (I2), it now follows from Theorem 2.9 that the functional $J(x)$ in (27) has a global minimizer on $\mathcal{D}$. Moreover, since $q_i > 1$ for all i , it follows from the complementation condition that $J(\cdot)$ is strictly convex and therefore such a minimizer is unique. $\blacksquare$

Under the same hypotheses of Theorem 4.5 one has that the solution of (27) is weakly stable under perturbations in the data y , in the parameters α_i and in the model operator T . More precisely we have the following result.

Lemma 4.6. *Let all the hypotheses of Theorem 4.5 hold. Let also $y, y_n \in \mathcal{Y}$, $T_n \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $n = 1, 2, \dots$, such that $y_n \rightarrow y$, $\{T_n\}$ is $\vec{L}$ -uniformly consistent for T and for each $i = 1, 2, \dots, N$, let $\{\alpha_i^n\}_{n=1}^\infty \subset \mathbb{R}^+$ such that $\alpha_i^n \rightarrow \alpha_i$ as $n \rightarrow \infty$. If x_n is a global minimizer of the functional*

$$J_n(x) \doteq \|T_n x - y_n\|^2 + \sum_{i=1}^N \alpha_i^n \|L_i x\|^{q_i}, \quad (28)$$

then $x_n \xrightarrow{w} \bar{x}$, where $\bar{x}$ is the unique minimizer of (27).

Proof. Let $W \doteq W_{\vec{L}, \vec{q}, \vec{\alpha}}$ as in Theorem 4.5. From the hypotheses it follows easily that $\{J_n\}$ is T - W -coercive and W -uniformly consistent for J .

Let x_n be the unique minimizer of J_n . Then $J_n(x_n) \leq J_n(\bar{x})$, $\forall n$. Therefore

$$\limsup_{n \rightarrow \infty} J_n(x_n) \leq \limsup_{n \rightarrow \infty} J_n(\bar{x}) = J(\bar{x}) < \infty, \quad (29)$$

where the equality follows from the W -uniform consistency of $\{J_n\}$ for J . But since $\{J_n\}$ is T - W -coercive it then follows that $\{x_n\}$ is T - W -bounded. We claim that $x_n \xrightarrow{w} \bar{x}$. In fact, suppose that is not the case. Then, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that no subsequence of $\{x_{n_j}\}$ converges weakly to $\bar{x}$. But since $\{x_{n_j}\}$ is T - W -bounded and $\mathcal{X}$ is reflexive, there exist $x^* \neq \bar{x}$ and $\{x_{n_{j_k}}\} \subset \{x_{n_j}\}$ such that $x_{n_{j_k}} \xrightarrow{w} x^*$. Following the same steps as in Theorem 4.5 we obtain that there exists a subsequence $\{x_{n_{j_{k_\ell}}}\} \subset \{x_{n_{j_k}}\}$ and $z_i \in \mathcal{Z}_i$, $i = 1, 2, \dots, N$, such that $L_i x_{n_{j_{k_\ell}}} \xrightarrow{w} z_i = L_i x^*$ as $\ell \rightarrow \infty$, $\forall i = 1, 2, \dots, N$, and

$$W(x^*) \leq \liminf_{\ell \rightarrow \infty} W(x_{n_{j_{k_\ell}}}). \quad (30)$$

Also, since $\{x_{n_{j_{k_\ell}}}\}$ is W -bounded and $\{J_n\}$ is W -uniformly consistent for J , it follows that

$$\lim_{\ell \rightarrow \infty} \left(J(x_{n_{j_{k_\ell}}}) - J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) = 0. \quad (31)$$

Hence

$$\begin{aligned} J(x^*) &= \|Tx^* - y\|^2 + W(x^*) \\ &\leq \liminf_{\ell \rightarrow \infty} \|Tx_{n_{j_{k_\ell}}} - y\|^2 + \liminf_{\ell \rightarrow \infty} W(x_{n_{j_{k_\ell}}}) \quad (\text{by (30)}) \\ &\leq \liminf_{\ell \rightarrow \infty} \left(\|Tx_{n_{j_{k_\ell}}} - y\|^2 + J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) \end{aligned}$$

$$\begin{aligned}
&= \liminf_{\ell \rightarrow \infty} J(x_{n_{j_{k_\ell}}}) \\
&\leq \limsup_{\ell \rightarrow \infty} J(x_{n_{j_{k_\ell}}}) \\
&= \limsup_{\ell \rightarrow \infty} \left[\left(J(x_{n_{j_{k_\ell}}}) - J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) + J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right] \\
&\leq \limsup_{\ell \rightarrow \infty} \left(J(x_{n_{j_{k_\ell}}}) - J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \right) + \limsup_{\ell \rightarrow \infty} J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \\
&= \limsup_{\ell \rightarrow \infty} J_{n_{j_{k_\ell}}}(x_{n_{j_{k_\ell}}}) \quad (\text{by (31)}) \\
&\leq J(\bar{x}) \quad (\text{by (29) since } \{x_{n_{j_{k_\ell}}}\} \subset \{x_n\})
\end{aligned}$$

Since $\bar{x}$ is the unique minimizer of J it would then follow that $x^* = \bar{x}$, contradicting our previous result that $x^* \neq \bar{x}$. This contradiction came from the assumption that x_n did not converge weakly to $\bar{x}$. Hence $x_n \xrightarrow{w} \bar{x}$. $\blacksquare$

Lemma 4.7. *Under the same hypotheses of Lemma 4.6, if T - $\vec{L}$ -bounded sets are compact in $\mathcal{D}$, then strong stability holds, i.e., $x_n \rightarrow \bar{x}$.*

Proof. Let x_n denote the global minimizer of J_n and $W = W_{\vec{L}, \vec{q}, \vec{\alpha}}$. In Lemma 4.6 it was proved that the sequence $\{x_n\}$ is T - W -bounded. Suppose that $x_n \not\rightarrow \bar{x}$. Then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that no subsequence of $\{x_{n_j}\}$ converges to $\bar{x}$. But since $\{x_{n_j}\}$ is T - W -bounded, now by compactness hypothesis there must exist $x^* \in \mathcal{D}$, $x^* \neq \bar{x}$, and a subsequence $\{x_{n_{j_k}}\} \subset \{x_{n_j}\}$ such that $x_{n_{j_k}} \rightarrow x^*$ as $k \rightarrow \infty$. Using the W -uniform consistency of $\{J_n\}$ for J and following similar steps as in Lemma 4.6 one obtains that $J(x^*) \leq J(\bar{x})$. Since $\bar{x}$ is the unique minimizer of J it would then follow that $x^* = \bar{x}$, contradicting our previous result that $x^* \neq \bar{x}$. Therefore we must have that $x_n \rightarrow \bar{x}$. $\blacksquare$

Remark 4.8. *Here again, for the case $q_i = 2 \forall i$, strong continuity of the solution of the functional $J(x)$ in (27) with respect to the data y and the parameters α_i follow without any further hypotheses than those in Lemma 4.6. This result follows easily from the fact that in such a case the unique global minimizer of (27) is given by $\bar{x} = \left(T^*T + \sum_{i=1}^N \alpha_i L_i^* L_i \right)^{-1} T^*y$. Thus, if x_n is the minimizer of (28) with $T_n = T \forall n$, then one has that*

$$\begin{aligned}
\bar{x} - x_n &= \left(T^*T + \sum_{i=1}^N \alpha_i L_i^* L_i \right)^{-1} \sum_{i=1}^N (\alpha_i^n - \alpha_i) L_i^* L_i x_n \\
&\quad + \left(T^*T + \sum_{i=1}^N \alpha_i L_i^* L_i \right)^{-1} T^*(y - y_n).
\end{aligned} \tag{32}$$

Now, from the complementation condition $\|Tx\|^2 + \sum_{i=1}^N \|L_i x\|^2 \geq k\|x\|^2, \forall x \in \mathcal{D}$, it follows easily that

$$0 \leq \left(T^*T + \sum_{i=1}^N \alpha_i L_i^* L_i \right)^{-1} \leq \frac{1}{k \min(1, \min_{1 \leq i \leq N} \alpha_i)}, \tag{33}$$

and also

$$\begin{aligned} & \left\| \left(T^*T + \sum_{i=1}^N \alpha_i L_i^* L_i \right)^{-1} \sum_{i=1}^N (\alpha_i^n - \alpha_i) L_i^* L_i x \right\| \\ & \leq \frac{\max_{1 \leq i \leq N} |\alpha_i^n - \alpha_i|}{\min_{1 \leq i \leq N} \alpha_i} \|x\|, \quad \forall x \in \mathcal{D}. \end{aligned} \quad (34)$$

Using (34) and (33) in (32) we obtain that

$$\|\bar{x} - x_n\| \leq \frac{\max_{1 \leq i \leq N} |\alpha_i^n - \alpha_i|}{\min_{1 \leq i \leq N} \alpha_i} \|x_n\| + \frac{\|T^*\|}{k \min(1, \min_{1 \leq i \leq N} \alpha_i)} \|y - y_n\|. \quad (35)$$

Now since by Lemma 4.6 $x_n \xrightarrow{w} \bar{x}$, it follows that $\{x_n\}$ is bounded. Since $y_n \rightarrow y$ and $\alpha_i^n \rightarrow \alpha_i \forall i = 1, 2, \dots, N$, as $n \rightarrow \infty$, it finally follows from (35) that $x_n \rightarrow x$.

5. Applications to Image Restoration

The purpose of this section is to present an application to a simple image restoration problem. The main objective is to show how the choice of the penalizer in a generalized Tikhonov-Phillips functional can affect the reconstructed image.

The basic mathematical model for image blurring is given by the following Fredholm integral equation

$$K f(x, y) \doteq \int_{\Omega} \int_{\Omega} k(x, y, x', y') f(x', y') dx' dy' = g(x, y), \quad (36)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $f \in \mathcal{X} \doteq L^2(\Omega)$ represents the original image and k is the so called “point spread function” (PSF). For the examples shown below we used a PSF of “atmospheric turbulence” type

$$k(x, y, x', y') = \frac{\kappa}{\pi} \exp \left(-\kappa \|(x, y) - (x', y')\|^2 \right), \quad (37)$$

with $\kappa = 6$. It is well known that with this PSF the operator K in (36) is compact with infinite dimensional range and therefore $K^\dagger$, the Moore-Penrose inverse of K , is unbounded.

Generalized Tikhonov-Phillips methods with different penalizers were used to obtain regularized solutions of the problem

$$K f = g. \quad (38)$$

The data g was contaminated with a 1% zero mean Gaussian noise (i.e. standard deviation of the order of 1% of $\|g\|_\infty$). Minimizers of functionals of the form

$$J_\alpha(f) = \|K f - \tilde{g}\|^2 + \alpha W(f) \quad (39)$$

were found for different penalizers $W(f)$, where $\tilde{g}$ represents the noisy version of g . In all cases the value of the regularization parameter α was approximated by using the L-curve method ([9], [12], [13]).

Figures 1(a) and 1(b) show the original image (unknown in real life problems) and the blurred noisy image which constitutes the data for the inverse problems,

respectively. Figures 1(c) and 1(d) show the reconstructions obtained with the classical Tikhonov-Phillips methods of order zero and one, corresponding to $W(f) = \|f\|^2$ and $W(f) = \|\nabla f\|^2$, respectively.

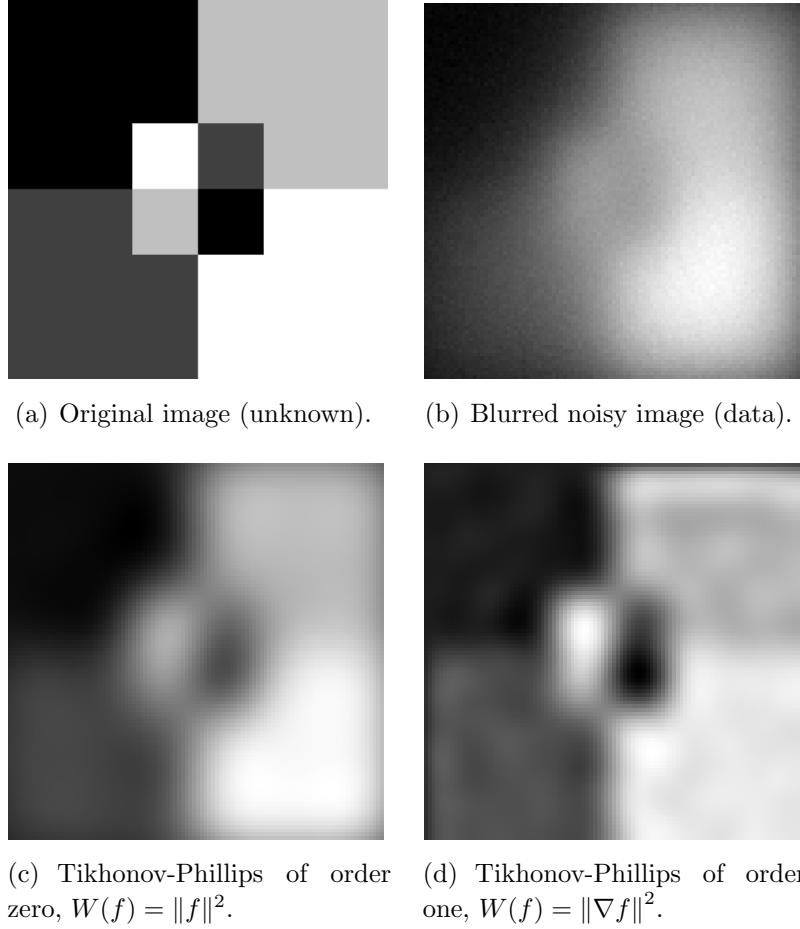


Figure 1. Original image (a), blurred noisy image (b) and regularized solutions obtained with the classical Tikhonov-Phillips methods of order zero (c) and one (d).

Figures 2(b) and 2(c) show the reconstructions obtained with a structural information penalizer of the form $W(f) = \|Lf\|^2$ where the operator L is constructed as in [14], including the information of the curve γ depicted in Figure 2(a), where it is expected that the original image have steep gradients. The operator L is constructed so as to capture this structural prior information. The discretization of L is given by $\int_{\Omega} \|A(x)\nabla f(x)\|^2 dx$ with $A(x) = I - (1 + c\|\nabla\gamma(x)\|^2)^{-1} \nabla\gamma(x) (\nabla\gamma(x))^T$, where c is a positive constant. In this way, if $\|\nabla\gamma(x)\|$ is large, the functional $W(f)$ penalizes only very mildly all intensity changes occurring in the direction of $\nabla\gamma(x)$ (see [14] for more details).

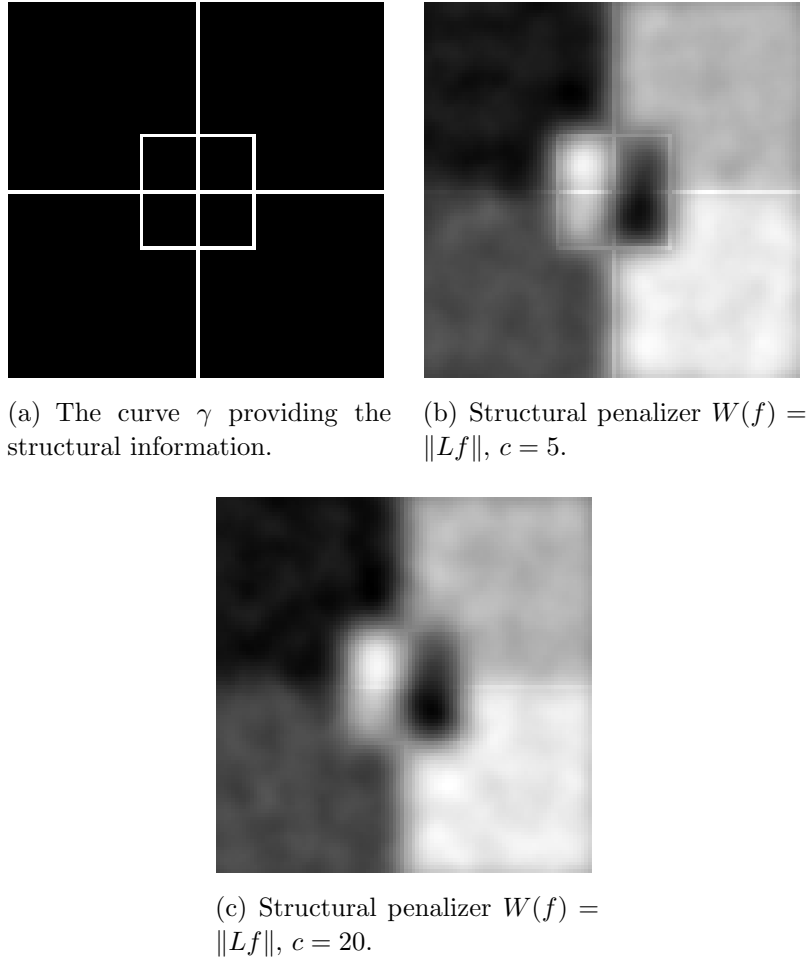


Figure 2. Structural information (a), reconstructed image with structural information penalizer and $c = 5$ (b) and $c = 20$ (c)

Figures 3(a) and 3(b) correspond to images reconstructed with hybrid Tikhonov-structural penalizers $W(f) = \frac{4}{5}\|f\|^2 + \frac{1}{5}\|Lf\|^2$, with $c = 5$ and $c = 20$, respectively.

A comparison of the images obtained with the different methods clearly show that the choice of the penalizer in Tikhonov-Phillips method can greatly affect the obtained approximated solution. In this particular case we observe how the classical order-zero method tends to smooth out boundaries and edges and, while the order-one method does a better job, the inclusion of the structural information through the operator L results in a significant improvement. Although the main objective of this article is theoretical in nature, providing sufficient conditions on the model operators and the penalizers for the existence, uniqueness and stability of solutions of the corresponding generalized Tikhonov-Phillips methods, the previous applications to image restoration were included to better emphasize the importance of the adequate choice of the penalizer.

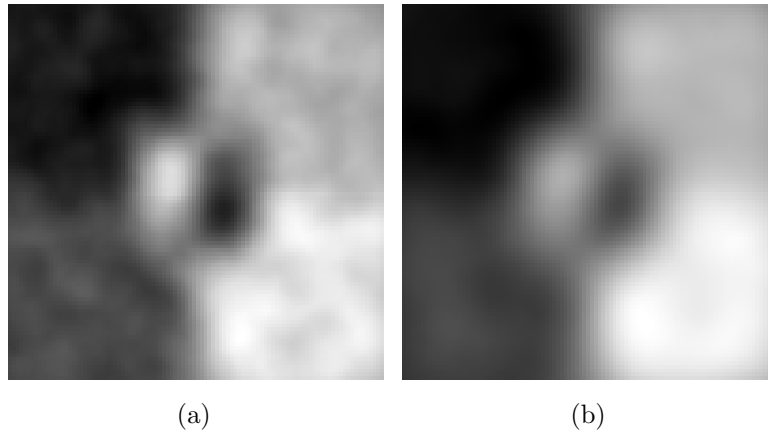


Figure 3. Reconstructed images with hybrid penalizers: $W(f) = \frac{4}{5}\|f\|^2 + \frac{1}{5}\|Lf\|^2$; $c = 5$ (a) and $c = 20$ (b)

6. Conclusions

In this article sufficient conditions on the penalizers in generalized Tikhonov-Phillips functionals guaranteeing existence, uniqueness and stability of the minimizers were found. The particular cases in which the penalizers are given by the bounded variation norm, by powers of seminorms and by linear combinations of powers of seminorms associated to closed operators, were studied. Several examples were presented and a few results on image restoration were shown to illustrate how the choice of the penalizer can greatly affect the regularized solutions.

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