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Noise Induced Switching in Delayed Systems

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ABSTRACT

We consider the problem of switching in stochastic systems with delayed feedback. A general variational formulation is derived for the switching rate in a stochastic differential delay equation where the noise source is of general form. The resulting equations of motion and boundary conditions describe the optimal escape path which maximizes the probability of escape. Analyzing the dynamics along the optimal path yields exponents of the distribution in terms of delay time, dissipation, and noise intensity. Theoretical predictions compare very well with numerical simulations in both additive and multiplicative noise cases, even outside of regions where the delay is assumed to be small.

I. INTRODUCTION

When considering dynamical systems with isolated feedback mechanisms or coupling devices to connect a network, there always exists a finite time for the signal to be transmitted from one device to another. Modeling such finite transmission times requires the inclusion of one or more delays in the dynamics. Evidence of the pervasiveness of delays can be seen in several reviews, such as the application of delay differential equations in many fields [1], where the effects of delay are seen in physics, biology, and engineering. The analysis of the effects of delay on stability and bifurcations is seen in several collections, such as [2–4]. Other areas involving delay occur in models involving populations such as epidemics [5–7], engineering such is machine tool cutting [8], and control [9].

Major applications and the effects of delay and noise on dynamical systems can be seen in experiments and models from nonlinear optics, such as laser self-feedback [10], delay coupled semi-conductor and solid state lasers [11–13], and fiber lasers [14]. In many of the coupled cases, zero -lag, also called isochronal, synchronization is an unstable state [12]. Combining delayed self-feedback along with delay coupling stabilizes the isochronal state in lasers [15], even in the presence of noise.

From a modeling perspective, the interaction of differential delay with noise leads one to study stochastic effects on infinite dimensional systems. In approximating the phase space, finite dimensional formulations based on coupled map lattices were used to initially examine approximations to steady state densities [16]. Here the authors employ a high dimensional Perron-Frobenius formulation to extract the density in the thermodynamic limit. However, the drawbacks to this formulation come from the high dimensionality of the problem making analysis of the distribution exponents difficult, and the restriction to the type of noise that may be considered. For small delays, the stochastic delay problem for Wiener processes has been re-formulated using a Fokker-Planck (FP) approach [17], where the new FP equations contain a conditionally averaged drift term in the approximation. The multivariate version of the perturbed problem is derived in general settings in [18, 19] to both Ito and Stratonovich formulations.

Noise induced switching was explored numerically in bistable potentials and has been considered in the presence of delayed feedback for systems modeling lasers to illustrate resonant behavior due to delayed feedback [20]. Stochastic resonance in bistable systems with delay and correlated noise sources was also studied using a perturbed Fokker-Planck approach for small delay [21]. In addition, others have investigated noise induced switching in bistable systems with delay, [21–23]. Overall, many of the above-mentioned techniques use a perturbed FP approach to describe small delay effects on the probability density function.

Rates of escape predicted from switching between co-existing stable states in the absence of delayed feedback is well investigated for a large variety of well-controlled systems and oscillators [24–31]. Fluctuations in these systems are usually due to thermal or externally applied Gaussian noise. An increasingly important role may be played also by more general noise sources, which typically are non-Gaussian. It may arise, for example, from just a few two-state discrete particles jumping at random between the states, often characterized as telegraph noise [32]. To reveal how switching in the presence of delay with general noise sources, we consider a different formulation from the usual perturbation procedure of the conditional averaged drift terms.

Here we consider predicting the escape rates from of a basin of attraction for stochastic systems with delayed feedback. The approach we take is to formulate the path which optimizes the probability of escape from a stable state with a basin boundary. The path which optimizes the probability of escape, the optimal path, will be found for general noise sources using a variational formulation, originally posed for driven systems in [33, 34], and recently

used to discuss Poisson induced switching in [35, 36].

The layout of the paper is such that in Sec. II we use a variational approach to determine the switching rate out of the basin of the attracting steady state. In Sec. III we apply our theory to a specific example with additive noise, while in Sec. IV we apply the theory to the case of multiplicative noise. In both cases we obtain excellent fit between numerical simulations and the theory. We finish the paper with a discussion in Sec. V.

II. GENERAL FORMULATION

We consider switching processes for the n -dimensional delay system posed as a Langevin problem:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{x}(t - \tau)) + \mathbf{g}(\mathbf{x}(t))\boldsymbol{\xi}(t), \quad (1)$$

where $\boldsymbol{\xi}(t)$ is a noise vector with independent components, and $\mathbf{g}(\mathbf{x}(t)) = \text{diag}\{g_1(t), g_2(t), \dots, g_n(t)\}$, a diagonal matrix, and $\tau > 0$. Since we are studying switching from a given state, we assume that in the absence of noise that there exists an attracting steady state, $\mathbf{x}_A$, and a saddle point, $\mathbf{x}_S$, where the latter is on the basin boundary of $\mathbf{x}_A$; i.e., $\mathbf{F}(\mathbf{x}_A, \mathbf{x}_A) = \mathbf{F}(\mathbf{x}_S, \mathbf{x}_S) = \mathbf{0}$.

General noise characterization of $\boldsymbol{\xi}$ may be accomplished through the characteristic functional [37], where

$$\tilde{\mathcal{P}}_{\boldsymbol{\xi}}[\mathbf{k}] = \left\langle \exp \left[i \int dt \mathbf{k}(t) \boldsymbol{\xi}(t) \right] \right\rangle_{\boldsymbol{\xi}}, \quad (2)$$

and where $\langle \dots \rangle_{\boldsymbol{\xi}}$ means averaging over $\boldsymbol{\xi}(t)$. For Gaussian noise, if D denotes the noise intensity, we characterize $\boldsymbol{\xi}(t)$ by its probability density functional $\mathcal{P}_{\boldsymbol{\xi}}[\boldsymbol{\xi}(t)] = \exp(-\mathcal{R}_{\boldsymbol{\xi}}/D)$,

$$\mathcal{R}_{\boldsymbol{\xi}}[\boldsymbol{\xi}(t)] = \frac{1}{4} \int dt dt' \boldsymbol{\xi}(t) \hat{\mathcal{F}}(t - t') \boldsymbol{\xi}(t'), \quad (3)$$

where $\hat{\mathcal{F}}(t - t')/2D$ is the inverse of the pair correlator of $\boldsymbol{\xi}(t)$. We also assume the noise intensity D is sufficiently small so that in our analysis sample paths will limit on an optimal path as $D \rightarrow 0$.

If we examine the tail of the distribution for a large fluctuation, assumed to be a rare event, then the probability of a large fluctuation is

$$\mathcal{P}_{\mathbf{x}}[\mathbf{x}] = \exp(-R/D), \quad R = \min \mathcal{R}[\mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\lambda}], \quad (4)$$

where

$$\mathcal{R}[\mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\lambda}] = \mathcal{R}_{\boldsymbol{\xi}}[\boldsymbol{\xi}(t)] + \int \boldsymbol{\lambda}(t) [\dot{\mathbf{x}}(t) - \mathbf{F}(\mathbf{x}, \mathbf{x}_\tau) - \mathbf{g}(\mathbf{x}(t))\boldsymbol{\xi}(t)] dt, \quad (5)$$

and $\mathbf{x}_\tau \equiv \mathbf{x}(t - \tau)$. The approach we take here is one of a Hamiltonian method, with added constraints [33, 34]. An alternative method is to use a Lagrangian description, but then one is forced to invoke an inverse function theorem to convert it a Hamiltonian. In many cases, as in general descriptions of finite population interactions [38], it is not possible to use the Lagrangian approach. In the general formulation above, the Lagrange multipliers, $\boldsymbol{\lambda}(t)$, correspond to the conjugate momenta in the Hamiltonian derivation. They will turn out to be a deterministic model of the effective force of noise on the system, which drives dynamics towards the saddle, $\mathbf{x}_S$, from the attracting state, $\mathbf{x}_A$. Once sufficiently past the saddle, the deterministic dynamics takes over and the process of escape is completed.

A. Equations of motion describing the optimal path

From here on, we concentrate on the case where we have a scalar state variable, x , and the noise is delta correlated in time. However, it is clear that the resulting equations of motion derived may easily be extended to the general

vector case. To determine the exponent, $\mathcal{R}$, we seek the equations which describe the maximum probability of the reaching the state at x_S if the initial state is x_A , where the exponent to extremize is now given by

$$\mathcal{R}[x, \xi, \lambda] = \frac{1}{2} \int \xi^2(t) dt + \int \lambda(t) [\dot{x}(t) - F(x, x_\tau) - g(x(t))\xi(t)] dt. \quad (6)$$

We derive the variation ($\delta\mathcal{R}$) by varying deviations from the path that minimizes $\mathcal{R}$. We start with the variation with respect to the noise ($\xi(t)$).

$$\frac{\delta\mathcal{R}}{\delta\xi} = \mathcal{R}[x, \xi + \eta, \lambda] - \mathcal{R}[x, \xi, \lambda] \quad (7)$$

$$= \frac{1}{2} \int [(\xi + \eta)^2 - \xi^2] dt - \int \lambda [g(\xi + \eta) - g\xi] dt \quad (8)$$

$$= \int (\xi - \lambda g)\eta dt + \mathcal{O}(\eta^2). \quad (9)$$

Since η is an arbitrary smooth function, we have

$$\xi = \lambda g. \quad (10)$$

Using similar arguments, we find the variation with respect to the Lagrange multiplier ($\lambda(t)$):

$$\dot{x} = F + g\xi. \quad (11)$$

Finally, we find the variation with respect to x . Here the delay is crucial in that it will introduce an advanced term in the equations of motion. The variation is given as:

$$\begin{aligned} \frac{\delta\mathcal{R}}{\delta x} &= \mathcal{R}[x + \eta, \xi, \lambda] - \mathcal{R}[x, \xi, \lambda] \\ &= \int \lambda(t) [\dot{\eta} - (F(x + \eta, x_\tau + \eta) - F(x, x_\tau) - [g(x + \eta) - g(x)]\xi)] dt \\ &= \int \lambda(t) \left[\dot{\eta} - \frac{\delta F(x, x_\tau)}{\delta x} \eta - \frac{\delta F(x, x_\tau)}{\delta x_\tau} \eta - \frac{\delta g(x, x_\tau)}{\delta x} \eta \xi \right] dt + \mathcal{O}(\eta^2). \end{aligned} \quad (12)$$

Using integration by parts in the usual way, and rescaling the limits we derive the equation of motion:

$$-\dot{\lambda} - \lambda(t) \frac{\partial F}{\partial x}(x, x_\tau) - \lambda(t + \tau) \frac{\partial F}{\partial x_\tau}(x_{-\tau}, x) - \lambda(t) \frac{\partial g}{\partial x}(x) \xi(t) = 0. \quad (13)$$

The three coupled equations derived above may be rewritten as a second order system:

$$\dot{x} = \lambda g^2(x) + F(x, x_\tau) \quad (14)$$

$$\dot{\lambda} = -\lambda^2 g(x) \frac{\partial g}{\partial x}(x) - \lambda \frac{\partial F}{\partial x}(x, x_\tau) - \lambda_{-\tau} \frac{\partial F}{\partial x_\tau}(x_{-\tau}, x), \quad (15)$$

which has Hamiltonian

$$H(x, x_\tau, \lambda) = \frac{\lambda^2 g^2(x)}{2} + \lambda F(x, x_\tau). \quad (16)$$

Notice the appearance of advanced terms in the equations of motion. The equations of motion in this formulation are given by:

$$\begin{aligned} \dot{x}_o &= \frac{\partial H}{\partial p}(x_o, p_o, x_{o\tau}) \\ \dot{p}_o &= -\frac{\partial H}{\partial x}(x_o, p_o, x_{o\tau}) - \frac{\partial H}{\partial x_\tau}(x_o(t + \tau), p_o(t + \tau), x_o(t)). \end{aligned} \quad (17)$$

An alternative derivation of Eq. 17 is presented in the appendix for arbitrary dimensions. When using the small noise techniques for n -dimensional stochastic systems without delay, the additional conjugate momenta doubles the phase space to $2n$. Attractors become saddles, and saddles remain saddles but in $2n$ -dimensional space. If delay is now included, we discover the equations of motion contain both retarded and advanced terms both of magnitude τ , effectively doubling the time history to 2τ as well as the phase space dimension to $2n$.

B. Boundary Conditions

Recall that in the absence of noise ($\xi(t) = 0$), F has two steady states: an attractor and a saddle; i.e., $F(x_A, x_A) = F(x_S, x_S) = 0$. Taking the limit as $t \rightarrow \infty$, we assume $x(t) \rightarrow x_A$, $\xi(t) \rightarrow 0$, and $\lambda(t) \rightarrow 0$. For escape from the basin of x_A , as $t \rightarrow \infty$, we assume $x(t) \rightarrow x_S$, $\xi(t) \rightarrow 0$, and $\lambda(t) \rightarrow 0$.

To verify the conjugate momenta have the right limits, we linearize Eq. (15) about steady state values:

$$-\dot{\Lambda} - \Lambda(t) \frac{\partial F}{\partial x}(\bar{x}, \bar{x}) - \Lambda(t + \tau) \frac{\partial F}{\partial x_\tau}(\bar{x}, \bar{x}) = 0, \quad (18)$$

where $\bar{x}$ is either x_A or x_S . Assuming $\Lambda = e^{at}$,

$$-a - \frac{\partial F}{\partial x}(\bar{x}, \bar{x}) - e^{a\tau} \frac{\partial F}{\partial x_\tau}(\bar{x}, \bar{x}) = 0. \quad (19)$$

We impose sufficient conditions on F so that the linearized solution has the correct asymptotic limits when the delay is small. More specifically, for the examples we consider in the following sections, we use Eq. (19) to compute the growth constant a and confirm that we obtain the correct limiting behavior under the influence of noise.

C. Computing the exponent along the optimal path

With the assumptions above for the equations of motion and asymptotic boundary conditions, computing the action $\mathcal{R}$ requires the solution to the two point boundary value problem. We suppose that the solution exists for the problem when $\tau = 0$, and that solutions for non-zero delay remain close to the zero delay solution. We do not necessarily need to assume the delay is small, as done in [19]. Then we may compute perturbations to the action.

In order to compute the correction to the action, assume that perturbations to the optimal path when $\tau \neq 0$ remain small. That is, if $x(t)$ is the solution when $\tau = 0$, then the perturbation $\delta_\tau x(t) \equiv x(t) - x(t - \tau)$ should remain small. We will need to put conditions on the growth of a general nonlinear function, F , so also assume $|F(x, x) - F(x, x_\tau)| \lesssim \kappa|x - x_\tau|$.

The action may now be expressed as the following perturbation problem:

$$\mathcal{R}[x, \xi, \lambda] = \mathcal{R}_0[x, \xi, \lambda] + \mathcal{R}_1[x, \xi, \lambda], \quad (20)$$

where

$$\mathcal{R}_0[x, \xi, \lambda] = \frac{1}{2} \int \xi^2(t) dt + \int \lambda(t) [\dot{x}(t) - F(x, x) - g(x(t))\xi(t)] dt \quad (21)$$

$$\mathcal{R}_1[x, \xi, \lambda] = \int \lambda(t) [F(x, x) - F(x, x_\tau)] dt. \quad (22)$$

The minimizing solution is solved by first considering the equations that minimize $\mathcal{R}_0$, denoted by $[x_o, \xi_o, \lambda_o]$. The first order correction is then evaluated at the zeroth order solution. The minimizing solutions satisfy

$$\dot{x}_o = \lambda_o g^2(x_o) + F(x_o, x_o) \quad (23)$$

$$\dot{\lambda}_o = -\lambda_o^2 g(x_o) \frac{\partial g}{\partial x}(x_o) - \lambda_o \frac{\partial F}{\partial x}(x_o, x_o) - \lambda_o \frac{\partial F}{\partial x_\tau}(x_o, x_o). \quad (24)$$

The general solution to Eq. (21) will have the form

$$\lambda_o(t) = -2 \frac{F(x_o(t), x_o(t))}{g^2(x_o(t))}, \text{ and } \dot{x}_o(t) = -F(x_o(t), x_o(t)). \quad (25)$$

In the next two sections we will consider a specific example of F for both additive ($g(x) = 1$) and multiplicative ($g(x) = \sqrt{x}$) noise. Using Eqs. (23)-(25) we compute the action, $\mathcal{R}$, and then the switching rate out of the attracting basin.

III. ADDITIVE NOISE SWITCHING

A natural place to discuss the escape rate is to consider additive noise; i.e., $g(x) \equiv 1$. For additive noise, either the Hamiltonian or Lagrangian formulation will work to describe the trajectory of the switching optimal path. For additive noise, the second order delay system in Eq. (14) is equivalent to the equations of motion derived from the Lagrangian, L , which is presented in the appendix. In the following example, we use the variational formulation to find the escape rate exponents for a scalar delay example.

A. Analysis for a given function F :

As an example, consider $F(x, x_\tau) = x(1 - x) - \gamma x_\tau$, where the steady states are $x_A = 1 - \gamma$, $x_S = 0$, and $0 \leq \gamma < 1$. We first analyze the stability near equilibria to verify that $\lambda(t)$ approaches 0 as $t \rightarrow \pm\infty$. Let $\Lambda(t)$ be the linearized solution to Eq. (19) about the steady states. Assuming $\Lambda(t) = \exp(at)$, for small τ we find that near x_A , $a = \frac{1-\gamma}{1-\tau\gamma} > 0$, implying $\Lambda(t) \rightarrow 0$ as $t \rightarrow -\infty$. On the other hand, in a neighborhood of x_S we find $a = -\frac{1-\gamma}{1-\tau\gamma} < 0$, and $\Lambda(t) \rightarrow 0$ as $t \rightarrow \infty$.

To compute the optimal path from x_A to x_S that minimizes $\mathcal{R}_0$, we use Eq. (25), which for $g(x) = 1$ are $\dot{x}_o(t) = -F(x_o(t), x_o(t))$, and $\lambda_o(t) = -2F(x_o(t), x_o(t))$. From Eq. (10) we also have that the optimal noise is given by $\xi_o(t) = \lambda_o(t)$. For the example $F(x, x_\tau)$ above, $\mathcal{R}_0$ has as its solution $x_o(t) = x_A/(1 + e^{x_A t})$. Notice that $\lim_{t \rightarrow \infty} x_o(t) = 0 = x_S$ and $\lim_{t \rightarrow -\infty} x_o(t) = 1 - \gamma = x_A$. In addition, it easy to show $\mathcal{R}_0 = x_A^3/3$.

Using $F(x, x) - F(x, x_\tau) = -\gamma(x - x_\tau)$, we now find the first order action,

$$\mathcal{R}_1 = -2\gamma \int_{-\infty}^{\infty} \dot{x}_o(t)(x_o(t) - x_o(t - \tau))dt. \quad (26)$$

Substituting for x_o , the first order results for the action are now given by,

$$\mathcal{R}_1 \approx -2\gamma \left(-\frac{x_A^2}{2} + \frac{x_A^2 e^{x_A \tau} (e^{x_A \tau} - 1 - x_A \tau)}{(1 - e^{x_A \tau})^2} \right). \quad (27)$$

The full action to first order in $\delta x(t)$ is then

$$\mathcal{R}(\tau) \approx \frac{x_A^3}{3} - 2\gamma \left(-\frac{x_A^2}{2} + \frac{x_A^2 e^{x_A \tau} (e^{x_A \tau} - 1 - x_A \tau)}{(1 - e^{x_A \tau})^2} \right). \quad (28)$$

Recall $x_A = 1 - \gamma$. Asymptotically, the expression for small τ becomes:

$$\mathcal{R}(\tau) \approx \frac{(1 - \gamma)^3}{3} (1 - \gamma \tau) + \mathcal{O}(\tau^2). \quad (29)$$

This is our relationship for the action on the delay.

B. Numerical Comparisons

We now compare the theoretical results with numerical simulations. In particular, we have computed the mean switching time as the noise drives the system out of the basin of attraction of x_A to $x < x_S$. In each of Figs. 1-3 the solid curve represents the theory, while the data points are the mean values taken over 1000 simulations. To insure that trajectory doesn't drift back to the original basin we require that the trajectory be driven well past x_S . More specifically, we define the switching time to be when $x < -0.2$ and find that this threshold gives consistently good results.

The switching rate is proportional to the probability of large fluctuations $\mathcal{P}_{\mathbf{x}}[\mathbf{x}]$, Eq. (4), and is given by

$$W_S = c \exp(-R/D), \quad (30)$$

where for R we use the minimum value given by Eq. (29). The switching time is simply the inverse of the rate and so $T_S = 1/W$. When plotting T_S on the $\log_{10}$ scale, the constant c corresponds to a vertical shift in the theoretical result. Because we are using Gaussian noise, we compute the prefactor c based on Kramer's theory [39] and find that it is given by $c = 2\pi(1 - \gamma\tau)/(1 - \gamma)$.

In Fig. 1, we plot the switching time as a function of the inverse of the noise intensity. We see that, as expected, when the noise intensity is increased the switching time decreases and we have excellent fit between the theory and simulations over four orders of magnitude. In the top curve the delay is set to zero so that $F(x, x_\tau) = F(x, x)$, while in the bottom curve the delay is $\tau = 0.5$. We see that even though τ is not appreciably small we still have excellent fit over the entire range of the noise intensity. Further, the effect of the delay is to decrease the switching time or, in other words, the delay increases the probability of switching.

In Fig. 2 we fix the noise intensity at two different values and vary the delay τ . As mentioned above, increasing the delay decreases the switching time and our theory based on small τ maintains good fidelity up almost $\tau = 1$; for $\tau > 1$ the theory and simulation results begin to diverge.

In Fig. 3 the delay is fixed but the strength of the delay term γ is varied from zero to almost one. Recall that $x_A = 1 - \gamma$ so that increasing γ moves the attractor closer saddle steady state. The effect is that the noise has to drive the system a shorter distance before switching occurs and, hence, the switching time decreases. We note that the theoretical curve deviates from the simulation results as γ gets close to 1; this is due to a singularity in the expression for the shift constant c .

Finally, the results for the switching time as a function of γ provide insight into why, for this system, the delay

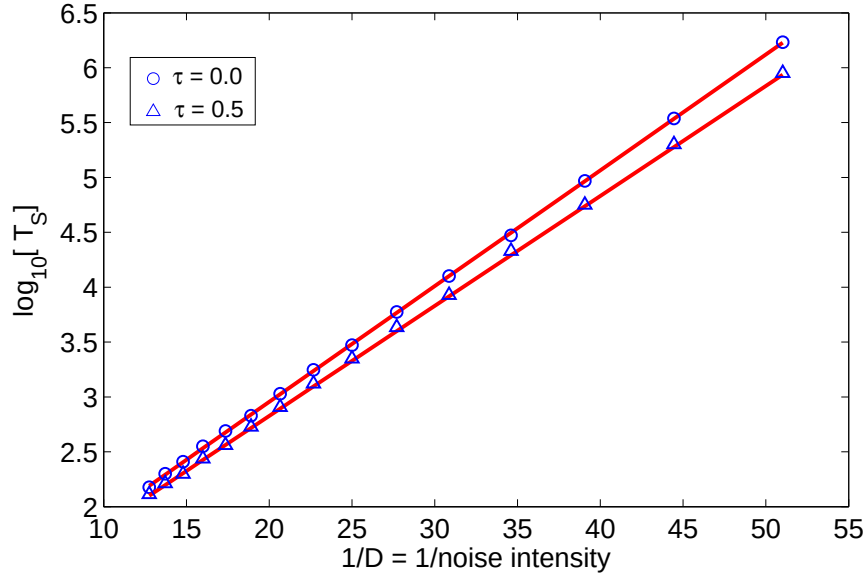


Figure 1: Log of escape time vs. inverse noise intensity ($\gamma = 0.1$). The solid curve represents the theory from Eqs. (29)-(30), while the data points are the mean values taken over 1000 simulations.

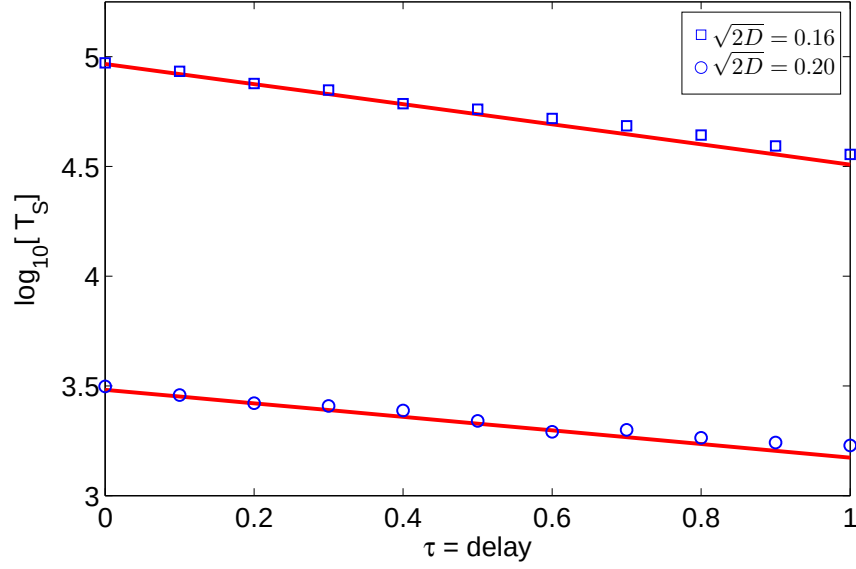


Figure 2: Log of escape time vs. delay ($\gamma = 0.1$). The solid curve represents the theory from Eqs. (29)-(30), while the data points are the mean values taken over 1000 simulations.

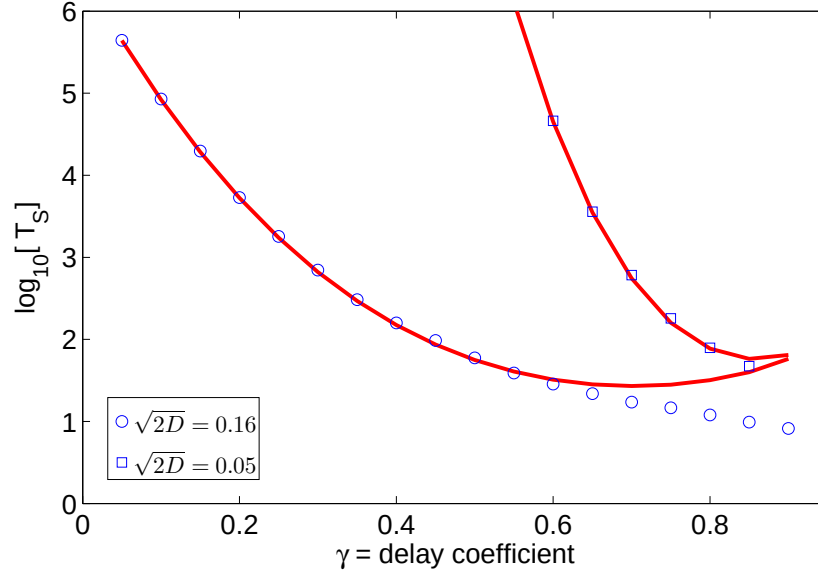


Figure 3: Log of escape time vs. dissipation ($\tau = 0.1$). The solid curve represents the theory from Eqs. (29)-(30), while the data points are the mean values taken over 1000 simulations.

decreases the switching time. For small x the deterministic repulsive force of the saddle is

$$\dot{x} \sim x - \gamma x(t - \tau). \quad (31)$$

For larger τ the second term takes values of x further back in the history of x , which because x is generally decreasing from x_A to $x_S = 0$, are larger. The second term being larger decreases the repulsive force of the saddle. Thus, as we

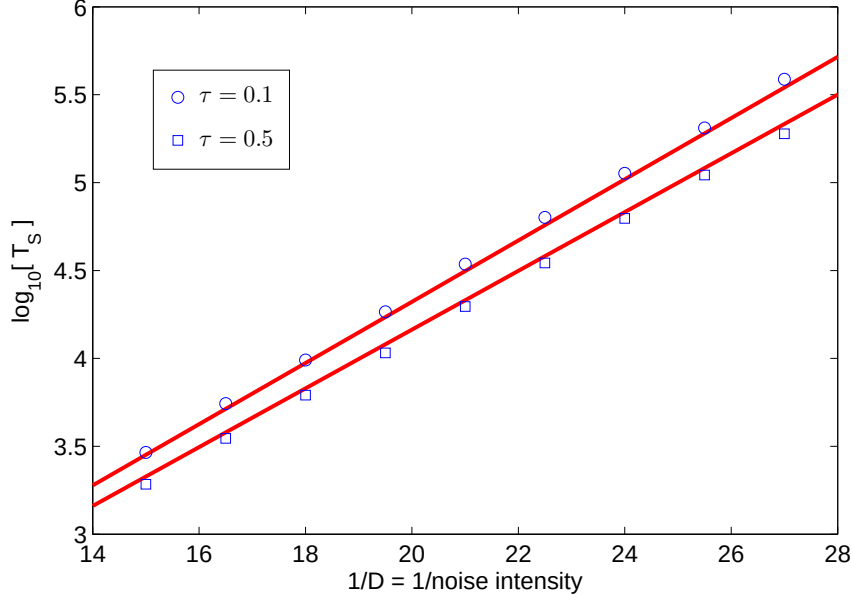


Figure 4: Log of escape time vs. inverse noise intensity ($\gamma = 0.1$). The solid curve represents the theory from Eqs. (30) and (35), while the data points are the mean values taken over 2000 simulations.

have found, the delay makes it easier for the noise to drive the system out of the basin of x_A and the switching time decreases.

IV. MULTIPLICATIVE NOISE SWITCHING

We now consider the switching rate for multiplicative noise to illustrate the generality of the variational approach. Let the multiplicative function be $g(x) = \sqrt{x}$. The path that minimizes Eq. (21) satisfies $\lambda_o(t) = -2F(x_o(t), x_o(t))/g^2(x_o(t))$ and $\dot{x}_o(t) = -F(x_o(t), x_o(t))$. For the example $F(x, x_\tau)$ above, $\mathcal{R}_0$ has as its solution $x_o(t) = x_A/(1 + e^{x_A t})$ and $\lambda_o(t) = -2(1 - x_o - \gamma)$. We evaluate

$$\mathcal{R}_0 = \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{\dot{x}_o - F(x_o, x_o)}{\sqrt{x_o}} \right)^2 dt = \frac{1}{2} \int_{-\infty}^{\infty} \frac{[-2F(x_o, x_o)]^2}{x_o} dt = x_A^2 \quad (32)$$

Using $F(x, x) - F(x, x_\tau) = -\gamma(x - x_\tau)$, we now find the first order action,

$$\mathcal{R}_1 = -2\gamma \int_{-\infty}^{\infty} \frac{\dot{x}_o(t)[x_o(t) - x_o(t - \tau)]}{x_o(t)} dt \quad (33)$$

After substituting for x_o and evaluating the integrals for $\mathcal{R}_1$ and $\mathcal{R}_\infty$, we find that

$$\mathcal{R}(\tau) \approx x_A^2 - 2\gamma \left(-x_A + \frac{x_A^2 \tau e^{x_A \tau}}{-1 + e^{x_A \tau}} \right). \quad (34)$$

Asymptotically, the expression for small τ becomes:

$$\mathcal{R}(\tau) \approx x_A^2(1 - \gamma\tau) + \mathcal{O}(\tau^2). \quad (35)$$

Again, we compare the theoretical results with numerical simulations. For multiplicative noise, we have computed the mean switching time as the noise drives the system out of the basin of attraction of x_A to $x < x_S$. In each of

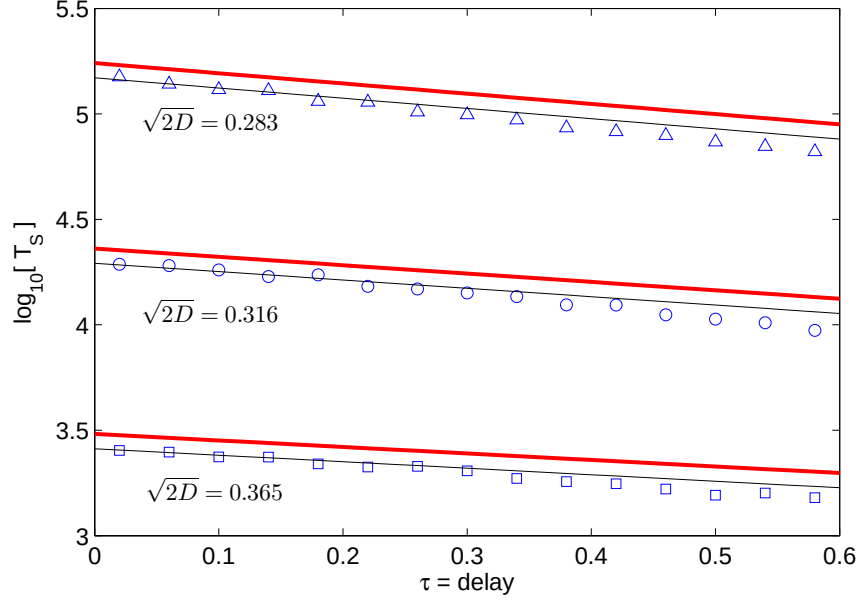


Figure 5: Log of escape time vs. delay ($\gamma = 0.1$). The bold solid curve represents the theory from Eqs. (30) and (35), and it is shifted vertically by -0.07 and plotted as the light solid curve to show agreement with the data in rate of change. The data points are the mean values taken over 2000 simulations.

Figs. 4-6 the solid curve represents the theory, while the data points are the mean values taken over 2000 simulations. To insure that trajectory doesn't drift back to the original basin we define the switching time to be when $x < -0.01$. To find the theoretical switching time, we approximate the switching rate using Eq. (30), with the minimum $\mathcal{R}$ given by Eq. (35), and the switching time is $T_S = 1/W$. Here we use the shift given in the case when the delay is small [40].

In Fig. 4, we plot the switching time for the multiplicative noise case as a function of the inverse of the noise intensity. As expected, when the noise intensity is increased the switching time decreases. In the top curve the delay is set to $\tau = 0.1$, while in the bottom curve the delay is $\tau = 0.5$. As before, the effect of the delay is to decrease the switching time or, in other words, the delay increases the probability of switching.

In Fig. 5, we fix the noise intensity at three different values and vary the delay τ . There is a small shift offsetting the data and the theory. In this figure, the theory was shifted vertically by -0.07 and replotted by the light black solid curve to show agreement with the data in rate of change. Our theory based on small τ maintains good fidelity up almost $\tau = 0.5$; for $\tau > 0.5$ the theory and simulation results begin to diverge.

In Fig. 6, the delay is fixed but the strength of the delay term γ is varied from zero to almost one. We note that the theoretical curve deviates from the simulation results as γ gets close to 1; this is due to a singularity in the expression for the shift constant c .

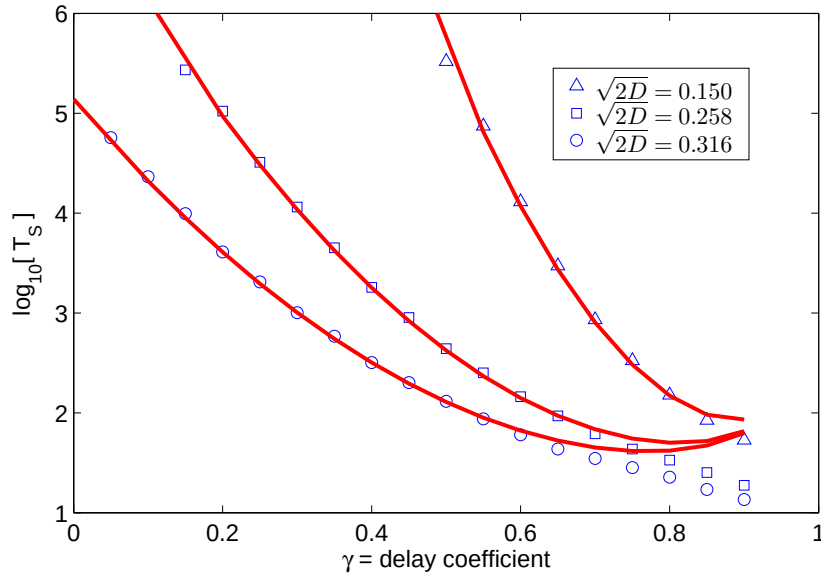


Figure 6: Log of escape time vs. dissipation ($\tau = 0.1$). The solid curve represents the theory from Eqs. (30) and (35), while the data points are the mean values taken over 2000 simulations.

V. CONCLUSIONS

We considered the problem of noise induced escape from a basin of attraction in stochastic systems having differential delay. The formulation for determining the escape was posed as a variational problem along an optimal path, which maximized the probability of escape. Since noise escape from a basin of attraction is a rare event, the optimized solution to the variational problem holds in the tail of the distribution. The scaling results derived are therefore based on using the optimized path to escape.

In contrast to many other approaches which consider Gaussian processes, the approach considered here is quite general in that it allows for any general noise, such as Poisson, Schott, or colored noise. It generalizes the problems of non-Gaussian noise induced escape considered in [35, 38, 41] by extending the formulation to include delay.

We applied the theory to scalar problems with both additive and multiplicative noise to explicitly compute the solution of the optimal path to escape. With the optimal solution in hand, we derived relations describing how the escape rate scales with respect to various parameters, such as delay, dissipation, and noise intensity. The perturbation theory used in the analysis was based upon a Melnikov approach. Here, the delay itself does not need to be small, but the escape path with delay is assumed to remain close to the path with no delay. Then, a first order correction to the integral along the optimal path may be computed.

The escape rate results found analytically were compared in each case to numerically simulated solutions. Even though a small delay approximation is used to explicitly compute the escape exponents, we find excellent agreement between theory and simulation for large delay, even when they are of order unity. Specifically, in some cases for the examples considered here, the escape rate agreement between theory and numerics holds for delays much longer than the relaxation time of the system.

Finally, in comparing the numerical simulations to the theory, we show where theory and numerical simulations

diverge. We expect that the approach considered here will shed light on many other escape processes where the stochastic forcing renders the problems non-Markovian in nature, as well as many other applications physical and biological experiments.

Acknowledgments

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- [1] T. Erneux, Applied Delay Differential Equations (Springer, 2009).
 - [2] R. E. Bellman and K. L. Cooke, Differential-Difference Equations (Academic Press, New York, 1963).
 - [3] A. Krasnovkii, Stability of motion and translation (Stanford University Press, Stanford, 1963).
 - [4] J. K. Hale, Theory of Functional Differential Equations (Springer-Verlag, New York, 1977).
 - [5] G. M. Wang, E. M. Sevick, E. Mittag, D. J. Searles, and D. J. Evans, Phys. Rev. Lett. **89**, 050601 (2002).
 - [6] M. Taylor and T. Carr, J. Math. Biol. **59**, 841 (2009).
 - [7] W. Guo, JOURNAL OF DYNAMICAL AND CONTROL SYSTEMS **16**, 259 (2010), ISSN 1079-2724.
 - [8] G. Stepan, T. Insperger, and R. Szalai, INTERNATIONAL JOURNAL OF BIFURCATION AND CHAOS **15**, 2783 (2005), ISSN 0218-1274, 5th EUROMECH Nonlinear Dynamics Conference (ENOC-2005), Eindhoven, NETHERLANDS, AUG 07-12, 2005.
 - [9] M. Bleich and J. Socolar, PHYSICAL REVIEW E **54**, R17 (1996), ISSN 1063-651X.
 - [10] F. ARECCHI, G. GIACOMELLI, A. LAPUCCI, and R. MEUCCI, PHYSICAL REVIEW A **43**, 4997 (1991), ISSN 1050-2947.
 - [11] T. W. Carr, I. B. Schwartz, M.-Y. Kim, and R. Roy, SIAM JOURNAL ON APPLIED DYNAMICAL SYSTEMS **5**, 699 (2006), ISSN 1536-0040.
 - [12] T. Heil, I. Fischer, W. Elsasser, J. Mulet, and C. Mirasso, PHYSICAL REVIEW LETTERS **86**, 795 (2001), ISSN 0031-9007.
 - [13] T. Heil, I. Fischer, W. Elsasser, B. Krauskopf, K. Green, and A. Gavrielides, PHYSICAL REVIEW E **67** (2003), ISSN 1063-651X.
 - [14] A. L. Franz, R. Roy, L. B. Shaw, and I. B. Schwartz, PHYSICAL REVIEW LETTERS **99** (2007), ISSN 0031-9007.
 - [15] I. B. Schwartz and L. B. Shaw, PHYSICAL REVIEW E **75** (2007), ISSN 1539-3755.
 - [16] J. Losson and M. C. Mackry, Physical Review E **52**, 115 (1995).
 - [17] S. Guillouezic, I. L'Heureux, and A. Longtin, PHYSICAL REVIEW E **59**, 3970 (1999), ISSN 1063-651X.
 - [18] T. Frank, PHYSICAL REVIEW E **66** (2002), ISSN 1063-651X.
 - [19] T. Frank, PHYSICAL REVIEW E **72** (2005), ISSN 1539-3755.
 - [20] C. Masoller, PHYSICAL REVIEW LETTERS **88** (2002), ISSN 0031-9007.

- [21] D. Wu and S. Zhu, PHYSICS LETTERS A **363**, 202 (2007), ISSN 0375-9601.
- [22] J. Houlihan, D. Goulding, T. Busch, C. Masoller, and G. Huyet, PHYSICAL REVIEW LETTERS **92** (2004), ISSN 0031-9007.
- [23] L. Tsimring and A. Pikovsky, PHYSICAL REVIEW LETTERS **87** (2001), ISSN 0031-9007.
- [24] L. J. Lapidus, D. Enzer, and G. Gabrielse, Phys. Rev. Lett. **83**, 899 (1999).
- [25] I. Siddiqi, R. Vijay, F. Pierre, C. M. Wilson, L. Frunzio, M. Metcalfe, C. Rigetti, R. J. Schoelkopf, M. H. Devoret, D. Vion, et al., Phys. Rev. Lett. **94**, 027005 (2005).
- [26] J. S. Aldridge and A. N. Cleland, Phys. Rev. Lett. **94**, 156403 (2005).
- [27] K. Kim, M. S. Heo, K. H. Lee, H. J. Ha, K. Jang, H. R. Noh, and W. Jhe, Phys. Rev. A **72**, 053402 (2005).
- [28] R. Gommers, P. Douglas, S. Bergamini, M. Goonasekera, P. H. Jones, and F. Renzoni, Phys. Rev. Lett. **94**, 143001 (2005).
- [29] C. Stambaugh and H. B. Chan, Phys. Rev. B **73**, 172302 (2006).
- [30] B. Abdo, E. Segev, O. Shtempluck, and E. Buks, J. Appl. Phys. **101**, 083909 (2007).
- [31] I. Serban and F. K. Wilhelm, Phys. Rev. Lett. **99**, 137001 (2007).
- [32] S. Kogan, Electronic Noise and Fluctuations in Solids, 2nd ed. (Cambridge University Press, Cambridge, 2008).
- [33] M. I. Dykman, Phys. Rev. A **42**, 2020 (1990).
- [34] M. I. Dykman, C. M. Maloney, V. N. Smelyanskiy, and M. Silverstein, Phys. Rev. E **57**, 5202 (1998).
- [35] L. Billings, M. I. Dykman, and I. B. Schwartz, Phys. Rev. E **78**, 051122 (2008).
- [36] L. Billings, I. Schwartz, M. McCary, A. N. Korotkov, and M. I. Dykman, Physical Review Letters **104**, 140601 (2010).
- [37] R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New-York, 1965).
- [38] M. I. Dykman, I. B. Schwartz, and A. S. Landsman, Phys. Rev. Lett. **101**, 078101 (2008).
- [39] C. W. Gardiner, Handbook of stochastic methods for physics, chemistry and the natural sciences (Springer, Berlin, 2005), 3rd ed.
- [40] E. GUARDIA and M. SAN MIGUEL, Physics Letters **109A**, 9 (1985).
- [41] I. B. Schwartz, L. Billings, M. Dykman, and A. Landsman, J. Stat. Mech. p. P01005 (2009).

VI. APPENDIX

A. Lagrangian formulation

The Lagrangian, L , and its corresponding second order equations of motion for a stochastic differential delay equation are given by:

$$L(x, \dot{x}, x_\tau) = \frac{1}{2}[\dot{x}(t) - F(x(t), x(t - \tau))]^2, \quad (36)$$

$$\frac{\partial L}{\partial x}(x, \dot{x}, x_\tau) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}, x_\tau) = -\frac{\partial L}{\partial x_\tau}(x_{-\tau}, \dot{x}_{-\tau}, x). \quad (37)$$

B. Derivation of Delayed Hamilton's Equations of Motion

When considering the Hamiltonian with delay, new equations will describe the momenta due to the addition of a time delay. This is now derived, and shows how the advanced time terms arise.

We begin by assuming the Lagrangian, $L(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}_\tau)$, of the systems is defined as a function of state vector $\mathbf{x}$, velocity $\dot{\mathbf{x}}$, and delayed state $\mathbf{x}_\tau \equiv \mathbf{x}(t - \tau)$. The action is defined as

$$J[\mathbf{x}, \dot{\mathbf{x}}] = \min \int_{-\infty}^{\infty} L(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}_\tau) dt. \quad (38)$$

The minimization is taken over all paths perturbed from the true minimum, assuming it exists. Defining the Hamiltonian using the usual Legendre transformation, we have the following minimization problem:

$$J_H[\mathbf{x}, \mathbf{p}] = \min \int_{-\infty}^{\infty} (\mathbf{p}\dot{\mathbf{x}} - H(\mathbf{x}, \mathbf{p}, \mathbf{x}_\tau)) dt, \quad (39)$$

where $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}_\tau)$. We assume sufficient hypotheses so that the inverse function theorem applies. We let the optimum path be given by $\mathbf{x} = \mathbf{x}_o(t)$, $\mathbf{p} = \mathbf{p}_o(t)$, and consider smooth perturbations from the path as $\boldsymbol{\eta}(t)$, $\boldsymbol{\xi}(t)$, respectively so that $\lim_{t \rightarrow \pm\infty} \boldsymbol{\eta}(t) = \mathbf{0}$, and similarly for $\boldsymbol{\xi}(t)$. That is, the perturbed paths satisfy the natural boundary conditions.

Consider the variation from the path:

$$\delta J_H[\mathbf{x}_o, \mathbf{p}_o] = J[\mathbf{x}_o + \boldsymbol{\eta}, \mathbf{p}_o + \boldsymbol{\xi}] - J[\mathbf{x}_o, \mathbf{p}_o] \quad (40)$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} [\mathbf{p}_o \dot{\boldsymbol{\eta}} + \dot{\mathbf{x}}_o \boldsymbol{\xi} - \frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}_o, \mathbf{p}_o, \mathbf{x}_{o\tau}) \boldsymbol{\eta} - \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}_o, \mathbf{p}_o, \mathbf{x}_{o\tau}) \boldsymbol{\xi} \\ &\quad - \frac{\partial H}{\partial \mathbf{x}_\tau}(\mathbf{x}_o, \mathbf{p}_o, \mathbf{x}_{o\tau}) \boldsymbol{\eta}_\tau] dt + O[|\boldsymbol{\eta}|^2 + |\boldsymbol{\xi}|^2]. \end{aligned} \quad (41)$$

Using the following identities,

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbf{p}_o \dot{\boldsymbol{\eta}} dt &= - \int_{-\infty}^{\infty} \dot{\mathbf{p}}_o \boldsymbol{\eta} dt, \\ \int_{-\infty}^{\infty} \frac{\partial H}{\partial \mathbf{x}_\tau}(\mathbf{x}_o, \mathbf{p}_o, \mathbf{x}_{o\tau}) \boldsymbol{\eta}_\tau dt &= \int_{-\infty}^{\infty} \frac{\partial H}{\partial \mathbf{x}_\tau}(\mathbf{x}_o(t + \tau), \mathbf{p}_o(t + \tau), \mathbf{x}_o(t)) \boldsymbol{\eta}(t) dt, \end{aligned}$$

we have that

$$\begin{aligned}
\delta J_H[\mathbf{x}_o, \mathbf{p}_0] &= \int_{-\infty}^{\infty} [(\dot{\mathbf{x}}_o - \frac{\partial \mathbf{H}}{\partial \mathbf{p}}(\mathbf{x}_o, \mathbf{p}_0, \mathbf{x}_{o\tau}))\boldsymbol{\xi}(t) \\
&\quad - (\dot{\mathbf{p}}_0 + \frac{\partial \mathbf{H}}{\partial \mathbf{x}}(\mathbf{x}_o, \mathbf{p}_0, \mathbf{x}_{o\tau}) + \frac{\partial \mathbf{H}}{\partial \mathbf{x}_\tau}(\mathbf{x}_o(t+\tau), \mathbf{p}_0(t+\tau), \mathbf{x}_o(t)))\boldsymbol{\eta}(t)]dt \\
&\quad + O[|\boldsymbol{\eta}|^2 + |\boldsymbol{\xi}|^2].
\end{aligned}$$

Since the perturbations to the path are arbitrary, we get Hamilton's equations of motion for a delayed stochastic system given by Eq. (17).

$$\begin{aligned}
\dot{\mathbf{x}}_o &= \frac{\partial \mathbf{H}}{\partial \mathbf{p}}(\mathbf{x}_o, \mathbf{p}_0, \mathbf{x}_{o\tau}) \\
\dot{\mathbf{p}}_0 &= -\frac{\partial \mathbf{H}}{\partial \mathbf{x}}(\mathbf{x}_o, \mathbf{p}_0, \mathbf{x}_{o\tau}) - \frac{\partial \mathbf{H}}{\partial \mathbf{x}_\tau}(\mathbf{x}_o(t+\tau), \mathbf{p}_0(t+\tau), \mathbf{x}_o(t)).
\end{aligned} \tag{42}$$