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Three-Dimensional Shape Optimization in Viscous Incompressible Flows

Final Performance Report

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Executive Summary

The integral equation constrained optimization approach to finding three-dimensional minimum-drag shapes for bodies translating in viscous incompressible fluid has been developed. The approach relies on the theory of generalized analytic functions for obtaining efficient integral equations for 3D boundary-value problems with linear partial differential equations (PDEs), e.g., the Stokes and Oseen equations for viscous incompressible fluid, linearized equations of magnetohydrodynamics (MHD) etc. It assumes the following steps: (i) identifying a class of generalized analytic functions related to the PDEs; (ii) representing corresponding fields, e.g., the fluid velocity field, the electromagnetic field etc., in terms of generalized analytic functions from the identified class; and (iii) reformulating boundary-value problems for the involved fields as boundary-value problems for generalized analytic functions, which could then be reduced to integral equations based on the generalized Cauchy integral formula. Solutions of the obtained integral equations can be represented by finite function series with series coefficients determined by quadratic error minimization. The framework of generalized analytic functions has been developed in application to 3D Stokes and Oseen flows, two-phase Stokes flows and 3D magnetohydrodynamic flows governed by linearized MHD equations.

In shape optimization problems, the suggested approach replaces 3D boundary-value problems with governing PDEs by corresponding boundary integral equations. Minimum-drag shapes, represented in finite function series form, are then found by the adjoint equation-based method with a gradient-based algorithm, in which the gradient for shape series coefficients is determined analytically. Compared to PDE constrained optimization coupled with the finite element method (FEM), the approach reduces dimensionality of the flow problems, solves the issue with region truncation in exterior problems, finds minimum-drag shapes in semi-analytical form, and has fast convergence. Its efficiency has been demonstrated in solving three drag minimization problems under the Oseen approximation of the Navier-Stokes equations for different Reynolds numbers: (i) for a body of constant volume, (ii) for a torpedo with only fore and aft noses being optimized, and (iii) for a body of constant volume following another body of fixed shape (e.g. torpedo chasing a target). The minimum-drag shapes in problem (i) are in good agreement with the existing optimality conditions and conform to those obtained by PDE constrained optimization. Problem (ii) has shown that the minimum-drag shape for the torpedo is fore-aft-symmetric, whereas problem (iii) has revealed that the minimum-drag shape for the trailing body is only slightly more prolate than the one in problem (i) for same Reynolds number.

For MHD flows in the presence of aligned magnetic fields, the necessary optimality condition for the minimum-drag shapes subject to a volume constraint has been obtained analytically. It has been shown that regardless of magnitudes of Hartmann number, Reynolds number, and magnetic Reynolds number, the minimum-drag shapes are fore-and-aft symmetric and have conic endpoints with the angle of $2\pi/3$. For fixed Reynolds and magnetic Reynolds numbers, the drag reduction as a function of the Cowling number is smallest at 1. In considered MHD problems, the drag coefficients for the minimum-drag shapes and minimum-drag spheroids are sufficiently close.

The project involved Postdoctoral Associate Anton Molyboha from the Department of Mathematical Sciences, Stevens Institute of Technology.

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Shape Optimization in Magnetohydrodynamics

1 Introduction

Industrial applications of magnetohydrodynamics (MHD) include but are not limited to MHD flow control schemes for hypersonic vehicles [19], torpedo drag reduction with MHD boundary-layer control [2], and MHD propulsion systems [25].

The steady flow of an electrically conducting viscous incompressible fluid in the presence of magnetic field and with neglected thermal effects can be characterized by three independent parameters: Hartmann number M , Reynolds number R , and magnetic Reynolds number R_m . The Hartmann number is the ratio of the Lorentz force to the viscous force in the Navier-Stokes equations (when $M = 0$, the velocity field is uncoupled from the electromagnetic field), whereas the magnetic Reynolds number is interpreted as the ratio of magnetic advection to magnetic diffusion in the combination of Ohm's and Ampere's laws (when $R_m = 0$, the magnetic field is uncoupled from the velocity field). The ratio of the magnetic forces to the inertial forces is characterized by the Cowling number S ,¹ which can be expressed as $M^2/(R_m R)$ when $R \neq 0$ and $R_m \neq 0$.

An MHD problem that has attracted much of the attention is arguably the one of an electrically conducting flow past a nonmagnetic sphere in the presence of a uniform magnetic field being aligned with the undisturbed flow; see e.g. [7, 12, 27, 13, 14]. For this problem, Figure 1 shows the drag for the sphere normalized to the sphere Stokes drag as a function of S for $R = R_m = 1$ (curve a); $R = 1, R_m = 3$ (curve b); $R = R_m = 2$ (curve c); $R = 3, R_m = 1$ (curve d); and $R = R_m = 3$ (curve e). At $S = 1$, the drag attains minimum and is nonsmooth; see [13, 12]. Namely the fact that the sphere drag at $S = 1$ is nonsmooth is remarkable. It poses the research questions: If the sphere is replaced by an arbitrary nonmagnetic body of revolution, what is body's shape that has the smallest drag subject to a volume constraint and how does it depend on S ? Are there any qualitative differences in the minimum-drag shapes for $S < 1$, $S = 1$, and $S > 1$? How does drag reduction depend on S ? Answering these questions is the subject of this study.

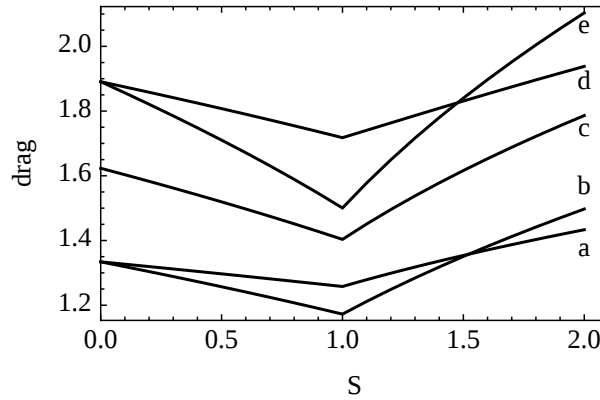


Figure 1: Drag for the unit sphere normalized to the sphere Stokes drag as a function of S for $R = R_m = 1$ (curve a); $R = 1, R_m = 3$ (curve b); $R = R_m = 2$ (curve c); $R = 3, R_m = 1$ (curve d); and $R = R_m = 3$ (curve e).

The challenge in addressing the posed questions is that the traditional PDE constrained optimization approach coupled with the finite element method (FEM) is slowly converging and inaccurate in general. The second deficiency is attributed to the fact that being applied to external problems, the approach truncates and discretizes an external domain and finds minimum-drag shapes point wisely.

Our approach reduces the MHD problem to boundary integral equations, derives the optimality condition for the minimum-drag shapes analytically, and obtains minimum-drag shapes in a functional series form. To

¹This number is also denoted by Co and is also called “pressure” number in [27, 13].

this end, the MHD equations are linearized, and the velocity, pressure and magnetic fields in and out the body are represented by four generalized analytic functions. Under the assumption that the fluid and body are both nonmagnetic and have same magnetic permeability, the axially symmetric MHD problem is reduced to integral equations for the boundary values of two generalized analytic functions based on the generalized Cauchy integral formula. The Hartmann number is assumed to be nonzero since when $M = 0$, the velocity field is uncoupled from the magnetic field. With $M \neq 0$, three cases are studied separately: (a) $R_m \neq 0$, $R_m R \neq M^2$ ($S \neq 1$); (b) $R_m = 0$; and (c) $R_m R = M^2$ ($S = 1$). The reason for this is that in each case, solving the MHD problem involves different number of generalized analytic functions from two classes: r -analytic functions and H -analytic functions. This is the mathematical explanation of why the case $S = 1$ is special.

1.1 r -Analytic and H -Analytic Functions

A function $F(x, y) = U(x, y) + iV(x, y)$ with $i = \sqrt{-1}$ is called generalized analytic or pseudo-analytic if its real and imaginary parts, U and V , respectively, satisfy the so-called Carleman or Bers-Vekua system [4, 26]

$$\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} + aU + bV = 0, \quad \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + cU + dV = 0, \quad (1)$$

where $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$, and $d = d(x, y)$ are real-valued functions. For example, for $a \equiv b \equiv c \equiv d \equiv 0$, the system (1) defines ordinary analytic functions. Generalized analytic functions arise in various areas of applied mathematics including hydrodynamics, gas dynamics, theory of elasticity, heat transfer, electromagnetism, quantum mechanics, etc.; see [6, 26, 21, 1]. Their theory has been extensively developed since the mid-20th century and extends the majority of results for ordinary analytic functions, e.g. the formal powers [5, 4] and the Cauchy integral formula [4, 26, 21, 8].

Several important classes of generalized analytic functions arise from the relationship

$$\text{curl } \Phi + [\mathbf{a} \times \Phi] = -\text{grad } \Psi, \quad \text{div } \Phi = 0 \quad (2)$$

for a vector field Φ and scalar field Ψ , where $\mathbf{a}$ is a known vector function. This relationship is frequently encountered in problems of applied mathematics. For example, for $\Psi = 0$ and $\mathbf{a} = 0$, (2) simplifies to an *irrotational solenoidal* field Φ found in *electrostatics* and *ideal fluid*, whereas for $\mathbf{a} = 0$, it defines so-called *related potentials* Ψ and Φ , e.g. the pressure and vorticity in the Stokes equations, electric potential and magnetic field in conductive materials, etc.; see [31]. For $\mathbf{a} \neq 0$ and $\Psi \neq 0$, (2) arises in the Maxwell equations for quasi-stationary electromagnetic fields, in the Oseen equations and linearized MHD equations; see [30, Example 4].

Let (r, ϕ, z) be a cylindrical coordinate system with the basis $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{k})$. In the axially symmetric case with the z axis being the axis of revolution, let $\Phi = \Phi_r(r, z)\mathbf{e}_r + \Phi_z(r, z)\mathbf{k}$, $\Psi = 0$, and $\mathbf{a} = -2\lambda\mathbf{k}$, where λ is a constant. Then for $U = e^{-\lambda z}\Phi_z$ and $V = e^{-\lambda z}\Phi_r$, (2) reduces to

$$\frac{\partial U}{\partial r} = \left(\frac{\partial}{\partial z} - \lambda \right) V, \quad \left(\frac{\partial}{\partial z} + \lambda \right) U = - \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) V, \quad (3)$$

which is a special case of (1) and implies that

$$(\Delta_0 - \lambda^2)U = 0 \quad \text{and} \quad (\Delta_1 - \lambda^2)V = 0, \quad (4)$$

where Δ_k denotes the so-called k -harmonic operator: $\Delta_k \equiv \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2 - k^2/r^2$. The functions U and V satisfying (3) form an H -analytic function $G = U + iV$ of a complex variable $\zeta = r + iz$. For $\lambda = 0$, (3) defines a so-called r -analytic function and is arguably the most studied system among various classes of generalized analytic functions. It arises in the axially symmetric theory of elasticity [21, 1] and in the axially symmetric Stokes and Oseen flows [35, 34, 31, 28, 29, 30]. Since an r -analytic function satisfies (4) for $\lambda = 0$, it is also referred to as 0-harmonically analytic function [28]. Both classes of r -analytic and H -analytic functions are instrumental in constructing solutions to axially symmetric MHD problems.

1.2 Generalized Cauchy Integral Formula and Series Representation

Let G^+ be a generalized analytic function in a bounded *open* region $\mathcal{D}^+$ in the *right-half* rz -plane ($\mathcal{D}^+$ may contain parts of the z axis). The boundary of $\mathcal{D}^+$ is assumed to be a piece-wise smooth positively oriented curve ℓ , which is either closed or open with the endpoints lying on the z axis.² Let $\mathcal{D}^-$ be the complement of $\mathcal{D}^+ \cup \ell$ in the right-half rz -plane ($\mathcal{D}^-$ is unbounded), and let G^- be a generalized analytic function in $\mathcal{D}^-$ that vanishes at infinity. For convenience, an arbitrary function $f(r, z)$ will be denoted by $f(\zeta)$ without assuming its analyticity. Let $G^\pm(\zeta)$ satisfy the Hölder condition³ on ℓ , and let ℓ' be the reflection of ℓ over the z axis. Then under the assumption of the symmetry condition $G^\pm(-\bar{\zeta}) = \overline{G^\pm(\zeta)}$, the Cauchy integral formula for $G^\pm$ is given by [1, 21, 26]:

$$G^\pm(\zeta) = \pm \frac{1}{2\pi i} \oint_{\ell \cup \ell'} G^\pm(\tau) \mathcal{W}(\zeta, \tau) d\tau, \quad \zeta \in \mathcal{D}^\pm, \quad (5)$$

where $\zeta = r + iz$, $\tau = r_1 + iz_1$ and $\mathcal{W}(\zeta, \tau) \equiv \mathcal{W}(r, z; r_1, z_1)$ is a generalized Cauchy kernel. If the boundary ℓ is smooth, then on ℓ , $G^\pm(\zeta)$ satisfies the generalized Sokhotski-Plemelj formula:

$$G^\pm(\zeta) = \frac{1}{2} G^\pm(\zeta) \pm \frac{1}{2\pi i} \oint_{\ell \cup \ell'} G^\pm(\tau) \mathcal{W}(\zeta, \tau) d\tau, \quad \zeta \in \ell. \quad (6)$$

If ℓ is piece-wise smooth (has salient points), in particular if the endpoints of ℓ lie on the z axis and the angle between ℓ and the z axis at one of the endpoints is not $\pi/2$ (conic endpoint), then at a salient point of ℓ , the coefficient $1/2$ in the generalized Sokhotski-Plemelj formula (6) is replaced by a finite function of the angle between tangents at the salient point; see [1, (31.13a), (31.13b)] and [33, (3.7)].

Let $\mathcal{W}_r$ and $\mathcal{W}_H$ denote the generalized Cauchy kernels for r -analytic and H -analytic functions, respectively. For r -analytic functions, $\mathcal{W}_r$ is given in [1, [28, Theorem 2] and [33, Theorem 3.2]. Theorem 1 and Corollary 4 in [30] present two forms for $\mathcal{W}_H$ with $\lambda > 0$. The next result extends the Cauchy integral formula for H -analytic functions for arbitrary $\lambda \neq 0$.

Theorem 1 *Let $G^\pm$ be an H -analytic function in $\mathcal{D}^\pm$ that satisfies (3) with $\lambda \neq 0$ and satisfies the Hölder condition on ℓ . Suppose $G^\pm(-\bar{\zeta}) = \overline{G^\pm(\zeta)}$. Then the generalized Cauchy integral formula (5) takes the form*

$$G^\pm(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} G(\tau) \mathcal{W}_H(\zeta, \tau) d\tau = \frac{1}{2\pi i} \int_{\ell} K_1(\zeta, \tau, \lambda) \operatorname{Re} [G^\pm(\tau) d\tau] + i K_2(\zeta, \tau, \lambda) \operatorname{Im} [G^\pm(\tau) d\tau], \quad \zeta \in \mathcal{D}^\pm, \quad (7)$$

where

$$\begin{aligned} K_1(\zeta, \tau, \lambda) &= -r_1 \int_0^\pi \frac{e^{-|\lambda| \rho(\zeta, \tau, t)}}{\rho(\zeta, \tau, t)} \left(\frac{(\bar{\zeta} - r_1 \cos t + iz_1)(1 + |\lambda| \rho(\zeta, \tau, t))}{\rho(\zeta, \tau, t)^2} - i\lambda \right) dt, \\ K_2(\zeta, \tau, \lambda) &= -r_1 \int_0^\pi \left[\frac{e^{-|\lambda| \rho(\zeta, \tau, t)}}{\rho(\zeta, \tau, t)^2} (r \cos t - r_1)(1 + |\lambda| \rho(\zeta, \tau, t)) + i \frac{(z_1 - z) \cos t}{\rho(\zeta, \tau, t)^2} \right. \\ &\quad \left. + i \frac{e^{-|\lambda| \rho(\zeta, \tau, t)} - e^{\lambda(z_1 - z)}}{\Delta(\zeta, \tau, t)} \left(\cos t + \frac{(r - r_1 \cos t)(r_1 - r \cos t)(2\Delta(\zeta, \tau, t) - z_1 + z)}{\rho(\zeta, \tau, t)^2 \Delta(\zeta, \tau, t)} \right) \right. \\ &\quad \left. + i e^{-|\lambda| \rho(\zeta, \tau, t)} \left(2\lambda \cos t + |\lambda| \frac{(r - r_1 \cos t)(r_1 - r \cos t)}{\rho(\zeta, \tau, t) \Delta(\zeta, \tau, t)} \right) \right] \frac{dt}{\rho(\zeta, \tau, t)}, \\ \rho(\zeta, \tau, t) &= \sqrt{r^2 + r_1^2 - 2rr_1 \cos t + (z - z_1)^2}, \\ \Delta(\zeta, \tau, t) &= z_1 - z + \rho(\zeta, \tau, t) \operatorname{sign} \lambda. \end{aligned}$$

²Open segments of the z axis that $\mathcal{D}^+$ may contain are not parts of ℓ .

³This condition means that for some parametrization $\zeta(t)$ of ℓ , the boundary value $f(\zeta(t))$ satisfies $|f(\zeta(t_2)) - f(\zeta(t_1))| \leq c|t_2 - t_1|^\beta$ for all t_1 and t_2 , some $\beta \in (0, 1]$, and nonnegative constant c .

Proof. The proof is similar to that of Theorem 1 in [30]. $\square$

For the region exterior to a sphere, Example 5 and Proposition 2 in [30] provide series representations for r -analytic functions and H -analytic functions with $\lambda > 0$, respectively. The next proposition generalizes the representation for H -analytic functions for any $\lambda \neq 0$.

Proposition 1 (H -analytic function in the region exterior to a sphere) *For the region exterior to a sphere centered at the origin, an H -analytic function satisfying (3) for $\lambda \neq 0$ and vanishing at infinity can be represented in the spherical coordinates (R, ϑ, φ) by*

$$H(R, \vartheta) = R^{-\frac{1}{2}} \sum_{n=1}^{\infty} A_n \left(L_n(\cos \vartheta) K_{n+\frac{1}{2}}(|\lambda| R) - \text{sign } \lambda L_{-n}(\cos \vartheta) K_{n-\frac{1}{2}}(|\lambda| R) \right), \quad (8)$$

where $L_n(\cos \vartheta) = n P_n(\cos \vartheta) - i P_n^{(1)}(\cos \vartheta)$, $P_n^{(k)}(\cos \vartheta)$ is the associated Legendre polynomial of the first kind of order n and rank k (for $k = 0$, the superscript is omitted), $K_{n+\frac{1}{2}}(\cdot)$ is a modified spherical Bessel function of the third kind (see [3, Sec. 7.2.6]), and real-valued coefficients A_n are such that the corresponding series (8) converge for all $\vartheta \in [0, \pi]$ and R greater than or equal to the radius of the sphere.

Proof. The proof is analogous to the proof of Proposition 2 in [30]. $\square$

2 Magnetohydrodynamics

The steady flow of an electrically conducting viscous incompressible fluid is governed by the MHD equations

$$\begin{cases} \rho(\mathbf{U} \cdot \text{grad}) \mathbf{U} = -\text{grad } \wp + \rho \nu \Delta \mathbf{U} + \mu [\mathbf{J} \times \mathbf{H}], \\ \text{curl } \mathbf{H} = 4\pi \mathbf{J}, \quad \mathbf{J} = \sigma (\mathbf{E} + \mu [\mathbf{U} \times \mathbf{H}]), \\ \text{curl } \mathbf{E} = 0, \quad \text{div } \mathbf{U} = 0, \quad \text{div } \mathbf{H} = 0, \end{cases} \quad (9)$$

where $\mathbf{U}$ is the fluid velocity, $\wp$ is the pressure in the fluid, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively, $\mathbf{J}$ is the current density, ν is the kinematic viscosity, ρ is the fluid density, μ is the magnetic permeability, and σ is the conductivity; see [22, 23, 16].

Suppose a nonmagnetic solid body of revolution is immersed in the uniform flow in the presence of magnetic field. At infinity, the flow and magnetic field are assumed to be constant and aligned with the z -axis, i.e. $\mathbf{U}|_{\infty} = V_{\infty} \mathbf{k}$ and $\mathbf{H}|_{\infty} = H_{\infty} \mathbf{k}$, where V_{∞} and H_{∞} are constants. Also, the body's axis of revolution is parallel to the z -axis. Under different assumptions, this MHD problem was considered in [9, 13, 14, 15, 11, 24, 10]. Let $\mathbf{u}$ be the disturbance of the velocity field: $\mathbf{U} = V_{\infty}(\mathbf{k} + \mathbf{u})$, and let $\mathbf{h}^+$ and $\mathbf{h}^-$ be the disturbances of the magnetic field in and out the body, respectively: $\mathbf{H} = H_{\infty}(\mathbf{k} + \mathbf{h}^+)$ in the body and $\mathbf{H} = H_{\infty}(\mathbf{k} + \mathbf{h}^-)$ out the body. On the boundary S of the body and at infinity, $\mathbf{u}$ and $\wp$ satisfy the conditions

$$\mathbf{u}|_S = -\mathbf{k}, \quad \mathbf{u}|_{\infty} = 0, \quad \wp|_{\infty} = 0. \quad (10)$$

Inside the body, the electric field is zero, whereas the magnetic field satisfies $\text{curl } \mathbf{H} = 0$ and $\text{div } \mathbf{H} = 0$, or in terms of the disturbance $\mathbf{h}^+$,

$$\text{curl } \mathbf{h}^+ = 0, \quad \text{div } \mathbf{h}^+ = 0. \quad (11)$$

As in [9, 14], it is assumed that the fluid and body are nonmagnetic and have same magnetic permeability. In this case, the magnetic field is continuous across the boundary of the body⁴ and also $\mathbf{h}^-$ vanishes at infinity:

$$\mathbf{h}^+|_S = \mathbf{h}^-|_S, \quad \mathbf{h}^-|_{\infty} = 0. \quad (12)$$

⁴If the fluid and body have different magnetic permeability then across the body's boundary, the tangential component of the magnetic field $\mathbf{H}$ and the normal component of the magnetic induction $\mathbf{B}$ should be continuous.

The problem (9)–(12) is axially symmetric. In this case, the velocity, pressure and electromagnetic field are independent of the angular coordinate φ , i.e.

$$\mathbf{u} = u_r(r, z)\mathbf{e}_r + u_z(r, z)\mathbf{k}, \quad \wp = \wp(r, z), \quad \mathbf{h}^\pm = h_r^\pm(r, z)\mathbf{e}_r + h_z^\pm(r, z)\mathbf{k}, \quad \mathbf{E} = E_\varphi(r, z)\mathbf{e}_\varphi.$$

The equation $\text{curl } \mathbf{E} = 0$ implies that $rE_\varphi = c$, where c is a constant. Since $[\mathbf{u} \times \mathbf{H}]$ and $\text{curl } \mathbf{H}$ vanish at infinity, we have $c = 0$, and consequently, $\mathbf{E} = 0$.

Rescaling the linear dimensions, velocity, pressure, and magnetic field by a , V_∞ , $V_\infty \rho v/a$, and H_∞ , respectively, where a is the half of the diameter of the body, and assuming $\mathbf{u}$ and $\mathbf{h}^-$ to be small, we can rewrite equations (9) without $\mathbf{E}$ and $\mathbf{J}$ in the dimensionless linearized form

$$\begin{cases} R(\mathbf{k} \cdot \text{grad})\mathbf{u} = -\text{grad } \wp + \Delta \mathbf{u} + M^2[(\mathbf{u} - \mathbf{h}^-) \times \mathbf{k}] \times \mathbf{k}, \\ \text{curl } \mathbf{h}^- = R_m[(\mathbf{u} - \mathbf{h}^-) \times \mathbf{k}], \\ \text{div } \mathbf{u} = 0, \quad \text{div } \mathbf{h}^- = 0. \end{cases} \quad (13)$$

where $R = V_\infty a/v$ is the Reynolds number, $R_m = 4\pi V_\infty a \mu \sigma$ is the magnetic Reynolds number, and $M = \mu H_\infty a \sqrt{\sigma/(\rho v)}$ is the Hartmann number.

For $M = 0$, the first equation in (13) reduces to the Oseen equations, and the problem for the velocity and pressure becomes uncoupled from the magnetic field. For solving the Oseen equations, see e.g. [30]. For $M \neq 0$, there are three cases to analyze:

- (a) $R_m \neq 0$ and $R_m R \neq M^2$: A solution to (13) is represented by two H -analytic functions and one r -analytic function, whereas a solution to (11) is given by a single r -analytic function.
- (b) $R_m = 0$: The magnetic field is constant everywhere, i.e. $\mathbf{h}^\pm = 0$, and a solution to (13) is represented by two H -analytic functions. (The case of $R = 0$ yields no simplification.)
- (c) $R_m R = M^2$ ($S = 1$): A solution to (13) is represented by one H -analytic function and two r -analytic functions, and a solution to (11) is given by a single r -analytic function.

Let ℓ be the positively oriented cross section of the body in the right-half rz -plane ($r > 0$), and let ℓ' be the reflection of ℓ over the z axis. The curve ℓ divides the right-half rz -plane into interior bounded region $\mathcal{D}^+$ and exterior unbounded region $\mathcal{D}^-$.

Theorem 2 (solution representation, case (a)) *In the axially symmetric case with $M \neq 0$ and $R_m R \neq M^2$, a solution to (13) and (11) is given by*

$$u_z + i u_r = \frac{1}{\lambda_1 - \lambda_2} \left((1 - 2\lambda_2 \varkappa) e^{\lambda_1 z} G_1 - (1 - 2\lambda_1 \varkappa) e^{\lambda_2 z} G_2 \right) - \frac{1}{R_m R - M^2} G_3^-, \quad (14)$$

$$h_z^- + i h_r^- = -\frac{2\varkappa}{\lambda_1 - \lambda_2} \left(\lambda_2 e^{\lambda_1 z} G_1 - \lambda_1 e^{\lambda_2 z} G_2 \right) - \frac{1}{R_m R - M^2} G_3^-, \quad (15)$$

$$h_z^+ + i h_r^+ = G_3^+, \quad (16)$$

where G_1 and G_2 are H -analytic functions in $\mathcal{D}^-$ satisfying (3) with $\lambda = \lambda_{1,2} = \frac{R+R_m \pm \sqrt{(R-R_m)^2 + 4M^2}}{4}$, respectively, and vanishing at infinity; G_3^+ and G_3^- are r -analytic functions in $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively, with G^- vanishing at infinity; and $\varkappa = R_m / (R_m R - M^2)$.

In this case, the pressure and scalar vortex function $\omega = \mathbf{e}_\varphi \cdot \text{curl } \mathbf{u}$ (i.e. $\text{curl } \mathbf{u} = \omega \mathbf{e}_\varphi$) are determined by

$$\wp = \frac{2(R\varkappa - 1)}{\lambda_1 - \lambda_2} \left(\lambda_2 e^{\lambda_1 z} \text{Re}[G_1] - \lambda_1 e^{\lambda_2 z} \text{Re}[G_2] \right) + \frac{R}{R_m R - M^2} \text{Re}[G_3^-], \quad (17)$$

$$\omega = \frac{1}{\lambda_1 - \lambda_2} \left((R - 2\lambda_2) e^{\lambda_1 z} \text{Im}[G_1] - (R - 2\lambda_1) e^{\lambda_2 z} \text{Im}[G_2] \right). \quad (18)$$

Proof. With the identity $(\mathbf{k} \cdot \text{grad}) \mathbf{u} = -\text{curl}[\mathbf{k} \times \mathbf{u}]$, the first equation in (13) can be rewritten in the form

$$\text{curl}(\boldsymbol{\omega} - R[\mathbf{k} \times \mathbf{u}]) - M^2[\mathbf{k} \times [\mathbf{k} \times (\mathbf{u} - \mathbf{h}^-)]] + \text{grad} \wp = 0, \quad (19)$$

where $\boldsymbol{\omega} = \text{curl} \mathbf{u}$ is the vorticity. The continuity equation $\text{div} \mathbf{u} = 0$ implies that $\mathbf{u}$ and $\boldsymbol{\omega}$ satisfy the identity

$$\text{curl}[\mathbf{k} \times \mathbf{u}] + \text{grad}(\mathbf{k} \cdot \mathbf{u}) - [\mathbf{k} \times \boldsymbol{\omega}] = 0, \quad (20)$$

whereas $\text{div} \mathbf{h}^- = 0$ implies a similar identity for $\mathbf{h}^-$

$$[\mathbf{k} \times \text{curl} \mathbf{h}^-] = \text{curl}[\mathbf{k} \times \mathbf{h}^-] + \text{grad}(\mathbf{k} \cdot \mathbf{h}^-),$$

with which the second equation in (13) can be represented in the form:

$$\text{curl}[\mathbf{k} \times \mathbf{h}^-] + \text{grad}(\mathbf{k} \cdot \mathbf{h}^-) + R_m[\mathbf{k} \times [\mathbf{k} \times (\mathbf{u} - \mathbf{h}^-)]] = 0. \quad (21)$$

Forming a linear combination of (19), (20), and (21) with constant weights β_1 , β_2 , and β_3 , respectively, we have

$$\begin{aligned} \text{curl}(\beta_1 \boldsymbol{\omega} + [\mathbf{k} \times ((\beta_2 - \beta_1 R) \mathbf{u} + \beta_3 \mathbf{h}^-)]) - [\mathbf{k} \times (\beta_2 \boldsymbol{\omega} + (\beta_1 M^2 - \beta_3 R_m) [\mathbf{k} \times (\mathbf{u} - \mathbf{h}^-)])] \\ + \text{grad}(\beta_1 \wp + \beta_2 (\mathbf{k} \cdot \mathbf{u}) + \beta_3 (\mathbf{k} \cdot \mathbf{h}^-)) = 0. \end{aligned} \quad (22)$$

Let $\beta_3 = -(\beta_2 - \beta_1 R)$ and $(\beta_2 - \beta_1 R)/\beta_1 = (\beta_1 M^2 - \beta_3 R_m)/\beta_2$, then $(\beta_2/\beta_1)^2 - (R + R_m)\beta_2/\beta_1 + R_m R - M^2 = 0$, whence $\beta_2/\beta_1 = 2\lambda_k$, $k = 1, 2$, and (22) becomes a particular case of (2):

$$\text{curl} \boldsymbol{\Lambda}_k - 2\lambda_k [\mathbf{k} \times \boldsymbol{\Lambda}_k] + \text{grad} \Psi_k = 0, \quad \text{div} \boldsymbol{\Lambda}_k = 0, \quad k = 1, 2, \quad (23)$$

where

$$\boldsymbol{\Lambda}_k = \boldsymbol{\omega} + (2\lambda_k - R) [\mathbf{k} \times (\mathbf{u} - \mathbf{h}^-)], \quad \Psi_k = \wp + 2\lambda_k (\mathbf{k} \cdot \mathbf{u}) + (R - 2\lambda_k) (\mathbf{k} \cdot \mathbf{h}^-). \quad (24)$$

In the axially symmetric case, (23) reduces to the system (3) determining H -analytic functions. Let $\boldsymbol{\Lambda}_k = \Lambda_k \mathbf{e}_\varphi$, $k = 1, 2$, and $\boldsymbol{\omega} = \omega \mathbf{e}_\varphi$. In this case,

$$\Psi_k + i\Lambda_k = 2e^{\lambda_k z} G_k, \quad k = 1, 2, \quad (25)$$

where G_k are H -analytic functions defined in the theorem.

On the other hand, if $\beta_2 = 0$ and $\beta_1 M^2 - \beta_3 R_m = 0$ then (22) becomes the relationship for *related potentials*:

$$\text{curl} \boldsymbol{\Lambda}_3 + \text{grad} \Psi_3 = 0, \quad \text{div} \boldsymbol{\Lambda}_3 = 0, \quad (26)$$

where

$$\boldsymbol{\Lambda}_3 = R_m \boldsymbol{\omega} + [\mathbf{k} \times (M^2 \mathbf{h}^- - R_m R \mathbf{u})], \quad \Psi_3 = R_m \wp + M^2 (\mathbf{k} \cdot \mathbf{h}^-). \quad (27)$$

In the axially symmetric case, $\boldsymbol{\Lambda}_3 = \Lambda_3 \mathbf{e}_\varphi$ and (26) determines an r -analytic function G_3^- in $\mathcal{D}^-$:

$$\Psi_3 + i\Lambda_3 = G_3^-. \quad (28)$$

Equations (23)–(28) can be reformulated in the complex form

$$\begin{aligned} 2e^{\lambda_k z} G_k &= \wp + i\omega + 2\lambda_k (u_z + iu_r) - iRu_r + (R - 2\lambda_k)(h_z^- + ih_r^-), \quad k = 1, 2, \\ G_3^- &= R_m(\wp + i\omega) - iRR_mu_r + M^2(h_z^- + ih_r^-), \end{aligned}$$

from which the representations (14), (15), (17), and (18) follow. The condition $M \neq 0$ guarantees that $\lambda_1 \neq \lambda_2$.

Finally, in the axially symmetric case, $\mathbf{h}^+$, being an *irrotational solenoidal field* (see (11)), can be represented by (16). $\square$

Corollary 1 (solution representation: case (b)) *In the axially symmetric case with $M \neq 0$ and $R_m = 0$, the magnetic disturbance is zero, i.e. $\mathbf{h}^\pm \equiv 0$, and the velocity, pressure, and vorticity that solve (13) can be represented by*

$$\begin{aligned} u_z + i u_r &= \frac{1}{\lambda_1 - \lambda_2} \left(e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2 \right), \\ \wp + i \omega &= e^{\lambda_1 z} \left(G_1 - \frac{R}{2(\lambda_1 - \lambda_2)} \bar{G}_1 \right) + e^{\lambda_2 z} \left(G_2 + \frac{R}{2(\lambda_1 - \lambda_2)} \bar{G}_2 \right), \end{aligned} \quad (29)$$

where G_1 and G_2 are H -analytic functions satisfying (3) with $\lambda = \lambda_{1,2} = \frac{1}{4} \left(R \pm \sqrt{R^2 + 4M^2} \right)$, respectively, and vanishing at infinity.

Detail. For $R_m = 0$, (15) reduces to $h_z^- + i h_r^- = M^{-2} G_3^-$ which with (16) and (12) implies $M^{-2} G_3^- = G_3^+$ on ℓ . Consequently, since G_3^- vanishes at infinity, by the Sokhotski-Plemelj formula (6), $G_3^\pm \equiv 0$ in $\mathcal{D}^\pm$, respectively, and (29) follows from (14), (17), and (18). $\square$

Corollary 2 *For $M \neq 0$ and $R_m = R = 0$ the representation (29) simplifies to*

$$u_z + i u_r = \frac{1}{M} \left(e^{Mz/2} G_1 - e^{-Mz/2} G_2 \right), \quad \wp + i \omega = e^{Mz/2} G_1 + e^{-Mz/2} G_2,$$

where G_1 and G_2 are H -analytic functions satisfying (3) with $\lambda = M/2$ and $\lambda = -M/2$, respectively. It is similar to the solution form suggested by Chester [9] in the case of $R = R_m = 0$.

Corollary 3 *For $M = 0$ and $R \neq 0$, the representation (29) reduces to the representation (22)–(23) in [30] for the velocity, pressure and vorticity for the axially symmetric Oseen flow of nonconducting fluid.*

Theorem 3 (solution representation, case (c)) *In the axially symmetric case with $M \neq 0$ and $R_m R = M^2$, a solution to (13) and (11) is given by*

$$u_z + i u_r = \frac{1}{R + R_m} \left(\frac{2R}{R + R_m} e^{\lambda z} G_1 + \left(R_m \left(z - \frac{i}{2} r \right) - \frac{R}{R + R_m} \right) G_2 + R_m G_3^- \right), \quad (30)$$

$$h_z^- + i h_r^- = \frac{R_m}{R + R_m} \left(-\frac{2}{R + R_m} e^{\lambda z} G_1 + \left(z - \frac{i}{2} r + \frac{1}{R + R_m} \right) G_2 + G_3^- \right), \quad (31)$$

$$h_z^+ + i h_r^+ = G_3^+, \quad (32)$$

where $\lambda = (R + R_m)/2$, G_1 is an H -analytic function in $\mathcal{D}^-$ that satisfies (3) with λ and vanishes at infinity; G_2 and G_3^- are r -analytic functions in $\mathcal{D}^-$ and vanishing at infinity; and G_3^+ is an r -analytic function in $\mathcal{D}^+$.

In this case, the pressure and vortex function are determined by

$$\wp = \frac{R_m R}{R + R_m} \left(\frac{2}{R + R_m} e^{\lambda z} \text{Re}[G_1] - \text{Re} \left[\left(z - \frac{i}{2} r + \frac{1}{R + R_m} \right) G_2 \right] - \text{Re}[G_3^-] \right) + \text{Re}[G_2], \quad (33)$$

$$\omega = \frac{1}{R + R_m} \left(2 \text{Re}^{\lambda z} \text{Im}[G_1] + R_m \text{Im}[G_2] \right). \quad (34)$$

Proof. The proof is partially based on the proof of Theorem 2. When $R_m R = M^2$, the relationships (23)–(25) hold for $\lambda_1 = (R + R_m)/2$ and $\lambda_2 = 0$. For $\lambda_1 = (R + R_m)/2$, (25) simplifies to

$$2e^{(R+R_m)z/2} G_1 = \wp + i \omega + (R + R_m)(u_z + i u_r) - i R u_r - R_m(h_z^- + i h_r^-), \quad (35)$$

where G_1 is the H -analytic function defined in this theorem. The case of $\lambda_2 = 0$ means that (23) reduces to the relationship (26) for the related potentials. Consequently, a different approach is required. Since $R_m R = M^2$, multiplying (21) by R and adding to (19), we obtain

$$\operatorname{curl}(\boldsymbol{\omega} - R[\mathbf{k} \times (\mathbf{u} - \mathbf{h}^-)]) + \operatorname{grad}(\wp + R(\mathbf{k} \cdot \mathbf{h}^-)) = 0,$$

which with the second equation in (13) and conditions $\operatorname{div} \mathbf{u} = 0$ and $\operatorname{div} \mathbf{h}^- = 0$ can be rewritten as

$$\Delta(\mathbf{u} + (R/R_m)\mathbf{h}^-) = \operatorname{grad}(\wp + R(\mathbf{k} \cdot \mathbf{h}^-)), \quad \operatorname{div}(\mathbf{u} + (R/R_m)\mathbf{h}^-) = 0. \quad (36)$$

The system (36) is similar to the Stokes equations for the viscous incompressible fluid. In the axially symmetric case, its solution is given by Proposition 7 in [28]

$$\begin{aligned} u_z + i u_r + \frac{R}{R_m}(h_z^- + i h_r^-) &= \left(z - \frac{i}{2}r\right) G_2 + G_3^-, \\ \wp + i \omega - i R u_r + R(h_z^- + i h_r^-) &= G_2, \end{aligned} \quad (37)$$

where G_2 and G_3^- are the r -analytic functions defined in this theorem. Observe that the conditions $M \neq 0$ and $R_m R = M^2$ guarantee that $R_m \neq 0$. The representations (30), (31), (33), and (34) follow from (35) and (37), whereas (32) remains same as in Theorem 2. $\square$

The advantage of the representations (14)–(18), (29), and (30)–(34) compared to the existing forms, e.g. in [9], is that these representations, being linear combinations of the generalized analytic functions, involve no derivatives of unknown functions and represent simultaneously all three: the velocity, vorticity and pressure. This fact considerably simplifies deriving boundary integral equations based on the generalized Cauchy integral formula.

3 Boundary-value Problems and Integral Equations

3.1 Case (a): $M \neq 0$, $R_m \neq 0$, and $R_m R \neq M^2$

For $M \neq 0$ and $R_m R \neq M^2$, equations (13) with the boundary conditions (10) and (12) reduce to the boundary-value problem for the four generalized analytic functions G_1 , G_2 , and $G_3^\pm$ defined in Theorem 2:

$$\begin{aligned} (1 - 2\lambda_2 \varkappa) e^{\lambda_1 z} G_1 - (1 - 2\lambda_1 \varkappa) e^{\lambda_2 z} G_2 - \frac{\lambda_1 - \lambda_2}{R_m R - M^2} G_3^- &= \lambda_2 - \lambda_1, \quad \zeta \in \ell, \\ e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2 &= (\lambda_2 - \lambda_1)(G_3^+ + 1), \quad \zeta \in \ell. \end{aligned} \quad (38)$$

Proposition 2 *The boundary-value problem (38) has a unique solution.*

Proof. The proposition is equivalent to the fact that the homogenous problem (38) has only zero solution, which with the representations (14)–(18) corresponds to (13) with the zero boundary conditions, i.e. $\mathbf{u}|_S = 0$.

Let $\mathbb{D}^+$ and $\mathbb{D}^-$ be the regions bounded by the surface S and exterior to S , respectively. The first equation in (13) can be recast in the form

$$\operatorname{curl} \boldsymbol{\omega} + \operatorname{grad} \wp + R \frac{\partial \mathbf{u}}{\partial z} - M^2 [\mathbf{k} \times [\mathbf{k} \times (\mathbf{u} - \mathbf{h}^-)]] = 0. \quad (39)$$

With the identities

$$\begin{aligned} \mathbf{u} \cdot \operatorname{curl} \boldsymbol{\omega} &= \operatorname{div}(\boldsymbol{\omega} \times \mathbf{u}) + |\boldsymbol{\omega}|^2, \\ \mathbf{u} \cdot \operatorname{grad} \wp &= \operatorname{div}(\wp \mathbf{u}) - \wp \operatorname{div} \mathbf{u} = \operatorname{div}(\wp \mathbf{u}), \end{aligned}$$

the scalar product of (39) and $\mathbf{u}$ takes the form

$$\operatorname{div} \left([\boldsymbol{\omega} \times \mathbf{u}] + \wp \mathbf{u} + \frac{R}{2} |\mathbf{u}|^2 \mathbf{k} \right) + |\boldsymbol{\omega}|^2 + M^2 u_r (u_r - h_r^-) = 0. \quad (40)$$

On the other hand, with the identity

$$[\mathbf{k} \times \mathbf{h}^-] \cdot \operatorname{curl} \mathbf{h}^- = \operatorname{div} \left(\frac{1}{2} |\mathbf{h}^-|^2 - \mathbf{h}^- (\mathbf{k} \cdot \mathbf{h}^-) \right), \quad (41)$$

the scalar product of the second equation in (13) and $[\mathbf{k} \times \mathbf{h}^-]$ results in

$$h_r^- (h_r^- - u_r) - \frac{1}{R_m} \operatorname{div} \left(\frac{1}{2} |\mathbf{h}^-|^2 - \mathbf{h}^- (\mathbf{k} \cdot \mathbf{h}^-) \right) = 0, \quad (42)$$

provided that $R_m \neq 0$. The linear combination (40) + $M^2(42)$ reduces to

$$\operatorname{div} \left([\boldsymbol{\omega} \times \mathbf{u}] + \wp \mathbf{u} + \frac{R}{2} |\mathbf{u}|^2 \mathbf{k} - \frac{M^2}{R_m} \left(\frac{1}{2} |\mathbf{h}^-|^2 - \mathbf{h}^- (\mathbf{k} \cdot \mathbf{h}^-) \right) \right) + |\boldsymbol{\omega}|^2 + M^2 (u_r - h_r^-)^2 = 0. \quad (43)$$

Since $\operatorname{curl} \mathbf{h}^+ = 0$ in $\mathbb{D}^+$, we can also write

$$[\mathbf{k} \times \mathbf{h}^+] \cdot \operatorname{curl} \mathbf{h}^+ = \operatorname{div} \left(\frac{1}{2} |\mathbf{h}^+|^2 - \mathbf{h}^+ (\mathbf{k} \cdot \mathbf{h}^+) \right) = 0. \quad (44)$$

Let $\mathbb{D}_R$ be the region bounded by the body's surface S and by a sphere S_R with large radius R and center at the origin, so that $\mathbb{D}_R \rightarrow \mathbb{D}^-$ as $R \rightarrow \infty$. Applying the divergence theorem to the linear combination of volume integrals: $\iiint_{\mathbb{D}_R} (43) dV - M^2/R_m \iiint_{\mathbb{D}^+} (44) dV$, and using the boundary conditions $\mathbf{u} = 0$ and $\mathbf{h}^+ = \mathbf{h}^-$ on S , we obtain

$$\mathcal{F} = I_R + \iiint_{\mathbb{D}_R} \left(|\boldsymbol{\omega}|^2 + M^2 (u_r - h_r^-)^2 \right) dV, \quad (45)$$

where

$$I_R = \iint_{S_R} \mathbf{n} \cdot \left([\boldsymbol{\omega} \times \mathbf{u}] + \wp \mathbf{u} + \frac{R}{2} |\mathbf{u}|^2 \mathbf{k} - \frac{M^2}{R_m} \left(\frac{1}{2} |\mathbf{h}^-|^2 - \mathbf{h}^- (\mathbf{k} \cdot \mathbf{h}^-) \right) \right) dS$$

with $\mathbf{n}$ being the outward normal. Note that (45) holds for multiply connected $\mathbb{D}_R$, since $\mathbf{u}$, $\boldsymbol{\omega}$, $\wp$, and $\mathbf{h}^\pm$ are all single-valued functions.⁵

Next we show that $I_R \rightarrow 0$ as $R \rightarrow \infty$. Let C_R be the positively oriented cross section of S_R in the rz -half plane, i.e. C_R is a semicircle, and let $\partial/\partial s$ and $\partial/\partial n$ be the tangential and normal derivatives for C_R , respectively. With the identities

$$\frac{\partial r}{\partial s} = \frac{\partial z}{\partial n}, \quad \frac{\partial r}{\partial n} = -\frac{\partial z}{\partial s}, \quad (46)$$

and the surface element $dS = r ds d\phi$, where ds is the differential of the length of C_R , we have

$$I_R = 2\pi \operatorname{Re} \left[\int_{C_R} \left((u_z + i u_r) \left(\wp + i \omega + \frac{R}{2} (u_z - i u_r) \right) + \frac{M^2}{2R_m} (h_z^- + i h_r^-)^2 \right) r d\zeta \right]. \quad (47)$$

Now with the representations (14)–(15) and (17)–(18), the integral (47) reduces to

$$I_R = 2\pi \operatorname{Re} \left[\int_{C_R} \left(k_1 e^{2\lambda_1 z} G_1^2 + k_2 e^{2\lambda_2 z} G_2^2 - \frac{(G_3^-)^2}{2R_m(R_m R - M^2)} \right) r d\zeta \right],$$

⁵If $\mathbb{D}_R$ is multiply connected, we can make crosscuts in $\mathbb{D}_R$ to make $\mathbb{D}_R$ simply connected, and since $\mathbf{u}$, $\boldsymbol{\omega}$, $\wp$, and $\mathbf{h}^\pm$ are single-valued, they have the same values on the banks of a crosscut.

where $k_1 = (R/2 - 2\lambda_2 + 2\kappa\lambda_2^2) / (\lambda_1 - \lambda_2)^2$ and $k_2 = (R/2 - 2\lambda_1 + 2\kappa\lambda_1^2) / (\lambda_1 - \lambda_2)^2$. Observe that $\lambda_1 \neq \lambda_2$ since $M \neq 0$.

The representation (8) implies that in the spherical coordinates (R, ϑ, φ) related to the cylindrical coordinates in the ordinary way, $G_i(R, \vartheta) = f_i(\vartheta)R^{-1}e^{-|\lambda_i|R} + O(R^{-2}e^{-|\lambda_i|R})$ as $R \rightarrow \infty$ for $i = 1, 2$, where $f_i(\vartheta)$ is a bounded complex-valued function with $|f_i(\vartheta)| < K$ for some constant K and $\vartheta \in [0, \pi]$, whereas Example 5 in [30] implies that $G_3^- = O(R^{-2})$ as $R \rightarrow \infty$. Thus, with $\zeta = Re^{i\vartheta}$, $\vartheta \in [0, \pi]$, we can evaluate

$$\begin{aligned} |I_R| &\leq 2\pi K^2 \int_0^\pi \left(\sum_{j=1}^2 |k_j| e^{-2R(|\lambda_j| - \lambda_j \cos \vartheta)} \right) \sin \vartheta d\vartheta + o(1) \\ &= 2\pi K^2 \sum_{j=1}^2 \frac{|k_j|}{2|\lambda_j|R} \left(1 - e^{-4|\lambda_j|R} \right) + o(1) \rightarrow 0 \quad \text{as } R \rightarrow \infty, \end{aligned}$$

where $\lambda_i \neq 0$ provided that $R_m R \neq M^2$. Consequently, since $\mathcal{F} \equiv 0$, passing R to infinity in (45), we obtain $\iiint_{\mathbb{D}^-} (|\omega|^2 + M^2(u_r - h_r^-)^2) dV = 0$ so that $\omega = 0$ and $u_r = h_r^-$ in $\mathbb{D}^-$, which along with the representations (14), (15), and (18) imply that $\text{Im } G_j = 0$, $j = 1, 2$, in $\mathcal{D}^-$. It follows from the system (3) that $\text{Re } G_j = c_j e^{-\lambda_j z}$, $i = 1, 2$, where c_i are real-valued constants. But since G_1 and G_2 vanish at infinity, $c_j = 0$, $j = 1, 2$, and thus, $G_1 = 0$ and $G_2 = 0$ in $\mathcal{D}^-$. Then, the first and second equations of the homogeneous problem (38) imply $G_3^- = 0$ and $G_3^+ = 0$ on ℓ , and it follows from the Cauchy integral formula (5) for r -analytic functions that $G_3^\pm = 0$ in $\mathcal{D}^\pm$, respectively. $\square$

The next proposition is considered from the mathematical point of view only. It will be used in determining homogeneous solutions to integral equations that follow from the boundary-value problem (38).

Proposition 3 (homogeneous conjugate boundary-value problem, case (a)) *Let G_1 and G_2 be H -analytic functions in $\mathcal{D}^+$, and let $G_3^\pm$ be r -analytic functions in $\mathcal{D}^\pm$, respectively, with G^- vanishing at infinity. Under the assumptions $M \neq 0$, $R_m \neq 0$, and $R_m R \neq M^2$, the homogeneous conjugate boundary-value problem*

$$\begin{aligned} (1 - 2\lambda_2 \kappa) e^{\lambda_1 z} G_1 - (1 - 2\lambda_1 \kappa) e^{\lambda_2 z} G_2 - \frac{\lambda_1 - \lambda_2}{R_m R - M^2} G_3^+ &= 0, & \zeta \in \ell, \\ e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2 - (\lambda_2 - \lambda_1) G_3^- &= 0, & \zeta \in \ell, \end{aligned} \quad (48)$$

has the solution

$$G_1 = c e^{-\lambda_1 z}, \quad G_2 = c e^{-\lambda_2 z}, \quad G_3^+ = 2R_m c, \quad G_3^- = 0, \quad (49)$$

where c is an arbitrary real-valued constant, and λ_1 and λ_2 are defined in Theorem 2.

Proof. The formulas (14)–(18), in which G_3^+ and G_3^- are interchanged and G_1 , G_2 , and $G_3^\pm$ are determined in this proposition, represent $\mathbf{u}$, $\wp$, ω , and $\mathbf{h}^+$ that satisfy (13) in $\mathbb{D}^+$ and represent $\mathbf{h}^-$ that satisfy (11) in $\mathbb{D}^-$. In this case, (13) in $\mathbb{D}^+$ and (11) in $\mathbb{D}^-$ will be called conjugate equations for original (13) and (11). The homogeneous boundary-value problem (48) corresponds to the conjugate equations with the boundary conditions $\mathbf{u} = 0$ and $\mathbf{h}^+ = \mathbf{h}^-$ on S , where $\mathbf{h}^-$ vanishes at infinity. For the conjugate equations, the proof of Proposition 4 can be repeated so that

$$\mathcal{F} = I_R + \iiint_{\mathbb{D}^+} (|\omega|^2 + M^2(u_r - h_r^+)^2) dV = 0, \quad (50)$$

where

$$I_R = -\frac{M^2}{R_m} \iint_{S_R} \mathbf{n} \cdot \left(\frac{1}{2} |\mathbf{h}^-|^2 - \mathbf{h}^- (\mathbf{k} \cdot \mathbf{h}^-) \right) dS.$$

As in the proof of Proposition 4, it can be shown that $|I_R| \rightarrow 0$ as $R \rightarrow \infty$, and passing R to infinity in (50), we have $\omega = 0$ and $u_r = h_r^+$ in $\mathbb{D}^+$, and the representations (14)–(18) with interchanged G_3^+ and G_3^- imply that $\text{Im } G_1 = 0$ and $\text{Im } G_2 = 0$ in $\mathcal{D}^+$, and it follows from the system (3) that $\text{Re } G_j = c_j e^{-\lambda_j z}$, $j = 1, 2$, where c_i are real-valued constants. Then the second equation in (48) implies that $G_3^- = (c_1 - c_2)/(\lambda_2 - \lambda_1)$ on ℓ , and it follows from the Cauchy integral formula (5) for r -analytic functions that $G_3^- = (c_1 - c_2)/(\lambda_2 - \lambda_1)$ in $\mathcal{D}^-$, and consequently, G_3^- vanishes at infinity only if $c_1 = c_2$, whence $G_3^- = 0$ in $\mathcal{D}^-$. Finally, the first equation in (48) implies that $G_3^+ = 2R_m c$ on ℓ , where $c = c_1 = c_2$, and by the Cauchy integral formula (5) for r -analytic functions, we obtain $G_3^+ = 2R_m c$ in $\mathcal{D}^+$. $\square$

The problem (38) can be reduced to integral equations for the boundary values of G_1 and G_2 based on the Cauchy integral formula for generalized analytic functions.

Theorem 4 (integral equations, case (a)) *Let $M \neq 0$, $R_m \neq 0$, and $R_m R \neq M^2$. In this case, (38) yields two integral equations for the boundary values of $F_k(\zeta) = e^{\lambda_k z} G_k(\zeta)$, $k = 1, 2$:*

$$\frac{1 - 2\kappa\lambda_k}{2\kappa(\lambda_1 - \lambda_2)} (F_1(\zeta) - F_2(\zeta)) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(\mathcal{W}_r(\zeta, \tau) - e^{\lambda_k(z - z_1)} \mathcal{W}_H(\zeta, \tau, \lambda_k) \right) F_k(\tau) d\tau = -\frac{1}{2\kappa}, \quad k = 1, 2, \quad (51)$$

where $\zeta \in \ell$ and λ_1 and λ_2 are defined in Theorem 2. A solution to (51) is determined up to a real-valued constant c , i.e. $F_k(\zeta) = c$, $k = 1, 2$, are a homogeneous solution to (51). Let $\hat{F}_k(\zeta)$, $k = 1, 2$, solve (51), then $G_1(\zeta)$ and $G_2(\zeta)$ on ℓ are determined by

$$G_k(\zeta) = \left(\hat{F}_k(\zeta) - c \right) e^{-\lambda_k z}, \quad k = 1, 2, \quad \zeta \in \ell, \quad (52)$$

where

$$c = \frac{1}{2} \hat{F}_k(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \hat{F}_k(\tau) e^{\lambda_k(z - z_1)} \mathcal{W}_H(\zeta, \tau, \lambda_k) d\tau, \quad \zeta \in \ell. \quad (53)$$

Proof. The necessary and sufficient conditions for $G_1(\zeta)$, $G_2(\zeta)$, and $G_3^\pm(\zeta)$, $\zeta \in \ell$, to be boundary values for the corresponding generalized analytic functions are given by the generalized Sokhotski-Plemelj formulae (6):

$$\frac{1}{2} G_k(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} G_k(\tau) \mathcal{W}_H(\zeta, \tau, \lambda_k) d\tau = 0, \quad k = 1, 2, \quad (54a)$$

$$\frac{1}{2} G_3^\pm(\zeta) \mp \frac{1}{2\pi i} \oint_{\ell \cup \ell'} G_3^\pm(\tau) \mathcal{W}_r(\zeta, \tau) d\tau = 0. \quad (54b)$$

Expressing $G_3^+(\zeta)$ and $G_3^-(\zeta)$ from (38) in terms of $G_1(\zeta)$ and $G_2(\zeta)$, then substituting them into corresponding equations in (54b) and solving the latter for the integral terms, we have

$$\frac{1 - 2\kappa\lambda_k}{2\kappa(\lambda_1 - \lambda_2)} (F_1(\zeta) - F_2(\zeta)) + \frac{1}{2} F_k(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} F_k(\tau) \mathcal{W}_r(\zeta, \tau) d\tau = -\frac{1}{2\kappa}, \quad k = 1, 2.$$

Then (54a) is multiplied by $e^{\lambda_k z}$ and is recast in terms of $F_k(\zeta)$:

$$\frac{1}{2} F_k(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} F_k(\tau) e^{\lambda_k(z - z_1)} \mathcal{W}_H(\zeta, \tau, \lambda_k) d\tau = 0, \quad k = 1, 2.$$

Subtracting this equation from the previous one for corresponding k , we obtain (51).

Now we will show that the solution to the boundary-value problem (38) is determined by (52) and (53) and that a homogenous solution to (51) is $F_k(\zeta) = c$, $k = 1, 2$. It is known that if a boundary-value problem for ordinary analytic functions is reduced to integral equations, then a homogeneous solution to those integral

equations is a solution to the homogeneous conjugate boundary-value problem; see [18]. Thus, the approach is to reduce (51) to the homogeneous conjugate boundary-value problem (48).

Let $\widehat{F}_k(\zeta)$, $k = 1, 2$, solve (51). Then let $\Theta_k^+(\zeta)$, $k = 1, 2$, be H -analytic functions in $\mathcal{D}^+$ determined by the generalized Cauchy-type integral

$$\Theta_k^+(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \widehat{F}_k(\tau) e^{-\lambda_k z_1} \mathcal{W}_H(\zeta, \tau, \lambda_k) d\tau, \quad k = 1, 2, \quad \zeta \in \mathcal{D}^+, \quad (55)$$

and let $\Phi^\pm(\zeta)$ be r -analytic functions in $\mathcal{D}^\pm$, respectively, also determined by the generalized Cauchy-type integrals

$$\begin{aligned} \Phi^+(\zeta) &= \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left((1 - 2\lambda_1 \varkappa) \widehat{F}_2(\tau) - (1 - 2\lambda_2 \varkappa) \widehat{F}_1(\tau) + \lambda_2 - \lambda_1 \right) \mathcal{W}_r(\zeta, \tau) d\tau, \quad \zeta \in \mathcal{D}^+, \\ \Phi^-(\zeta) &= \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(\widehat{F}_2(\tau) - \widehat{F}_1(\tau) \right) \mathcal{W}_r(\zeta, \tau) d\tau, \quad \zeta \in \mathcal{D}^-. \end{aligned}$$

With the introduced functions Θ_1^+ , Θ_2^+ , and $\Phi^\pm$ and the corresponding generalized Sokhotski-Plemelj formulas (6) for these functions, the difference of equations (51) for $k = 1$ and $k = 2$ is given by

$$e^{\lambda_2 z} \Theta_2^+(\zeta) - e^{\lambda_1 z} \Theta_1^+(\zeta) = \Phi^-(\zeta), \quad \zeta \in \ell.$$

Similarly, the linear combination $(1 - 2\lambda_2 \varkappa) \cdot (51)|_{k=1} - (1 - 2\lambda_1 \varkappa) \cdot (51)|_{k=2}$ reduces to

$$(1 - 2\lambda_1 \varkappa) e^{\lambda_2 z} \Theta_2^+(\zeta) - (1 - 2\lambda_2 \varkappa) e^{\lambda_1 z} \Theta_1^+(\zeta) = \Phi^+(\zeta), \quad \zeta \in \ell.$$

Observe that the last two equations are equivalent to the homogeneous conjugate boundary-value problem (48) with $\Theta_k^+ = G_k$, $k = 1, 2$, $\Phi^+ = -\frac{\lambda_1 - \lambda_2}{R_m R - M^2} G_3^+$, and $\Phi^- = (\lambda_1 - \lambda_2) G_3^-$. Consequently, by Proposition 3, the only solution these equations have is $\Theta_k^+ = c e^{-\lambda_k z}$, $k = 1, 2$, $\Phi^+ = 2(\lambda_2 - \lambda_1) \varkappa c$, and $\Phi^- = 0$, where c is a real-valued constant. In this case, the representation (55) can be rearranged in the form

$$\frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(\widehat{F}_k(\tau) - c \right) e^{-\lambda_k z_1} \mathcal{W}_H(\zeta, \tau, \lambda_k) d\tau = 0 \quad k = 1, 2, \quad \zeta \in \mathcal{D}^+.$$

For ζ approaching ℓ within $\mathcal{D}^+$, the above equation reduces to the generalized Sokhotski-Plemelj formula

$$\frac{1}{2} \left(\widehat{F}_k(\zeta) - c \right) e^{-\lambda_k z} + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(\widehat{F}_k(\tau) - c \right) e^{-\lambda_k z_1} \mathcal{W}_H(\zeta, \tau, \lambda_k) d\tau = 0 \quad k = 1, 2, \quad \zeta \in \ell. \quad (56)$$

which is the necessary and sufficient condition for $\left(\widehat{F}_k(\zeta) - c \right) e^{-\lambda_k z}$, $\zeta \in \ell$, to be the boundary value of an H -analytic function in $\mathcal{D}^-$ that vanishes at infinity. Thus, the solution (52)–(53) follows from (56).

Finally, let $\widetilde{F}_k(\zeta)$, $k = 1, 2$, $\zeta \in \ell$ be another solution to (51). Similarly, we can show that $G_k(\zeta) = \left(\widetilde{F}_k(\zeta) - \widetilde{c} \right) e^{-\lambda_k z}$, $k = 1, 2$, $\zeta \in \ell$, solve the boundary-value problem (38), where $\widetilde{c}$ is a real-valued constant. However, by Proposition 2, (38) has a unique solution, and consequently, $\widehat{F}_k(\zeta) - \widetilde{F}_k(\zeta) = c - \widetilde{c}$, $k = 1, 2$, $\zeta \in \ell$, which means that a solution to the integral equations (51) is determined up to a real-valued constant, and the proof is finished. $\square$

Several remarks are in order.

Remark 1 (logarithmic singularity) *The kernels in (51) have logarithmic singularity.*

Remark 2 (conic endpoint) *If ℓ is piece-wise smooth (has salient points), in particular has conic endpoints, the integral equations (51) hold for all $\zeta \in \ell$ except for salient points and conic endpoints, i.e. almost everywhere.*

Remark 3 (multi-connected body) *The integral equations (51) hold for multiply connected regions, e.g. in the MHD problem for a torus.*

The integral equations (51) can be solved by the quadratic error minimization method. Let ℓ be parametrized by $\zeta = \zeta(t) \in C^1[-1, 1]$, and let $F_k = F_k(t)$, $t \in [0, 1]$, $k = 1, 2$ be unknown boundary values. For brevity, the integral equations (51) are represented by

$$\mathcal{A}_k(F_1, F_2) = f_k, \quad t \in [-1, 1], \quad k = 1, 2,$$

where $\mathcal{A}_k$ are corresponding integral operators, and $f_1(t) = f_2(t) = -1/(2\pi)$. The functions $F_k(t)$ can be approximated by finite functional series

$$F_1(t) = \sum_{k=1}^n (a_k + \mathbf{i} b_k) T_{k-1}(t), \quad F_2(t) = \sum_{k=1}^n c_k T_k(t) + \mathbf{i} d_k T_{k-1}(t), \quad (57)$$

where $T_k(t)$ is the Chebyshev polynomial of the first kind and a_k , b_k , c_k , and d_k , $k = 1, \dots, n$, are real-valued coefficients. Observe that the real part in the series approximating F_2 contains no constant term, whereas the series for F_1 does. This is because same real-valued constant in places of F_1 and F_2 is a homogeneous solution to (51).

Unknown coefficients a_k , b_k , c_k , and d_k , $k = 1, \dots, n$, can be found by minimizing the quadratic error in satisfying (51), i.e.

$$\min_{a_k, b_k, c_k, d_k} \sum_{k=1}^2 \|\mathcal{A}_k(F_1, F_2) - f_k\|^2 \quad (58)$$

with the inner product and norm for complex-valued functions $f(t)$ and $g(t)$ introduced by⁶

$$\langle f, g \rangle = \operatorname{Re} \left\{ \int_{-1}^1 f(t) \overline{g(t)} dt \right\}, \quad \|f\| = \sqrt{\langle f, f \rangle}. \quad (59)$$

Since $\mathcal{A}_k$, $k = 1, 2$, are linear operators, the problem (58) is unconstrained quadratic optimization and reduces to a system of linear algebraic equations for finding a_k , b_k , c_k , and d_k , $k = 1, \dots, n$.

3.2 Case (b): $M \neq 0$ and $R_m = 0$

Corollary 1 implies that for $M \neq 0$ and $R_m = 0$, the boundary conditions (10) reduce to the problem for determining boundary values of G_1 and G_2

$$e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2 = \lambda_2 - \lambda_1, \quad \zeta \in \ell, \quad (60)$$

where both G_1 and G_2 vanish at infinity.

Proposition 4 *The boundary-value problem (60) has a unique solution.*

Proof. The proof is similar to the proof of Proposition 4. In this case, Corollary 1 shows that the disturbance of the magnetic field in and out the body is zero, i.e. $\mathbf{h}^\pm = 0$. With $\mathbf{h}^- = 0$, the functional (45) and the surface integral (47) reduce to

$$\mathcal{F} = I_R + \iiint_{\mathbb{D}_R} (|\omega|^2 + M^2 u_r^2) dV, \quad (61)$$

⁶These functions are viewed as two-dimensional vector-functions from the direct sum $\mathcal{L}^2([-1, 1]) \oplus \mathcal{L}^2([-1, 1])$.

and

$$I_R = 2\pi \operatorname{Re} \left[\int_{C_R} \left((u_z + i u_r) \left(\wp + i \omega + \frac{R}{2} (u_z - i u_r) \right) \right) r d\zeta \right],$$

respectively. Using the representation (29), we obtain

$$I_R = \frac{2\pi}{\lambda_1 - \lambda_2} \operatorname{Re} \left[\int_{C_R} r \left(e^{2\lambda_1 z} G_1^2 - e^{2\lambda_2 z} G_2^2 \right) d\zeta \right],$$

and the rest of the proof is analogous to the proof of Proposition 4. $\square$

Proposition 5 (homogeneous conjugate boundary-value problem, case (b)) *Let G_1 and G_2 be H -analytic functions in $\mathcal{D}^+$. Under the assumption $M \neq 0$, the homogeneous conjugate boundary-value problem*

$$e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2 = 0, \quad \zeta \in \ell \quad (62)$$

has the solution

$$G_1 = c e^{-\lambda_1 z}, \quad G_2 = c e^{-\lambda_2 z}, \quad (63)$$

where c is an arbitrary real-valued constant, and λ_1 and λ_2 are defined in Corollary 1.

Proof. The proof is similar to the proof of Proposition 3. $\square$

With the generalized Cauchy integral formula for H -analytic functions, the boundary-value problem (10) readily reduces to an integral equation.

Theorem 5 (integral equations, case (b)) *Let $M \neq 0$ and $R_m = 0$, and let $F_1(\zeta) = e^{\lambda_1 z} G_1(\zeta)$. The boundary-value problem (60) reduces to the integral equation for the boundary value of F_1 :*

$$\frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(e^{\lambda_2(z-z_1)} \mathcal{W}_H(\zeta, \tau, \lambda_2) - e^{\lambda_1(z-z_1)} \mathcal{W}_H(\zeta, \tau, \lambda_1) \right) F_1(\tau) d\tau = \lambda_2 - \lambda_1, \quad \zeta \in \ell. \quad (64)$$

A solution to (64) is determined up to a real-valued constant c . Let $\widehat{F}_1(\zeta)$ solve (64), then $G_1(\zeta)$ in (60) is given by

$$G_1(\zeta) = \left(\widehat{F}_1(\zeta) - c \right) e^{-\lambda_1 z}, \quad \zeta \in \ell, \quad (65)$$

where

$$c = \frac{1}{2} \widehat{F}_1(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \widehat{F}_1(\tau) e^{\lambda_1(z-z_1)} \mathcal{W}_H(\zeta, \tau, \lambda_1) d\tau, \quad \zeta \in \ell. \quad (66)$$

Proof. The integral equation (64) is derived similarly to (51). In the second part of the proof that determines a homogeneous solution to (64), H -analytic functions Θ_k^+ , $k = 1, 2$, in $\mathcal{D}^+$ are introduced by the generalized Cauchy-type integrals

$$\begin{aligned} \Theta_1^+(\zeta) &= \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \widehat{F}_1(\tau) e^{-\lambda_1 z_1} \mathcal{W}_H(\zeta, \tau, \lambda_k) d\tau, \quad \zeta \in \mathcal{D}^+, \\ \Theta_2^+(\zeta) &= \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(\widehat{F}_1(\tau) + \lambda_1 - \lambda_2 \right) e^{-\lambda_2 z_1} \mathcal{W}_H(\zeta, \tau, \lambda_2) d\tau, \quad \zeta \in \mathcal{D}^+, \end{aligned}$$

where $\widehat{F}_1$ is a solution to (64). With these functions and corresponding generalized Sokhotski-Plemelj formulas (6), the integral equation (64) reduces to the homogenous conjugate boundary-value problem (62), whose solution is given by (63). The rest of the proof is analogous to that of Theorem 4. $\square$

The integral equation (64) can be solved by the quadratic error minimization method outlined in the end of Section 3.1. Remarks 1–3 also hold for (64).

3.3 Case (c): $M \neq 0$ and $R_m R = M^2$

For $M \neq 0$ and $R_m R = M^2$, the boundary conditions (10) reduce to the boundary-value problem for the four generalized analytic functions G_1 , G_2 , and $G_3^\pm$ defined in Theorem 3:

$$\begin{aligned} 2e^{\lambda z} G_1 - G_2 + (R + R_m)(G_3^+ + 1) &= 0, \quad \zeta \in \ell, \\ R_m \left(\left(z - \frac{i}{2} r \right) G_2 + G_3^- + 1 \right) - R G_3^+ &= 0, \quad \zeta \in \ell, \end{aligned} \quad (67)$$

where $\lambda = (R + R_m)/2$.

Proposition 6 *The boundary-value problem (67) has a unique solution.*

Proof. The assumptions $M \neq 0$ and $R_m R = M^2$ imply $R_m \neq 0$, and consequently, as in the proof of Proposition 4, we obtain the functional (45) with the surface integral (47). With the representations (30)–(34), the integral (47) reduces to

$$I_R = 2\pi \operatorname{Re} \left[\int_{C_R} \left((u_z + i u_r) G_2 + \frac{R}{2(R + R_m)} (2e^{\lambda z} G_1 - G_2)^2 \right) r d\zeta \right].$$

Since in this case, G_1 and G_2 are H -analytic and r -analytic functions, respectively, we have $G_1(R, \vartheta) = f(\vartheta) R^{-1} e^{-|\lambda|R} + O(R^{-2} e^{-|\lambda|R})$ as $R \rightarrow \infty$, where $f(\vartheta)$ is a bounded complex-valued function, and $G_2 = O(R^{-2})$ as $R \rightarrow \infty$. The rest of the proof is completely analogous to the proof of Proposition 4. $\square$

Proposition 7 (homogeneous conjugate boundary-value problem, case (c)) *Let G_1 and G_2 be H -analytic and r -analytic functions in $\mathcal{D}^+$, respectively, and let $G_3^\pm$ be r -analytic functions in $\mathcal{D}^\pm$, respectively, with G^- vanishing at infinity. Under the assumptions $M \neq 0$ and $R_m R = M^2$, the homogeneous conjugate boundary-value problem*

$$\begin{aligned} 2e^{\lambda z} G_1 - G_2 + (R + R_m) G_3^- &= 0, \quad \zeta \in \ell, \\ R_m \left(\left(z - \frac{i}{2} r \right) G_2 + G_3^+ \right) - R G_3^- &= 0, \quad \zeta \in \ell, \end{aligned} \quad (68)$$

has the solution

$$G_1 = c e^{-\lambda z}, \quad G_2 = 2c, \quad G_3^+ = -c(2z - i r), \quad G_3^- = 0, \quad (69)$$

where c is an arbitrary real-valued constant, and $\lambda = (R + R_m)/2$.

Proof. The proof is similar to the proof of Proposition 3. $\square$

Theorem 6 (integral equations, case (c)) *Let $M \neq 0$ and $R_m R = M^2$, and let $\lambda = (R + R_m)/2$, then the boundary-value problem (67) reduces to two integral equations for $F_1(\zeta) = e^{\lambda z} G_1(\zeta)$ and $F_2(\zeta) = G_2(\zeta)$, $\zeta \in \ell$:*

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(e^{\lambda(z-z_1)} \mathcal{W}_H(\zeta, \tau, \lambda) - \mathcal{W}_r(\zeta, \tau) \right) F_1(\tau) d\tau + F_1(\zeta) - \frac{1}{2} F_2(\zeta) &= 0, \\ \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(z_1 - z + \frac{i}{2}(r - r_1) \right) F_2(\tau) \mathcal{W}_r(\zeta, \tau) d\tau + \frac{R}{R_m(R + R_m)} (2F_1(\zeta) - F_2(\zeta)) &= -\frac{R + R_m}{R_m}. \end{aligned} \quad (70)$$

Equations (70) determine F_1 and F_2 up to c and $2c$, where c is a real-valued constant. Let $\widehat{F}_k$, $k = 1, 2$, solve (70), then the solution to the boundary-value problem (67) is given by

$$G_1(\zeta) = (\widehat{F}_1(\zeta) - c) e^{-\lambda z}, \quad G_2(\zeta) = \widehat{F}_2(\zeta) - 2c, \quad \zeta \in \ell, \quad (71)$$

where

$$2c = \frac{1}{2} \widehat{F}_2(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \widehat{F}_2(\tau) \mathcal{W}_r(\zeta, \tau) d\tau, \quad \zeta \in \ell. \quad (72)$$

Proof. The derivation of (70) is similar to that of (51); see the proof of Theorem 4. In the second part of the proof that determines a homogeneous solution, an H -analytic function Θ^+ in $\mathcal{D}^+$, r -analytic functions Φ_1^+ and Φ_2^+ in $\mathcal{D}^+$, and an r -analytic function Φ_2^- in $\mathcal{D}^-$ are introduced by the generalized Cauchy-type integrals

$$\begin{aligned}\Theta^+(\zeta) &= \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \widehat{F}_1(\tau) e^{-\lambda z_1} \mathcal{W}_H(\zeta, \tau, \lambda) d\tau, \quad \zeta \in \mathcal{D}^+, \\ \Phi_1^+(\zeta) &= \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \widehat{F}_2(\tau) \mathcal{W}_r(\zeta, \tau) d\tau, \quad \zeta \in \mathcal{D}^+, \\ \Phi_2^+(\zeta) &= \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(\left(z_1 - \frac{i}{2} r_1 \right) \widehat{F}_2(\tau) + \frac{R}{R_m(R+R_m)} (2\widehat{F}_1(\tau) - \widehat{F}_2(\tau)) + \frac{R+R_m}{R_m} \right) \mathcal{W}_r(\zeta, \tau) d\tau, \quad \zeta \in \mathcal{D}^+, \\ \Phi_2^-(\zeta) &= \frac{1}{2\pi i} \oint_{\ell \cup \ell'} (2\widehat{F}_1(\tau) - \widehat{F}_2(\tau)) \mathcal{W}_r(\zeta, \tau) d\tau, \quad \zeta \in \mathcal{D}^-, \end{aligned}$$

where $\widehat{F}_k$, $k=1,2$, are a solution to (70). With these functions and corresponding generalized Sokhotski-Plemelj formulas (6), the integral equations (70) reduce to the homogeneous conjugate boundary-value problem (68), whose solution is given by (69). The rest of the proof continues as in the proof of Theorem 4. $\square$

The integral equation (70) can be solved by the quadratic error minimization method outlined in the end of Section 3.1. Remarks 1–3 hold for (70) as well.

3.4 Drag

The drag (force) exerted on the body by the electrically conducting fluid in the presence of the magnetic field has the mechanic and magnetic components

$$\mathbf{F} = \iint_S \mathbf{P}_n dS - \mu \iiint_{\mathbb{D}^-} [\mathbf{J} \times \mathbf{H}] dV,$$

where $\mathbf{P}_n = \rho \mathbf{v} (2\partial \mathbf{u} / \partial n + [\mathbf{n} \times \text{curl} \mathbf{u}]) - \wp \mathbf{n}$, with $\mathbf{n}$ being the outward normal to S , and $\mathbf{J} = \frac{1}{4\pi} \text{curl} \mathbf{H}$.

In the axially symmetric case, $\text{div} \mathbf{u} = 0$ and the boundary conditions (10) imply

$$\frac{\partial \mathbf{u}}{\partial n} = -[\mathbf{n} \times \text{curl} \mathbf{u}] \quad \text{on } S, \quad (73)$$

see e.g. the proof of Proposition 8 in [28]. Also, the second term in $\mathbf{F}$ vanishes.

Proposition 8 *If at infinity, the flow and magnetic field are constant and parallel to the body's axis of revolution, and if also the fluid and body have the same magnetic permeability, the magnetic force has no direct contribution to the drag.*

Proof. In the axially symmetric case, both the total mechanic and magnetic forces are parallel to the body's axis of revolution, and consequently, it is sufficient to show that the projection of the magnetic force onto $\mathbf{k}$ vanishes. Let $\mathbb{D}_R$ be the region bounded by the body's surface S and sphere S_R with large radius R and center at the origin. Using the identity (41) and the divergence theorem, we obtain

$$\iiint_{\mathbb{D}_R} \mathbf{k} \cdot [\mathbf{J} \times \mathbf{H}] dV = -\frac{H_\infty^2}{4\pi} \iiint_{\mathbb{D}_R} [\mathbf{k} \times \mathbf{h}^-] \cdot \text{curl} \mathbf{h}^- dV = -\frac{H_\infty^2}{4\pi} (I_R - I),$$

where

$$I = \iint_S \mathbf{n} \cdot \left((\mathbf{k} \cdot \mathbf{h}^-) \mathbf{h}^- - \frac{1}{2} |\mathbf{h}^-|^2 \mathbf{k} \right) dS, \quad I_R = \iint_{S_R} \mathbf{n} \cdot \left((\mathbf{k} \cdot \mathbf{h}^-) \mathbf{h}^- - \frac{1}{2} |\mathbf{h}^-|^2 \mathbf{k} \right) dS.$$

As in the proof of Proposition 4, we can show that in all three cases (a), (b), and (c), $I_R \rightarrow 0$ as $R \rightarrow \infty$. On the other hand, since $\mathbf{h}^+ = \mathbf{h}^-$ on S and $\text{curl } \mathbf{h}^+ = 0$ in $\mathbb{D}^+$, using the identity similar to (41) for $\mathbf{h}^+$, we have

$$I = \iint_S \mathbf{n} \cdot \left((\mathbf{k} \cdot \mathbf{h}^+) \mathbf{h}^+ - \frac{1}{2} |\mathbf{h}^+|^2 \mathbf{k} \right) dS = - \iiint_{\mathbb{D}^+} [\mathbf{k} \times \mathbf{h}^+] \cdot \text{curl } \mathbf{h}^+ dV = 0.$$

Consequently, $\iiint_{\mathbb{D}^+} \mathbf{k} \cdot [\mathbf{J} \times \mathbf{H}] dV \rightarrow 0$ as $R \rightarrow \infty$. □

In view of (73) and Proposition 8, the drag in terms of dimensionalized $\mathbf{u}$ and $\wp$ takes the form

$$F_z = -V_\infty \rho \nu a \iint_S \mathbf{k} \cdot ([\mathbf{n} \times \text{curl } \mathbf{u}] + \wp \mathbf{n}) dS, \quad (74)$$

which, as in the proof of Proposition 11 in [28], reduces to

$$F_z = -2\pi V_\infty \rho \nu a \text{Re} \left[\int_\ell r (\wp + i\omega) d\zeta \right]. \quad (75)$$

Let $C_D = -F_z / (6\pi V_\infty \rho \nu a)$ be a dimensionless drag coefficient, where $6\pi V_\infty \rho \nu a$ is the drag of the sphere with radius a in the Stokes flow. Theorem 2, Corollary 1, and Theorem 3 and the corresponding boundary-value problems (38), (60), and (67) imply that

$$\wp + i\omega = \begin{cases} \frac{1}{\lambda_1 - \lambda_2} \left((R - 2\lambda_2) e^{\lambda_1 z} G_1 - (R - 2\lambda_1) e^{\lambda_2 z} G_2 \right) + R, & \zeta \in \ell \text{ case (a),} \\ 2 \left(e^{\lambda_1 z} G_1 + \lambda_1 \right), & \zeta \in \ell \text{ case (b),} \\ \frac{1}{R + R_m} \left(2R e^{\lambda z} G_1 + R_m G_2 \right) + R, & \zeta \in \ell \text{ case (c).} \end{cases}$$

Now with this representation for $\wp + i\omega$ and the fact that $\text{Re} \left[\int_\ell r d\zeta \right] = 0$, (75) leads to the following result.

Proposition 9 (drag) *Let $M \neq 0$, then the drag coefficient C_D is determined as follows.*

(a) *If $R_m \neq 0$ and $R_m R \neq M^2$, then*

$$C_D = \frac{1}{3(\lambda_1 - \lambda_2)} \text{Re} \left[\int_\ell r \left((R - 2\lambda_2) e^{\lambda_1 z} G_1 - (R - 2\lambda_1) e^{\lambda_2 z} G_2 \right) d\zeta \right]. \quad (76)$$

where G_1 and G_2 are defined in Theorem 2.

(b) *If $R_m = 0$, then*

$$C_D = \frac{2}{3} \text{Re} \left[\int_\ell r e^{\lambda_1 z} G_1 d\zeta \right], \quad (77)$$

where G_1 and λ_1 are defined in Corollary 1.

(c) *If $R_m R = M^2$, then*

$$C_D = \frac{1}{3(R + R_m)} \text{Re} \left[\int_\ell r \left(2R e^{\lambda z} G_1 + R_m G_2 \right) d\zeta \right], \quad (78)$$

where G_1 and G_2 are defined in Theorem 3.

4 Axially Symmetric MHD Problems

4.1 Sphere

The purpose of solving the MHD problem for sphere is three-fold: (i) verifying solution representations in terms of generalized analytic functions by comparing to the existing solutions for sphere; (ii) testing accuracy of solutions to the integral equations, and (iii) obtaining drag for various R , R_m , and M to compare to the drag of minimum-drag shapes.

Let S_a be the sphere with radius a and center at the origin, and let (R, ϑ, ϕ) be the spherical coordinates related to the cylindrical coordinates in the ordinary way.

4.1.1 Case (a): $M \neq 0$, $R_m \neq 0$, and $R_m R \neq M^2$

In the region exterior to S_a , the functions G_1 and G_2 in Theorem 2 can be represented by series similar to (8):

$$G_k(R, \vartheta) = \sqrt{\frac{a}{R}} \sum_{n=1}^{\infty} A_{k,n} N_n(\cos \vartheta, R, \lambda_k), \quad k = 1, 2, \quad (79)$$

where $A_{k,n}$ are unknown real-valued coefficients and

$$N_n(t, R, \lambda) = L_n(t) \frac{K_{n+\frac{1}{2}}(|\lambda| R)}{K_{n+\frac{1}{2}}(|\lambda| a)} - (\text{sign } \lambda) L_{-n}(t) \frac{K_{n-\frac{1}{2}}(|\lambda| R)}{K_{n+\frac{1}{2}}(|\lambda| a)},$$

whereas $G_3^{\pm}$ can be represented by

$$G_3^{\pm}(R, \vartheta) = \sum_{n=1}^{\infty} B_n^{\pm} (R/a)^{\pm n-1} L_{\mp n}(\cos \vartheta), \quad (80)$$

where $B_n^{\pm}$ are unknown real-valued coefficients; see [30, Example 5].

With the orthogonality property

$$\langle L_n(t), L_m(t) \rangle = 2m \delta_{mn} \quad (81)$$

for all integer n and m , where δ_{mn} is the Kronecker delta, and the representations (79) and (80), the inner product of the first equation in (38) with $L_{-m-1}(t)$ for $m \geq 0$ and the inner product of the second equation in (38) with $L_m(t)$ for $m \geq 1$ reduce to a real-valued infinite linear system for $A_{1,n}$ and $A_{2,n}$:

$$\begin{aligned} (1 - 2\lambda_2 \varkappa) \sum_{n=1}^{\infty} \xi_{mn}(\lambda_1) A_{1,n} - (1 - 2\lambda_1 \varkappa) \sum_{n=1}^{\infty} \xi_{mn}(\lambda_2) A_{2,n} &= 2(\lambda_1 - \lambda_2) \delta_{m0}, & m \geq 0, \\ \sum_{n=1}^{\infty} \eta_{mn}(\lambda_1) A_{1,n} - \sum_{n=1}^{\infty} \eta_{mn}(\lambda_2) A_{2,n} &= 0, & m \geq 1, \end{aligned} \quad (82)$$

where $\xi_{mn}(\lambda) = \langle e^{\lambda a t} N_n(t, a, \lambda), L_{-m-1}(t) \rangle$ and $\eta_{mn}(\lambda) = \langle e^{\lambda a t} N_n(t, a, \lambda), L_m(t) \rangle$. The system (82) can be truncated at some large m and solved numerically.

With the second equation in (82), the drag coefficient (76) reduces to

$$C_D = \frac{2}{3} \sum_{n=1}^{\infty} A_{1,n} \eta_{1n}(\lambda_1).$$

Figure 1 shows C_D as a function of S for five pairs: $R = R_m = 1$; $R = 1$, $R_m = 3$; $R = R_m = 2$; $R = 3$, $R_m = 1$; and $R = R_m = 3$.

4.1.2 Case (b): $M \neq 0$ and $R_m = 0$

As in the previous case, the functions G_1 and G_2 in Corollary 1 can be represented by the series (79). With the orthogonality property (81) and representations (79), the inner product of (60) with $e^{-\lambda_2 at} L_m(t)$ for $m \geq 1$ and the inner product of (60) with $e^{-\lambda_1 at} L_m(t)$ for $m \geq 1$, reduce to a real-valued infinite linear system for $A_{1,n}$ and $A_{2,n}$:

$$\begin{aligned} \sum_{n=1}^{\infty} \tilde{\xi}_{mn} A_{1,n} - 2mA_{2,m} &= b_{2,m}, \quad m \geq 1, \\ 2mA_{1,m} - \sum_{n=1}^{\infty} \tilde{\eta}_{mn} A_{2,n} &= b_{1,m}, \quad m \geq 1, \end{aligned} \quad (83)$$

where in this case, $\tilde{\xi}_{mn} = \langle e^{\lambda_1 at} N_n(t, a, \lambda_1), e^{-\lambda_2 at} L_m(t) \rangle$, $\tilde{\eta}_{mn} = \langle e^{\lambda_2 at} N_n(t, a, \lambda_2), e^{-\lambda_1 at} L_m(t) \rangle$, and $b_{k,m} = \langle \lambda_2 - \lambda_1, e^{-\lambda_k at} L_m(t) \rangle$, $k = 1, 2$. Similarly, the system (83) can be truncated at some large m and solved numerically.

The drag coefficient (77) reduces to

$$C_D = \frac{2}{3} \sum_{n=1}^{\infty} A_{1,n} \langle e^{\lambda_1 at} N_n(t, a, \lambda_1), L_1(t) \rangle.$$

Figure 2 depicts C_D as a function of R for $R_m = 0$ and $M = 0, 1, 2, 3$, and as a function of M for $R_m = 0$ and $R = 0, 1, 2, 3$. Numerical results show that for $R = R_m = 0$, the drag is in good agreement with Chester's formula [9]: $C_D = 1 + \frac{3}{8}M + \frac{7}{960}M^2 - \frac{43}{7680}M^3 + O(M^4)$ for small M up to 0.25; see curve a on Figure 2b.

4.1.3 Case (c): $M \neq 0$ and $R_m R = M^2$ ($S = 1$)

In this case, the H -analytic function G_1 and the r -analytic function G_2 in Theorem 3 are represented by the series

$$G_1(R, \vartheta) = \sqrt{\frac{a}{R}} \sum_{n=1}^{\infty} A_{1,n} N_n(\cos \vartheta, R, \lambda), \quad G_2(R, \vartheta) = \sum_{n=1}^{\infty} A_{2,n} (R/a)^{-n-1} L_n(\cos \vartheta), \quad (84)$$

where $N_n(\cos \vartheta, R, \lambda)$ is introduced in Section 4.1.1 and $\lambda = (R + R_m)/2$, whereas the r -analytic functions $G_3^{\pm}$ are represented by (80).

With the orthogonality property (81) and representations (84) and (80), the inner products of the first equation in (67) with $L_m(t)$ for $m \geq 1$ and with $L_{-m-1}(t)$ for $m \geq 0$ reduce to a real-valued infinite linear system

$$\begin{aligned} \sum_{n=1}^{\infty} \eta_{mn}(\lambda) A_{1,n} - mA_{2,m} &= 0, \quad m \geq 1, \\ \sum_{n=1}^{\infty} \xi_{mn}(\lambda) A_{1,n} - (R + R_m)(m+1)B_{m+1}^+ &= (R + R_m)\delta_{m0}, \quad m \geq 0, \end{aligned} \quad (85)$$

where $\eta_{mn}(\lambda)$ and $\xi_{mn}(\lambda)$ are those in Section 4.1.1.

Similarly, with (84) and (80) and the orthogonality property for the Legendre polynomials, the inner products of the second equation in (67) with $L_m(t)$ for $m \geq 1$ and with $L_{-m-1}(t)$ for $m \geq 0$ reduce to

$$\begin{aligned} a \left(\frac{m-1}{2m-1} A_{2,m-1} + \frac{3(m+1)}{2m+1} A_{2,m+1} \right) + 2B_m^- &= 0, \quad m \geq 1, \\ R_m a \frac{(m+1)(m+2)}{(2m+1)(2m+3)} A_{2,m+1} - R(m+1)B_{m+1}^+ &= -R_m \delta_{m0}, \quad m \geq 0, \end{aligned} \quad (86)$$

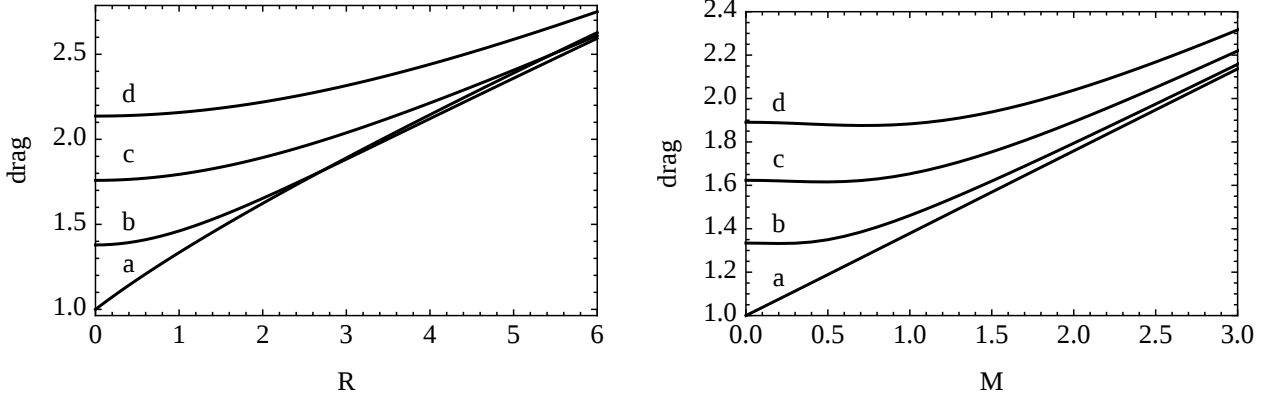
It follows from the system (85) and the second equation in (86) that

$$\sum_{n=1}^{\infty} \left(\frac{a R_m (R + R_m) (m+2)}{R (2m+1) (2m+3)} \eta_{(m+1)n}(\lambda) - \xi_{mn}(\lambda) \right) A_{1,n} = -\frac{(R + R_m)^2}{R} \delta_{m0}, \quad m \geq 0.$$

The drag coefficient (78) simplifies to the formula

$$C_D = \frac{2}{3} \sum_{n=1}^{\infty} A_{1,n} \eta_{1n}(\lambda),$$

which is used to evaluate C_D at $S = 1$ in Figure 1 and which for various small R and R_m agrees with Gotoh's formula [14]: $C_D \approx 1 + \frac{3}{8}R - (\frac{19}{320}R^2 + \frac{2}{15}RR_m) + (\frac{1}{7680}(213R^3 + 256R^2R_m - 704RR_m^2) + \frac{1}{30}RR_m(R + 4R_m))$.



(a) curves a, b, c, d correspond to $M = 0, 1, 2, 3$

(b) curves a, b, c, d correspond to $R = 0, 1, 2, 3$

Figure 2: Drag for the unit sphere normalized to the sphere's Stokes drag $6\pi\nu\rho V_\infty$ for $R_m = 0$ as functions of R and M .

4.2 Minimum-drag Spheroids

This section solves the axially symmetric MHD problem for solid nonmagnetic spheroids having the volume of a unit sphere, i.e. $4\pi/3$, and among those finds the spheroids that have the smallest drag for given R , R_m , and M . In the rz -half plane ($r \geq 0$), the cross section ℓ of spheroids is parametrized by $r(t) = a \cos t$, $z(t) = a^{-2} \sin t$, $t \in [-\pi/2, \pi/2]$, with $a \in (0, 1]$. Then for fixed R , R_m , and M , the spheroid drag becomes a function of a , and the minimum of this function along with the value of a at which the minimum is attained is found by the modified bisection method.

Let C_D and C_D^* be the drag coefficients for the minimum-drag spheroid and unit sphere, respectively, for same R , R_m , and S , and let κ be the axes ratio of the corresponding minimum-drag spheroid. Figure 3 shows the drag ratio C_D/C_D^* and κ as functions of $S \in [0, 2]$ for the three pairs $R = R_m = 1$, $R = R_m = 2$, and $R = R_m = 3$. As a function of S , C_D/C_D^* has maximum at $S = 1$ (and is nonsmooth at $S = 1$), which suggests that drag reduction is smallest for $S = 1$ and is more significant for $S \gg 1$, whereas the shortest minimum-drag spheroids appear to be not at $S = 1$. The found minimum-drag spheroids will be used as initial approximations for the minimum-drag shapes (Section 8).

5 Optimality Condition for Minimum-drag Shape

This section derives necessary optimality conditions for the shape of the body that has the smallest drag and the volume of a unit sphere in the MHD problem considered in Section 2.

Let body's surface S divide the space into the interior and exterior regions $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively. Since the drag $\mathbf{F}$ is parallel to the z axis, minimizing the absolute value of $\mathbf{F}$ is equivalent to minimizing $-\mathbf{k} \cdot \mathbf{F}$, and in view of (74), the shape optimization problem is formulated by

$$\min_S \mathcal{E}, \quad \mathcal{E} = \mathbf{k} \cdot \iint_S ([\mathbf{n} \times \text{curl } \mathbf{u}] + \wp \mathbf{n}) dS \quad (87)$$

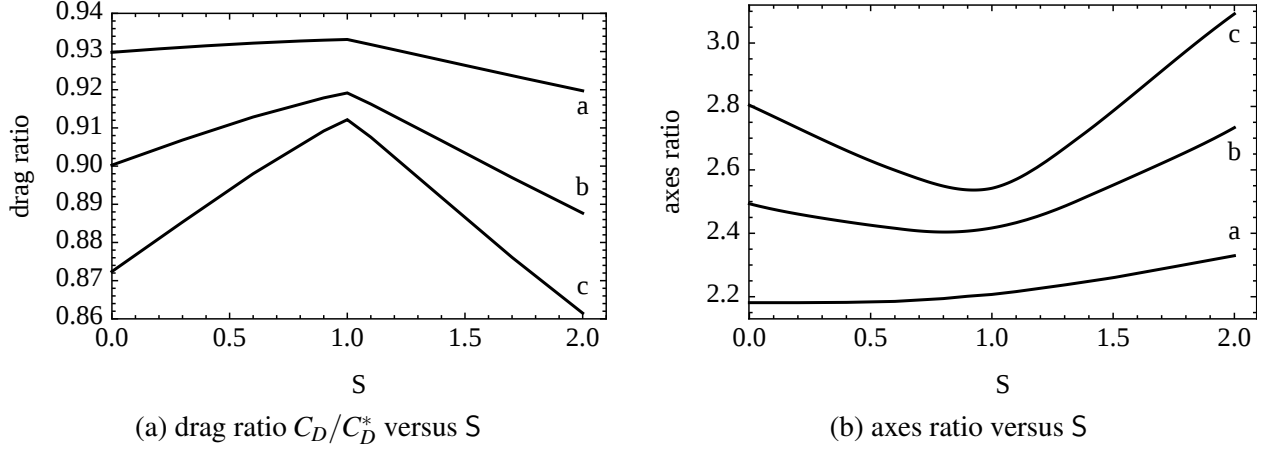


Figure 3: The drag ratio C_D/C_D^* for the minimum-drag spheroid and corresponding spheroid's axes ratio as functions of S for $R = R_m = 1$ (curve a); $R = R_m = 2$ (curve b); and $R = R_m = 3$ (curve c).

subject to (11) and (13) reformulated as

$$\begin{cases} \text{curl curl } \mathbf{u} + \text{grad } \wp + R \frac{\partial \mathbf{u}}{\partial z} - M^2 [\mathbf{k} \times [\mathbf{k} \times (\mathbf{u} - \mathbf{h}^-)]] = 0, \\ \text{curl } \mathbf{h}^- = R_m [(\mathbf{u} - \mathbf{h}^-) \times \mathbf{k}], \\ \text{div } \mathbf{u} = 0, \quad \text{div } \mathbf{h}^- = 0. \end{cases} \quad (88)$$

with the boundary conditions (12) and (10).

To derive necessary optimality conditions for (87)–(88), we use Mironov's shape variation approach [17]. Let $\mathbf{r}$ be the radius vector describing the optimal shape S , and let the shape variation S_ϵ be determined by

$$\mathbf{r}_\epsilon = \mathbf{r} + \epsilon f(\mathbf{r}) \mathbf{n},$$

where ϵ is a positive small number, $\mathbf{n}$ is the outward normal, and $f(\mathbf{r})$ is any continuous bounded scalar function such that $\iint_S f dS = 0$. The condition on f follows from the volume variation: if $\mathbb{D}_\epsilon^+$ is the region bounded by the surface S_ϵ , then

$$\iiint_{\mathbb{D}_\epsilon^+} dV = \iiint_{\mathbb{D}^+} dV + \iiint_{\mathbb{D}_\epsilon^+ - \mathbb{D}^+} dV = \iiint_{\mathbb{D}^+} dV + \epsilon \iint_S f dS + o(\epsilon) = 4\pi/3,$$

and since $\iiint_{\mathbb{D}^+} dV = 4\pi/3$, f should satisfy $\iint_S f dS = 0$.

Let $\mathbf{u}$, $\wp$, and $\mathbf{h}^\pm$ be the velocity disturbance, pressure, and magnetic field disturbances in and out the body for the *optimal* shape, and let their variations be given by

$$\mathbf{u}_\epsilon = \mathbf{u} + \epsilon \mathbf{u}_1 + o(\epsilon), \quad \wp_\epsilon = \wp + \epsilon \wp_1 + o(\epsilon), \quad \mathbf{h}_\epsilon^\pm = \mathbf{h}^\pm + \epsilon \mathbf{h}_1^\pm + o(\epsilon),$$

where $\mathbf{u}_1$, $\wp_1$, and $\mathbf{h}_1^\pm$ satisfy (88).

The boundary condition (10) should hold for $\mathbf{u}_\epsilon$ on S_ϵ :

$$\mathbf{u}_\epsilon = -\mathbf{k} \quad \text{on } S_\epsilon,$$

which implies

$$\mathbf{u}_\epsilon = \mathbf{u}(\mathbf{r} + \epsilon f(\mathbf{r}) \mathbf{n}) + \epsilon \mathbf{u}_1(\mathbf{r} + \epsilon f(\mathbf{r}) \mathbf{n}) + o(\epsilon) = -\mathbf{k},$$

whence

$$\mathbf{u}(\mathbf{r}) + \varepsilon \left(f(\mathbf{r}) \frac{\partial \mathbf{u}(\mathbf{r})}{\partial n} + \mathbf{u}_1(\mathbf{r}) \right) + o(\varepsilon) = -\mathbf{k}$$

and, consequently,

$$f \frac{\partial \mathbf{u}}{\partial n} + \mathbf{u}_1 = 0 \quad \text{on } S,$$

or, equivalently,

$$\mathbf{u}_1 = -f \frac{\partial \mathbf{u}}{\partial n} = f [\mathbf{n} \times \text{curl } \mathbf{u}] \quad \text{on } S. \quad (89)$$

Similarly, the boundary condition (12) should hold for $\mathbf{h}_\varepsilon^\pm$ on S_ε :

$$\mathbf{h}_\varepsilon^+ = \mathbf{h}_\varepsilon^- \quad \text{on } S_\varepsilon,$$

which implies

$$\mathbf{h}^+(\mathbf{r} + \varepsilon f(\mathbf{r}) \mathbf{n}) + \varepsilon \mathbf{h}_1^+(\mathbf{r} + \varepsilon f(\mathbf{r}) \mathbf{n}) + o(\varepsilon) = \mathbf{h}^-(\mathbf{r} + \varepsilon f(\mathbf{r}) \mathbf{n}) + \varepsilon \mathbf{h}_1^-(\mathbf{r} + \varepsilon f(\mathbf{r}) \mathbf{n}) + o(\varepsilon),$$

whence

$$\mathbf{h}^+(\mathbf{r}) + \varepsilon \left(f(\mathbf{r}) \frac{\partial \mathbf{h}^+(\mathbf{r})}{\partial n} + \mathbf{h}_1^+(\mathbf{r}) \right) + o(\varepsilon) = \mathbf{h}^-(\mathbf{r}) + \varepsilon \left(f(\mathbf{r}) \frac{\partial \mathbf{h}^-(\mathbf{r})}{\partial n} + \mathbf{h}_1^-(\mathbf{r}) \right) + o(\varepsilon)$$

and, consequently,

$$\mathbf{h}_1^+ - \mathbf{h}_1^- = -f \left(\frac{\partial \mathbf{h}^+}{\partial n} - \frac{\partial \mathbf{h}^-}{\partial n} \right) \quad \text{on } S. \quad (90)$$

The variation of the functional $\mathcal{E}$ defined in (87) can be determined as follows:

$$\begin{aligned} \mathcal{E}_\varepsilon &= \mathbf{k} \cdot \iint_{S_\varepsilon} ([\mathbf{n} \times \text{curl } \mathbf{u}_\varepsilon] + \wp_\varepsilon \mathbf{n}) dS \\ &= \underbrace{\mathbf{k} \cdot \iint_S ([\mathbf{n} \times \text{curl } \mathbf{u}_\varepsilon] + \wp_\varepsilon \mathbf{n}) dS}_{=I_\varepsilon} + \underbrace{\mathbf{k} \cdot \iint_{S_\varepsilon - S} ([\mathbf{n} \times \text{curl } \mathbf{u}_\varepsilon] + \wp_\varepsilon \mathbf{n}) dS}_{=J_\varepsilon} = I_\varepsilon + J_\varepsilon, \end{aligned} \quad (91)$$

with

$$I_\varepsilon = \mathcal{E} + \varepsilon \mathbf{k} \cdot \iint_S ([\mathbf{n} \times \text{curl } \mathbf{u}_1] + \wp_1 \mathbf{n}) dS + o(\varepsilon), \quad (92)$$

and

$$\begin{aligned} J_\varepsilon &= \mathbf{k} \cdot \iiint_{\mathbb{D}_\varepsilon} (\text{curl curl } \mathbf{u} + \text{grad } \wp + \varepsilon (\text{curl curl } \mathbf{u}_1 + \text{grad } \wp_1)) \text{sign } f dV + o(\varepsilon) \\ &= \varepsilon \mathbf{k} \cdot \iint_S (\text{curl curl } \mathbf{u} + \text{grad } \wp) f dS + o(\varepsilon) = -\varepsilon R \iint_S \frac{\partial u_z}{\partial z} f dS + o(\varepsilon). \end{aligned} \quad (93)$$

where $\mathbb{D}_\varepsilon$ is the region bounded by the surfaces S and S_ε , and in obtaining (93), we used the divergence theorem, the first equation in (88) and the fact that $\mathbf{k} \cdot [\mathbf{k} \times [\mathbf{k} \times (\mathbf{u} - \mathbf{h}^-)]] = 0$.

A further transformation of the functional I_ε relies on the equations adjoint to (88).

Proposition 10 (adjoint equations) *Let $\mathbf{w} \in C^2(\mathbb{D}^-)$, $q \in C^1(\mathbb{D}^-)$, $\mathbf{g}^\pm \in C^1(\mathbb{D}^\pm)$, and $v^\pm \in C^1(\mathbb{D}^\pm)$ satisfy the adjoint equations*

$$\begin{cases} \text{curl curl } \mathbf{w} - R \frac{\partial \mathbf{w}}{\partial z} - M^2 [\mathbf{k} \times [\mathbf{k} \times \mathbf{w}]] + \text{grad } q + R_m [\mathbf{g}^- \times \mathbf{k}] = 0, & \text{div } \mathbf{w} = 0, \\ \text{curl } \mathbf{g}^- + \text{grad } v^- = R_m [\mathbf{g}^- \times \mathbf{k}] - M^2 [\mathbf{k} \times [\mathbf{k} \times \mathbf{w}]], \end{cases} \quad (94)$$

in $\mathbb{D}^-$ and

$$\operatorname{curl} \mathbf{g}^+ + \operatorname{grad} v^+ = 0 \quad \text{in } \mathbb{D}^+ \quad (95)$$

with the boundary conditions on S and conditions at infinity to be

$$\begin{aligned} \mathbf{w}|_S &= \mathbf{k}, & \mathbf{g}^+|_S &= \mathbf{g}^+|_S, & v^+|_S &= v^+|_S, \\ \mathbf{w}|_\infty &= 0, & \mathbf{g}^-|_\infty &= 0, & v^-|_\infty &= 0, & q|_\infty &= 0. \end{aligned} \quad (96)$$

Then

$$\begin{aligned} \mathbf{k} \cdot \iint_S ([\mathbf{n} \times \operatorname{curl} \mathbf{u}_1] + p_1 \mathbf{n}) dS &= \iint_S \mathbf{u}_1 \cdot ([\mathbf{n} \times \operatorname{curl} \mathbf{w}] + q \mathbf{n} - R(\mathbf{n} \cdot \mathbf{k}) \mathbf{k}) dS \\ &+ \iint_S \mathbf{n} \cdot ([(\mathbf{h}_1^+ - \mathbf{h}_1^-) \times \mathbf{g}^-] - (\mathbf{h}_1^+ - \mathbf{h}_1^-) v^-) dS, \end{aligned} \quad (97)$$

provided that

$$\begin{aligned} I_R &= \iint_{S_R} \mathbf{n} \cdot ([\operatorname{curl} \mathbf{u}_1 \times \mathbf{w}] + [\mathbf{u}_1 \times \operatorname{curl} \mathbf{w}] + p_1 \mathbf{w} - q \mathbf{u}_1 \\ &+ R(\mathbf{w} \cdot \mathbf{u}_1) \mathbf{k} + [\mathbf{h}_1^- \times \mathbf{g}^-] - \mathbf{h}_1^- \cdot v^-) dS \rightarrow 0 \quad \text{as } R \rightarrow \infty, \end{aligned}$$

where S_R is the sphere centered at the origin and having large radius R .

Proof. Since $\mathbf{u}_1$, $\wp_1$, and $\mathbf{h}_1^-$ satisfy (88), we can write

$$\begin{aligned} 0 &= \mathbf{w} \cdot \left(\operatorname{curl} \operatorname{curl} \mathbf{u}_1 + R \frac{\partial \mathbf{u}_1}{\partial z} - M^2 [\mathbf{k} \times [\mathbf{k} \times (\mathbf{u}_1 - \mathbf{h}_1^-)]] + \operatorname{grad} \wp \right) - q \operatorname{div} \mathbf{u}_1 \\ &+ \mathbf{g}^- \cdot (\operatorname{curl} \mathbf{h}_1^- - R_m [(\mathbf{u}_1 - \mathbf{h}_1^-) \times \mathbf{k}]) - v^- \operatorname{div} \mathbf{h}_1^- \\ &= \operatorname{div} ([\operatorname{curl} \mathbf{u}_1 \times \mathbf{w}] + [\mathbf{u}_1 \times \operatorname{curl} \mathbf{w}] + p_1 \mathbf{w} - q \mathbf{u}_1 + R(\mathbf{w} \cdot \mathbf{u}_1) \mathbf{k} + [\mathbf{h}_1^- \times \mathbf{g}^-] - \mathbf{h}_1^- \cdot v^-) \\ &+ \mathbf{u}_1 \cdot \left(\operatorname{curl} \operatorname{curl} \mathbf{w} - R \frac{\partial \mathbf{w}}{\partial z} - M^2 [\mathbf{k} \times [\mathbf{k} \times \mathbf{w}]] + \operatorname{grad} q + R_m [\mathbf{g}^- \times \mathbf{k}] \right) - p_1 \operatorname{div} \mathbf{w} \\ &+ \mathbf{h}_1^- \cdot (\operatorname{curl} \mathbf{g}^- + \operatorname{grad} v^- - R_m [\mathbf{g}^- \times \mathbf{k}] + M^2 [\mathbf{k} \times [\mathbf{k} \times \mathbf{w}]]), \end{aligned}$$

which in view of the adjoint equations (94) implies

$$\operatorname{div} ([\operatorname{curl} \mathbf{u}_1 \times \mathbf{w}] + [\mathbf{u}_1 \times \operatorname{curl} \mathbf{w}] + p_1 \mathbf{w} - q \mathbf{u}_1 + R(\mathbf{w} \cdot \mathbf{u}_1) \mathbf{k} + [\mathbf{h}_1^- \times \mathbf{g}^-] - \mathbf{h}_1^- \cdot v^-) = 0 \quad \text{in } \mathbb{D}^-. \quad (98)$$

On the other hand, since $\mathbf{h}^+$ satisfies $\operatorname{curl} \mathbf{h}^+ = 0$ and $\operatorname{div} \mathbf{h}^+ = 0$, we have

$$0 = \mathbf{g}^+ \cdot \operatorname{curl} \mathbf{h}_1^+ - v^+ \operatorname{div} \mathbf{h}_1^+ = \operatorname{div} ([\mathbf{h}_1^+ \times \mathbf{g}^+] - v^+ \mathbf{h}_1^+) + \mathbf{h}_1^+ \cdot (\operatorname{curl} \mathbf{g}^+ + \operatorname{grad} v^+),$$

which with the adjoint equation (95) reduces to

$$\operatorname{div} ([\mathbf{h}_1^+ \times \mathbf{g}^+] - v^+ \mathbf{h}_1^+) = 0 \quad \text{in } \mathbb{D}^+. \quad (99)$$

Let $\mathbb{D}_R$ be the region bounded by the surface S and sphere S_R . Integrating (98) and (99) over $\mathbb{D}_R$ and $\mathbb{D}^+$, respectively, and then using the divergence theorem and adding the resulting equations, we obtain

$$\begin{aligned} I_R - \iint_S \mathbf{n} \cdot ([\operatorname{curl} \mathbf{u}_1 \times \mathbf{w}] + [\mathbf{u}_1 \times \operatorname{curl} \mathbf{w}] + p_1 \mathbf{w} - q \mathbf{u}_1 + R(\mathbf{w} \cdot \mathbf{u}_1) \mathbf{k} \\ + [\mathbf{h}_1^- \times \mathbf{g}^-] - \mathbf{h}_1^- \cdot v^- - [\mathbf{h}_1^+ \times \mathbf{g}^+] + v^+ \mathbf{h}_1^+) dS = 0, \end{aligned}$$

which with the boundary conditions (96) and the assumption that $I_R \rightarrow 0$ as $R \rightarrow \infty$ can be rearranged in the form (97). $\square$

With the relationship (97) and the boundary conditions (89)–(90), the functional (92) takes the form

$$I_\varepsilon = \mathcal{E} + \varepsilon \iint_S \left([\mathbf{n} \times \text{curl } \mathbf{u}] \cdot [\mathbf{n} \times \text{curl } \mathbf{w}] + R(\mathbf{n} \cdot \mathbf{k}) \frac{\partial u_z}{\partial n} \right) f dS \\ - \varepsilon \iint_S \mathbf{n} \cdot \left(\left[\left(\frac{\partial \mathbf{h}^+}{\partial n} - \frac{\partial \mathbf{h}^-}{\partial n} \right) \times \mathbf{g}^- \right] - \left(\frac{\partial \mathbf{h}^+}{\partial n} - \frac{\partial \mathbf{h}^-}{\partial n} \right) v^- \right) f dS + o(\varepsilon). \quad (100)$$

Now since $\partial \mathbf{u} / \partial s = 0$ on S , we observe that

$$(\mathbf{n} \cdot \mathbf{k}) \frac{\partial u_z}{\partial n} = \frac{\partial z}{\partial n} \frac{\partial u_z}{\partial n} = \frac{\partial z}{\partial n} \left(\frac{\partial u_z}{\partial r} \frac{\partial r}{\partial n} + \frac{\partial u_z}{\partial z} \frac{\partial z}{\partial n} \right) = \frac{\partial r}{\partial n} \left(\frac{\partial u_z}{\partial r} \frac{\partial z}{\partial n} - \frac{\partial u_z}{\partial z} \frac{\partial r}{\partial n} \right) + \frac{\partial u_z}{\partial z} \\ = \frac{\partial r}{\partial n} \frac{\partial u_z}{\partial s} + \frac{\partial u_z}{\partial z} = \frac{\partial u_z}{\partial z} \quad \text{on } S. \quad (101)$$

Let $\widehat{\mathbf{h}} = \mathbf{h}^+ - \mathbf{h}^-$ on S . Then (12) and the conditions $\text{div } \mathbf{h}^\pm = 0$ in $\mathbb{D}^\pm \cup S$ imply that $\widehat{\mathbf{h}} = 0$ and $\text{div } \widehat{\mathbf{h}} = 0$ on S . In this case, $\widehat{\partial \mathbf{h}} / \partial n = -[\mathbf{n} \times \text{curl } \widehat{\mathbf{h}}]$ on S ; see the proof of Proposition 11 in [28]. On the other hand, it follows from (11), the second equation in (13), and (10) that $\text{curl } \widehat{\mathbf{h}} = R_m [\mathbf{h}^- \times \mathbf{k}]$ on S . Thus,

$$\frac{\partial \mathbf{h}^+}{\partial n} - \frac{\partial \mathbf{h}^-}{\partial n} = -R_m [\mathbf{n} \times [\mathbf{h}^- \times \mathbf{k}]] \quad \text{on } S. \quad (102)$$

Also, in the axially symmetric case, $\text{curl } \mathbf{u} = \omega(r, z) \mathbf{e}_\varphi$, $\text{curl } \mathbf{w} = \omega^*(r, z) \mathbf{e}_\varphi$, and $\mathbf{g}^- = g^-(r, z) \mathbf{e}_\varphi$. Finally, in view of (93) and (100) with (101)–(102), (91) reduces to

$$\mathcal{E}_\varepsilon = \mathcal{E} + \varepsilon \iint_S (\omega \omega^* + R_m h_r^- g^-) f dS + o(\varepsilon).$$

The necessary optimality condition requires $\lim_{\varepsilon \rightarrow 0} (\mathcal{E}_\varepsilon - \mathcal{E}) / \varepsilon = 0$, which yields $\iint_S (\omega \omega^* + R_m h_r^- g^-) f dS$. However, since f should satisfy $\iint_S f dS = 0$, we conclude that $\omega \omega^* + R_m h_r^- g^-$ is constant on S .

Consequently, the following result has been proved.

Theorem 7 *Under the assumptions of Proposition 10, the necessary optimality condition for the minimum-drag shape subject to the volume constraint is given by*

$$\omega \omega^* + R_m h_r^- g^- = \text{const} \quad \text{on } S, \quad (103)$$

where $\omega = \mathbf{e}_\varphi \cdot \text{curl } \mathbf{u}$, $\omega^* = \mathbf{e}_\varphi \cdot \text{curl } \mathbf{w}$ and $g^- = \mathbf{e}_\varphi \cdot \mathbf{g}^-$, and $\mathbf{w}$ and $\mathbf{g}^-$ are solutions of the adjoint equations (94)–(95) with the boundary conditions (96).

6 Solving Adjoint MHD Equations

A solution to the adjoint equations (94)–(95) with the boundary conditions (96) can be also constructed in terms of generalized analytic functions. In the axially symmetric case with the axis of revolution coinciding with the z axis, the functions $\mathbf{w}$, q , $\mathbf{g}^\pm$, and $v^\pm$ have the following representations:

$$\mathbf{w} = w_r(r, z) \mathbf{e}_r + w_z(r, z) \mathbf{k}, \quad q = q(r, z), \quad \mathbf{g}^\pm = g^\pm(r, z) \mathbf{e}_\varphi, \quad v^\pm = v^\pm(r, z),$$

where the third formula implies $\text{div } \mathbf{g}^\pm = 0$.

6.1 Case (a): $M \neq 0$ and $R_m R \neq M^2$

Let ℓ be the positively oriented cross section of S in the right-half rz -plane, and let ℓ' be the reflection of ℓ over the z axis.

Theorem 8 (solution representation for the adjoint equations, case (a)) *Let λ_1 and λ_2 be as those defined in Theorem 2. In the axially symmetric case with $M \neq 0$ and $R_m R \neq M^2$, a solution to the adjoint equations (94)–(95) is given by*

$$\begin{aligned} w_z + i w_r &= \frac{1}{\lambda_1 - \lambda_2} \left((2\lambda_2 \varkappa - 1) e^{-\lambda_1 z} G_1 - (2\lambda_1 \varkappa - 1) e^{-\lambda_2 z} G_2 \right) + \varkappa G_3^-, \\ v^- + i g^- &= (1 - R \varkappa) \left(\frac{2}{\lambda_1 - \lambda_2} \left(\lambda_2 e^{-\lambda_1 z} G_1 - \lambda_1 e^{-\lambda_2 z} G_2 \right) + G_3^- \right), \\ v^+ + i g^+ &= G_3^+, \\ q &= \frac{2(R \varkappa - 1)}{\lambda_1 - \lambda_2} \left(\lambda_2 e^{-\lambda_1 z} \operatorname{Re}[G_1] - \lambda_1 e^{-\lambda_2 z} \operatorname{Re}[G_2] \right) + R \varkappa \operatorname{Re}[G_3^-], \\ \omega^* &= \frac{1}{\lambda_1 - \lambda_2} \left((R - 2\lambda_2) e^{-\lambda_1 z} \operatorname{Im}[G_1] - (R - 2\lambda_1) e^{-\lambda_2 z} \operatorname{Im}[G_2] \right). \end{aligned} \quad (104)$$

where G_1 and G_2 are H -analytic functions in $\mathcal{D}^-$ that satisfy (3) with $\lambda = -\lambda_{1,2}$, respectively, and vanish at infinity; $G_3^\pm$ are r -analytic functions in $\mathcal{D}^\pm$, respectively, with G_3^- vanishing at infinity.

Proof. The proof is similar to that of Theorem 2. □

With the representations (104), the boundary conditions (96) reduce to the boundary-value problem for the generalized analytic functions G_1 , G_2 , and $G_3^\pm$ defined in Theorem 8:

$$\begin{aligned} (1 - 2\lambda_2 \varkappa) e^{-\lambda_1 z} G_1 - (1 - 2\lambda_1 \varkappa) e^{-\lambda_2 z} G_2 - (\lambda_1 - \lambda_2) \varkappa G_3^- &= \lambda_2 - \lambda_1, \quad \zeta \in \ell, \\ e^{-\lambda_1 z} G_1 - e^{-\lambda_2 z} G_2 &= (\lambda_2 - \lambda_1) \left(\frac{R_m}{M^2} G_3^+ + 1 \right), \quad \zeta \in \ell. \end{aligned} \quad (105)$$

The problem (105) can be reduced to integral equations for the boundary values of G_1 and G_2 based on the generalized Cauchy integral formula.

Theorem 9 (adjoint integral equations, case (a)) *Let $M \neq 0$, $R_m \neq 0$, and $R_m R \neq M^2$. In this case, (105) yields two integral equations for the boundary values of $F_k(\zeta) = e^{-\lambda_k z} G_k(\zeta)$, $k = 1, 2$:*

$$\frac{1 - 2\varkappa \lambda_k}{2\varkappa(\lambda_1 - \lambda_2)} (F_1(\zeta) - F_2(\zeta)) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(\mathcal{W}_r(\zeta, \tau) - e^{\lambda_k(z_1 - z)} \mathcal{W}_H(\zeta, \tau, -\lambda_k) \right) F_k(\tau) d\tau = -\frac{1}{2\varkappa}, \quad k = 1, 2, \quad (106)$$

where $\zeta \in \ell$ and λ_1 and λ_2 are defined in Theorem 2. A solution to (106) is determined up to a real-valued constant c , i.e. $F_k(\zeta) = c$, $k = 1, 2$, are a homogeneous solution to (106). Let $\hat{F}_k(\zeta)$, $k = 1, 2$, solve (106), then $G_k(\zeta)$, $k = 1, 2$, are determined by

$$G_k(\zeta) = \left(\hat{F}_k(\zeta) - c \right) e^{\lambda_k z}, \quad k = 1, 2, \quad \zeta \in \ell, \quad (107)$$

where

$$c = \frac{1}{2} \hat{F}_k(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \hat{F}_k(\tau) e^{\lambda_k(z_1 - z)} \mathcal{W}_H(\zeta, \tau, -\lambda_k) d\tau, \quad \zeta \in \ell. \quad (108)$$

Proof. The proof is completely analogous to the proof of Theorem 4. $\square$

In fact, the integral equations (51) and (106) are very similar. The only difference is in the sign of λ_k in the kernels. The next proposition shows that solutions of (51) and (106) are closely related.

Proposition 11 *Let $\bar{\ell}$ be the reflection of the curve ℓ over the r axis, and let $F_k(\zeta)$, $k = 1, 2$, be a solution to the integral equations (51) for $\zeta \in \bar{\ell}$, then $\tilde{F}_k(\zeta) = \overline{F_k(\bar{\zeta})}$, $k = 1, 2$, $\zeta \in \ell$, is a solution to the integral equations (106).*

Proof. With the change of variables $\tau = \bar{\tau}^*$, $\tau^* \in \bar{\ell}$, and $\zeta = \bar{\zeta}^*$, $\zeta^* \in \bar{\ell}$, in (106) and with the properties $\mathcal{W}_r(\bar{\zeta}, \bar{\tau}) = \overline{\mathcal{W}_r(\zeta, \tau)}$ and $\mathcal{W}_H(\bar{\zeta}, \bar{\tau}, -\lambda) = \overline{\mathcal{W}_H(\zeta, \tau, \lambda)}$, the integral equations (106) reduce to (51) for unknown functions $\tilde{F}_k(\zeta^*) = \overline{F_k(\bar{\zeta}^*)}$, $k = 1, 2$, $\zeta^* \in \bar{\ell}$. $\square$

6.2 Case (b): $M \neq 0$ and $R_m = 0$

We recall that for $R_m = 0$, the magnetic field is uncoupled from the velocity field but is involved in determining the velocity field, and Corollary 1 shows that the magnetic field is zero in this case. For the adjoint equations (94)–(95), when $R_m = 0$, $\mathbf{w}$ and q are uncoupled from $\mathbf{g}^\pm$ and $v^\pm$, and their representation follows directly from (104).

Corollary 4 (solution representation for the adjoint equations, case (b)) *Let λ_1 and λ_2 be as those defined in Corollary 1. In the axially symmetric case with $M \neq 0$ and $R_m = 0$, a solution to the adjoint equations (94)–(95) is given by*

$$\begin{aligned} w_z + i w_r &= -\frac{1}{\lambda_1 - \lambda_2} \left(e^{-\lambda_1 z} G_1 - e^{-\lambda_2 z} G_2 \right), \\ q + i \omega^* &= e^{-\lambda_1 z} \left(G_1 - \frac{R}{2(\lambda_1 - \lambda_2)} \overline{G_1} \right) + e^{-\lambda_2 z} \left(G_2 + \frac{R}{2(\lambda_1 - \lambda_2)} \overline{G_2} \right), \\ v^- + i g^- &= \frac{2}{\lambda_1 - \lambda_2} \left(\lambda_2 e^{-\lambda_1 z} G_1 - \lambda_1 e^{-\lambda_2 z} G_2 \right) + G_3^-, \\ v^+ + i g^+ &= G_3^+, \end{aligned} \quad (109)$$

where G_1 , G_2 , and $G_3^\pm$ are defined in Theorem 8.

In this case, both equations in (105) reduce to the same boundary-value problem

$$e^{-\lambda_1 z} G_1 - e^{-\lambda_2 z} G_2 = \lambda_2 - \lambda_1, \quad \zeta \in \ell. \quad (110)$$

Once G_1 and G_2 are found from (110), the boundary values of $G_3^\pm$ can be found based on the last two formulas in (109) and the fact that $v^+ + i g^+ = v^- + i g^-$ on ℓ . Namely, $G_3^+ - G_3^- = \frac{2}{\lambda_2 - \lambda_1} (\lambda_2 e^{-\lambda_1 z} G_1 - \lambda_1 e^{-\lambda_2 z} G_2)$ on ℓ , and $G_3^\pm$ are determined by the Cauchy integral formula (5) for r -analytic functions.

Theorem 10 (adjoint integral equations, case (b)) *Let $M \neq 0$ and $R_m = 0$, and let $F_1(\zeta) = e^{-\lambda_1 z} G_1(\zeta)$, where λ_1 is defined in Corollary 1, then (110) reduces to the integral equation for the boundary value of F_1 :*

$$\frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(e^{\lambda_2(z_1 - z)} \mathcal{W}_H(\zeta, \tau, -\lambda_2) - e^{\lambda_1(z_1 - z)} \mathcal{W}_H(\zeta, \tau, -\lambda_1) \right) F_1(\tau) d\tau = \lambda_2 - \lambda_1, \quad \zeta \in \ell. \quad (111)$$

A solution to (64) is determined up to a real-valued constant c . Let $\hat{F}_1(\zeta)$ solve (111), then $G_1(\zeta)$ in (110) is given by

$$G_1(\zeta) = \left(\hat{F}_1(\zeta) - c \right) e^{\lambda_1 z}, \quad \zeta \in \ell, \quad (112)$$

where

$$c = \frac{1}{2} \widehat{F}_1(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \widehat{F}_1(\tau) e^{\lambda_1(z_1 - z)} \mathcal{W}_H(\zeta, \tau, -\lambda_1) d\tau, \quad \zeta \in \ell. \quad (113)$$

Proof. The proof is analogous to the proof of Theorem 5. $\square$

Proposition 12 *Let $\bar{\ell}$ be the reflection of the curve ℓ over the r axis, and let $F_1(\zeta)$ be a solution to the integral equation (64) for $\zeta \in \bar{\ell}$, then $\widehat{F}_1(\zeta) = \overline{F_1(\bar{\zeta})}$, $\zeta \in \ell$, is a solution to the integral equation (111).*

Proof. The proof is similar to that of Proposition 11. $\square$

6.3 Case (c): $M \neq 0$ and $R_m R = M^2$

Theorem 11 (solution representation for the adjoint equations, case (c)) *Let $\lambda = (R + R_m)/2$. In the axially symmetric case with $M \neq 0$ and $R_m R = M^2$, a solution to the adjoint equations (94)–(95) is given by*

$$\begin{aligned} w_z + i w_r &= \frac{1}{R + R_m} \left(-\frac{2R}{R + R_m} e^{-\lambda z} G_1 + \left(R_m \left(z - \frac{i}{2} r \right) + \frac{R}{R + R_m} \right) G_2 + R_m G_3^- \right), \\ v^- + i g^- &= -\frac{R_m R}{R + R_m} \left(\frac{2}{R + R_m} e^{-\lambda z} G_1 + \left(z - \frac{i}{2} r - \frac{1}{R + R_m} \right) G_2 + G_3^- \right), \\ v^+ + i g^+ &= G_3^+, \\ q &= \frac{R_m R}{R + R_m} \left(\frac{2}{R + R_m} e^{-\lambda z} \operatorname{Re}[G_1] + \operatorname{Re} \left[\left(z - \frac{i}{2} r - \frac{1}{R + R_m} \right) G_2 \right] + \operatorname{Re}[G_3^-] \right) + \operatorname{Re}[G_2], \\ \omega^* &= \frac{1}{R + R_m} \left(2 \operatorname{Re} e^{-\lambda z} \operatorname{Im}[G_1] + R_m \operatorname{Im}[G_2] \right), \end{aligned} \quad (114)$$

where G_1 is an H -analytic function in $\mathcal{D}^-$ satisfying (3) with λ and vanishing at infinity; G_2 and G_3^- are r -analytic functions in $\mathcal{D}^-$ and vanishing at infinity; and G_3^+ is an r -analytic function in $\mathcal{D}^+$.

Proof. The proof is similar to the proof of Theorem 3. $\square$

In this case, the boundary conditions (96) reduce to the boundary-value problem

$$\begin{aligned} 2e^{-\lambda z} G_1 - G_2 + (R + R_m) \left(\frac{1}{R} G_3^+ + 1 \right) &= 0, \quad \zeta \in \ell, \\ R_m \left(\left(z - \frac{i}{2} r \right) G_2 + G_3^- - 1 \right) + G_3^+ &= 0, \quad \zeta \in \ell. \end{aligned} \quad (115)$$

Theorem 12 (adjoint integral equations, case (c)) *Let $M \neq 0$ and $R_m R = M^2$, and let $\lambda = (R + R_m)/2$, then the boundary-value problem (115) reduces to two integral equations for $F_1(\zeta) = e^{-\lambda z} G_1(\zeta)$ and $F_2(\zeta) = G_2(\zeta)$:*

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(e^{\lambda(z_1 - z)} \mathcal{W}_H(\zeta, \tau, -\lambda) - \mathcal{W}_r(\zeta, \tau) \right) F_1(\tau) d\tau + F_1(\zeta) - \frac{1}{2} F_2(\zeta) &= 0, \\ \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left(z_1 - z + \frac{i}{2} (r - r_1) \right) F_2(\tau) \mathcal{W}_r(\zeta, \tau) d\tau - \frac{R}{R_m(R + R_m)} (2F_1(\zeta) - F_2(\zeta)) &= \frac{R + R_m}{R_m}, \end{aligned} \quad (116)$$

where $\zeta \in \ell$. Equations (116) determine F_1 and F_2 up to c and $2c$, where c is a real-valued constant. Let $\widehat{F}_1$ and $\widehat{F}_2$ solve (116), then the solution to the boundary-value problem (115) is given by

$$G_1(\zeta) = \left(\widehat{F}_1(\zeta) - c \right) e^{\lambda z}, \quad G_2(\zeta) = \widehat{F}_2(\zeta) - 2c, \quad \zeta \in \ell, \quad (117)$$

where

$$2c = \frac{1}{2} \widehat{F}_2(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \widehat{F}_2(\tau) \mathcal{W}_r(\zeta, \tau) d\tau, \quad \zeta \in \ell. \quad (118)$$

Proof. The proof is similar to that of Theorem 6. $\square$

Proposition 13 *Let $\bar{\ell}$ be the reflection of the curve ℓ over the r axis, and let $F_k(\zeta)$, $k = 1, 2$, be a solution to the integral equations (70) for $\zeta \in \bar{\ell}$, then $\widetilde{F}_k(\zeta) = \overline{F_k(\bar{\zeta})}$, $k = 1, 2$, $\zeta \in \ell$, are a solution to the integral equations (116).*

Proof. The proof is similar to that of Proposition 11. $\square$

7 Asymptotic Behavior at Conic Endpoint

It is known that for a nonconducting viscous incompressible fluid under the Stokes and Oseen approximations, the minimum-drag shapes subject to a volume constraint have conic endpoints with the angle between the axis of revolution and the tangent at the endpoint to be $2\pi/3$; see [20, 17]. This section analyzes the asymptotic behavior of a solution to the MHD problem (10)–(13) near the vicinity of a conic endpoint.

Let one of the endpoints of ℓ lie on the z axis and have coordinates $r = 0$, $z = c$, and let (ρ, θ) be the local polar coordinates with the pole at $r = 0$, $z = c$ and with the angle θ counted from the z axis. In this case, (ρ, θ) are related to (r, z) by $r = \rho \sin \theta$ and $z = c + \rho \cos \theta$, $\theta \in [0, \pi]$. Let θ_0 be the angle between the z axis and the tangent to ℓ at the endpoint. It is assumed that ℓ is in the cone $\theta \geq \theta_0$.

7.1 Cases (a) and (b): $M \neq 0$ and $R_m R \neq M^2$

Proposition 14 (asymptotic behavior of G_1 and G_2) *Let $M \neq 0$ and $R_m R \neq M^2$, and let G_1 and G_2 be H -analytic functions defined in Theorem 2. The asymptotic behavior of G_1 and G_2 on ℓ in the vicinity of the endpoint is given by*

$$G_k = \begin{cases} b_k + o(\rho^0), & \theta_0 < 2\pi/3, \\ a_k (\ln \rho + 2 \ln(\cos(\theta_0/2)) - i \tan(\theta_0/2)) + b_k + o(\rho^0), & \theta_0 = 2\pi/3, \\ a_k \rho^{\alpha-1} M_{\alpha-1}(\cos \theta) + b_k + o(\rho^0), & \theta_0 > 2\pi/3, \end{cases} \quad (119)$$

as $\rho \rightarrow 0$, where $a_k, b_k \in \mathbb{R}$ are some constants, $M_\alpha(\cos \theta) = (\alpha + 1) P_\alpha(\cos \theta) + i P_\alpha^{(1)}(\cos \theta)$, and α is the single zero of the equation

$$(1 + \cos^2 \theta_0) P_\alpha(\cos \theta_0) P_\alpha^{(1)}(\cos \theta_0) + \sin \theta_0 \cos \theta_0 \left([P_\alpha^{(1)}(\cos \theta_0)]^2 + \alpha(\alpha + 1) [P_\alpha(\cos \theta_0)]^2 \right) = 0 \quad (120)$$

in the interval $(0, 1)$ for given θ_0 .⁷

Proof. The proof is conducted similarly to the proof of Proposition 1 in [32]. First, let $R_m \neq 0$.

In the local polar coordinates (ρ, θ) , the H -analytic functions G_1 and G_2 and r -analytic functions $G_3^\pm$ can be represented in the form

$$\begin{aligned} G_k &= a_k \rho^{\alpha-1} M_{\alpha-1}(\cos \theta) + b_k + \rho^\alpha (c_k M_\alpha(\cos \theta) - \lambda_k a_k P_\alpha(\cos \theta)) + o(\rho^\alpha), & 0 < \alpha < 1, \\ G_3^\pm &= a_3^\pm \rho^{\alpha-1} M_{\alpha-1}(\cos \theta) + b_3^\pm + c_3^\pm \rho^\alpha M_\alpha(\cos \theta) + o(\rho^\alpha), & 0 < \alpha < 1, \end{aligned} \quad (121)$$

⁷Equation (120) reduces to identity for $\alpha = 0$ and $\alpha = 1$.

as $\rho \rightarrow 0$, where $k = 1, 2$, and $a_1, b_1, c_1, a_2, b_2, c_2, a_3^\pm, b_3^\pm, c_3^\pm$ are real constants. Substituting (121) into the boundary-value problem (38) with $e^{\lambda_k z} = e^{\lambda_k \rho \cos \theta + O(\rho^2)}$ and equating corresponding coefficients at $\rho^{\alpha-1}$, ρ^0 , and ρ^α , we obtain six equations:

$$\begin{aligned} (1 - 2\lambda_2 \varkappa) a_1 e^{\lambda_1 c} - (1 - 2\lambda_1 \varkappa) a_2 e^{\lambda_2 c} - \frac{\lambda_1 - \lambda_2}{R_m R - M^2} a_3^- &= 0, \\ (1 - 2\lambda_2 \varkappa) b_1 e^{\lambda_1 c} - (1 - 2\lambda_1 \varkappa) b_2 e^{\lambda_2 c} - \frac{\lambda_1 - \lambda_2}{R_m R - M^2} b_3^- &= \lambda_2 - \lambda_1, \\ e^{\lambda_1 c} a_1 - e^{\lambda_2 c} a_2 &= (\lambda_2 - \lambda_1) a_3^+, \\ e^{\lambda_1 c} b_1 - e^{\lambda_2 c} b_2 &= (\lambda_2 - \lambda_1) (b_3^+ + 1), \\ \left((1 - 2\lambda_2 \varkappa) c_1 e^{\lambda_1 c} - (1 - 2\lambda_1 \varkappa) c_2 e^{\lambda_2 c} - \frac{\lambda_1 - \lambda_2}{R_m R - M^2} c_3^- \right) M_\alpha(t) \\ + \left((1 - 2\lambda_2 \varkappa) \lambda_1 a_1 e^{\lambda_1 c} - (1 - 2\lambda_1 \varkappa) \lambda_2 a_2 e^{\lambda_2 c} \right) (t M_{\alpha-1}(t) - P_\alpha(t)) &= 0, \end{aligned} \quad (122)$$

$$\left(e^{\lambda_1 c} c_1 - e^{\lambda_2 c} c_2 - (\lambda_2 - \lambda_1) c_3^+ \right) M_\alpha(t) + \left(e^{\lambda_1 c} \lambda_1 a_1 - e^{\lambda_2 c} \lambda_2 a_2 \right) (t M_{\alpha-1}(t) - P_\alpha(t)) = 0, \quad (123)$$

where $t = \cos \theta_0$. In fact, (122) and (123) are complex-valued equations and equivalent to a system of four equations which can have nonzero a_1 and a_2 only if system's determinant is zero, i.e.,

$$\operatorname{Re}[M_\alpha(t)] \operatorname{Im}[t M_{\alpha-1}(t) - P_\alpha(t)] - \operatorname{Im}[M_\alpha(t)] \operatorname{Re}[t M_{\alpha-1}(t) - P_\alpha(t)] = 0. \quad (124)$$

With the relationships $P_{\alpha-1}(t) = t P_\alpha(t) - \alpha^{-1} \sqrt{1-t^2} P_\alpha^{(1)}(t)$ and $P_{\alpha-1}^{(1)}(t) = t P_\alpha^{(1)}(t) + \alpha \sqrt{1-t^2} P_\alpha(t)$, the condition (124) reduces to (120), which has a zero α in $(0, 1)$ only for $\theta_0 > 2\pi/3$. The zero is single on $(0, 1)$, and $\alpha \rightarrow 1-$ as $\theta_0 \rightarrow 2\pi/3+$.

Now when $\alpha \rightarrow 1$, $\rho^{\alpha-1} M_{\alpha-1}(\cos \theta)$ becomes equivalent to ρ^0 in (121). In this case, G_1 , G_2 , and $G_3^\pm$ are represented by

$$\begin{aligned} G_k &= (A_k(\theta) + B_k(\theta) \rho) \ln \rho + C_k(\theta) + D_k(\theta) \rho + o(\rho), \quad k = 1, 2, \\ G_3^\pm &= (A_3^\pm(\theta) + B_3^\pm(\theta) \rho) \ln \rho + C_3^\pm(\theta) + D_3^\pm(\theta) \rho + o(\rho), \end{aligned} \quad (125)$$

with complex-valued functions $A_1, B_1, C_1, A_2, B_2, C_2, A_3^\pm, B_3^\pm, C_3^\pm$ given by

$$\begin{aligned} A_k(\theta) &= a_k, \quad B_k(\theta) = b_k M_1(\cos \theta) - a_k \lambda_k \cos \theta, \quad C_k(\theta) = c_k + a_k \psi_1^-(\theta), \quad k = 1, 2, \\ D_k(\theta) &= d_k M_1(\cos \theta) - b_k \psi_2^-(\theta) + \lambda_k (a_k (1 - 2 \cos \theta \ln(\cos \frac{\theta}{2})) - c_k \cos \theta), \quad k = 1, 2, \\ A_3^\pm(\theta) &= a_3^\pm, \quad B_3^\pm(\theta) = b_3^\pm M_1(\cos \theta), \quad C_3^\pm(\theta) = c_3^\pm + a_3^\pm \psi_1^\pm(\theta), \quad D_3^\pm(\theta) = d_3^\pm M_1(\cos \theta) - b_3^\pm \psi_2^\pm(\theta) \end{aligned}$$

where

$$\begin{aligned} \psi_1^-(\theta) &= 2 \ln(\cos(\theta/2)) - i \tan(\theta/2), \quad \psi_2^-(\theta) = -\psi_1^-(\theta) M_1(\cos \theta) + 1 - i \cos \theta \tan(\theta/2) \\ \psi_1^+(\theta) &= 2 \ln(\sin(\theta/2)) + i \cot(\theta/2), \quad \psi_2^+(\theta) = -\psi_1^+(\theta) M_1(\cos \theta) - 1 + i \cos \theta \cot(\theta/2), \end{aligned}$$

and $a_1, b_1, c_1, a_2, b_2, c_2, a_3^\pm, b_3^\pm, c_3^\pm$ are real constants. Observe that $A_1, B_1, C_1, A_2, B_2, C_2, A_3^-, B_3^-, C_3^-$ are finite for $\theta \in [0, \pi]$, whereas A_3^+, B_3^+, C_3^+ are finite for $\theta \in (0, \pi]$.

For brevity, let $\mathbf{a}_k = (a_k, b_k, c_k)$, $k = 1, 2$, and $\mathbf{a}_3^\pm = (a_3^\pm, b_3^\pm, c_3^\pm)$. Substituting (125) into (38) with $e^{\lambda_k z} = e^{\lambda_k \rho \cos \theta + O(\rho^2)}$ and equating corresponding coefficients at $\ln \rho$, $\rho \ln \rho$, ρ^0 , and ρ , we obtain after some transformations: $a_3^+ = 0, b_3^+ = 0$, and

$$\begin{aligned} (1 - 2\lambda_2 \varkappa) e^{\lambda_1 c} \mathbf{a}_1 - (1 - 2\lambda_1 \varkappa) e^{\lambda_2 c} \mathbf{a}_2 - \frac{\lambda_1 - \lambda_2}{R_m R - M^2} \mathbf{a}_3^- &= (0, 0, \lambda_1 - \lambda_2), \\ e^{\lambda_1 c} \mathbf{a}_1 - e^{\lambda_2 c} \mathbf{a}_2 - (\lambda_2 - \lambda_1) \mathbf{a}_3^+ &= (0, 0, \lambda_2 - \lambda_1), \end{aligned}$$

$$\begin{aligned} & \left((1 - 2\lambda_2 \varkappa) d_1 e^{\lambda_1 c} - (1 - 2\lambda_1 \varkappa) d_2 e^{\lambda_2 c} - \frac{\lambda_1 - \lambda_2}{R_m R - M^2} d_3^- \right) M_1(\cos \theta_0) \\ & + \left((1 - 2\lambda_2 \varkappa) \lambda_1 a_1 e^{\lambda_1 c} - (1 - 2\lambda_1 \varkappa) \lambda_2 a_2 e^{\lambda_2 c} \right) (1 - i \cos \theta_0 \tan(\theta_0/2)) = 0, \end{aligned} \quad (126)$$

$$\left(d_1 e^{\lambda_1 c} - d_2 e^{\lambda_2 c} - (\lambda_2 - \lambda_1) d_3^+ \right) M_1(\cos \theta_0) + \left(\lambda_1 a_1 e^{\lambda_1 c} - \lambda_2 a_2 e^{\lambda_2 c} \right) (1 - i \cos \theta_0 \tan(\theta_0/2)) = 0. \quad (127)$$

As in the previous case, (126)–(127) is a system of four scalar equations which can have nonzero a_1 and a_2 only if system's determinant is zero:

$$\operatorname{Re}[M_1(\cos \theta_0)] \operatorname{Im}[1 - i \cos \theta_0 \tan(\theta_0/2)] - \operatorname{Im}[M_1(\cos \theta_0)] \operatorname{Re}[1 - i \cos \theta_0 \tan(\theta_0/2)] = 0, \quad (128)$$

which reduces to $2 \cos^2 \theta_0 - \cos \theta_0 - 1 = 0$. This condition implies $\theta_0 = 0$ and $\theta_0 = 2\pi/3$, which proves (119) for $\theta_0 = 2\pi/3$. For $R_m = 0$, the proof is completely analogous. $\square$

7.2 Case (c): $M \neq 0$ and $R_m R = M^2$

Proposition 15 (asymptotic behavior of G_1 and G_2) *Let $M \neq 0$ and $R_m R = M^2$, and let G_1 and G_2 be those defined in Theorem 3. The asymptotic behavior of G_1 and G_2 on ℓ in the vicinity of the endpoint is given by (119)–(120).*

Proof. The proof is similar to that of Proposition 14. Though there are some differences. In the vicinity of the conic endpoint, let G_1 and $G_3^\pm$ be represented by (121) with $\lambda_1 = (R + R_m)/2$. However, G_2 here is an r -analytic function and is represented by

$$G_2 = a_2 \rho^{\alpha-1} M_{\alpha-1}(\cos \theta) + b_2 + c_2 \rho^\alpha M_\alpha(\cos \theta) + o(\rho^\alpha), \quad 0 < \alpha < 1.$$

Substituting these representations into (67) and equating corresponding coefficients at ρ^α , ρ^0 , and ρ^α , we have

$$\begin{aligned} 2a_1 e^{\lambda_1 c} - a_2 + (R + R_m) a_3^+ &= 0, & R_m (c a_2 + a_3^-) - R a_3^+ &= 0, \\ 2b_1 e^{\lambda_1 c} - b_2 + (R + R_m) (b_3^+ + 1) &= 0, & R_m (c b_2 + b_3^- + 1) - R b_3^+ &= 0, \\ \left(2c_1 e^{\lambda_1 c} - c_2 + (R + R_m) c_3^+ \right) M_\alpha(t) + 2\lambda_1 a_1 e^{\lambda_1 c} (t M_{\alpha-1}(t) - P_\alpha(t)) &= 0, \\ (R_m (c c_2 + c_3^-) - R c_3^+) M_\alpha(t) + \frac{1}{2} R_m a_2 M_1(t) M_{\alpha-1}(t) &= 0, \end{aligned} \quad (129)$$

where $t = \cos \theta_0$. The complex-valued equations (129) have nonzero a_1 and a_2 if the determinant of each equation is zero. The determinant of the first equation in (129) is given by (124), whereas the determinant of the second takes the form

$$\operatorname{Re}[M_\alpha(t)] \operatorname{Im}[M_1(t) M_{\alpha-1}(t)] - \operatorname{Im}[M_\alpha(t)] \operatorname{Re}[M_1(t) M_{\alpha-1}(t)] = 0.$$

Remarkably, as (124), this condition also reduces to (120), which proves (119) for $0 < \alpha < 1$.

When $\alpha \rightarrow 1$, the functions G_1 , G_2 , and $G_3^\pm$ are represented by (125) with $\lambda_1 = (R + R_m)/2$ and with complex-valued functions A_1 , B_1 , C_1 , $A_3^\pm$, $B_3^\pm$, $C_3^\pm$ as defined in the proof of Proposition 14, and

$$A_2(\theta) = a_2, \quad B_2(\theta) = b_2 M_1(\cos \theta), \quad C_2(\theta) = c_2 + a_2 \Psi_1^-(\theta), \quad D_2(\theta) = d_2 M_1(\cos \theta) - b_2 \Psi_2^-(\theta).$$

Substituting these representations into (67) and equating corresponding coefficients at $\ln \rho$, $\rho \ln \rho$, ρ^0 , and ρ , we obtain after some transformations: $a_3^+ = 0$, $b_3^+ = 0$, $2e^{\lambda_1 c} a_1 - a_2 = 0$, $2e^{\lambda_1 c} b_1 - b_2 = 0$, $2e^{\lambda_1 c} c_1 - c_2 + (R + R_m)(c_3^+ + 1) = 0$, $c a_2 + a_3^- = 0$, $a_2/2 + c b_2 + b_3^- = 0$, $R_m (c c_2 + c_3^- + 1) - R c_3^+ = 0$, and

$$\begin{aligned} & \left(2e^{\lambda_1 c} d_1 - d_2 \right) M_1(\cos \theta_0) + 2\lambda_1 a_1 e^{\lambda_1 c} (1 - i \cos \theta_0 \tan(\theta_0/2)) = 0, \\ & (R_m (c_2/2 + c d_2 + d_3^-) - R d_3^+) M_1(\cos \theta_0) + \frac{1}{2} a_2 (1 - i \cos \theta_0 \tan(\theta_0/2)) = 0. \end{aligned} \quad (130)$$

The complex-valued equations (130) have nonzero a_1 and a_2 if (128) holds, and the rest of the proof is similar to that of Proposition 14. $\square$

7.3 Adjoint Equations

The asymptotic behavior of solutions to the adjoint equations (94)–(96) near the conic endpoint on the boundary is analyzed similarly.

Proposition 16 (asymptotic behavior of G_1 and G_2 for the adjoint equations) *Let $M \neq 0$, and let G_1 and G_2 be defined as in either Theorem 8 or Theorem 11. In both cases, the asymptotic behavior of G_1 and G_2 on ℓ in the vicinity of the endpoint is given by (119)–(120).*

Proof. The proof is completely analogous to the proofs of Propositions 14 and 15. $\square$

8 Analysis of Minimum-drag Shape

The results obtained in Sections 5–7 have the following implications.

With the representations (104), (109), and (114), we can prove that in Proposition 10, $I_R \rightarrow 0$ as $R \rightarrow \infty$. The proof is similar to showing that $I_R \rightarrow 0$ as $R \rightarrow \infty$ in the proof of Proposition 2, and consequently, the assumption of Proposition 10 holds true. With the representations for solutions of the adjoint equations obtained in Section 6, the optimality condition (103) is specialized for each case (a), (b), and (c).

Corollary 5 (optimality condition)

- (a) *In case (a) (i.e. $M \neq 0$, $R_m R \neq M^2$, $R_m \neq 0$), let G_1 and G_2 be those defined in Theorem 2, and let G_1^* and G_2^* be those introduced in Theorem 8. The representations (18) and (104) with the boundary conditions (38) and (105) and with the identity $(R - 2\lambda_1)(R - 2\lambda_2) + M^2 = 0$ imply that on ℓ , the optimality condition (103) simplifies to*

$$\omega \omega^* + R_m h_r^- g^- = \frac{2}{\lambda_1 - \lambda_2} ((R - 2\lambda_2) \operatorname{Im}[G_1] \operatorname{Im}[G_1^*] - (R - 2\lambda_1) \operatorname{Im}[G_2] \operatorname{Im}[G_2^*]) = \text{const.} \quad (131)$$

- (b) *In case (b) (i.e. $M \neq 0$, $R_m = 0$), let G_1 and G_2 be those defined in Corollary 1, and let G_1^* and G_2^* be those introduced in Corollary 4. The representations (29) and (109) with the boundary conditions (60) and (110) imply that on ℓ , the optimality condition (103) reduces to*

$$\omega \omega^* + R_m h_r^- g^- = 4 \operatorname{Im}[G_1] \operatorname{Im}[G_1^*] = \text{const.} \quad (132)$$

Observe that in this case, $\operatorname{Im}[G_1] = \operatorname{Im}[G_2]$ and $\operatorname{Im}[G_1^] = \operatorname{Im}[G_2^*]$ on ℓ , and under these conditions, (132) follows from (131).*

- (c) *In case (c) (i.e. $M \neq 0$, $R_m R = M^2$), let G_1 and G_2 be those defined in Theorem 3, and let G_1^* and G_2^* be those introduced in Theorem 11. The representations (34) and (114) with the boundary conditions (67) and (115) imply that on ℓ , the optimality condition (103) simplifies to*

$$\omega \omega^* + R_m h_r^- g^- = \frac{1}{R + R_m} (4R \operatorname{Im}[G_1] \operatorname{Im}[G_1^*] + R_m \operatorname{Im}[G_2] \operatorname{Im}[G_2^*]) = \text{const.} \quad (133)$$

Corollary 6 *If $R_m = 0$ and $R = 0$, the adjoint equations (94) for $\mathbf{w}$ and q coincide with (13) for $\mathbf{u}$ and $\wp$. In this case, the boundary conditions (10) and (96) imply $\mathbf{w} = -\mathbf{u}$ and $\omega^* = -\omega$, and thus, the optimality condition (103) simplifies to $\omega = \text{const}$ on S , or $e^{Mz/2} \operatorname{Im}[G_1] = \text{const}$ on ℓ , where G_1 is defined in Corollary 2.*

Corollary 7 *Propositions 11–13 imply that if ℓ is fore-and-aft symmetric, i.e. symmetric with respect to the r axis, and ℓ admits a parametrization $\zeta = \zeta(t)$, $t \in [-1, 1]$, such that $\overline{\zeta(t)} = \zeta(-t)$, then if $F_k(t)$, $t \in [-1, 1]$, $k = 1, 2$, are a solution to (51), or (64), or (70), then $\overline{F_k(-t)}$, $t \in [-1, 1]$, $k = 1, 2$, are a solution to (106) or (111) or (116), respectively.*

Now if an optimal shape is to be found iteratively and the iteration process starts from a fore-and-aft symmetric shape, e.g. sphere or spheroid, then at each iteration, the optimality conditions (131)–(133) are symmetric with respect to the r axis, which means that the optimal shape will be fore-and-aft symmetric. In this case, (131)–(133) take the form $v(t) = \text{const}$ for $t \in [-1, 1]$, where

$$v(t) = \begin{cases} -\frac{2}{\lambda_1 - \lambda_2} ((R - 2\lambda_2)V_1(t)V_1(-t) - (R - 2\lambda_1)V_2(t)V_2(-t)) & \text{in case (a),} \\ -4V_1(t)V_1(-t) & \text{in case (b),} \\ -\frac{1}{R + R_m} (4RV_1(t)V_1(-t) + R_m V_2(t)V_2(-t)) & \text{in case (c),} \end{cases} \quad (134)$$

and $V_k(t) = \text{Im}[F_k(t)]$, $k = 1, 2$.

Propositions 14 and 15 do not specify whether a_1 and a_2 for G_1 and G_2 in (119) are related. Since the kernels in the integral equations (51) and (70) have logarithmic singularities, the integrals in (51) and (70) with G_1 and G_2 behaving as (119) in the vicinity of a conic endpoint are integrable and finite. This implies that $F_1(\zeta) - F_2(\zeta)$ in (51) and $2F_1(\zeta) - G_2(\zeta)$ in (70) are finite, and consequently, in (119), $a_1 = a_2$ for cases (a) and (b), and $2a_1 = a_2$ for case (c). The same is true for G_1^* and G_2^* .

Corollary 8 (conic endpoints) *In the setting of Corollary 7, suppose $t = \pm 1$ correspond to conic endpoints. Then in the vicinity of $t = \pm 1$, $v(t)$ behaves as $-4V_1(t)V_1(-t)$ in all three cases (a)–(c) with V_1 being defined respectively. Then (119) implies that $-4V_1(t)V_1(-t) = \text{const} \neq 0$ in the vicinity of ± 1 only if $\theta_0 = 2\pi/3$.*

Now we proceed with finding the minimum-drag shape in semi-analytical form. Let ζ be parametrized by

$$\zeta(t) = r(t) + iz(t) = \gamma \left(\cos(\pi t/2) \sum_{j=0}^{n_1} a_j T_{2j}(t) + i \sin(\pi t/2) \sum_{j=0}^{n_2} b_j T_{2j}(t) \right), \quad t \in [-1, 1], \quad (135)$$

where $T_j(t)$ is the Chebyshev polynomial of the first kind; $a_0, \dots, a_{n_1}$, $b_0, \dots, b_{n_2}$ are real coefficients; and $\gamma = \gamma(a_0, \dots, a_{n_1}, b_0, \dots, b_{n_2})$ is the multiplier introduced to satisfy the volume constraint $\pi \int_{-1}^1 r^2 z' dt = 4\pi/3$ identically, i.e. if $\zeta(t) = \gamma(\hat{r}(t) + i\hat{z}(t))$, then

$$\gamma = \left(\frac{3}{4} \int_{-1}^1 \hat{r}^2 \hat{z}' dt \right)^{-1/3}. \quad (136)$$

The choice of the parametrization (135)–(136) is explained in [33].

Let $\tilde{v}(t) = v(t) - \frac{1}{2} \int_{-1}^1 v(t) dt$, where the second term in the right-hand side is the average of $v(t)$ on $t \in [-1, 1]$, and let $\zeta_k = r_k + iz_k$ and $\tilde{v}_k$ be the shape ζ and function $\tilde{v}$, respectively, at step k . Then ζ_k can be updated as in Pironneau's iterative procedure [20]:

$$\zeta_{k+1} = \zeta_k + \varepsilon_k \tilde{v}_k \frac{\partial \zeta_k}{\partial n}. \quad (137)$$

This procedure reduces to finding optimal $a_0, \dots, a_{n_1}$, $b_0, \dots, b_{n_2}$ in (135) as described by Algorithm 1 in [33], in which u_n^* needs to be replaced by $\tilde{v}$, and initial a_0 and b_0 should correspond to the axes of the minimum-drag spheroid found in Section 4.2 (for given R , R_m , and S) while initial $a_1, \dots, a_{n_1}$, $b_1, \dots, b_{n_2}$ should be set zero.

Figure 4a shows three minimum-drag shapes, having the volume of the unit sphere, for $R = R_m = 3$ and $S = 0, 1, 2$. In the case of $S = 0$, which corresponds to the Oseen equations, the minimum-drag shape was obtained by the approach presented in [32]. For the three shapes, Tables 1 and 2 present corresponding drag coefficient C_D , the constant in the optimality condition (103), error $\|\cdot\|$ in satisfying (103), and parameters in the representation (135)–(136). Among those three shapes, the one for $S = 1$ is shortest. Let C_D^* be the drag coefficient for the unit sphere. Figure 4b interpolates the ratio C_D/C_D^* for $R = R_m = 3$ based on three obtained values for $S = 0, 1, 2$. Figures 3a and 4b show that the drag ratios C_D/C_D^* for the minimum-drag spheroids and minimum-drag shapes for same R, R_m , and S are sufficiently close.

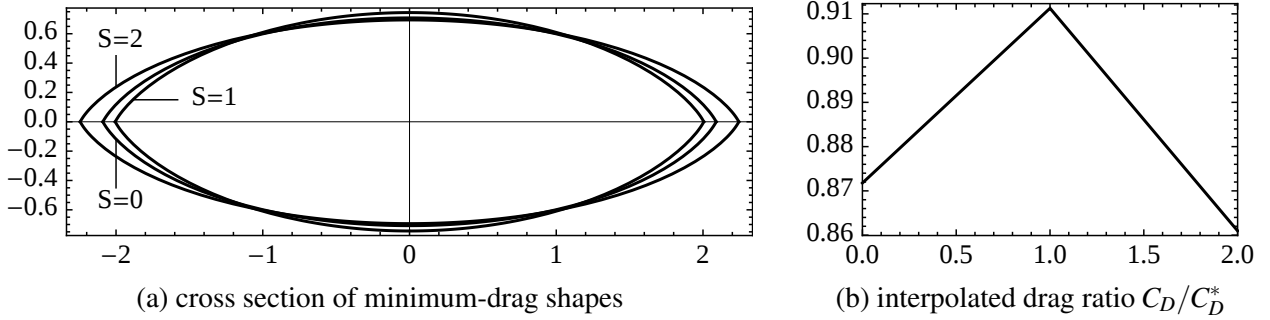


Figure 4: Minimum-drag shapes and the interpolated drag ratio C_D/C_D^* for $R = R_m = 3$ and $S = 0, 1, 2$.

Table 1: Drag coefficient C_D , the constant in the optimality condition (103), and error $\|\cdot\|$ in satisfying (103) for the minimum-drag shapes for $R = R_m = 3$ and $S = 0, 1$, and 2.

	$S = 0$	$S = 1$	$S = 2$
C_D	1.64828	1.36736	1.81103
constant	-3.63472×10^{-1}	-2.35380	-3.66957
error	1.58348×10^{-3}	3.94289×10^{-3}	4.91428×10^{-3}

9 Conclusions

This work has developed an approach of generalized analytic functions to the MHD problem of an electrically conducting viscous incompressible flow past a solid body of revolution under the assumptions that the applied magnetic field and body's axis of revolution are aligned with the flow at infinity and that the body and fluid have same magnetic permeability. In the three complementary cases: (a) $R_m \neq 0, R_m R \neq M^2$ ($S \neq 1$); (b) $R_m = 0$; and (c) $R_m R = M^2$ ($S = 1$) all assuming $M \neq 0$, the velocity field, pressure and magnetic fields in and out the immersed body are represented by four generalized analytic functions. In particular, in case (a), the fields are represented by two H -analytic functions and two r -analytic functions; in case (b), the magnetic field is constant everywhere, and the velocity and pressure are represented by two H -analytic functions; and in case (c), the fields are represented by one H -analytic function and three r -analytic functions. r -Analytic functions, whose real and imaginary parts are harmonic, are related to the Stokes equations, whereas H -analytic functions have the real and imaginary parts satisfying the modified Helmholtz equation and are related to the Oseen equations. Namely the difference in the number of r -analytic and H -analytic functions in the solutions in cases (a) and (c) contributes to the peculiarity of the case of $S = 1$.

In all three cases (a), (b), and (c), the MHD problem has been reduced to integral equations for unknown boundary values of the involved generalized analytic functions based on the generalized Cauchy integral for-

Table 2: Parameters in the representation (135)–(136) for the minimum-drag shapes for $R = R_m = 3$ and $S = 0, 1$, and 2 .

	$S = 0$	$S = 1$	$S = 2$
γ	1.01196396246	1.01366399904	1.01131424639
a_0	$6.29221950407 \times 10^{-1}$	$6.50248611262 \times 10^{-1}$	$5.93267526768 \times 10^{-1}$
a_1	$-9.43328552499 \times 10^{-2}$	$-9.23964575126 \times 10^{-2}$	$-1.17578599834 \times 10^{-1}$
a_2	$-1.37355103545 \times 10^{-2}$	$-6.7092264183 \times 10^{-3}$	$-2.61378349707 \times 10^{-2}$
a_3	$1.98202204631 \times 10^{-3}$	$2.34809221062 \times 10^{-3}$	$-1.57412245298 \times 10^{-3}$
a_4	—	$4.22269308778 \times 10^{-4}$	$-3.89459211608 \times 10^{-5}$
a_5	—	$-1.08999300292 \times 10^{-4}$	$-2.62129253651 \times 10^{-5}$
b_0	2.00004216653	1.87568864891	2.13349213677
b_1	$8.69011301008 \times 10^{-2}$	$1.00707086887 \times 10^{-1}$	$8.57434199435 \times 10^{-2}$
b_2	$1.39653034146 \times 10^{-4}$	$2.70040905434 \times 10^{-4}$	$-1.52613634178 \times 10^{-3}$
b_3	$2.83047975928 \times 10^{-3}$	$1.85753309456 \times 10^{-3}$	$1.31075784337 \times 10^{-3}$
b_4	—	$-1.0430097082 \times 10^{-4}$	$-1.10494890615 \times 10^{-5}$
b_5	—	$-7.96297414857 \times 10^{-5}$	$-8.91048112023 \times 10^{-6}$

mula. Solutions to the integral equations have been represented by a finite functional series with series coefficients determined by quadratic error minimization and have been shown to coincide with the series-form solutions for sphere with high accuracy.

The necessary optimality condition (103) for minimum-drag shapes subject to the volume constraint has been obtained by Mironov's shape variation approach. In the absence of the magnetic field, it reduces to those for the Stokes and Oseen equations, and coincides with the one for the Navier-Stokes equations. It has been shown analytically that regardless of magnitudes of R , R_m , and S , the minimum-drag shapes have conic endpoints with the angle of $2\pi/3$ counted from the axis of revolution. Remarkably, the minimum-drag shapes for the Stokes, Oseen, and Navier-Stokes equations have the same property. The numerical analysis has shown that the minimum-drag shapes are fore-and-aft symmetric and that for $R = R_m = 3$, the minimum-drag shape for $S = 1$ has the largest drag ratio and smallest aspect ratio. Also, for same R , R_m , and S , the drag coefficients for the minimum-drag spheroids and minimum-drag shapes are sufficiently close.

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