

Detection of a Ferromagnetic Microwire

by Frank J. Crowne

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14. ABSTRACT This report, which supplements the material presented in the <i>IEEE Transactions in Magnetics</i> publication entitled "Two-Frequency Excitation of a Ferromagnetic Microwire," describes a simple configuration of antenna beams, polarizations, and wire placement calculations, which are used in calculating the response of a magnetic wire to ferromagnetic resonance (FMR) excitation by an external magnetic field.					
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1. Introduction

As described in the publication “Two-Frequency Excitation of a Ferromagnetic Microwire” (1), the remote sensing of ferroelectric microwires based on ferromagnetic resonance (FMR) has been proposed as a way to passively collect information from a novel type of “radio frequency identification (RFID) card” that can be attached to a target under surveillance (2).

Implementation of this proposal requires a careful study of the external scattering problem of electromagnetic (EM) waves from the microwire, which is the topic of this technical report. A detailed description of the physics of ferromagnetic microwires can be found in reference 1 and ARL-TR-6114 (3).

2. The Wire Problem

Assume that a ferroelectric microwire located at the coordinate origin and oriented vertical to the ground, i.e., along the “z-axis”, is illuminated by two external magnetic fields, which we refer to as “signal” and “pump” fields, both of which are the magnetic components of linearly polarized EM beams. The “illumination” geometries correspond to radar applications: if both beams originate from the same antenna, we speak of a “monostatic” geometry, while for signals originating from different antennas the geometry is referred to as “bistatic.” In this report, we discuss the monostatic scenario only, i.e., we assume that both beams originate from the same antenna and propagate horizontally and parallel to one another along, e.g., the x-axis. The geometry is shown in figure 1.

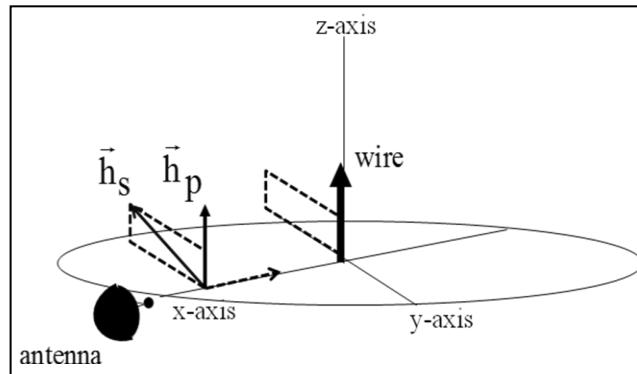


Figure 1 Monostatic geometry for wire illumination—propagation vector.

The two fields are characterized by both spatial and temporal phase information, i.e., if $(\mathcal{H}_{px}, \mathcal{H}_{py}, \mathcal{H}_{pz})$ and $(\mathcal{H}_{sx}, \mathcal{H}_{sy}, \mathcal{H}_{sz})$ are real magnitudes of the (strong) pump field and the (weak) signal field, the amplitudes of the rectangular components of the fields are given by

$$h_{px}(t) = \text{Re} \left(\mathcal{H}_{px} e^{-i(\omega t + \theta_{px})} \right), \quad h_{py}(t) = \text{Re} \left(\mathcal{H}_{py} e^{-i(\omega t + \theta_{py})} \right), \text{ and}$$

$$h_{pz}(t) = \text{Re} \left(\mathcal{H}_{pz} e^{-i(\omega t + \theta_{pz})} \right), \text{ where spatial phase information associated with the wave}$$

polarization is included by letting $\theta_{px} \neq \theta_{py} \neq \theta_{pz}$. For our case, we assume that the pump field

at the antenna is linearly polarized along the z-axis, so that $h_{px} = 0$, $h_{py} = 0$, and $h_{pz} = \mathcal{H}_{pc}$,

where $\mathcal{H}_{pc}$ is the total pump-field magnitude. In the same way, the (weak) signal field at the antenna is linearly polarized in the yz plane, with its electric field vector making an angle φ with the z-axis, having components $h_{sx} = 0$, $h_{sy} = \mathcal{H}_{sc} \cos \varphi$, and $h_{sz} = \mathcal{H}_{sc} \sin \varphi$, where $\mathcal{H}_{sc}$ is the total signal-field magnitude. Then, the wave components at the target are

$$h_p(t) = \text{Re} \left[\begin{pmatrix} 0 \\ 0 \\ \mathcal{H}_{pc} \end{pmatrix} e^{-i(\beta x + \omega t)} \right] \quad h_s(t) = \text{Re} \left[\begin{pmatrix} 0 \\ \mathcal{H}_{sc} \cos \varphi \\ \mathcal{H}_{sc} \sin \varphi \end{pmatrix} e^{-i(\beta x + \omega t)} \right] \quad (1)$$

Following reference 1, we introduce the following complex fields (italics)

$$\begin{aligned} h_{px} &= 0 & h_{sx} &= 0 \\ h_{py} &= 0 & h_{sy} &= \mathcal{H}_{sc} \cos \varphi \\ h_{pz} &= \mathcal{H}_{pc} & h_{sz} &= \mathcal{H}_{sc} \sin \varphi \end{aligned} \quad (2)$$

and the circular field components, defined as follows:

$$\begin{aligned}
h_{px} + ih_{py} &= \mathcal{H}_{px} e^{-i\theta_{px}} + i\mathcal{H}_{py} e^{-i\theta_{py}}, \quad h_{pz} = \mathcal{H}_{pz} e^{-i\theta_{pz}} \\
\mathcal{H}_{px} &= 0, \quad \mathcal{H}_{py} = 0, \quad \mathcal{H}_{pz} = \mathcal{H}_{pc}, \quad \theta_{pz} = 0 \\
h_{sx} + ih_{sy} &= \mathcal{H}_{sx} e^{-i\theta_{sx}} + i\mathcal{H}_{sy} e^{-i\theta_{sy}}, \quad h_{sz} = \mathcal{H}_{sz} e^{-i\theta_{sz}} \\
\mathcal{H}_{sx} &= 0, \quad \mathcal{H}_{sy} = \mathcal{H}_{sc} \cos\varphi, \quad \mathcal{H}_{sz} = \mathcal{H}_{sc} \sin\varphi, \quad \theta_y = 0, \quad \theta_z = 0 \\
\Rightarrow \begin{cases} h_{px} + ih_{py} = 0 \\ h_{sx} + ih_{sy} = i\mathcal{H}_{sc} \cos\varphi \\ (h_{px} - ih_{py})^* = 0 \\ (h_{sx} - ih_{sy})^* = i\mathcal{H}_{sc} \cos\varphi \end{cases} & \quad (3)
\end{aligned}$$

In order to estimate the power radiated by FMR excitation of the microwire, it is necessary first to describe the dynamics of its magnetization in the presence of an external time-dependent magnetic field. These dynamics are governed by the Landau-Lifshits-Gilbert (LLG) equation (4):

$$\frac{\partial \vec{M}}{\partial t} = \gamma \vec{M} \times \vec{H}_{eff}(t) - \frac{\alpha}{|\vec{M}_{st}|} \vec{M} \times \frac{\partial \vec{M}}{\partial t} \quad (4)$$

The material parameters of the wire include $\vec{M}_{st}$, the static magnetization of the material; γ , the gyromagnetic ratio; and α , the damping constant. $\vec{H}_{eff}(t)$ is an effective magnetic field, which includes the DC anisotropy field $\vec{H}_a$ that fixes the direction of $\vec{M}_{st}$ in the material, the depolarization field (5) due to the body's shape, and the external time-dependent applied field. Note that in the absence of an ac magnetic field, there is no depolarization field, and so $\vec{M}_{st} \times \vec{H}_a = 0$ under DC conditions, i.e., there is no DC torque. The first term is the torque exerted by the effective magnetic field $\vec{H}_{eff}(t)$, while the second term gives rise to damping via eddy currents (6). The form of this equation implies that $\vec{M} \cdot \frac{\partial \vec{M}}{\partial t} = 0 \Rightarrow |\vec{M}|^2$ is constant in time.

The RF magnetic field $\vec{h}(t)$ generates an RF magnetization $\vec{m}$ that under remote-sensing conditions is small compared to $\vec{M}_{st}$. The dynamics of this magnetization, in particular its decay with time, is strongly affected by the conservation of length of the total magnetization vector, whose endpoint is constrained to lie on a spherical surface at all times.

Let us further specify the perturbed magnetization and effective magnetic field in the LLG equation as follows: $\vec{M} = \vec{M}_{st} + \vec{m}$, $\vec{H}_{eff} = \vec{H}_a + \vec{h} - N\vec{m}$, where $\vec{M}_{st} \times \vec{H}_a = 0$ and

$$\mathbf{N} = \begin{pmatrix} \tilde{N} & 0 & 0 \\ 0 & \tilde{N} & 0 \\ 0 & 0 & N_z \end{pmatrix} \quad (5)$$

is the depolarization tensor; for a long wire, $\tilde{N} = 1/2$ and $N_z \approx 0$. As described in reference 1, we can introduce a scaled dimensionless magnetization $\vec{g}$ such that $\vec{m} = |\vec{M}_{st}| \vec{g}$. In general, this magnetization has three components g_x, g_y, g_z , where g_z is the component of $\vec{g}$ parallel to the wire and (g_x, g_y) is a two-dimensional vector perpendicular to the wire, with the latter small compared to the former. As shown in reference 1, under weak-signal conditions the perpendicular components of the dimensionless magnetization can be expressed in terms of a single complex number G at each illuminating frequency, expressions for which are

$$\begin{cases} G_{1s} = \frac{\gamma(h_{sx} + ih_{sy})}{\omega_B - \omega_s - i\alpha\omega_s} e^{-i\omega_s t} + \frac{\gamma(h_{sx} - ih_{sy})^*}{\omega_B + \omega_s + i\alpha\omega_s} e^{i\omega_s t} \\ G_{1p} = \frac{\gamma(h_{px} + ih_{py})}{\omega_B - \omega_p - i\alpha\omega_p} e^{-i\omega_p t} + \frac{\gamma(h_{px} - ih_{py})^*}{\omega_B + \omega_p + i\alpha\omega_p} e^{i\omega_p t} \end{cases} \quad (6)$$

In the geometry we have chosen, these expressions become

$$G_{1s} = \frac{\gamma}{\omega_B} \mathcal{H}_{sc} \cos\varphi \left[\frac{i}{1 - \xi_s - i\alpha\xi_s} e^{-i\omega_s t} + \frac{i}{1 + \xi_s + i\alpha\xi_s} e^{i\omega_s t} \right] \quad (7)$$

$$G_{1p} = 0$$

while

$$g_{z1,s} = 0$$

$$g_{z1,p} = 0$$

3. Nonlinear Response: Second Order

From reference 1, we find that, to second order, the wire response functions satisfy the equations

$$\begin{aligned}
(\alpha - i) \frac{\partial}{\partial t} \mathcal{G}_2 + \omega_B \mathcal{G}_2 &= -\gamma \mathcal{G}_1 \\
(\alpha + i) \frac{\partial}{\partial t} \mathcal{G}_2^* + \omega_B \mathcal{G}_2^* &= -\gamma \mathcal{G}_1^*
\end{aligned} \tag{8}$$

where the “sources” appearing on the right side of these equations are independent of $\mathcal{G}_2$.

Introducing the notation

$$\vec{H}_{s,p} = \begin{pmatrix} h_{s,px} + ih_{s,py} \\ h_{s,px} - ih_{s,py} \end{pmatrix} = \begin{pmatrix} H_{s,p} \\ H_{s,p}^* \end{pmatrix} \tag{9}$$

and

$$\begin{aligned}
\Gamma_{s,p+} &= \frac{\gamma}{2} \frac{1}{\omega_B - \omega_{s,p} - i\omega\alpha} & \Gamma_{s,p-} &= \frac{\gamma}{2} \frac{1}{\omega_B + \omega_{s,p} - i\omega\alpha} \\
\Gamma_{s,p+}^* &= \frac{\gamma}{2} \frac{1}{\omega_B - \omega_{s,p} + i\omega\alpha} & \Gamma_{s,p-}^* &= \frac{\gamma}{2} \frac{1}{\omega_B + \omega_{s,p} + i\omega\alpha}
\end{aligned} \tag{10}$$

we obtain the following expressions for the source terms:

$$\begin{aligned}
-\gamma \mathcal{G}_1 &= -\frac{\gamma}{2} \left[\begin{aligned} &\left(h_{sz} \Gamma_{s+} H_{s+} \right) e^{-2i\omega_s t} + \left(h_{sz}^* \Gamma_{s-}^* H_{s-}^* \right) e^{+2i\omega_s t} \\ &+ \left(h_{sz} \Gamma_{p+} H_{p+} + h_{pz} \Gamma_{s+} H_{s+} \right) e^{-i\omega_{s+p} t} + \left(h_{sz}^* \Gamma_{p-}^* H_{p-}^* + h_{pz}^* \Gamma_{s-}^* H_{s-}^* \right) e^{+i\omega_{s+p} t} \\ &+ h_{sz} \Gamma_{s-}^* H_{s-}^* + h_{pz} \Gamma_{p-}^* H_{p-}^* + h_{sz}^* \Gamma_{s+} H_{s+} + h_{pz}^* \Gamma_{p+} H_{p+} \\ &+ \left(h_{sz} \Gamma_{p-}^* H_{p-}^* + h_{pz}^* \Gamma_{s+} H_{s+} \right) e^{-i\omega_{s-p} t} + \left(h_{pz} \Gamma_{s-}^* H_{s-}^* + h_{sz}^* \Gamma_{p+} H_{p+} \right) e^{+i\omega_{s-p} t} \\ &+ h_{pz} \Gamma_{p+} H_{p+} e^{-2i\omega_p t} + h_{pz}^* \Gamma_{p-}^* H_{p-}^* e^{+2i\omega_p t} \end{aligned} \right] \\
-\gamma \mathcal{G}_1^* &= -\frac{\gamma}{2} \left[\begin{aligned} &\left(h_{sz} \Gamma_{s-} H_{s-} \right) e^{-2i\omega_s t} + \left(h_{sz}^* \Gamma_{s+}^* H_{s+}^* \right) e^{+2i\omega_s t} + \\ &+ \left(h_{sz} \Gamma_{p-} H_{p-} + h_{pz} \Gamma_{s-} H_{s-} \right) e^{-i\omega_{s+p} t} + \left(h_{sz}^* \Gamma_{p+}^* H_{p+}^* + h_{pz}^* \Gamma_{s+}^* H_{s+}^* \right) e^{+i\omega_{s+p} t} \\ &+ h_{sz} \Gamma_{s+}^* H_{s+}^* + h_{pz} \Gamma_{p+}^* H_{p+}^* + h_{sz}^* \Gamma_{s-} H_{s-} + h_{pz}^* \Gamma_{p-} H_{p-} \\ &+ \left(h_{pz}^* \Gamma_{s-} H_{s-} + h_{sz} \Gamma_{p+}^* H_{p+}^* \right) e^{-i\omega_{s-p} t} + \left(h_{sz}^* \Gamma_{p-} H_{p-} + h_{pz} \Gamma_{s+}^* H_{s+}^* \right) e^{+i\omega_{s-p} t} \\ &+ h_{pz} \Gamma_{p-} H_{p-} e^{-2i\omega_p t} + h_{pz}^* \Gamma_{p+}^* H_{p+}^* e^{+2i\omega_p t} \end{aligned} \right] \tag{11}
\end{aligned}$$

Let us write the second-order response in the following spectral form:

$$\mathcal{G}_2 = \mathcal{G}_{2,2s} + \mathcal{G}_{2,2p} + \mathcal{G}_{2,s+p} + \mathcal{G}_{2,s-p} + \mathcal{G}_{2,0}. \quad (12)$$

Here $\mathcal{G}_{2,2s}$ is the response at the 2nd harmonic of the signal field, $\mathcal{G}_{2,2s}$ is the response at the 2nd harmonic of the pump field, $\mathcal{G}_{2,s+p}$ is the response at the sum of the signal and pump frequencies, $\mathcal{G}_{2,s-p}$ is the response at the difference of the signal and pump frequencies, and $\mathcal{G}_{2,0}$ is a DC response arising from rectification of the pump and signal. We introduce the following convention: if A is a sum of multi-frequency terms, we denote the sum of all the terms in A at the same frequency q by $[A]_q$.

The response $\mathcal{G}_{2,2s}$ at the 2nd harmonic frequency $2s$ is computed as follows:

$$\begin{aligned} (\alpha - i) \frac{\partial}{\partial t} \mathcal{G}_{2,2s} + \omega_B \mathcal{G}_{2,2s} &= [-\gamma h_z \mathcal{G}_1]_{2s} \\ &= -\frac{\gamma}{2} \left[\left(h_{sz} \Gamma_{s+} H_{s+} \right) e^{-2i\omega_s t} + \left(h_{sz} \Gamma_{s-} H_{s-} \right)^* e^{+2i\omega_s t} \right] \square -\frac{\gamma}{2} \left[S_{2,2s+} e^{-2i\omega_s t} + S_{2,2s-}^* e^{+2i\omega_s t} \right] \quad (13) \\ (\alpha + i) \frac{\partial}{\partial t} \mathcal{G}_{2,2s}^* + \omega_B \mathcal{G}_{2,2s}^* &= [-\gamma h_z \mathcal{G}_1^*]_{2s} \\ &= -\frac{\gamma}{2} \left[\left(h_{sz} \Gamma_{s-} H_{s-} \right) e^{+2i\omega_s t} + \left(h_{sz} \Gamma_{s+} H_{s+} \right)^* e^{-2i\omega_s t} \right] \square -\frac{\gamma}{2} \left[S_{2,2s-} e^{-2i\omega_s t} + S_{2,2s+}^* e^{+2i\omega_s t} \right] \\ \Rightarrow \mathcal{G}_{2,2s} &= -\frac{\gamma}{2} \left[\frac{S_{2,2s+}}{\omega_B - \omega_{2s} - i\alpha\omega_{2s}} e^{-2i\omega_s t} + \frac{S_{2,2s-}^*}{\omega_B + \omega_{2s} + i\alpha\omega_{2s}} e^{+2i\omega_s t} \right] \\ &= -\frac{\gamma}{2} \left[\frac{h_{sz} \Gamma_{s+} H_{s+}}{\omega_B - \omega_{2s} - i\alpha\omega_{2s}} e^{-2i\omega_s t} + \frac{\left(h_{sz} \Gamma_{s-} H_{s-} \right)^*}{\omega_B + \omega_{2s} + i\alpha\omega_{2s}} e^{+2i\omega_s t} \right] \\ &= -\frac{\gamma}{2} \left[\frac{\Gamma_{s+}}{\omega_B - \omega_{2s} - i\alpha\omega_{2s}} h_{sz} H_{s+} e^{-2i\omega_s t} + \frac{\Gamma_{s-}^*}{\omega_B + \omega_{2s} + i\alpha\omega_{2s}} h_{sz}^* H_{s-}^* e^{+2i\omega_s t} \right] \\ &= -\left(\frac{\gamma}{2} \right)^2 \left[\frac{h_{sz} H_{s+}}{(\omega_B - \omega_{2s} - i\alpha\omega_{2s})(\omega_B - \omega_s - i\alpha\omega_s)} e^{-2i\omega_s t} + \frac{h_{sz}^* H_{s-}^*}{(\omega_B + \omega_{2s} + i\alpha\omega_{2s})(\omega_B + \omega_s + i\alpha\omega_s)} e^{+2i\omega_s t} \right] \\ &= -\left(\frac{\gamma}{2\omega_B} \right)^2 \left[\frac{h_{sz} (h_{sx} + ih_{sy})}{(1 - \xi_{2s} - i\alpha\xi_{2s})(1 - \xi_s - i\alpha\xi_s)} e^{-2i\omega_s t} + \frac{h_{sz}^* (h_{sx} - ih_{sy})^*}{(1 + \xi_{2s} + i\alpha\xi_{2s})(1 + \xi_s + i\alpha\xi_s)} e^{+2i\omega_s t} \right] \quad (14) \end{aligned}$$

where we introduce the dimensionless frequencies $\xi_i = \frac{\omega_i}{\omega_B}$. Likewise, the response $\mathcal{G}_{2,2p}$ at the 2nd harmonic frequency $2p$ is

$$\begin{aligned}
(\alpha - i) \frac{\partial}{\partial t} \mathcal{G}_{2,2p} + \omega_B \mathcal{G}_{2,2p} &= [-\gamma_z \mathcal{G}_1]_{2p} \\
&= -\frac{\gamma}{2} \left[\left(h_{pz} \Gamma_{p+} H_{p+} \right) e^{-2i\omega_p t} + \left(h_{pz} \Gamma_{p-} H_{p-} \right)^* e^{+2i\omega_p t} \right] - \frac{\gamma}{2} \left[S_{2,2p+} e^{-2i\omega_p t} + S_{2,2p-}^* e^{+2i\omega_p t} \right] \quad (15) \\
(\alpha + i) \frac{\partial}{\partial t} \mathcal{G}_{2,2p}^* + \omega_B \mathcal{G}_{2,2p}^* &= [-\gamma_z \mathcal{G}_1^*]_{2p} \\
&= -\frac{\gamma}{2} \left[\left(h_{pz} \Gamma_{p-} H_{p-} \right) e^{-2i\omega_p t} + \left(h_{pz} \Gamma_{p+} H_{p+} \right)^* e^{+2i\omega_p t} \right] - \frac{\gamma}{2} \left[S_{2,2p-} e^{-2i\omega_p t} + S_{2,2p+}^* e^{+2i\omega_p t} \right] \\
\Rightarrow \mathcal{G}_{2,2p} &= -\frac{\gamma}{2} \left[\frac{S_{2,2p+}}{\omega_B - \omega_{2p} - i\alpha\omega_{2p}} e^{-2i\omega_p t} + \frac{S_{2,2p-}^*}{\omega_B + \omega_{2p} + i\alpha\omega_{2p}} e^{+2i\omega_p t} \right] \\
&= -\frac{\gamma}{2} \left[\frac{h_{pz} \Gamma_{p+} H_{p+}}{\omega_B - \omega_{2p} - i\alpha\omega_{2p}} e^{-2i\omega_p t} + \frac{\left(h_{pz} \Gamma_{p-} H_{p-} \right)^*}{\omega_B + \omega_{2p} + i\alpha\omega_{2p}} e^{+2i\omega_p t} \right] \\
&= -\frac{\gamma}{2} \left[\frac{\Gamma_{p+}}{\omega_B - \omega_{2p} - i\alpha\omega_{2p}} h_{pz} H_{p+} e^{-2i\omega_p t} + \frac{\Gamma_{p-}^*}{\omega_B + \omega_{2p} + i\alpha\omega_{2p}} h_{pz}^* H_{p-}^* e^{+2i\omega_p t} \right] \\
&= -\left(\frac{\gamma}{2} \right)^2 \left[\frac{h_{pz} H_{p+}}{\left(\omega_B - \omega_{2p} - i\alpha\omega_{2p} \right) \left(\omega_B - \omega_{p-} - i\alpha\omega_p \right)} e^{-2i\omega_p t} + \frac{h_{pz}^* H_{p-}^*}{\left(\omega_B + \omega_{2p} + i\alpha\omega_{2p} \right) \left(\omega_B + \omega_{p+} + i\alpha\omega_p \right)} e^{+2i\omega_p t} \right] \\
&= -\left(\frac{\gamma}{2\omega_B} \right)^2 \left[\frac{h_{pz} (h_{px} + ih_{py})}{(1 - \xi_{2s} - i\alpha\xi_{2s})(1 - \xi_p - i\alpha\xi_p)} e^{-2i\omega_p t} + \frac{h_{pz}^* (h_{px} - ih_{py})^*}{(1 + \xi_{2p} + i\alpha\xi_{2p})(1 + \xi_p + i\alpha\xi_p)} e^{+2i\omega_p t} \right] \quad (16)
\end{aligned}$$

The DC response $\mathcal{G}_{2,0}$ is

$$\begin{aligned}
(\alpha - i) \frac{\partial}{\partial t} \mathcal{G}_{2,0} + \omega_B \mathcal{G}_{2,0} &= [-\mathcal{H}_z \mathcal{G}_1]_0 \\
[-\mathcal{H}_z \mathcal{G}_1]_0 &= -\frac{\gamma}{2} \left[h_{sz} \Gamma_{s-}^* H_{s-}^* + h_{pz} \Gamma_{p-}^* H_{p-}^* + h_{sz}^* \Gamma_{s+} H_{s+} + h_{pz}^* \Gamma_{p+} H_{p+} \right] \\
&= -\frac{\gamma}{2} \left[h_{sz} \frac{\gamma}{2} \frac{1}{\omega_B + \omega_s + i\alpha\omega_s} H_{s-}^* + h_{pz} \frac{\gamma}{2} \frac{1}{\omega_B + \omega_p + i\alpha\omega_p} H_{p-}^* \right. \\
&\quad \left. + h_{sz}^* \frac{\gamma}{2} \frac{1}{\omega_B - \omega_s - i\alpha\omega_s} H_{s+} + h_{pz}^* \frac{\gamma}{2} \frac{1}{\omega_B - \omega_p - i\alpha\omega_p} H_{p+} \right] \\
\Rightarrow \mathcal{G}_{2,0} &= -\left(\frac{\gamma}{2\omega_B} \right) \left(\frac{\gamma}{2} \right) \left[h_{sz} \frac{1}{\omega_B + \omega_s + i\alpha\omega_s} H_{s-}^* + h_{sz}^* \frac{1}{\omega_B - \omega_s - i\alpha\omega_s} H_{s+} \right. \\
&\quad \left. + h_{pz} \frac{1}{\omega_B + \omega_p + i\alpha\omega_p} H_{p-}^* + h_{pz}^* \frac{1}{\omega_B - \omega_p - i\alpha\omega_p} H_{p+} \right] \\
&= -\left(\frac{\gamma}{2\omega_B} \right)^2 \left[h_{sz} \frac{1}{1 + \xi_s + i\alpha\xi_s} H_{s-}^* + h_{sz}^* \frac{1}{1 - \xi_s - i\alpha\xi_s} H_{s+} \right. \\
&\quad \left. + h_{pz} \frac{1}{1 + \xi_p + i\alpha\xi_p} H_{p-}^* + h_{pz}^* \frac{1}{1 - \xi_p - i\alpha\xi_p} H_{p+} \right] \tag{17}
\end{aligned}$$

The sum-frequency response $\mathcal{G}_{2,s+p}$ is

$$\begin{aligned}
(\alpha - i) \frac{\partial}{\partial t} \mathcal{G}_{2,s+p} + \omega_B \mathcal{G}_{2,s+p} &= [-\gamma h_z \mathcal{G}_1]_{s+p} \\
&= -\frac{\gamma}{2} \left[\left(h_{sz} \Gamma_{p+} H_{p+} + h_{pz} \Gamma_{s+} H_{s+} \right) e^{-i\omega_{s+p} t} \right. \\
&\quad \left. + \left(h_{sz} \Gamma_{p-} H_{p-} + h_{pz} \Gamma_{s-} H_{s-} \right)^* e^{+i\omega_{s+p} t} \right] \square - \frac{\gamma}{2} \left[S_{2,s+p+} e^{-i\omega_{s+p} t} + S_{2,s+p-}^* e^{+i\omega_{s+p} t} \right] \quad (18)
\end{aligned}$$

$$\begin{aligned}
(\alpha + i) \frac{\partial}{\partial t} \mathcal{G}_{2,s+p}^* + \omega_B \mathcal{G}_{2,s+p}^* &= [-\gamma h_z \mathcal{G}_1^*]_{s+p} \\
&= -\frac{\gamma}{2} \left[\left(h_{sz} \Gamma_{p-} H_{p-} + h_{pz} \Gamma_{s-} H_{s-} \right) e^{-i\omega_{s+p} t} \right. \\
&\quad \left. + \left(h_{sz} \Gamma_{p+} H_{p+} + h_{pz} \Gamma_{s+} H_{s+} \right)^* e^{+i\omega_{s+p} t} \right] \square - \frac{\gamma}{2} \left[S_{2,s+p-} e^{-i\omega_{s+p} t} + S_{2,s+p+}^* e^{+i\omega_{s+p} t} \right] \\
\Rightarrow \mathcal{G}_{2,s+p} &= -\frac{\gamma}{2} \left[\frac{S_{2,s+p+}}{\omega_B - \omega_{s+p} - i\alpha\omega_{s+p}} e^{-i\omega_{s+p} t} + \frac{S_{2,s+p-}^*}{\omega_B + \omega_{s+p} + i\alpha\omega_{s+p}} e^{+i\omega_{s+p} t} \right] \\
&= -\frac{\gamma}{2} \left[\frac{h_{sz} \Gamma_{p+} H_{p+} + h_{pz} \Gamma_{s+} H_{s+}}{\omega_B - \omega_{s+p} - i\alpha\omega_{s+p}} e^{-i\omega_{s+p} t} + \frac{\left(h_{sz} \Gamma_{p-} H_{p-} + h_{pz} \Gamma_{s-} H_{s-} \right)^*}{\omega_B + \omega_{s+p} + i\alpha\omega_{s+p}} e^{+i\omega_{s+p} t} \right] \quad (19)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\gamma}{2} \left[\left\{ \frac{\Gamma_{p+}}{\omega_B - \omega_{s+p} - i\alpha\omega_{s+p}} h_{sz} H_{p+} \right\} e^{-i\omega_{s+p} t} + \left\{ \frac{\Gamma_{p-}^*}{\omega_B + \omega_{s+p} + i\alpha\omega_{s+p}} h_{sz}^* H_{p-}^* \right\} e^{+i\omega_{s+p} t} \right. \\
&\quad \left. + \left\{ \frac{\Gamma_{s+}}{\omega_B - \omega_{s+p} - i\alpha\omega_{s+p}} h_{pz} H_{s+} \right\} e^{-i\omega_{s+p} t} + \left\{ \frac{\Gamma_{s-}^*}{\omega_B + \omega_{s+p} + i\alpha\omega_{s+p}} h_{pz}^* H_{s-}^* \right\} e^{+i\omega_{s+p} t} \right] \\
&= -\left(\frac{\gamma}{2}\right)^2 \left[\left\{ \frac{h_{sz} H_{p+}}{(\omega_B - \omega_{s+p} - i\alpha\omega_{s+p})(\omega_B - \omega_p - i\alpha\omega_p)} \right\} e^{-i\omega_{s+p} t} + \left\{ \frac{h_{sz}^* H_{p-}^*}{(\omega_B + \omega_{s+p} + i\alpha\omega_{s+p})(\omega_B + \omega_p + i\alpha\omega_p)} \right\} e^{+i\omega_{s+p} t} \right. \\
&\quad \left. + \left\{ \frac{h_{pz} H_{s+}}{(\omega_B - \omega_{s+p} - i\alpha\omega_{s+p})(\omega_B - \omega_s - i\alpha\omega_s)} \right\} e^{-i\omega_{s+p} t} + \left\{ \frac{h_{pz}^* H_{s-}^*}{(\omega_B + \omega_{s+p} + i\alpha\omega_{s+p})(\omega_B + \omega_s + i\alpha\omega_s)} \right\} e^{+i\omega_{s+p} t} \right] \quad (20) \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \frac{h_{sz} (h_{px} + ih_{py})}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_p - i\alpha\xi_p)} \right\} e^{-i\omega_{s+p} t} + \left\{ \frac{h_{sz}^* (h_{px} - ih_{py})^*}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_p + i\alpha\xi_p)} \right\} e^{+i\omega_{s+p} t} \right. \\
&\quad \left. + \left\{ \frac{h_{pz} (h_{sx} + ih_{sy})}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_s - i\alpha\xi_s)} \right\} e^{-i\omega_{s+p} t} + \left\{ \frac{h_{pz}^* (h_{sx} - ih_{sy})^*}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_s + i\alpha\xi_s)} \right\} e^{+i\omega_{s+p} t} \right]
\end{aligned}$$

Finally, the difference-frequency response $G_{2,s-p}$ is

$$\begin{aligned}
& (\alpha - i) \frac{\partial}{\partial t} G_{2,s-p} + \omega_B G_{2,s-p} = [-\gamma h_z G_1]_{s-p} \\
& = -\frac{\gamma}{2} \left[\left(h_{sz} \Gamma_{p+}^* H_{p+}^* + h_{pz}^* \Gamma_{s-} H_{s-} \right) e^{-i\omega_{s-p} t} \right. \\
& \quad \left. + \left(h_{sz} \Gamma_{p-}^* H_{p-}^* + h_{pz}^* \Gamma_{s+} H_{s+} \right) e^{+i\omega_{s-p} t} \right] - \frac{\gamma}{2} \left[S_{2,s-p+} e^{-i\omega_{s-p} t} + S_{2,s-p-}^* e^{+i\omega_{s-p} t} \right] \quad (21)
\end{aligned}$$

$$\begin{aligned}
& (\alpha + i) \frac{\partial}{\partial t} G_{2,s-p}^* + \omega_B G_{2,s-p}^* = [-\gamma h_z G_1^*]_{s-p} \\
& = -\frac{\gamma}{2} \left[\left(h_{sz} \Gamma_{p-}^* H_{p-}^* + h_{pz}^* \Gamma_{s+} H_{s+} \right) e^{-i\omega_{s+p} t} \right. \\
& \quad \left. + \left(h_{sz} \Gamma_{p+}^* H_{p+}^* + h_{pz}^* \Gamma_{s-} H_{s-} \right) e^{+i\omega_{s+p} t} \right] - \frac{\gamma}{2} \left[S_{2,s-p-} e^{-i\omega_{s+p} t} + S_{2,s-p+}^* e^{+i\omega_{s+p} t} \right] \\
& \Rightarrow G_{2,s-p} = -\frac{\gamma}{2} \left[\frac{S_{2,s-p+}}{\omega_B - \omega_{s-p} - i\alpha\omega_{s-p}} e^{-i\omega_{s-p} t} + \frac{S_{2,s-p-}^*}{\omega_B + \omega_{s-p} + i\alpha\omega_{s-p}} e^{+i\omega_{s-p} t} \right] \\
& = -\frac{\gamma}{2} \left[\frac{h_{sz} \Gamma_{p+}^* H_{p+}^* + h_{pz}^* \Gamma_{s-} H_{s-}}{\omega_B - \omega_{s-p} - i\alpha\omega_{s-p}} e^{-i\omega_{s-p} t} + \frac{\left(h_{sz} \Gamma_{p-}^* H_{p-}^* + h_{pz}^* \Gamma_{s+} H_{s+} \right)^*}{\omega_B + \omega_{s-p} + i\alpha\omega_{s-p}} e^{+i\omega_{s-p} t} \right] \\
& = -\frac{\gamma}{2} \left\{ \left[\frac{\Gamma_{p+}^*}{\omega_B - \omega_{s-p} - i\alpha\omega_{s-p}} h_{sz} H_{p+}^* \right. \right. \\
& \quad \left. \left. + \frac{\Gamma_{s-}}{\omega_B - \omega_{s-p} - i\alpha\omega_{s-p}} h_{pz}^* H_{s-} \right] e^{-i\omega_{s-p} t} + \left[\frac{\Gamma_{p-}}{\omega_B + \omega_{s-p} + i\alpha\omega_{s-p}} h_{sz}^* H_{p-} \right. \right. \\
& \quad \left. \left. + \frac{\Gamma_{s+}^*}{\omega_B + \omega_{s-p} + i\alpha\omega_{s-p}} h_{pz} H_{s+}^* \right] e^{+i\omega_{s-p} t} \right\} \\
& = -\left(\frac{\gamma}{2}\right)^2 \left\{ \left[\frac{h_{sz} H_{p+}^*}{(\omega_B - \omega_{s-p} - i\alpha\omega_{s-p})(\omega_B - \omega_{p+} + i\alpha\omega_{p+})} \right. \right. \\
& \quad \left. \left. + \frac{h_{pz}^* H_{s-}}{(\omega_B - \omega_{s-p} - i\alpha\omega_{s-p})(\omega_B + \omega_{s-} - i\alpha\omega_{s-})} \right] e^{-i\omega_{s-p} t} + \left[\frac{h_{sz}^* H_{p-}}{(\omega_B + \omega_{s-p} + i\alpha\omega_{s-p})(\omega_B + \omega_{p+} + i\alpha\omega_{p+})} \right. \right. \\
& \quad \left. \left. + \frac{h_{pz} H_{s+}^*}{(\omega_B + \omega_{s-p} + i\alpha\omega_{s-p})(\omega_B + \omega_{s+} + i\alpha\omega_{s+})} \right] e^{+i\omega_{s-p} t} \right\} \\
& = -\left(\frac{\gamma}{2\omega_B}\right)^2 \left\{ \left[\frac{h_{sz} (h_{px} + ih_{py})^*}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 - \xi_{p+} + i\alpha\xi_{p+})} \right. \right. \\
& \quad \left. \left. + \frac{h_{pz}^* (h_{sx} - ih_{sy})}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 + \xi_{s-} - i\alpha\xi_{s-})} \right] e^{-i\omega_{s-p} t} + \left[\frac{h_{sz}^* (h_{px} - ih_{py})}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 + \xi_{p+} + i\alpha\xi_{p+})} \right. \right. \\
& \quad \left. \left. + \frac{h_{pz} (h_{sx} + ih_{sy})^*}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 - \xi_{s+} + i\alpha\xi_{s+})} \right] e^{+i\omega_{s-p} t} \right\} \quad (22)
\end{aligned}$$

The various numerators of these terms are evaluated according to our scenario: for the sum-frequency response $G_{2,s+p}$,

$$h_{px} + ih_{py} = \mathcal{H}_{px} e^{-i\theta_{px}} + i\mathcal{H}_{py} e^{-i\theta_{py}}, h_{pz} = \mathcal{H}_{pz} e^{-i\theta_{sz}}$$

$$\mathcal{H}_{px} = 0, \mathcal{H}_{py} = 0, \mathcal{H}_{pz} = \mathcal{H}_{pc}, \theta_{pz} = 0$$

$$h_{sx} + ih_{sy} = \mathcal{H}_{sx} e^{-i\theta_{sx}} + i\mathcal{H}_{sy} e^{-i\theta_{sy}}, h_{sz} = \mathcal{H}_{sz} e^{-i\theta_{sz}}$$

$$\mathcal{H}_{sx} = 0, \mathcal{H}_{sy} = \mathcal{H}_{sc} \cos\varphi, \mathcal{H}_{sz} = \mathcal{H}_{sc} \sin\varphi, \theta_y = 0, \theta_z = 0$$

$$\Rightarrow \begin{cases} h_{sz}(h_{px} + ih_{py}) = \mathcal{H}_{sc} \sin\varphi(0) \\ h_{pz}(h_{sx} + ih_{sy}) = \mathcal{H}_{pc}(i\mathcal{H}_{sc} \cos\varphi) \\ h_{sz}^*(h_{px} - ih_{py})^* = \mathcal{H}_{sc} \sin\varphi(0) \\ h_{pz}^*(h_{sx} - ih_{sy})^* = \mathcal{H}_{pc}(i\mathcal{H}_{sc} \cos\varphi) \end{cases} \quad (23)$$

$$\Rightarrow \mathcal{G}_{2,s+p} = -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \frac{(0)\mathcal{H}_{sc} \sin\varphi}{(1-\xi_{s+p} - i\alpha\xi_{s+p})(1-\xi_p - i\alpha\xi_p)} + \frac{\mathcal{H}_{pc}(i\mathcal{H}_{sc} \cos\varphi)}{(1-\xi_{s+p} - i\alpha\xi_{s+p})(1-\xi_s - i\alpha\xi_s)} \right\} e^{-i\omega_{s+p}t} + \left\{ \frac{(0)\mathcal{H}_{sc} \sin\varphi}{(1+\xi_{s+p} + i\alpha\xi_{s+p})(1+\xi_p + i\alpha\xi_p)} + \frac{\mathcal{H}_{pc}(i\mathcal{H}_{sc} \cos\varphi)}{(1+\xi_{s+p} + i\alpha\xi_{s+p})(1+\xi_s + i\alpha\xi_s)} \right\} e^{+i\omega_{s+p}t} \right] \quad (24)$$

$$= -\left(\frac{\gamma}{2\omega_B}\right)^2 \mathcal{H}_{pc} \mathcal{H}_{sc} \cos\varphi \left[\left\{ \frac{i}{(1-\xi_{s+p} - i\alpha\xi_{s+p})(1-\xi_s - i\alpha\xi_s)} \right\} e^{-i\omega_{s+p}t} + \left\{ \frac{i}{(1+\xi_{s+p} + i\alpha\xi_{s+p})(1+\xi_s + i\alpha\xi_s)} \right\} e^{+i\omega_{s+p}t} \right]$$

and for the difference-frequency response $\mathcal{G}_{2,s-p}$,

$$h_{px} + ih_{py} = \mathcal{H}_{px} e^{-i\theta_{px}} + i\mathcal{H}_{py} e^{-i\theta_{py}}, h_{pz} = \mathcal{H}_{pz} e^{-i\theta_{sz}}$$

$$\mathcal{H}_{px} = 0, \mathcal{H}_{py} = 0, \mathcal{H}_{pz} = \mathcal{H}_{pc}, \theta_{pz} = 0$$

$$h_{sx} + ih_{sy} = \mathcal{H}_{sx} e^{-i\theta_{sx}} + i\mathcal{H}_{sy} e^{-i\theta_{sy}}, h_{sz} = \mathcal{H}_{sz} e^{-i\theta_{sz}}$$

$$\mathcal{H}_{sx} = 0, \mathcal{H}_{sy} = \mathcal{H}_{sc} \cos\varphi, \mathcal{H}_{sz} = \mathcal{H}_{sc} \sin\varphi, \theta_y = 0, \theta_z = 0$$

$$\Rightarrow \begin{cases} (h_{px} + ih_{py})^* h_{sz} = (0)\mathcal{H}_{sc} \sin\varphi \\ h_{pz}^*(h_{sx} - ih_{sy}) = \mathcal{H}_{pc}(-i\mathcal{H}_{sc} \cos\varphi) \\ (h_{px} - ih_{py}) h_{sz}^* = (0)\mathcal{H}_{sc} \sin\varphi \\ h_{pz}(h_{sx} + ih_{sy})^* = \mathcal{H}_{pc}(-i\mathcal{H}_{sc} \cos\varphi) \end{cases} \quad (25)$$

$$\begin{aligned}
\Rightarrow G_{2,s-p} &= -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \frac{(0)\mathcal{H}_{sc}\sin\varphi}{(1-\xi_{s-p}-i\alpha\xi_{s-p})(1-\xi_p+i\alpha\xi_p)} \right\} e^{-i\omega_{s-p}t} + \left\{ \frac{(0)\mathcal{H}_{sc}\sin\varphi}{(1+\xi_{s-p}+i\alpha\xi_{s-p})(1+\xi_p-i\alpha\xi_p)} \right\} e^{+i\omega_{s-p}t} \right] \\
&\quad + \left\{ \frac{\mathcal{H}_{pc}(-i\mathcal{H}_{sc}\cos\varphi)}{(1-\xi_{s-p}-i\alpha\xi_{s-p})(1+\xi_{s-p}-i\alpha\xi_s)} \right\} e^{-i\omega_{s-p}t} + \left\{ \frac{\mathcal{H}_{pc}(-i\mathcal{H}_{sc}\cos\varphi)}{(1+\xi_{s-p}+i\alpha\xi_{s-p})(1-\xi_s+i\alpha\xi_s)} \right\} e^{+i\omega_{s-p}t} \right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2 \mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi \left[\left\{ \frac{-i}{(1-\xi_{s-p}-i\alpha\xi_{s-p})(1+\xi_{s-p}-i\alpha\xi_s)} \right\} e^{-i\omega_{s-p}t} + \left\{ \frac{-i}{(1+\xi_{s-p}+i\alpha\xi_{s-p})(1-\xi_s+i\alpha\xi_s)} \right\} e^{+i\omega_{s-p}t} \right]
\end{aligned} \tag{26}$$

For the z – component terms, we find

$$\begin{aligned}
g_{z2,s} &= -\frac{1}{2}\mathcal{G}_1^*\mathcal{G}_1 = -\frac{1}{2}\mathcal{G}_{1,s}^*\mathcal{G}_{1,s} = -\frac{1}{2}\left(\frac{\gamma}{\omega_B}\right)^2 \mathcal{H}_{sc}^2 \cos^2\varphi \left| \frac{i}{1-\xi_s-i\alpha\xi_s} e^{-i\omega_s t} + \frac{i}{1+\xi_s+i\alpha\xi_s} e^{i\omega_s t} \right|^2 \\
&= -\frac{1}{2}\left(\frac{\gamma}{\omega_B}\right)^2 \mathcal{H}_{sc}^2 \cos^2\varphi \left\{ \left| \frac{1}{(1-\xi_s)^2 + \alpha^2 \xi_s^2} + \frac{1}{(1+\xi_s)^2 + \alpha^2 \xi_s^2} \right|^2 \right. \\
&\quad \left. + \frac{1}{1-\xi_s-i\alpha\xi_s} \frac{1}{1+\xi_s-i\alpha\xi_s} e^{-2i\omega_s t} + \frac{1}{1-\xi_s+i\alpha\xi_s} \frac{1}{1+\xi_s+i\alpha\xi_s} e^{2i\omega_s t} \right\}
\end{aligned} \tag{27}$$

$$g_{z2,p} = -\frac{1}{2}\mathcal{G}_{1,p}^* \mathcal{G}_{1,p}$$

which do not contain sum or difference contributions.

4. Nonlinear Response: Third Order

When the pump field polarization is linear and “vertical,” the third-order equation reads

$$\begin{aligned}
(\alpha - i) \frac{\partial}{\partial t} \mathcal{G}_3 + \omega_B \mathcal{G}_3 &= -\gamma \mathcal{h}_z \mathcal{G}_2 \\
(\alpha + i) \frac{\partial}{\partial t} \mathcal{G}_3^* + \omega_B \mathcal{G}_3^* &= -\gamma \mathcal{h}_z \mathcal{G}_2^*
\end{aligned} \tag{28}$$

where $\mathcal{G}_2$ is the second-order response

$$\begin{aligned}
G_2 &= G_{2,2s} + G_{2,2p} + G_{2,s+p} + G_{2,s-p} + G_{2,0} \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\frac{h_{sz}(h_{sx} + ih_{sy})}{(1 - \xi_{2s} - i\alpha\xi_{2s})(1 - \xi_s - i\alpha\xi_s)} e^{-2i\omega_s t} + \frac{h_{sz}^*(h_{sx} - ih_{sy})}{(1 + \xi_{2s} + i\alpha\xi_{2s})(1 + \xi_s + i\alpha\xi_s)} e^{+2i\omega_s t} \right] \\
&\quad - \left(\frac{\gamma}{2\omega_B}\right)^2 \left[\frac{h_{pz}(h_{px} + ih_{py})}{(1 - \xi_{2s} - i\alpha\xi_{2s})(1 - \xi_p - i\alpha\xi_p)} e^{-2i\omega_p t} + \frac{h_{pz}^*(h_{px} - ih_{py})}{(1 + \xi_{2p} + i\alpha\xi_{2p})(1 + \xi_p + i\alpha\xi_p)} e^{+2i\omega_p t} \right] \\
&\quad - \left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \frac{h_{sz}(h_{px} + ih_{py})}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_p - i\alpha\xi_p)} \right\} e^{-i\omega_{s+p} t} + \left\{ \frac{h_{sz}^*(h_{px} - ih_{py})}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_p + i\alpha\xi_p)} \right\} e^{+i\omega_{s+p} t} \right. \\
&\quad \left. + \left\{ \frac{h_{pz}(h_{sx} + ih_{sy})}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_s - i\alpha\xi_s)} \right\} e^{-i\omega_{s+p} t} + \left\{ \frac{h_{pz}^*(h_{sx} - ih_{sy})}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_s + i\alpha\xi_s)} \right\} e^{+i\omega_{s+p} t} \right] \\
&\quad - \left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \frac{h_{sz}(h_{px} + ih_{py})^*}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 - \xi_p + i\alpha\xi_p)} \right\} e^{-i\omega_{s-p} t} + \left\{ \frac{h_{sz}^*(h_{px} - ih_{py})}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 + \xi_p + i\alpha\xi_p)} \right\} e^{+i\omega_{s-p} t} \right. \\
&\quad \left. + \left\{ \frac{h_{pz}^*(h_{sx} - ih_{sy})}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 + \xi_s - i\alpha\xi_s)} \right\} e^{-i\omega_{s-p} t} + \left\{ \frac{h_{pz}(h_{sx} + ih_{sy})^*}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 - \xi_s + i\alpha\xi_s)} \right\} e^{+i\omega_{s-p} t} \right] \quad (29) \\
&\quad - \frac{\gamma}{2\omega_B} \left[h_{sz} \Gamma_{s-}^* H_{s-}^* + h_{pz} \Gamma_{p-}^* H_{p-}^* + h_{sz}^* \Gamma_{s+} H_{s+} + h_{pz}^* \Gamma_{p+} H_{p+} \right]
\end{aligned}$$

Denoting the quantities with dimensionless denominators by $\mathbb{Q}$, we write

$$G_2 = -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\begin{aligned} &\mathbb{Q}_{2s+} e^{-2i\omega_s t} + \mathbb{Q}_{2s-} e^{+2i\omega_s t} + \mathbb{Q}_0 + \mathbb{Q}_{2p+} e^{-2i\omega_p t} + \mathbb{Q}_{2p-} e^{+2i\omega_p t} \\ &+ \mathbb{Q}_{s+p+} e^{-i\omega_{s+p} t} + \mathbb{Q}_{s+p-} e^{+i\omega_{s+p} t} + \mathbb{Q}_{s-p+} e^{-i\omega_{s-p} t} + \mathbb{Q}_{s-p-} e^{+i\omega_{s-p} t} \end{aligned} \right] \quad (30)$$

where

$$\begin{aligned}
\Box_{2s+} &= \frac{h_{sz}(h_{sx} + ih_{sy})}{(1 - \xi_{2s} - i\alpha\xi_{2s})(1 - \xi_s - i\alpha\xi_s)} \rightarrow \frac{\mathcal{H}_{sc}\sin\varphi(+i\mathcal{H}_{sc}\cos\varphi)}{(1 - \xi_{2s} - i\alpha\xi_{2s})(1 - \xi_s - i\alpha\xi_s)} = \frac{i\mathcal{H}_{sc}^2\sin\varphi\cos\varphi}{(1 - \xi_{2s} - i\alpha\xi_{2s})(1 - \xi_s - i\alpha\xi_s)} \\
\Box_{2s-} &= \frac{h_{sz}^*(h_{sx} - ih_{sy})}{(1 + \xi_{2s} + i\alpha\xi_{2s})(1 + \xi_s + i\alpha\xi_s)} \rightarrow \frac{\mathcal{H}_{sc}\sin\varphi(+i\mathcal{H}_{sc}\cos\varphi)}{(1 + \xi_{2s} + i\alpha\xi_{2s})(1 + \xi_s + i\alpha\xi_s)} = \frac{i\mathcal{H}_{sc}^2\sin\varphi\cos\varphi}{(1 + \xi_{2s} + i\alpha\xi_{2s})(1 + \xi_s + i\alpha\xi_s)} \\
\Box_0 &= \frac{h_{sz}(h_{sx} - ih_{sy})}{1 + \xi_s + i\alpha\xi_s} + \frac{h_{pz}(h_{px} - ih_{py})}{1 + \xi_p + i\alpha\xi_p} + \frac{h_{sz}^*(h_{sx} + ih_{sy})}{1 - \xi_s - i\alpha\xi_s} + \frac{h_{pz}^*(h_{px} + ih_{py})}{1 - \xi_p - i\alpha\xi_p} \\
&\rightarrow \frac{\mathcal{H}_{sc}\sin\varphi(+i\mathcal{H}_{sc}\cos\varphi)}{1 + \xi_s + i\alpha\xi_s} + \frac{\mathcal{H}_{pc}(0)^*}{1 + \xi_p + i\alpha\xi_p} + \frac{\mathcal{H}_{sc}\sin\varphi(+i\mathcal{H}_{sc}\cos\varphi)}{1 - \xi_s - i\alpha\xi_s} + \frac{\mathcal{H}_{pc}(0)}{1 - \xi_p - i\alpha\xi_p} \\
&= \mathcal{H}_{sc}^2\sin\varphi\cos\varphi \left\{ \frac{i}{1 + \xi_s + i\alpha\xi_s} + \frac{i}{1 - \xi_s - i\alpha\xi_s} \right\} \\
\Box_{2p+} &= \frac{h_{pz}(h_{px} + ih_{py})}{(1 - \xi_{2s} - i\alpha\xi_{2s})(1 - \xi_p - i\alpha\xi_p)} \rightarrow \frac{\mathcal{H}_{pc}(0)}{(1 - \xi_{2s} - i\alpha\xi_{2s})(1 - \xi_p - i\alpha\xi_p)} = 0 \\
\Box_{2p-} &= \frac{h_{pz}^*(h_{px} - ih_{py})}{(1 + \xi_{2p} + i\alpha\xi_{2p})(1 + \xi_p + i\alpha\xi_p)} \rightarrow \frac{\mathcal{H}_{pc}(0)}{(1 + \xi_{2p} + i\alpha\xi_{2p})(1 + \xi_p + i\alpha\xi_p)} = 0 \\
\Box_{s+p+} &= \frac{h_{sz}(h_{px} + ih_{py})}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_p - i\alpha\xi_p)} + \frac{h_{pz}(h_{sx} + ih_{sy})}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_s - i\alpha\xi_s)} \\
&\rightarrow \frac{\mathcal{H}_{sc}\sin\varphi(0)}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_p - i\alpha\xi_p)} + \frac{\mathcal{H}_{pc}(+i\mathcal{H}_{sc}\cos\varphi)}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_s - i\alpha\xi_s)} = \mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi \frac{i}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_s - i\alpha\xi_s)} \\
\Box_{s+p-} &= \frac{h_{sz}^*(h_{px} - ih_{py})}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_p + i\alpha\xi_p)} + \frac{h_{pz}^*(h_{sx} - ih_{sy})}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_s + i\alpha\xi_s)} \\
&\rightarrow \frac{\mathcal{H}_{sc}\sin\varphi(0)}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_p + i\alpha\xi_p)} + \frac{\mathcal{H}_{pc}(+i\mathcal{H}_{sc}\cos\varphi)}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_s + i\alpha\xi_s)} = \mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi \frac{i}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_s + i\alpha\xi_s)} \\
\Box_{s-p+} &= \frac{h_{sz}(h_{px} + ih_{py})}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 - \xi_p + i\alpha\xi_p)} + \frac{h_{pz}^*(h_{sx} - ih_{sy})}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 + \xi_s - i\alpha\xi_s)} \\
&\rightarrow \frac{\mathcal{H}_{sc}\sin\varphi(0)}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 - \xi_p + i\alpha\xi_p)} + \frac{\mathcal{H}_{pc}(-i\mathcal{H}_{sc}\cos\varphi)}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 + \xi_s - i\alpha\xi_s)} = \mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi \frac{-i}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 + \xi_s - i\alpha\xi_s)} \\
\Box_{s-p-} &= \frac{h_{sz}^*(h_{px} - ih_{py})}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 + \xi_p + i\alpha\xi_p)} + \frac{h_{pz}(h_{sx} + ih_{sy})}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 - \xi_s + i\alpha\xi_s)} \\
&\rightarrow \frac{\mathcal{H}_{sc}\sin\varphi(0)}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 + \xi_p + i\alpha\xi_p)} + \frac{\mathcal{H}_{pc}(-i\mathcal{H}_{sc}\cos\varphi)}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 - \xi_s + i\alpha\xi_s)} = \mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi \frac{-i}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 - \xi_s + i\alpha\xi_s)}
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\rightarrow \frac{\mathcal{H}_{sc}\sin\varphi(0)}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 - \xi_p + i\alpha\xi_p)} + \frac{\mathcal{H}_{pc}(-i\mathcal{H}_{sc}\cos\varphi)}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 + \xi_s - i\alpha\xi_s)} = \mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi \frac{-i}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 + \xi_s - i\alpha\xi_s)} \\
&\rightarrow \frac{\mathcal{H}_{sc}\sin\varphi(0)}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 + \xi_p + i\alpha\xi_p)} + \frac{\mathcal{H}_{pc}(-i\mathcal{H}_{sc}\cos\varphi)}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 - \xi_s + i\alpha\xi_s)} = \mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi \frac{-i}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 - \xi_s + i\alpha\xi_s)}
\end{aligned} \tag{32}$$

We obtain the general source terms from the spectral decomposition of $h_z G_2$:

$$\begin{aligned}
h_z G_2 &= \text{Re} \left\{ h_{az} e^{-i\omega_a t} + h_{bz} e^{-i\omega_b t} \right\} \cdot G_2 \\
&= \frac{1}{2} \begin{pmatrix} h_{sz} e^{-i\omega_s t} \\ +h_{pz} e^{-i\omega_p t} \\ +h_{sz}^* e^{+i\omega_s t} \\ +h_{pz}^* e^{+i\omega_p t} \end{pmatrix} \begin{bmatrix} \square_{2s+} e^{-2i\omega_s t} + \square_{2s-} e^{+2i\omega_s t} + \square_0 + \square_{2p+} e^{-2i\omega_p t} + \square_{2p-} e^{+2i\omega_p t} \\ +\square_{s+p+} e^{-i\omega_{s+p} t} + \square_{s+p-} e^{+i\omega_{s+p} t} + \square_{s-p+} e^{-i\omega_{s-p} t} + \square_{s-p-} e^{+i\omega_{s-p} t} \end{bmatrix} \quad (33) \\
&= h_{sz} e^{-i\omega_s t} \begin{bmatrix} \square_{2s+} e^{-2i\omega_s t} + \square_{2s-} e^{+2i\omega_s t} + \square_0 + \square_{2p+} e^{-2i\omega_p t} + \square_{2p-} e^{+2i\omega_p t} \\ +\square_{s+p+} e^{-i\omega_{s+p} t} + \square_{s+p-} e^{+i\omega_{s+p} t} + \square_{s-p+} e^{-i\omega_{s-p} t} + \square_{s-p-} e^{+i\omega_{s-p} t} \end{bmatrix} \\
&= h_{sz} \square_{2s+} e^{-3i\omega_s t} + h_{sz} \square_{2s-} e^{+i\omega_s t} + h_{sz} \square_0 e^{-i\omega_s t} + h_{sz} \square_{2p+} e^{-i(2\omega_p + \omega_s)t} + h_{sz} \square_{2p-} e^{+i(2\omega_p - \omega_s)t} \\
&\quad + h_{sz} \square_{s+p+} e^{-i(2\omega_s + \omega_p)t} + h_{sz} \square_{s+p-} e^{+i\omega_p t} + h_{sz} \square_{s-p+} e^{-i(2\omega_s - \omega_p)t} + h_{sz} \square_{s-p-} e^{-i\omega_p t} \\
&\quad h_{pz} e^{-i\omega_p t} \begin{bmatrix} \square_{2s+} e^{-2i\omega_s t} + \square_{2s-} e^{+2i\omega_s t} + \square_0 + \square_{2p+} e^{-2i\omega_p t} + \square_{2p-} e^{+2i\omega_p t} \\ +\square_{s+p+} e^{-i\omega_{s+p} t} + \square_{s+p-} e^{+i\omega_{s+p} t} + \square_{s-p+} e^{-i\omega_{s-p} t} + \square_{s-p-} e^{+i\omega_{s-p} t} \end{bmatrix} \\
&= h_{pz} \square_{2s+} e^{-i(2\omega_s + \omega_p)t} + h_{pz} \square_{2s-} e^{+i(2\omega_s - \omega_p)t} + h_{pz} \square_0 e^{-i\omega_p t} + h_{pz} \square_{2p+} e^{-3i\omega_p t} + h_{pz} \square_{2p-} e^{+i\omega_p t} \\
&\quad + h_{pz} \square_{s+p+} e^{-i(2\omega_p + \omega_s)t} + h_{pz} \square_{s+p-} e^{+i\omega_s t} + h_{pz} \square_{s-p+} e^{-i\omega_s t} + h_{pz} \square_{s-p-} e^{-i(2\omega_p - \omega_s)t} \quad (34)
\end{aligned}$$

$$\begin{aligned}
& h_{sz}^* e^{+i\omega_s t} \left[\begin{aligned} & \square_{2s+} e^{-2i\omega_s t} + \square_{2s-} e^{+2i\omega_s t} + \square_0 + \square_{2p+} e^{-2i\omega_p t} + \square_{2p-} e^{+2i\omega_p t} \\ & + \square_{s+p+} e^{-i\omega_{s+p} t} + \square_{s+p-} e^{+i\omega_{s+p} t} + \square_{s-p+} e^{-i\omega_{s-p} t} + \square_{s-p-} e^{+i\omega_{s-p} t} \end{aligned} \right] \\
& = h_{sz}^* \square_{2s+} e^{-i\omega_s t} + h_{sz}^* \square_{2s-} e^{+3i\omega_s t} + h_{sz}^* \square_0 e^{+i\omega_s t} + h_{sz}^* \square_{2p+} e^{-i(2\omega_p - \omega_s)t} + h_{sz}^* \square_{2p-} e^{+i(2\omega_p + \omega_s)t} \\
& + h_{sz}^* \square_{s+p+} e^{-i\omega_p t} + h_{sz}^* \square_{s+p-} e^{+i(2\omega_s + \omega_p)t} + h_{sz}^* \square_{s-p+} e^{+i\omega_p t} + h_{sz}^* \square_{s-p-} e^{+i(2\omega_s - \omega_p)t} \\
& h_{pz}^* e^{+i\omega_p t} \left[\begin{aligned} & \square_{2s+} e^{-2i\omega_s t} + \square_{2s-} e^{+2i\omega_s t} + \square_0 + \square_{2p+} e^{-2i\omega_p t} + \square_{2p-} e^{+2i\omega_p t} \\ & + \square_{s+p+} e^{-i\omega_{s+p} t} + \square_{s+p-} e^{+i\omega_{s+p} t} + \square_{s-p+} e^{-i\omega_{s-p} t} + \square_{s-p-} e^{+i\omega_{s-p} t} \end{aligned} \right] \\
& = h_{pz}^* \square_{2s+} e^{-i(2\omega_s - \omega_p)t} + h_{pz}^* \square_{2s-} e^{+i(2\omega_s + \omega_p)t} + h_{pz}^* \square_0 e^{+i\omega_p t} + h_{pz}^* \square_{2p+} e^{-i\omega_p t} + h_{pz}^* \square_{2p-} e^{+3i\omega_p t} \\
& + h_{pz}^* \square_{s+p+} e^{-i\omega_s t} + h_{pz}^* \square_{s+p-} e^{+i(2\omega_p + \omega_s)t} + h_{pz}^* \square_{s-p+} e^{+i(2\omega_p - \omega_s)t} + h_{pz}^* \square_{s-p-} e^{+i\omega_s t} \quad (35)
\end{aligned}$$

For ease of grouping, let us color each term according to its frequency, writing

$$\begin{aligned}
h_z G_2 = & \textcolor{red}{h_{sz}^* \square_{2s+} e^{-3i\omega_s t}} + \textcolor{green}{h_{sz}^* \square_{2s-} e^{+i\omega_s t}} + \textcolor{green}{h_{sz}^* \square_0 e^{-i\omega_s t}} + \textcolor{violet}{h_{sz}^* \square_{2p+} e^{-i(2\omega_p + \omega_s)t}} \\
& + \textcolor{orange}{h_{sz}^* \square_{2p-} e^{+i(2\omega_p - \omega_s)t}} + \textcolor{cyan}{h_{sz}^* \square_{s+p+} e^{-i(2\omega_s + \omega_p)t}} + \textcolor{blue}{h_{sz}^* \square_{s+p-} e^{+i\omega_p t}} + \textcolor{blue}{h_{sz}^* \square_{s-p+} e^{-i(2\omega_s - \omega_p)t}} \\
& + \textcolor{blue}{h_{sz}^* \square_{s-p-} e^{-i\omega_p t}} + \textcolor{cyan}{h_{pz}^* \square_{2s+} e^{-i(2\omega_s + \omega_p)t}} + \textcolor{blue}{h_{pz}^* \square_{2s-} e^{+i(2\omega_s - \omega_p)t}} + \textcolor{blue}{h_{pz}^* \square_0 e^{-i\omega_p t}} \\
& + \textcolor{green}{h_{pz}^* \square_{2p+} e^{-3i\omega_p t}} + \textcolor{blue}{h_{pz}^* \square_{2p-} e^{+i\omega_p t}} + \textcolor{violet}{h_{pz}^* \square_{s+p+} e^{-i(2\omega_p + \omega_s)t}} + \textcolor{green}{h_{pz}^* \square_{s+p-} e^{+i\omega_s t}} \\
& + \textcolor{green}{h_{pz}^* \square_{s-p+} e^{-i\omega_s t}} + \textcolor{orange}{h_{pz}^* \square_{s-p-} e^{-i(2\omega_p - \omega_s)t}} + \textcolor{green}{h_{sz}^* \square_{2s+} e^{-i\omega_s t}} + \textcolor{red}{h_{sz}^* \square_{2s-} e^{+3i\omega_s t}} \\
& + \textcolor{green}{h_{sz}^* \square_0 e^{+i\omega_s t}} + \textcolor{orange}{h_{sz}^* \square_{2p+} e^{-i(2\omega_p - \omega_s)t}} + \textcolor{violet}{h_{sz}^* \square_{2p-} e^{+i(2\omega_p + \omega_s)t}} \\
& + \textcolor{blue}{h_{sz}^* \square_{s+p+} e^{-i\omega_p t}} + \textcolor{cyan}{h_{sz}^* \square_{s+p-} e^{+i(2\omega_s + \omega_p)t}} + \textcolor{blue}{h_{sz}^* \square_{s-p+} e^{+i\omega_p t}} + \textcolor{blue}{h_{sz}^* \square_{s-p-} e^{+i(2\omega_s - \omega_p)t}} \\
& + \textcolor{blue}{h_{pz}^* \square_{2s+} e^{-i(2\omega_s - \omega_p)t}} + \textcolor{cyan}{h_{pz}^* \square_{2s-} e^{+i(2\omega_s + \omega_p)t}} + \textcolor{blue}{h_{pz}^* \square_0 e^{+i\omega_p t}} + \textcolor{blue}{h_{pz}^* \square_{2p+} e^{-i\omega_p t}} + \textcolor{green}{h_{pz}^* \square_{2p-} e^{+3i\omega_p t}} \quad (36) \\
& + \textcolor{green}{h_{pz}^* \square_{s+p+} e^{-i\omega_s t}} + \textcolor{violet}{h_{pz}^* \square_{s+p-} e^{+i(2\omega_p + \omega_s)t}} + \textcolor{orange}{h_{pz}^* \square_{s-p+} e^{+i(2\omega_p - \omega_s)t}} + \textcolor{green}{h_{pz}^* \square_{s-p-} e^{+i\omega_s t}}
\end{aligned}$$

$$\begin{aligned}
&= h_{sz} \square_{2s+} e^{-3i\omega_s t} + h_{sz}^* \square_{2s-} e^{+3i\omega_s t} \\
&+ h_{sz} \square_{2s-} e^{+i\omega_s t} + h_{sz} \square_0 e^{-i\omega_s t} + h_{pz} \square_{s+p-} e^{+i\omega_s t} + h_{pz} \square_{s-p+} e^{-i\omega_s t} \\
&+ h_{sz}^* \square_{2s+} e^{-i\omega_s t} + h_{sz}^* \square_0 e^{+i\omega_s t} + h_{pz}^* \square_{s+p+} e^{-i\omega_s t} + h_{pz}^* \square_{s-p-} e^{+i\omega_s t} \\
&+ h_{sz} \square_{s+p-} e^{+i\omega_p t} + h_{pz} \square_{2p-} e^{+i\omega_p t} + h_{sz}^* \square_{s-p+} e^{+i\omega_p t} + h_{pz}^* \square_0 e^{+i\omega_p t} \\
&+ h_{sz} \square_{s-p-} e^{-i\omega_p t} + h_{pz} \square_0 e^{-i\omega_p t} + h_{sz}^* \square_{s+p+} e^{-i\omega_p t} + h_{pz}^* \square_{2p+} e^{-i\omega_p t} \\
&+ h_{pz}^* \square_{s+p-} e^{+i(2\omega_p + \omega_s)t} + h_{sz}^* \square_{2p-} e^{+i(2\omega_p + \omega_s)t} + h_{pz} \square_{s+p+} e^{-i(2\omega_p + \omega_s)t} + h_{sz} \square_{2p+} e^{-i(2\omega_p + \omega_s)t} \\
&+ h_{sz} \square_{2p-} e^{+i(2\omega_p - \omega_s)t} + h_{pz} \square_{s-p-} e^{-i(2\omega_p - \omega_s)t} + h_{sz}^* \square_{2p+} e^{-i(2\omega_p - \omega_s)t} + h_{pz}^* \square_{s-p+} e^{+i(2\omega_p - \omega_s)t} \\
&+ h_{sz} \square_{s+p+} e^{-i(2\omega_s + \omega_p)t} + h_{pz} \square_{2s+} e^{-i(2\omega_s + \omega_p)t} + h_{sz}^* \square_{s+p-} e^{+i(2\omega_s + \omega_p)t} + h_{pz}^* \square_{2s-} e^{+i(2\omega_s + \omega_p)t} \\
&+ h_{sz} \square_{s-p+} e^{-i(2\omega_s - \omega_p)t} + h_{sz}^* \square_{s-p-} e^{+i(2\omega_s - \omega_p)t} + h_{pz} \square_{2s-} e^{+i(2\omega_s - \omega_p)t} + h_{pz}^* \square_{2s+} e^{-i(2\omega_s - \omega_p)t} \quad (37) \\
&+ h_{pz} \square_{2p+} e^{-3i\omega_p t} + h_{pz}^* \square_{2p-} e^{+3i\omega_p t} \\
&= h_{sz} \square_{2s+} e^{-3i\omega_s t} + h_{sz}^* \square_{2s-} e^{+3i\omega_s t} \\
&+ \left(h_{pz}^* \square_{s-p-} + h_{sz} \square_{2s-} + h_{pz} \square_{s+p-} + h_{sz}^* \square_0 \right) e^{+i\omega_s t} + \left(h_{pz} \square_{s-p+} + h_{sz}^* \square_{2s+} + h_{sz} \square_0 + h_{pz}^* \square_{s+p+} \right) e^{-i\omega_s t} \\
&+ \left(h_{sz} \square_{s+p-} + h_{pz} \square_{2p-} + h_{sz}^* \square_{s-p+} + h_{pz}^* \square_0 \right) e^{+i\omega_p t} + \left(h_{sz} \square_{s-p-} + h_{pz} \square_0 + h_{sz}^* \square_{s+p+} + h_{pz}^* \square_{2p+} \right) e^{-i\omega_p t} \\
&+ \left(h_{pz}^* \square_{s+p-} + h_{sz}^* \square_{2p-} \right) e^{+i(2\omega_p + \omega_s)t} + \left(h_{pz} \square_{s+p+} + h_{sz} \square_{2p+} \right) e^{-i(2\omega_p + \omega_s)t} \\
&+ \left(h_{sz} \square_{2p-} + h_{pz}^* \square_{s-p+} \right) e^{+i(2\omega_p - \omega_s)t} + \left(h_{pz} \square_{s-p-} + h_{sz}^* \square_{2p+} \right) e^{-i(2\omega_p - \omega_s)t} \\
&+ \left(h_{sz} \square_{s+p+} + h_{pz} \square_{2s+} \right) e^{-i(2\omega_s + \omega_p)t} + \left(h_{sz}^* \square_{s+p-} + h_{pz}^* \square_{2s-} \right) e^{+i(2\omega_s + \omega_p)t} \\
&+ \left(h_{sz} \square_{s-p+} + h_{pz}^* \square_{2s+} \right) e^{-i(2\omega_s - \omega_p)t} + \left(h_{sz}^* \square_{s-p-} + h_{pz} \square_{2s-} \right) e^{+i(2\omega_s - \omega_p)t} \\
&+ h_{pz} \square_{2p+} e^{-3i\omega_p t} + h_{pz}^* \square_{2p-} e^{+3i\omega_p t} \quad (38)
\end{aligned}$$

Then, the spectral decomposition can finally be written

$$\begin{aligned}
h_z \mathcal{G}_2 = & h_{sz} \square_{2s+} e^{-3i\omega_s t} + h_{sz}^* \square_{2s-} e^{+3i\omega_s t} \\
& + \left(h_{pz} \square_{s+p-} + h_{sz} \square_{2s-} + h_{pz}^* \square_{s-p-} + h_{sz}^* \square_0 \right) e^{+i\omega_s t} + \left(h_{sz} \square_0 + h_{pz} \square_{s-p+} + h_{pz}^* \square_{s+p+} + h_{sz}^* \square_{2s+} \right) e^{-i\omega_s t} \\
& + \left(h_{sz} \square_{s+p-} + h_{pz} \square_{2p-} + h_{sz}^* \square_{s-p+} + h_{pz}^* \square_0 \right) e^{+i\omega_p t} + \left(h_{pz} \square_0 + h_{sz} \square_{s-p-} + h_{sz}^* \square_{s+p+} + h_{pz}^* \square_{2p+} \right) e^{-i\omega_p t} \\
& + \left(h_{pz}^* \square_{s+p-} + h_{sz}^* \square_{2p-} \right) e^{+i(2\omega_p + \omega_s)t} + \left(h_{pz} \square_{s+p+} + h_{sz} \square_{2p+} \right) e^{-i(2\omega_p + \omega_s)t} \\
& + \left(h_{pz}^* \square_{s-p+} + h_{sz} \square_{2p-} \right) e^{+i(2\omega_p - \omega_s)t} + \left(h_{pz} \square_{s-p-} + h_{sz}^* \square_{2p+} \right) e^{-i(2\omega_p - \omega_s)t} \\
& + \left(h_{sz}^* \square_{s+p-} + h_{pz}^* \square_{2s-} \right) e^{+i(2\omega_s + \omega_p)t} + \left(h_{sz} \square_{s+p+} + h_{pz} \square_{2s+} \right) e^{-i(2\omega_s + \omega_p)t} \\
& + \left(h_{sz}^* \square_{s-p-} + h_{pz} \square_{2s-} \right) e^{+i(2\omega_s - \omega_p)t} + \left(h_{sz} \square_{s-p+} + h_{pz}^* \square_{2s+} \right) e^{-i(2\omega_s - \omega_p)t} \\
& + h_{pz} \square_{2p+} e^{-3i\omega_p t} + h_{pz}^* \square_{2p-} e^{+3i\omega_p t}
\end{aligned} \tag{39}$$

Because we are interested only in the components that arise from mixing of the pump and signal, we solve for the sideband (intermod) responses only:

$$(\alpha - i) \frac{\partial}{\partial t} \mathcal{G}_3 \Big|_{\text{sideband}} + \omega_B \mathcal{G}_3 \Big|_{\text{sideband}} = \frac{\gamma}{2} \left\{ \begin{array}{l} S_{3,\text{sp}2\text{p},+} e^{-i(\omega_s + 2\omega_p)t} + S_{3,\text{sp}2\text{p},-} e^{+i(\omega_s + 2\omega_p)t} \\ + S_{3,\text{sm}2\text{p},+} e^{-i(\omega_s - 2\omega_p)t} + S_{3,\text{sm}2\text{p},-} e^{+i(\omega_s - 2\omega_p)t} \end{array} \right\} \tag{40}$$

where

$$\begin{aligned}
S_{3,\text{s}+2\text{p},+} &= h_{pz} \square_{s+p+} + h_{sz} \square_{2p+} = h_{pz} \square_{s+p+} \\
S_{3,\text{s}+2\text{p},-} &= h_{pz}^* \square_{s+p-} + h_{sz}^* \square_{2p-} = h_{pz}^* \square_{s+p-} \\
S_{3,\text{s}-2\text{p},+} &= h_{pz}^* \square_{s-p+} + h_{sz} \square_{2p-} = h_{pz}^* \square_{s-p+} \\
S_{3,\text{s}-2\text{p},-} &= h_{pz} \square_{s-p-} + h_{sz}^* \square_{2p+} = h_{pz} \square_{s-p-}
\end{aligned} \tag{41}$$

Let $\mathcal{G}_3 \Big|_{\text{sideband}} = \mathcal{G}_{3,\text{sp}2\text{p},+} + \mathcal{G}_{3,\text{sp}2\text{p},-} + \mathcal{G}_{3,\text{sm}2\text{p},+} + \mathcal{G}_{3,\text{sm}2\text{p},-}$. Then for the upper sideband, we have

$$\begin{aligned}
G_{2,s+p} &= -\frac{\gamma}{2} \left[\frac{S_{2,s+p,+}}{\omega_B - \omega_{s+p} - i\alpha\omega_{s+p}} e^{-i\omega_{s+p}t} + \frac{S_{2,s+p,-}^*}{\omega_B + \omega_{s+p} + i\alpha\omega_{s+p}} e^{+i\omega_{s+p}t} \right] \\
&= -\left(\frac{\gamma}{2\omega_B} \right)^2 \left[\left\{ \frac{h_{sz}(h_{px} + ih_{py})}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_p - i\alpha\xi_p)} + \frac{h_{pz}(h_{sx} + ih_{sy})}{(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_s - i\alpha\xi_s)} \right\} e^{-i\omega_{s+p}t} + \left\{ \frac{h_{sz}^*(h_{px} - ih_{py})^*}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_p + i\alpha\xi_p)} + \frac{h_{pz}^*(h_{sx} - ih_{sy})^*}{(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_s + i\alpha\xi_s)} \right\} e^{+i\omega_{s+p}t} \right] \\
&= -\left(\frac{\gamma}{2\omega_B} \right)^2 \left[\square_{s+p+} e^{-i\omega_{s+p}t} + \square_{s+p-} e^{+i\omega_{s+p}t} \right] \\
\Rightarrow G_{3,s+2p} &= -\frac{\gamma}{2} \left[\frac{S_{3,s+2p,+}}{\omega_B - \omega_{s+2p} - i\alpha\omega_{s+2p}} e^{-i\omega_{s+2p}t} + \frac{S_{3,s+2p,-}^*}{\omega_B + \omega_{s+2p} + i\alpha\omega_{s+2p}} e^{+i\omega_{s+2p}t} \right] \\
&= -\frac{\gamma}{2} \left(\frac{\gamma}{2\omega_B} \right)^2 \left[\frac{h_{pz}}{\omega_B - \omega_{s+2p} - i\alpha\omega_{s+2p}} \square_{s+p+} e^{-i\omega_{s+2p}t} + \frac{h_{pz}^*}{\omega_B + \omega_{s+2p} + i\alpha\omega_{s+2p}} \square_{s+p-} e^{+i\omega_{s+2p}t} \right] \\
&= -\left(\frac{\gamma}{2\omega_B} \right)^3 \mathcal{H}_{pc}^2 \mathcal{H}_{sc} \cos\varphi \left[\frac{i}{(1 - \xi_{s+2p} - i\alpha\xi_{s+2p})(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_s - i\alpha\xi_s)} e^{-i\omega_{s+2p}t} + \frac{i}{(1 + \xi_{s+2p} + i\alpha\xi_{s+2p})(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_s + i\alpha\xi_s)} e^{+i\omega_{s+2p}t} \right] \\
&= -\left(\frac{\gamma}{2\omega_B} \right)^3 \mathcal{H}_{pc}^2 \mathcal{H}_{sc} \cos\varphi \left[\frac{i}{(1 - \xi_{s+2p} - i\alpha\xi_{s+2p})(1 - \xi_{s+p} - i\alpha\xi_{s+p})(1 - \xi_s - i\alpha\xi_s)} e^{-i\omega_{s+2p}t} + \frac{i}{(1 + \xi_{s+2p} + i\alpha\xi_{s+2p})(1 + \xi_{s+p} + i\alpha\xi_{s+p})(1 + \xi_s + i\alpha\xi_s)} e^{+i\omega_{s+2p}t} \right] \tag{42}
\end{aligned}$$

Likewise, for the lower sideband, we have

$$\begin{aligned}
G_{2,s-p} &= -\frac{\gamma}{2} \left[\frac{S_{2,s-p,+}}{\omega_B - \omega_{s-p} - i\alpha\omega_{s-p}} e^{-i\omega_{s-p}t} + \frac{S_{2,s-p,-}^*}{\omega_B + \omega_{s-p} + i\alpha\omega_{s-p}} e^{+i\omega_{s-p}t} \right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \frac{h_{sz}(h_{px} + ih_{py})^*}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 - \xi_p + i\alpha\xi_p)} + \frac{h_{pz}^*(h_{sx} - ih_{sy})}{(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 + \xi_s - i\alpha\xi_s)} \right\} e^{-i\omega_{s-p}t} + \left\{ \frac{h_{sz}^*(h_{px} - ih_{py})}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 + \xi_p + i\alpha\xi_p)} + \frac{h_{pz}(h_{sx} + ih_{sy})^*}{(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 - \xi_s + i\alpha\xi_s)} \right\} e^{+i\omega_{s-p}t} \right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\square_{s-p+} e^{-i\omega_{s-p}t} + \square_{s-p-} e^{+i\omega_{s-p}t} \right] \\
\Rightarrow G_{2,s-2p} &= -\frac{\gamma}{2} \left[\frac{S_{3,s-2p,+}}{\omega_B - \omega_{s-2p} - i\alpha\omega_{s-2p}} e^{-i\omega_{s-2p}t} + \frac{S_{3,s-2p,-}^*}{\omega_B + \omega_{s-2p} + i\alpha\omega_{s-2p}} e^{+i\omega_{s-2p}t} \right] \\
&= -\frac{\gamma}{2} \left(\frac{\gamma}{2\omega_B}\right)^2 \left[\frac{h_{pz}^* \square_{s-p+}}{\omega_B - \omega_{s-2p} - i\alpha\omega_{s-2p}} e^{-i\omega_{s-2p}t} + \frac{h_{pz} \square_{s-p-}}{\omega_B + \omega_{s-2p} + i\alpha\omega_{s-2p}} e^{+i\omega_{s-2p}t} \right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^3 \mathcal{H}_{pc}^2 \mathcal{H}_{sc} \cos\phi \left[\frac{-i}{(1 - \xi_{s-2p} - i\alpha\xi_{s-2p})(1 - \xi_{s-p} - i\alpha\xi_{s-p})(1 + \xi_s - i\alpha\xi_s)} e^{-i\omega_{s-2p}t} + \frac{-i}{(1 + \xi_{s-2p} + i\alpha\xi_{s-2p})(1 + \xi_{s-p} + i\alpha\xi_{s-p})(1 - \xi_s + i\alpha\xi_s)} e^{+i\omega_{s-2p}t} \right]
\end{aligned} \tag{43}$$

These expressions are the ones used in reference 1.

5. Conclusion

The complexity of these results, even for the idealized geometry chosen here, makes them problematic for use in deciphering experimental data from, e.g., pump illumination by a moving airframe. Nevertheless, the effort could be worthwhile if the signal returns were strong enough. Unfortunately, as reference 1 shows, this requires very high levels of pump power even over short distances.

6. References

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