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Tracking Targets with Specified Spectra Using the H-PMHT Algorithm

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PREFACE

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The H-PMHT algorithm is applied to sensor-level, multi-target tracking problems in which target and noise power spectra are assumed to be specified *a priori*. The resulting algorithm, based on the expectation-maximization method, is equivalent to a bank of iteratively reweighted smoothing filters. These results constitute a potentially important algorithm for tracking multiple targets in hyperspectral image sequences.

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TRACKING TARGETS WITH SPECIFIED SPECTRA USING THE H-PMHT ALGORITHM

1. INTRODUCTION

The function of sensor-level, multi-target tracking at the output of a spatial-spectral sensor signal processor (e.g., multi-band remote imaging systems, array beamformers) is to follow sources, or targets, as they traverse the sensor measurement space. If target and noise power spectra are specified *a priori*, as is assumed herein, the multi-target tracking problem is reduced to following targets as they move across the spatial component of the sensor measurement space. In this case, target spectra improve the ability to estimate target spatial location, especially in difficult scenarios such as crossing targets.

The H-PMHT (Histogram-Probabilistic Multi-Hypothesis Tracking) algorithm (see Streit [1], Walsh et al. [2], Streit et al. [3]) is extended to treat the case of targets and noise having specified spectra. The sensor is assumed to be a linear system; consequently, target and noise spectra are linearly superimposed at the sensor output. The histogram model in the H-PMHT algorithm is equivalent to linear superposition; that is, the foundation of the spectral H-PMHT algorithm is a model of linear superposition in the sensor output. Another assumption of the spectral H-PMHT algorithm is that successive "snapshots" of spatial-spectral sensor data are statistically independent when conditioned on the collection of target states. Conditional independence models are traditional for sequences of sensor data obtained from a measurement process. The discussion here assumes that the spatial-spectral distribution of each target separates into a product of spatial and spectral factors; however, the more general case in which target spatial extent is a deterministic function of frequency is treated in the appendix.

The presentation of the spectral H-PMHT algorithm assumes familiarity with the technical details and notation of Streit [1] and with the method of expectation-maximization (EM) as it is applied to multi-target tracking (see Streit and Luginbuhl [4]). Closely related work involving parameterized spectral mixture models in a tracking application includes Luginbuhl [5] and Luginbuhl and Willett [6, 7]. To facilitate comparisons with their work, a detailed algorithm is presented for Gaussian spectra.

2. MODIFIED SENSOR CELL STRUCTURE

The notation of Streit [1] is adopted here, but it is specialized to treat sensor cells that are multi-dimensional rectangles, i.e., Cartesian products of spatial and spectral cells. The sensor cells $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_S\}$ are the Cartesian products of the U disjoint spatial cells $\{\mathcal{D}_1, \ldots, \mathcal{D}_U\}$ and the V disjoint spectral cells $\{\mathcal{E}_1, \ldots, \mathcal{E}_V\}$. The particular choice of spatial and spectral cells is application dependent. The total number of sensor cells $S = UV$, and every cell $\mathcal{C}_\ell$ can be written in the form

$$\mathcal{C}_\ell = \mathcal{D}_i \times \mathcal{E}_j \tag{1}$$

for some (unique) choice of the cells $\mathcal{D}_i$ and $\mathcal{E}_j$. Let $\mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_U = R^{\dim \mathcal{D}}$ and $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_V = R^{\dim \mathcal{E}}$, where R denotes the real number line.

The set of sensor cells from which measurements are available at time t is denoted by $\mathcal{M}(t)$. This set may vary from scan to scan and may be an arbitrary subset of $\mathcal{C}$. Discussing general measurement sets $\mathcal{M}(t)$ requires cumbersome notation; instead, a special case is presented, and from this discussion the general $\mathcal{M}(t)$ case is easily treated. It is assumed that the number of cells $L(t)$ in $\mathcal{M}(t)$ is given by

$$L(t) = U(t)V(t), \quad 1 \le U(t) \le U, \quad 1 \le V(t) \le V,$$

and that $\mathcal{M}(t)$ is the Cartesian product of the spatial cells $\{D_1(t), \ldots, D_{U(t)}(t)\}$ and the spectral cells $\{E_1(t), \ldots, E_{V(t)}(t)\}$, where Roman fonts are used (in place of corresponding script fonts) to denote that these lists are linearly ordered. The $(i,j)^{th}$ cell in $\mathcal{M}(t)$ is thus $D_i(t) \times E_j(t)$. The sensor measurement vector at time t is denoted by

$$Z_t = \{z_{t,1,1}, \ldots, z_{t,U(t),V(t)}\},$$

where z_{tij} is the output spectral power of the sensor at time t in the cell $D_i(t) \times E_j(t)$. To facilitate the discussion of truncated cells, the linear ordering of the spatial cells is extended (arbitrarily) so that

$$\{D_1(t), \ldots, D_{U(t)}(t), D_{U(t)+1}(t), \ldots, D_U(t)\}$$

is an ordered list of all U cells, and similarly for the spectral cell ordering. These orderings change from scan to scan to accommodate different subsets $\mathcal{M}(t)$.

The spatial-spectral sensor cell structure in equation (1) facilitates simplifications to the basic equations of Streit [1]. Let X_t denote the set of target states at time t. The cell probability for the $(i,j)^{th}$ cell takes the form (*cf.* [1, equation (6)])

$$P_{ij}(X_t) = \int_{D_i(t) \times E_j(t)} f(u, v | X_t)\, du\, dv, \tag{2}$$

where the sample PDF $f(u, v|X_t)$ is defined over all $(u, v) \in R^{\dim \mathcal{D}} \times R^{\dim \mathcal{E}} = R^{\dim \mathcal{C}}$ by the mixture (*cf.* [1, equation (30)])

$$f(u, v|X_t) = \sum_{k=0}^{M} \pi_{tk} \, G_k(u, v|X_t), \tag{3}$$

and where the component $G_k(u, v|X_t)$ corresponds to target k if $k \geq 1$ and to noise if $k = 0$.

The total sensor probability (*cf.* [1, equation (7)]) at time t becomes

$$P(X_t) = \sum_{i=1}^{U(t)} \sum_{j=1}^{V(t)} P_{ij}(X_t). \tag{4}$$

The expected measurement $\bar{z}_{tij}$ takes the form (*cf.* [1, equation (55)])

$$\bar{z}_{tij} = \begin{cases} z_{tij}, & \text{for } 1 \leq i \leq U(t),\ 1 \leq j \leq V(t) \\ \|Z_t\| \, \frac{P_{ij}(X_t')}{P(X_t')}, & \text{for } U(t)+1 \leq i \leq U,\ V(t)+1 \leq j \leq V, \end{cases} \tag{5}$$

where

$$\|Z_t\| = \sum_{i=1}^{U(t)} \sum_{j=1}^{V(t)} z_{tij}.$$

The auxiliary functions become (*cf.* [1, equation (60)])

$$Q_{t\pi} = \sum_{k=0}^{M} \left[\sum_{i=1}^{U} \sum_{j=1}^{V} \frac{\bar{z}_{tij}}{P_{ij}(X_t')} \int_{D_i(t) \times E_j(t)} G_k(u, v|x_{tk}') \, du \, dv \right] \pi_{tk}' \log \pi_{tk} \tag{6}$$

and (*cf.* [1, equation (61)])

$$Q_{kX} = \sum_{t=1}^{T} \frac{\|Z_t\|}{P(X_t')} \log p_{\Xi_{t,k}|\Xi_{t-1,k}}(x_{tk}|x_{t-1,k}) + \sum_{t=1}^{T} \sum_{i=1}^{U} \sum_{j=1}^{V} \frac{\pi_{tk}' \, \bar{z}_{tij}}{P_{ij}(X_t')} \int_{D_i(t) \times E_j(t)} G_k(u, v|x_{tk}') \log G_k(u, v|x_{tk}) \, du \, dv. \tag{7}$$

If the noise component model is specified *a priori*, the auxiliary function Q_{kX} is needed only for $k = 1, \ldots, M$.

3. MODIFIED AUXILIARY FUNCTION

The spectral PDF of target k is denoted by $\mathcal{S}_k(v)$, so that

$$\int_{\mathcal{E}} \mathcal{S}_k(v)\, dv = 1.$$

The spectral PDF is equal to the traditional power spectrum normalized so that its integral over $\mathcal{E}$ is one. If the desired noise spectral PDF $\mathcal{S}_0(v)$ is white, it is constrained to integrate to one by making it constant over an appropriately specified finite subset of $\mathcal{E}$. To simplify the current discussion, target spatial and spectral characteristics are assumed to separate, i.e., the component $G_k(u, v|x_{tk})$ of the sample PDF factors into the form

$$G_k(u, v|x_{tk}) = \mathcal{S}_k(v)\, \mathcal{G}_k(u|x_{tk}), \quad k = 1, \ldots, M, \tag{8}$$

where $\mathcal{G}_k(u|x_{tk})$ is the spatial PDF of target k. The more general case in which the spatial PDF depends on frequency is discussed in the appendix. The product form (8) enables multiple integrals over $D_i(t) \times E_j(t)$ to be rewritten as products of integrals, so that

$$\int_{D_i(t)\times E_j(t)} G_k(u, v|x'_{tk})\, du\, dv = \int_{E_j(t)} \mathcal{S}_k(v)\, dv \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\, du \tag{9}$$

and, using the mixture (3) and the definition (2),

$$P_{ij}(X'_t) = \sum_{k=0}^{M} \pi'_{tk} \int_{E_j(t)} \mathcal{S}_k(v)\, dv \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\, du. \tag{10}$$

Similarly,

$$\begin{aligned} &\int_{D_i(t)\times E_j(t)} G_k(u, v|x'_{tk}) \log G_k(u, v|x_{tk})\, du\, dv \\ &\quad = \int_{E_j(t)} \mathcal{S}_k(v)\, dv \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk}) \log \mathcal{G}_k(u|x_{tk})\, du \\ &\qquad + \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\, du \int_{E_j(t)} \mathcal{S}_k(v) \log \mathcal{S}_k(v)\, dv. \end{aligned} \tag{11}$$

Substituting (9) into (6) gives

$$Q_{t\pi} = \sum_{k=0}^{M} \left[\sum_{i=1}^{U} \left(\sum_{j=1}^{V} \frac{\bar{z}_{tij} \int_{E_j(t)} \mathcal{S}_k(v)\, dv}{P_{ij}(X'_t)} \right) \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\, du \right] \pi'_{tk} \log \pi_{tk}, \tag{12}$$

and substituting (11) into (7) gives

$$Q_{kX} = \sum_{t=1}^{T} \frac{\|Z_t\|}{P(X_t')} \log p_{\Xi_{t,k}|\Xi_{t-1,k}}(x_{tk}|x_{t-1,k})$$
$$+ \sum_{t=1}^{T} \pi_{tk}' \sum_{i=1}^{U} \left(\sum_{j=1}^{V} \frac{\bar{z}_{tij} \int_{E_j} \mathcal{S}_k(v)\, dv}{P_{ij}(X_t')} \right) \int_{D_i(t)} \mathcal{G}_k(u|x_{tk}') \log \mathcal{G}_k(u|x_{tk})\, du. \quad (13)$$

The second term in (11) contributes an additional term to (13), but it is omitted because it depends on x_{tk}' and not on x_{tk} and is not needed in the M-step of the EM method.

4. LINEAR GAUSSIAN CASE WITH SPECIFIED TARGET SPECTRA

Estimates of $\{\widehat{\pi}_{tk}\}$ are obtained using a Lagrangian multiplier technique in the same manner as in Streit [1]. The result is, from (12), for $k = 0, 1, \ldots, M$,

$$\widehat{\pi}_{tk} = \frac{\pi'_{tk}}{\lambda_t} \sum_{i=1}^{U} \left(\sum_{j=1}^{V} \frac{\bar{z}_{tij} \int_{E_j(t)} \mathcal{S}_k(v)\, dv}{P_{ij}(X'_t)} \right) \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\, du, \tag{14}$$

where

$$\begin{aligned} \lambda_t &= \sum_{k=0}^{M} \pi'_{tk} \left[\sum_{i=1}^{U} \left(\sum_{j=1}^{V} \frac{\bar{z}_{tij} \int_{E_j(t)} \mathcal{S}_k(v)\, dv}{P_{ij}(X'_t)} \right) \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\, du \right] \\ &= \sum_{i=1}^{U} \sum_{j=1}^{V} \bar{z}_{tij}, \end{aligned} \tag{15}$$

an identity obtained by making the sum over k innermost and using (10).

Estimating the state variables requires assuming specific parametric forms. Linear Gaussian target and measurement process models are adopted here. These forms are, for $k = 1, \ldots, M$,

$$p_{\Xi_{t,k}|\Xi_{t-1,k}}(x_{tk}|x_{t-1,k}) = \mathcal{N}(x_{tk}; F_{t-1,k}\, x_{t-1,k}, Q_{t-1,k}), \tag{16}$$

and

$$\mathcal{G}_k(u|x_{tk}) = \mathcal{N}(u; H_{tk}\, x_{tk}, R_{tk}). \tag{17}$$

The noise component can be very general; here, the noise is assumed to be specified *a priori* at every time t. If the desired noise PDF $\mathcal{G}_0(u|x_{t0}) \equiv \mathcal{G}_0(u)$ is spatially white, it is constrained to integrate to one by making it constant over an appropriately specified finite subset of $\mathcal{D}$.

Estimates for the state variables can be obtained by setting the gradient of the auxiliary function Q_{kX} to zero and solving; however, an alternative approach is taken here because it exploits the Kalman filter as an efficient computational algorithm. Completing the square on the state variables $X(k) = \{x_{0k}, x_{1k}, \ldots, x_{Tk}\}$ of target k in the expression for Q_{kX} and exponentiating the result, as in Streit and Luginbuhl [4], gives the expression (*cf.* [1, equation (75)])

$$e^{Q_{kX}} \propto \prod_{t=1}^{T} \mathcal{N}(x_{tk}; F_{t-1,k}\, x_{t-1,k}, \widetilde{Q}_{t-1,k})\, \mathcal{N}(\widetilde{z}_{tk}; H_{tk}\, x_{tk}, \widetilde{R}_{tk}), \tag{18}$$

where (*cf.* [1, equations (70) and (71), respectively])

$$\widetilde{Q}_{tk} = \frac{P(X'_{t+1})}{\|Z_{t+1}\|} Q_{tk}, \qquad 0 \le t \le T-1,$$

$$\widetilde{R}_{tk} = \frac{R_{tk}}{\pi'_{tk}\,\nu_{tk}}, \qquad 1 \le t \le T,$$

and the synthetic spatial measurements are (*cf.* [1, equation (74)])

$$\widetilde{z}_{tk} = \frac{1}{\nu_{tk}} \sum_{i=1}^{U} \left(\sum_{j=1}^{V} \frac{\bar{z}_{tij} \int_{E_j(t)} S_k(v)\,dv}{P_{ij}(X'_t)} \right) \int_{D_i(t)} u\,\mathcal{N}(u; H_{tk}\,x'_{tk}, R_{tk})\,du, \tag{19}$$

where (*cf.* [1, equation (72)])

$$\nu_{tk} = \sum_{i=1}^{U} \left(\sum_{j=1}^{V} \frac{\bar{z}_{tij} \int_{E_j(t)} S_k(v)\,dv}{P_{ij}(X'_t)} \right) \int_{D_i(t)} \mathcal{N}(u; H_{tk}\,x'_{tk}, R_{tk})\,du. \tag{20}$$

State estimates $\widehat{X}(k) = \{\widehat{x}_{tk}\}$ are, therefore, efficiently computed via a Kalman filter because of the form of the likelihood function (18). Further details are given in Streit [1] for converting these expressions for a batch algorithm into a recursive algorithm for a sliding batch.

The synthetic measurement expression (19) is now rewritten as a convex combination of synthetic "broadband" data obtained by marginalizing over the synthetic target spectrum, defined for target k in spatial cell $D_i(t)$ as

$$\widetilde{S}_{tki}(j) = \bar{z}_{tij} \frac{\pi'_{tk} \int_{E_j(t)} S_k(v)\,dv \int_{D_i(t)} \mathcal{N}(u; H_{tk}\,x'_{tk}, R_{tk})\,du}{\sum_{\kappa=0}^{M} \pi'_{t\kappa} \int_{E_j(t)} S_\kappa(v)\,dv \int_{D_i(t)} \mathcal{N}(u; H_{t\kappa}\,x'_{t\kappa}, R_{t\kappa})\,du}, \qquad j = 1, \ldots, V. \tag{21}$$

The broadband power of the synthetic spectrum $\widetilde{S}_{tki}(j)$ is therefore

$$\|\widetilde{S}_{tki}\| = \sum_{j=1}^{V} \widetilde{S}_{tki}(j), \qquad i = 1, \ldots, U. \tag{22}$$

Referring to $\|\widetilde{S}_{tki}\|$ as target broadband power in spatial cell $D_i(t)$ is reasonable because it is non-negative for all indices t, k, and i, and because

$$\sum_{k=0}^{M} \|\widetilde{S}_{tki}\| = \sum_{j=1}^{V} \bar{z}_{tij}$$

is the total (expected) power in $D_i(t)$. The target synthetic spectra (21) are insightful functions that pervade the technical discussion and, thus, are of independent interest. They are also potentially important in applications.

The spatial cell centroid $\widetilde{z}_{tki}$ for target k is given by

$$\widetilde{z}_{tki} = \frac{\int_{D_i(t)} u\,\mathcal{N}(u; H_{tk}\,x'_{tk}, R_{tk})\,du}{\int_{D_i(t)} \mathcal{N}(u; H_{tk}\,x'_{tk}, R_{tk})\,du}, \qquad i = 1, \ldots, U. \tag{23}$$

Each centroid $\widetilde{z}_{tki}$ lies in the spatial cell $D_i(t)$ because it is the mean of a PDF whose support is $D_i(t)$. Multiplying and dividing in (19) and (20) by the integral over $D_i(t)$, substituting (20) into (19), and rewriting the result gives

$$\widetilde{z}_{tk} = \frac{\sum_{i=1}^{U} \left\|\widetilde{S}_{tki}\right\| \widetilde{z}_{tki}}{\sum_{i=1}^{U} \left\|\widetilde{S}_{tki}\right\|}. \tag{24}$$

Thus, from (24), the spatial synthetic measurement $\widetilde{z}_{tk}$ for each target k is a convex combination of the spatial cell centroids $\{\widetilde{z}_{tki} : i = 1, \ldots, U\}$, and the coefficients of the convex combination are proportional to the broadband powers $\left\{\left\|\widetilde{S}_{tki}\right\| : i = 1, \ldots, U\right\}$ of the k^{th} target's synthetic spectra.

Writing the updated mixing proportions (14) in terms of the target synthetic spectra gives

$$\widehat{\pi}_{tk} = \frac{\sum_{i=1}^{U} \sum_{j=1}^{V} \widetilde{S}_{tki}(j)}{\sum_{\kappa=0}^{M} \sum_{i=1}^{U} \sum_{j=1}^{V} \widetilde{S}_{t\kappa i}(j)}, \tag{25}$$

or

$$\widehat{\pi}_{tk} = \frac{\sum_{i=1}^{U} \sum_{j=1}^{V} \widetilde{S}_{tki}(j)}{\sum_{i=1}^{U} \sum_{j=1}^{V} \bar{z}_{tij}}, \tag{26}$$

results that are easily verified by direct substitution. Expression (25) is insightful but computationally inefficient; however, expression (26) is potentially useful in applications.

5. LINEAR GAUSSIAN CASE WITH PARAMETERIZED TARGET SPECTRA

In this section it is assumed that a parametric form of the target spectral PDF is specified, and that the spectral parameters are estimated from the measured data using the EM method. Let $\mathcal{S}_k(v; y_{tk})$ denote the spectral PDF of component k at time t, where y_{tk} is the spectral parameter vector. Thus, (8) becomes

$$G_k(u, v|x_{tk}; y_{tk}) = \mathcal{G}_k(u|x_{tk})\,\mathcal{S}_k(v; y_{tk}), \tag{27}$$

and (11) becomes

$$\begin{aligned}\int_{D_i(t)\times E_j(t)} & G_k(u, v|x'_{tk}; y'_{tk}) \log G_k(u, v|x_{tk}; y_{tk})\,du\,dv \\ &= \int_{E_j(t)} \mathcal{S}_k(v; y'_{tk})\,dv \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk}) \log \mathcal{G}_k(u|x_{tk})\,du \\ &\qquad + \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\,du \int_{E_j(t)} \mathcal{S}_k(v; y'_{tk}) \log \mathcal{S}_k(v; y_{tk})\,dv. \end{aligned} \tag{28}$$

Letting $Y'_t = \{y'_{t0}, y'_{t1}, \ldots, y'_{tM}\}$ denote the vector of the current parameter values for noise ($k = 0$) and targets ($k > 0$), the integral (9) becomes

$$\int_{D_i(t)\times E_j(t)} G_k(u, v|x'_{tk}; y'_{tk})\,du\,dv = \int_{E_j(t)} \mathcal{S}_k(v; y'_{tk})\,dv \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\,du, \tag{29}$$

and the total probability (10) becomes

$$P_{ij}(X'_t, Y'_t) = \sum_{k=0}^{M} \pi'_{tk} \int_{E_j(t)} \mathcal{S}_k(v; y'_{tk})\,dv \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\,du. \tag{30}$$

Using expressions (27)–(30) gives rise to slightly modified forms of the H-PMHT auxiliary functions (6)–(7). The auxiliary function for the mixing proportions is

$$Q_{t\pi} = \sum_{k=0}^{M} \left[\sum_{i=1}^{U} \left(\sum_{j=1}^{V} \frac{\bar{z}_{tij} \int_{E_j(t)} \mathcal{S}_k(v; y'_{tk})\,dv}{P_{ij}(X'_t, Y'_t)} \right) \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\,du \right] \pi'_{tk} \log \pi_{tk}, \tag{31}$$

and it leads to essentially the same estimator (14). The auxiliary function for the state variables is

$$\begin{aligned} Q_{kX} &= \sum_{t=1}^{T} \frac{\|Z_t\|}{P(X'_t)} \log p_{\Xi_{t,k}|\Xi_{t-1,k}}(x_{tk}|x_{t-1,k}) \\ &\quad + \sum_{t=1}^{T} \pi'_{tk} \sum_{i=1}^{U} \left(\sum_{j=1}^{V} \frac{\bar{z}_{tij} \int_{E_j} \mathcal{S}_k(v; y'_{tk})\,dv}{P_{ij}(X'_t, Y'_t)} \right) \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk}) \log \mathcal{G}_k(u|x_{tk})\,du \end{aligned} \tag{32}$$

so that, for linear Gaussian models (16)–(17), state estimates are computed via a Kalman smoothing filter in the same manner as in Streit [1]. The auxiliary function for the spectral parameters is

$$Q_{kY} = \sum_{t=1}^{T} \pi'_{tk} \sum_{j=1}^{V} \left(\sum_{i=1}^{U} \frac{\bar{z}_{tij} \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\, du}{P_{ij}(X'_t, Y'_t)} \right) \int_{E_j(t)} \mathcal{S}_k(v; y'_{tk}) \log \mathcal{S}_k(v; y_{tk})\, dv. \quad (33)$$

This expression is obtained from the second term in (28), but it is omitted if the spectral parameters are not estimated.

The spectral PDF of target k is now assumed to be Gaussian distributed; that is,

$$\mathcal{S}_k(v; y_{tk}) = \mathcal{N}(v; \mu_{tk}, S_{tk}), \quad (34)$$

where $y_{tk} = \{\mu_{tk}, S_{tk}\}$. The parametric form of the spatial dependence function $\mathcal{G}_k(u|x_{tk})$ is left unspecified here. The synthetic target spectra in the current step of the algorithm are given by

$$\widetilde{\mathcal{S}}_{tki}(j) = \bar{z}_{tij} \frac{\pi'_{tk} \int_{E_j(t)} \mathcal{N}(v; \mu'_{tk}, S'_{tk})\, dv \int_{D_i(t)} \mathcal{G}_k(u|x'_{tk})\, du}{\sum_{\kappa=0}^{M} \pi'_{t\kappa} \int_{E_j(t)} \mathcal{N}(v; \mu'_{t\kappa}, S'_{t\kappa})\, dv \int_{D_i(t)} \mathcal{G}_\kappa(u|x'_{t\kappa})\, du}.$$

If (34) holds, Q_{kY} is easily maximized by setting its gradients with respect to μ_{tk} and S_{tk} to zero and solving for $\widehat{\mu}_{tk}$ and $\widehat{S}_{tk}$. The resulting estimators separate hierarchically in the usual way, and they are given by

$$\widehat{\mu}_{tk} = \frac{1}{\lambda_{tk}} \sum_{i=1}^{U} \sum_{j=1}^{V} \widetilde{\mathcal{S}}_{tki}(j) \frac{\int_{E_j(t)} v\, \mathcal{N}(v; \mu'_{tk}, S'_{tk})\, dv}{\int_{E_j(t)} \mathcal{N}(v; \mu'_{tk}, S'_{tk})\, dv}, \quad (35)$$

and

$$\widehat{S}_{tk} = \frac{1}{\lambda_{tk}} \sum_{i=1}^{U} \sum_{j=1}^{V} \widetilde{\mathcal{S}}_{tki}(j) \frac{\int_{E_j(t)} (v - \widehat{\mu}_{tk})(v - \widehat{\mu}_{tk})^* \mathcal{N}(v; \mu'_{tk}, S'_{tk})\, dv}{\int_{E_j(t)} \mathcal{N}(v; \mu'_{tk}, S'_{tk})\, dv}, \quad (36)$$

where the normalization constant is given by

$$\lambda_{tk} = \sum_{i=1}^{U} \sum_{j=1}^{V} \widetilde{\mathcal{S}}_{tki}(j). \quad (37)$$

These estimators are readily interpreted as convex combinations of cell-level mean and variance contributions.

Explicit spectral parameter estimates can also be derived for other families of parameterizations of the target spectral PDFs. For example, the spectral PDF can be a Gaussian mixture. For further details on the use of mixtures to estimate spectral PDFs from histogram data, see Luginbuhl [5, chapter 2].

6. SUMMARY

The incorporation of specified target and noise power spectra directly into the multi-target tracking problem is accomplished using the H-PMHT algorithm. The synthetic H-PMHT measurements for this application are a pair of closely related quantities, namely, synthetic spatial measurements for each target and synthetic spectra for each target in every spatial cell. These synthetic data are used to write the spectral H-PMHT algorithm as a bank of recursively reweighted smoothing filters. The resulting spectral H-PMHT algorithm constitutes a potentially important extension of existing tracking methodologies that may be useful in several application areas—for example, hyperspectral data obtained by remote imaging systems.

APPENDIX
FREQUENCY DEPENDENT SPATIAL DENSITY FUNCTIONS

The general case in which the target spatial density varies with frequency is discussed in this appendix. It is assumed that the spectral cells $\{E_1(t), \ldots, E_{V(t)}(t)\}$ composing the measurement set $\mathcal{M}(t)$ are bounded for all t, a physically realistic sensor model. The factorization (8) is replaced (using Bayes Theorem) by the conditional product

$$G_k(u, v|x_{tk}) = \mathcal{S}_k(v)\, \mathcal{G}_k(u|v, x_{tk}). \tag{38}$$

Integrating (38) over the cell $D_i(t) \times E_j(t)$ gives (*cf.* (9))

$$\int_{D_i(t)\times E_j(t)} G_k(u, v|x'_{tk})\, du\, dv = \int_{E_j(t)} \left\{ \mathcal{S}_k(v) \int_{D_i(t)} \mathcal{G}_k(u|v, x'_{tk})\, du \right\} dv. \tag{39}$$

The term in braces in (39) is a function of v alone. It is assumed that this term is continuous on $\overline{E_j(t)}$, the closure of set $E_j(t)$. (This assumption is reasonable in many applications, but it does have potentially important implications. For example, if the spatial function $\mathcal{G}_k(u|v, x'_{tk})$ is a continuous function of v, the spectrum $\mathcal{S}_k(v)$ is continuous and so must correspond to a signal having no "idealized" narrowband components.) Applying the Mean Value Theorem of the calculus to the integral of this term gives, for some point $v_j(t) \in \overline{E_j(t)}$,

$$\int_{E_j(t)} \mathcal{S}_k(v) \int_{D_i(t)} \mathcal{G}_k(u|v, x'_{tk})\, du\, dv = |E_j(t)|\; \mathcal{S}_k(v_j(t)) \int_{D_i(t)} \mathcal{G}_k(u|v_j(t), x'_{tk})\, du, \tag{40}$$

where $|E_j(t)| \equiv \int_{E_j(t)} dv < \infty$. If the spectrum is constant over $E_j(t)$, then

$$|E_j(t)|\; \mathcal{S}_k(v_j(t)) = \int_{E_j(t)} \mathcal{S}_k(v)\, dv. \tag{41}$$

If the spectrum is not constant over $E_j(t)$, equation (41) is only an approximation, but one whose quality improves with decreasing size of the cell $E_j(t)$.

Substituting (41) into (40), and the result into (39) gives

$$\int_{D_i(t)\times E_j(t)} G_k(u, v|x'_{tk})\, du\, dv = \int_{E_j(t)} \mathcal{S}_k(v)\, dv \int_{D_i(t)} \mathcal{G}_k(u|v_j(t), x'_{tk})\, du. \tag{42}$$

The exact location of the point $v_j(t) \in \overline{E_j(t)}$ is unknown. Replacing it with an *a priori* specified point in $E_j(t)$, denoted by $\widehat{v}_j(t)$, gives the approximation

$$\int_{D_i(t)\times E_j(t)} G_k(u, v|x'_{tk})\, du\, dv \cong \int_{E_j(t)} \mathcal{S}_k(v)\, dv \int_{D_i(t)} \mathcal{G}_k(u|\widehat{v}_j(t), x'_{tk})\, du. \tag{43}$$

Because $\widehat{v}_j(t)$ is known, the approximation (43) may be used to replace the factorization of equation (9) throughout the discussion.

The primary impact of using the approximation (43) on the spectral H-PMHT algorithm is that spatial integrals now depend on the spectral index j via the parameter $\widehat{v}_j(t)$, so that certain double summations cannot be evaluated as efficiently as in the frequency-independent case. The numerical details depend on the parameterizations. For example, if the specified spatial form of $\mathcal{G}_k(u|\widehat{v}_j(t), x_{tk})$ is given by (*cf.* equation (17))

$$\mathcal{G}_k(u|\widehat{v}_j(t), x_{tk}) = \mathcal{N}\left(u; H_{tk}\, x_{tk},\; R_{tk}/\left(\gamma\, \widehat{v}_j(t)\right)^2\right), \tag{44}$$

where $\widehat{v}_j(t)$ is the midpoint of the cell $E_j(t)$ and γ is a specified constant, then the estimate of the synthetic target spectrum $\widetilde{\mathcal{S}}_{tki}(j)$ becomes (*cf.* equation (21)), for $j = 1, \ldots, V$,

$$\widetilde{\mathcal{S}}_{tki}(j) = \bar{z}_{tij}\, \frac{\pi'_{tk} \int_{E_j(t)} \mathcal{S}_k(v)\, dv \int_{D_i(t)} \mathcal{N}\left(u; H_{tk}\, x'_{tk},\; \left(\gamma\, \widehat{v}_j(t)\right)^{-2} R_{tk}\right) du}{\sum_{\kappa=0}^{M} \pi'_{t\kappa} \int_{E_j(t)} \mathcal{S}_\kappa(v)\, dv \int_{D_i(t)} \mathcal{N}\left(u; H_{t\kappa}\, x'_{t\kappa},\; \left(\gamma\, \widehat{v}_j(t)\right)^{-2} R_{tk}\right) du}. \tag{45}$$

The expected measurements $\bar{z}_{tij}$ are also modified by the parameterization (44); however, further details are omitted.

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