

# Frictional torque on a rotating disc

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## Abstract

Resistance to motion often includes a dry frictional term independent of the speed of an object and a fluid drag term varying linearly with speed in the viscous limit. (At higher speeds, quadratic drag can also occur.) Here, measurements are performed for an aluminium disc mounted on bearings that is given an initial twist and allowed to spin until it comes to a stop. It is found that a sum of both the dry and fluid resistive terms are needed to accurately fit the entire data. However, the speed-independent term alone suffices over any suitably limited range of motion, permitting one to use the standard kinematic equations for constant angular acceleration. The measurements and theory are appropriate for an introductory physics laboratory at the college level.

(Some figures may appear in colour only in the online journal)

In a recent paper, the rotational dynamics of a large cylindrical disc mounted on bearings were investigated [1]. A ‘let it roll’ experiment was used to quantify the frictional torque  $\tau_f$  by measuring the decrease in the angular speed  $\omega$  of the disc as a function of time  $t$  after giving it an initial spin  $\omega_0$ . Based on a previous paper [2], it was assumed that the resistive torque varies linearly with the speed,

$$\tau_f = b\omega, \quad (1)$$

where  $b$  is a constant coefficient. Using the rotational form of Newton’s second law,

$$\tau_f = -I \frac{d\omega}{dt}, \quad (2)$$

equation (1) is separated and integrated to obtain

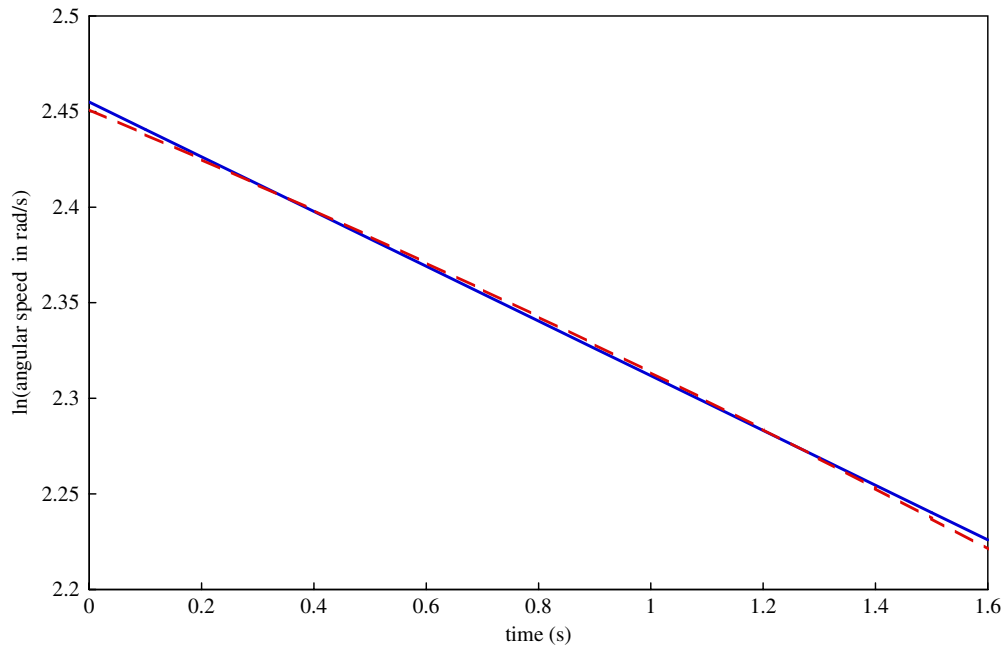
$$\omega(t) = \omega_0 \exp(-bt/I). \quad (3)$$

To test this model for the drag,  $\ln(\omega)$  was plotted versus  $t$ , resulting in a linear fit to the experimental data with an intercept of  $\ln(\omega_0)$  and a slope of  $-b/I$ . That straight-line fit has been reproduced as the solid curve in figure 1 over the same horizontal and vertical axis ranges used in [1].

However, in another recent paper [3] which referenced previous work on the damping of a physical pendulum [4], the frictional torque at an axle was instead assumed to be constant, independent of the angular speed of motion,

$$\tau_f = a \quad (4)$$

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| 14. ABSTRACT<br><b>Resistance to motion often includes a dry frictional term independent of the speed of an object and a fluid drag term varying linearly with speed in the viscous limit. (At higher speeds, quadratic drag can also occur.) Here measurements are performed for an aluminium disc mounted on bearings that is given an initial twist and allowed to spin until it comes to a stop. It is found that a sum of both the dry and fluid resistive terms are needed to accurately fit the entire data. However, the speed-independent term alone suffices over any suitably limited range of motion, permitting one to use the standard kinematic equations for constant angular acceleration. The measurements and theory are appropriate for an introductory physics laboratory at the college level.</b>                               |                                    |                                     |                                                                  |                                                     |                                    |
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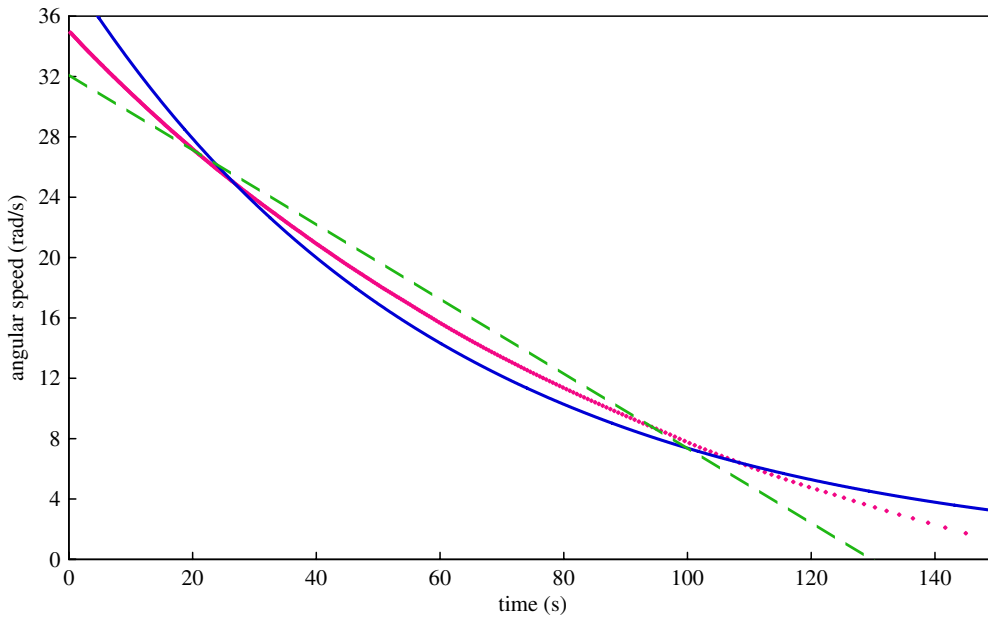
**Figure 1.** Natural logarithm of the angular speed as a function of time. The solid blue curve reproduces the fit to equation (3) from figure 4 of [1] using  $\omega_0 = 11.65 \text{ rad s}^{-1}$  and  $b/I = 0.1432 \text{ s}^{-1}$ . The dashed red curve is a plot of equation (5) using  $\omega_0 = 11.60 \text{ rad s}^{-1}$  and  $a/I = 1.489 \text{ s}^{-2}$ .

for as long as the system is rotating. If this expression is substituted into equation (2), one deduces

$$\omega(t) = \omega_0 - at/I \quad (5)$$

instead of equation (3). Equations (1) and (4) can be compared by setting  $a = b\omega_{\text{ave}}$ . The average angular speed over the linearized vertical range of the data in figure 1 is  $\omega_{\text{ave}} = 10.4 \text{ rad s}^{-1}$ . The fit values  $b = 6.50 \times 10^{-5} \text{ kg} \cdot \text{m}^2 \text{ s}^{-1}$  and  $I = 4.54 \times 10^{-4} \text{ kg} \cdot \text{m}^2$  from [1] then imply  $a/I = 1.489 \text{ s}^{-2}$ . A plot of equation (5) has been overlaid onto figure 1 as the dashed curve using this value of  $a/I$ . The difference between the solid blue and dashed red curves is much smaller than the jitter in the experimental data in figure 4 of [1]. Thus either a model of purely linear Stokes drag (equation (1)) or of purely constant dry friction (equation (4)) gives an acceptable fit to the 1.6 s range of data. To experimentally distinguish between these (or other) assumed forms of the resistive torque, less noisy measurements are required and over a longer timescale.

For this purpose, a standard ‘centripetal force apparatus’ (Sargent–Welch model CP93000) was used, consisting of a vertical rotating shaft to which a small metal flag was attached that passes through a photogate once each revolution. An aluminium disc was bolted to the top of the shaft; the disc had a mass of  $M = 1246 \pm 1 \text{ g}$  and a diameter of  $2R = 152.5 \pm 0.5 \text{ mm}$ , so that its moment of inertia is  $I = \frac{1}{2}MR^2 = 3.62 \pm 0.03 \text{ g} \cdot \text{m}^2$ . The photogate was connected to a computer running LoggerPro via a LabPro interface, and the times  $t_i$  that the leading edge of the flag crossed the light beam were recorded for revolutions  $i = 1, 2, 3, \dots$ . The angular speed (averaged over a revolution) was then determined from the time interval between adjacent crossings,  $\Delta t_{i,i+1} = t_{i+1} - t_i$ , as  $\omega_{i,i+1} = 2\pi / \Delta t_{i,i+1}$ . That angular speed is



**Figure 2.** Measured values of the angular speed of the disc as a function of time are indicated by the red dots, one per revolution. Best fits of these data points to equations (3) and (5) are indicated by the solid blue and dashed green curves, respectively.

plotted versus time  $t = (t_{i+1} + t_i)/2$ , averaged over the same interval, as the dots in figure 2 after giving the disc a vigorous initial spin by hand.

Equations (3) and (5) have been respectively fit to these data as exponential and linear trendlines in Excel, shown in figure 2. One sees that the equations give comparable but poor fits to the experimental points. In contrast, an excellent fit is achieved by assuming that both dry friction and fluid drag are simultaneously acting on the disc, so that

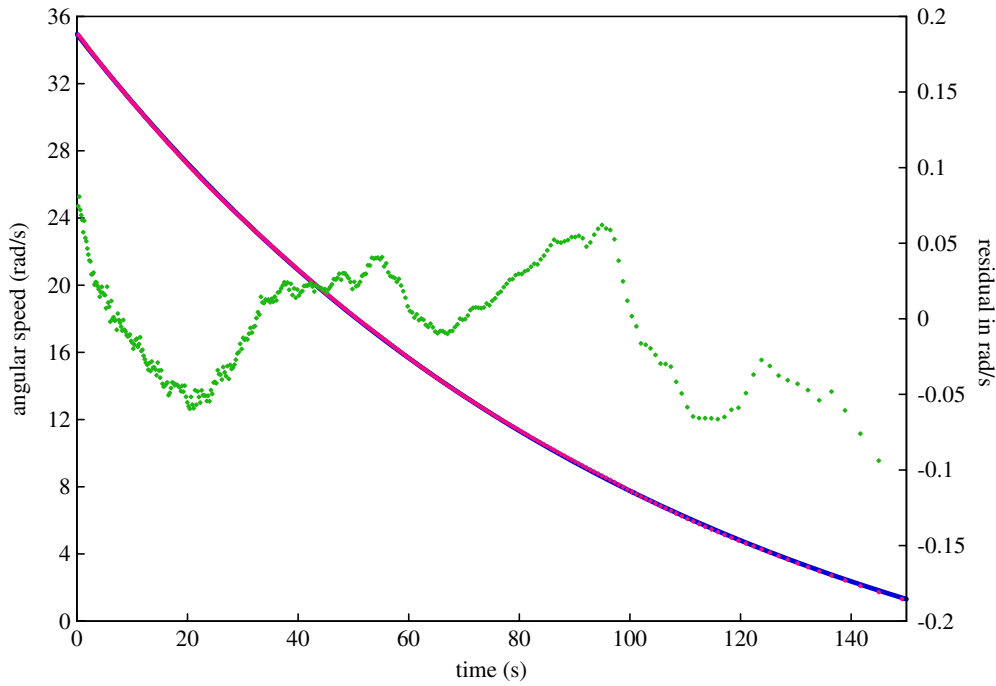
$$\tau_f = a + b\omega, \quad (6)$$

where  $a$  and  $b$  are constants. Setting the right-hand sides of equations (2) and (6) equal to each other gives

$$\omega(t) = (\omega_0 + a/b) \exp(-bt/I) - a/b \quad (7)$$

after separating variables and integrating. The blue curve in figure 3 is a plot of this equation with the fitted values  $\omega_0 = 34.94 \pm 0.06 \text{ rad s}^{-1}$ ,  $a/I = 0.08804 \pm 0.00015 \text{ s}^{-2}$ , and  $b/I = 0.009565 \pm 0.000010 \text{ s}^{-1}$ . (This nonlinear fit can be performed directly in LoggerPro, making it accessible to introductory physics students.) The blue curve passes right through the red data points in figure 3. The residuals, computed by subtracting the fit value from each experimental point, are plotted in green. One sees from them that the experimental and theoretical angular speeds agree to better than  $0.1 \text{ rad s}^{-1}$  across the entire range of data. There is no systematic trend in the residuals, which would indicate quadratic Newtonian drag or other simple damping terms are missing from equation (6).

Equation (7) by inspection reduces to equation (3) if one sets  $a = 0$ . It is slightly trickier to see how equation (7) reduces to (5) when  $b = 0$ . One easy way to proceed is to assume  $|bt/I| \ll 1$  so that  $\exp(-bt/I) \approx 1 - bt/I$ , expand the product of that pair of terms with  $\omega_0 + a/b$ , subtract off  $a/b$ , and then finally set  $b = 0$ .



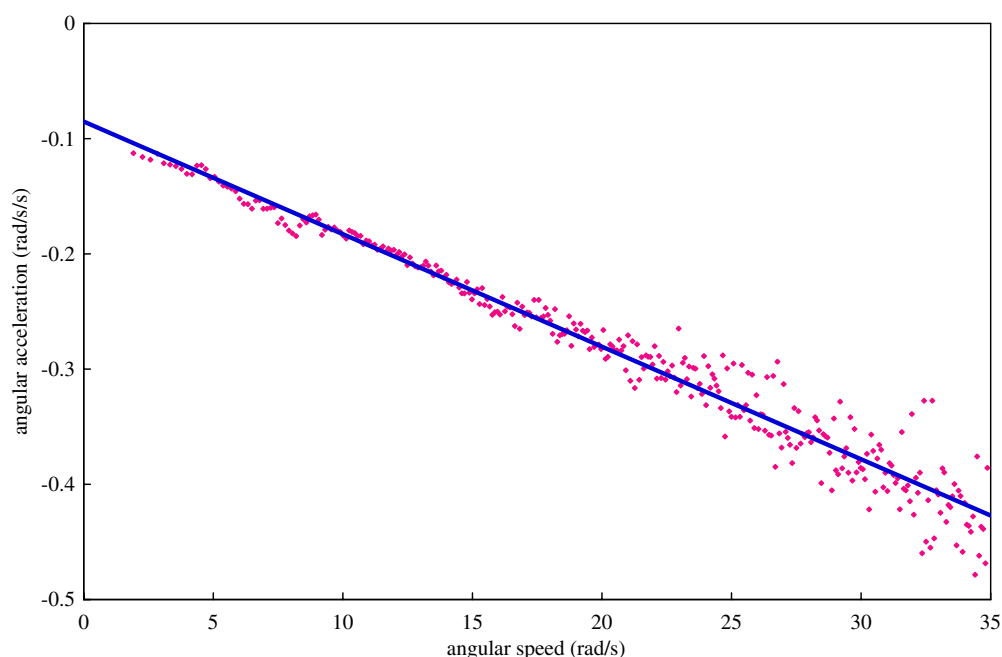
**Figure 3.** Graph of the angular speed of the disc as a function of time (left-hand vertical axis): the experimental measurements indicated by the red dots repeat the data presented in figure 2, while the continuous blue line is a fit to equation (7). The difference between the experimental and fitted values of the angular speeds are plotted as the green dots, one per revolution, on the scale of the right-hand vertical axis.

An alternative to the nonlinear fit of the data to equation (7) is to instead plot the angular acceleration  $\alpha \equiv d\omega/dt$  versus the angular speed  $\omega$ . Combining equations (2) and (6) implies

$$\alpha = -\frac{a}{I} - \frac{b}{I}\omega. \quad (8)$$

One thus expects a simple linear trend, where the intercept and slope directly give the two resistive coefficients and one can immediately see if either term can be neglected on the scale of the scatter in the data. Recalling that  $\omega_{i,i+1} = 2\pi/\Delta t_{i,i+1}$  where  $\Delta t_{i,i+1} = t_{i+1} - t_i$ , the angular acceleration is computed as  $\alpha = 2(\omega_{i+1,i+2} - \omega_{i,i+1})/(t_{i+2} - t_i)$  and plotted against  $\omega = (\omega_{i+1,i+2} + \omega_{i,i+1})/2$  as the red dots in figure 4. The blue curve is a fit of equation (8) to these experimental points. The resulting coefficients are  $a/I = 0.0852 \pm 0.0024 \text{ s}^{-2}$  and  $b/I = 0.00977 \pm 0.00011 \text{ s}^{-1}$ , similar to those listed following equation (7) above, but with substantially larger uncertainties. One can see that the scatter in the data is much larger than in figure 3, as a result of computing the second-order time derivative  $\alpha$ . Hence, from a numerical point of view, it is preferable to perform the nonlinear fit of figure 3. Nevertheless, linearizing the data as in figure 4 has the advantage that one can immediately evaluate the relative importance of the two drag coefficients by eye.

By examining equation (6), one can see that the linear damping term proportional to  $b$  can be neglected compared to the speed-independent term  $a$  over some range of angular speeds  $\Delta\omega$  of interest if  $\Delta\omega \ll a/b$ . For example, in figure 3 any data interval for which the angular speed changes by significantly less than  $9 \text{ rad s}^{-1}$  will be well fit using equation (4).



**Figure 4.** Experimental values (red dots) of  $\alpha$  versus  $\omega$  and a linear best fit (blue curve) of them to equation (8). The correlation coefficient for the fit is  $R = 0.981$ .

This reasoning explains why figure 1, in which  $\omega$  decreases by only  $2.5 \text{ rad s}^{-1}$ , is accurately fit by equation (5).

In conclusion, frictional torque terms proportional to both the first and zeroth powers of the angular speed are needed to accurately fit the deceleration of an undriven wheel rotating to rest (as evidenced by figures 2 and 3). However, over a limited range of angular speeds (such as in figure 1), it suffices to include only the speed-independent term. This simplified form for the resistive torque is analogous to the speed-independent model of the kinetic frictional force conventionally used for an unlubricated object translationally sliding to rest on a solid surface. It enables use of the standard constant angular acceleration equations of rotational kinematics. By using those equations and starting at a moderate angular speed (say  $5 \text{ rad s}^{-1}$  for the apparatus used here), the damping coefficient can be quickly estimated as

$$a/I = 2\theta_{\text{stop}}/t_{\text{stop}}^2 \quad (9)$$

by measuring the time  $t_{\text{stop}}$  and number of revolutions  $N_{\text{stop}} = \theta_{\text{stop}}/2\pi$  that it takes for the disc to come to a full stop. One can then proceed to more interesting experiments of the motion of the system driven by a string wrapped around the shaft and connected to a hanging weight over a pulley. In contrast, equation (3) would imply that the disc does not suddenly cease rotating but instead gradually tails off to slower and slower speeds, unlike what one actually observes.

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