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Perturbation Theory for the Landau-Lifshits-Gilbert Equation

by Frank Crowne

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This report supplements the material presented in the *IEEE Transactions in Magnetics* publication entitled "Two-Frequency Excitation of a Magnetic Microwire" and contains ancillary calculations for the response of a magnetic system to an external magnetic field, using the Landau-Lifshits-Gilbert (LLG) equation and a perturbation expansion in powers of the external field.

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Contents

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1. Introduction

Although the literature on nonlinear magnetic effects is extensive, there is little quantitative information available on multi-frequency excitation of ferromagnetic systems, even at the level of perturbation theory. The calculations presented in this technical report constitute an attempt to remedy this situation. Because it is in the nature of remote sensing to involve small probe fields, the analysis needed is an ideal use of perturbation theory, cumbersome though it may be. It is hoped that their derivation will help to lay the groundwork for evaluating the usefulness of ferromagnetic resonance (FMR)-based detection of the nonlinear excitation of magnetic objects in a true remote-sensing environment, i.e., where large magnetic fields of the sort used in laboratories or medical equipment are unavailable.

2. The Landau-Lifshits and Landau-Lifshits-Gilbert Equations

In order to estimate the power radiated by FMR excitation of a finite magnetized body, it is necessary first to describe the dynamics of the magnetic dipole moment per unit volume $\vec{M}$ (i.e., the magnetization) within the body in the presence of an external time-dependent magnetic field. This dynamic problem is governed by the Landau-Lifshits equation (*1*):

$$\frac{\partial \vec{M}}{\partial t} = \gamma \vec{M} \times \vec{H}_{eff} - \frac{\alpha}{\left|\vec{M}_{st}\right|} \vec{M} \times \frac{\partial \vec{M}}{\partial t} \tag{1}$$

where $\vec{M}_{st}$ is the static magnetization of the material, γ is the gyromagnetic ratio, α is the damping constant, and $\vec{H}_{eff}$ is an effective magnetic field consisting of the DC anisotropy field $\vec{H}_a$ that fixes the direction of $\vec{M}_{st}$ in the material, the depolarization field due to the body's shape, and the external time-dependent applied field. Note that in the absence of an AC magnetic field there is no depolarization field, and so $\vec{M}_{st} \times \vec{H}_a = 0$ under DC conditions, i.e., there is no DC torque. The first term is the torque exerted by the effective magnetic field $\vec{H}_{eff}$, while the second term gives rise to damping via eddy currents. The form of this equation implies that $\vec{M} \cdot \frac{\partial \vec{M}}{\partial t} = 0 \Rightarrow \left|\vec{M}\right|^2$ is constant in time.

Let us solve this problem approximately using perturbation theory. It is more convenient to first put this equation in Landau-Lifshits-Gilbert (LLG) form (*2*): if we define the torque vector

$\vec{\Omega} = \gamma \vec{M} \times \vec{H}_{eff}$ and a dimensionless vector damping rate $\vec{\mathrm{Q}} = \dfrac{\alpha}{\left|\vec{M}_{st}\right|}\vec{M}$, equation 1 can be rewritten as follows:

$$\frac{\partial \vec{M}}{\partial t} = \left(1+\alpha^2\right)^{-1}\left(\vec{\Omega} - \vec{\mathrm{Q}} \times \vec{\Omega}\right) \tag{2}$$

Assume a coordinate system with the *z*-axis along the wire axis. We start by defining a small perturbing radio frequency (RF) magnetic field $\vec{\mathrm{h}}$, which generates an RF magnetization $\vec{\mathrm{m}}$ that is small compared to $\vec{M}_{st}$. The decay dynamics of this magnetization is strongly affected by the conservation of length of the total magnetization vector, whose endpoint is constrained to lie on a spherical surface at all times. Let us further specify the perturbed magnetization and effective magnetic field in the LLG equation as follows:

$$\vec{M} = \vec{M}_{st} + \vec{\mathrm{m}} \tag{3}$$

and

$$\vec{H}_{eff} = \vec{H}_a + \vec{\mathrm{h}} - \mathrm{N}\vec{\mathrm{m}}\,, \tag{4}$$

where $\vec{M}_{st} \times \vec{H}_a = 0$ and

$$\mathrm{N} = \begin{pmatrix} \tilde{N} & 0 & 0 \\ 0 & \tilde{N} & 0 \\ 0 & 0 & N_z \end{pmatrix} \tag{5}$$

is the depolarization tensor. In this report, we consider a long wire parallel to the *z*-axis, for which $\tilde{N} = 1/2$ and $N_z \approx 0$. Following Antonenko et al. (*3*), we define a static susceptibility $\chi_{st} = \dfrac{\left|\vec{M}_{st}\right|}{\left|\vec{H}_a\right|}$. Then in rectangular coordinates with $\vec{M}_{st}$ and $\vec{H}_a$ along the positive *z* – axis, i.e., $\vec{M}_{st} = \left|\vec{M}_{st}\right|\hat{z}$ and $\vec{H}_a = H_a\hat{z}$, we obtain

$$\vec{H}_a = {\chi_{st}}^{-1}\left|\vec{M}_{st}\right|\hat{z} \Rightarrow \begin{cases} H_{eff,x} = \mathrm{h}_x - \tilde{N}\mathrm{m}_x \\ H_{eff,y} = \mathrm{h}_y - \tilde{N}\mathrm{m}_y \\ H_{eff,z} = \mathrm{h}_z + {\chi_{st}}^{-1}\left|\vec{M}_{st}\right| \end{cases} \tag{6}$$

$$\Rightarrow \vec{\Omega} = \gamma\left(\vec{M}_{st} + \vec{\mathrm{m}}\right) \times \vec{H}_{eff} = \gamma \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \mathrm{m}_x & \mathrm{m}_y & \left|\vec{M}_{st}\right| + \mathrm{m}_z \\ \mathrm{h}_x - \tilde{N}\mathrm{m}_x & \mathrm{h}_y - \tilde{N}\mathrm{m}_y & \mathrm{h}_z + {\chi_{st}}^{-1}\left|\vec{M}_{st}\right| \end{vmatrix}$$

$$= \gamma \begin{Bmatrix} \hat{x}\left[-\left|\vec{M}_{st}\right|\mathrm{h}_y + \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_{st}\right|\mathrm{m}_y + \mathrm{m}_y\mathrm{h}_z - \mathrm{m}_z\mathrm{h}_y + \tilde{N}\mathrm{m}_z\mathrm{m}_y \right] \\ + \hat{y}\left[\left|\vec{M}_{st}\right|\mathrm{h}_x - \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_{st}\right|\mathrm{m}_x + \mathrm{m}_z\mathrm{h}_x - \mathrm{m}_x\mathrm{h}_z - \tilde{N}\mathrm{m}_z\mathrm{m}_x \right] \\ + \hat{z}\left[\mathrm{m}_x\mathrm{h}_y - \mathrm{m}_y\mathrm{h}_x \right] \end{Bmatrix} \tag{7}$$

Introduce a scaled dimensionless magnetization $\vec{\mathrm{g}}$ such that $\vec{\mathrm{m}} = \left|\vec{M}_{st}\right|\vec{\mathrm{g}}$. Since $\vec{M}_{st} = \left|\vec{M}_{st}\right|\hat{z}$, the length constraint on $\vec{M}$ becomes

$$\begin{aligned} &\left|\vec{M}\right|^2 = \left|\vec{M}_{st} + \vec{\mathrm{m}}\right|^2 = \left|\vec{M}_{st}\right|^2 \Rightarrow \left|\hat{z} + \vec{\mathrm{g}}\right|^2 = \left(1 + \mathrm{g_z}\right)^2 + {\mathrm{g_x}}^2 + {\mathrm{g_y}}^2 = 1 \\ &\Rightarrow \mathrm{g_z} = -1 + \sqrt{1 - {\mathrm{g_x}}^2 - {\mathrm{g_y}}^2} \end{aligned} \tag{8}$$

Let $\vec{Q} = \alpha\left(\hat{z} + \vec{\mathrm{g}}\right)$. Then

$$\left|\vec{M}_{st}\right|\frac{\partial \vec{\mathrm{g}}}{\partial t} = \left(1 + \alpha^2\right)^{-1}\left(\vec{\Omega} - \vec{\mathrm{Q}} \times \vec{\Omega}\right) \tag{9}$$

and

$$\vec{\Omega} = \gamma\left|\vec{M}_{st}\right| \begin{Bmatrix} \hat{x}\left[-\mathrm{h}_y + \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_{st}\right|\mathrm{g}_y + \mathrm{g}_y\mathrm{h}_z - \mathrm{g}_z\mathrm{h}_y + \tilde{N}\left|\vec{M}_{st}\right|\mathrm{g}_z\mathrm{g}_y \right] \\ + \hat{y}\left[\mathrm{h}_x - \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_{st}\right|\mathrm{g}_x + \mathrm{g}_z\mathrm{h}_x - \mathrm{g}_x\mathrm{h}_z - \tilde{N}\left|\vec{M}_{st}\right|\mathrm{g}_z\mathrm{g}_x \right] \\ + \hat{z}\left[\mathrm{g}_x\mathrm{h}_y - \mathrm{g}_y\mathrm{h}_x \right] \end{Bmatrix} \tag{10}$$

Defining the constants $\Lambda = \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_s\right|$, $\Theta = \left|\vec{M}_s\right|\tilde{N}$ lets us write the torque vector as

$$\vec{\Omega} = \gamma\left|\vec{M}_{st}\right| \begin{Bmatrix} \hat{x}\left[-\mathrm{h}_y + \Lambda\mathrm{g}_y + \mathrm{g}_y\mathrm{h}_z - \mathrm{g}_z\mathrm{h}_y + \Theta\mathrm{g}_z\mathrm{g}_y \right] \\ + \hat{y}\left[\mathrm{h}_x - \Lambda\mathrm{g}_x + \mathrm{g}_z\mathrm{h}_x - \mathrm{g}_x\mathrm{h}_z - \Theta\mathrm{g}_z\mathrm{g}_x \right] \\ + \hat{z}\left[\mathrm{g}_x\mathrm{h}_y - \mathrm{g}_y\mathrm{h}_x \right] \end{Bmatrix} \tag{11}$$

The right side of the dynamic equation 2 can be treated as a "source" $\vec{S}$:

$$\vec{\mathrm{Q}}\times\vec{\Omega}=\alpha\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \mathrm{g}_x & \mathrm{g}_y & 1+\mathrm{g}_z\\ \Omega_x & \Omega_y & \Omega_z\end{vmatrix}\Rightarrow\begin{cases}\mathrm{S}_x=\left(\vec{\Omega}-\vec{\mathrm{Q}}\times\vec{\Omega}\right)_x=\Omega_x-\alpha\left(\mathrm{g}_y\Omega_z-\left[1+\mathrm{g}_z\right]\Omega_y\right)\equiv\left|\vec{M}_{st}\right|\hat{\mathrm{S}}_x\\ \mathrm{S}_y=\left(\vec{\Omega}-\vec{\mathrm{Q}}\times\vec{\Omega}\right)_y=\Omega_y-\alpha\left(\left[1+\mathrm{g}_z\right]\Omega_x-\mathrm{g}_x\Omega_z\right)\equiv\left|\vec{M}_{st}\right|\hat{\mathrm{S}}_y\\ \mathrm{S}_z=\left(\vec{\Omega}-\vec{\mathrm{Q}}\times\vec{\Omega}\right)_z=\Omega_z-\alpha\left(\mathrm{g}_x\Omega_y-\mathrm{g}_y\Omega_x\right)\equiv\left|\vec{M}_{st}\right|\hat{\mathrm{S}}_z\end{cases}\quad(12)$$

where

$$\hat{\mathrm{S}}_x=\gamma\begin{Bmatrix}-\mathrm{h}_y+\Lambda\mathrm{g}_y+\mathrm{g}_y\mathrm{h}_z-\mathrm{g}_z\mathrm{h}_y+\Theta\mathrm{g}_z\mathrm{g}_y\\ -\alpha\mathrm{g}_y\left(\mathrm{g}_x\mathrm{h}_y-\mathrm{g}_y\mathrm{h}_x\right)\\ +\alpha\left[1+\mathrm{g}_z\right]\left(\mathrm{h}_x-\Lambda\mathrm{g}_x+\mathrm{g}_z\mathrm{h}_x-\mathrm{g}_x\mathrm{h}_z-\Theta\mathrm{g}_z\mathrm{g}_x\right)\end{Bmatrix}$$

$$\hat{\mathrm{S}}_y=\gamma\begin{Bmatrix}\mathrm{h}_x-\Lambda\mathrm{g}_x+\mathrm{g}_z\mathrm{h}_x-\mathrm{g}_x\mathrm{h}_z-\Theta\mathrm{g}_z\mathrm{g}_x\\ -\alpha\left[1+\mathrm{g}_z\right]\left(-\mathrm{h}_y+\Lambda\mathrm{g}_y+\mathrm{g}_y\mathrm{h}_z-\mathrm{g}_z\mathrm{h}_y+\Theta\mathrm{g}_z\mathrm{g}_y\right)\\ +\alpha\mathrm{g}_x\left(\mathrm{g}_x\mathrm{h}_y-\mathrm{g}_y\mathrm{h}_x\right)\end{Bmatrix}\quad(13)$$

$$\hat{\mathrm{S}}_z=\gamma\begin{Bmatrix}\mathrm{g}_x\mathrm{h}_y-\mathrm{g}_y\mathrm{h}_x\\ -\alpha\mathrm{g}_x\left(\mathrm{h}_x-\Lambda\mathrm{g}_x+\mathrm{g}_z\mathrm{h}_x-\mathrm{g}_x\mathrm{h}_z-\Theta\mathrm{g}_z\mathrm{g}_x\right)\\ +\alpha\mathrm{g}_y\left(-\mathrm{h}_y+\Lambda\mathrm{g}_y+\mathrm{g}_y\mathrm{h}_z-\mathrm{g}_z\mathrm{h}_y+\Theta\mathrm{g}_z\mathrm{g}_y\right)\end{Bmatrix}$$

Note that the vector $\hat{\vec{\mathrm{S}}}$ has the dimensions of frequency.

It is advantageous to write the transverse system variables in vector form:

$$\vec{\mathrm{G}}=\begin{pmatrix}\mathrm{g}_x\\ \mathrm{g}_y\end{pmatrix}\qquad\vec{\mathrm{H}}=\begin{pmatrix}\mathrm{h}_x\\ \mathrm{h}_y\end{pmatrix}\quad(14)$$

Then, the equation for the transverse components of the source vector can be written as

$$\frac{\partial\vec{\mathrm{G}}}{\partial t}=\left(1+\alpha^2\right)^{-1}\hat{\vec{\mathrm{S}}}\quad(15)$$

where

$$\hat{\vec{\mathrm{S}}}=\begin{pmatrix}\hat{\mathrm{S}}_x\\ \hat{\mathrm{S}}_y\end{pmatrix}=\gamma\begin{pmatrix}\alpha\left[1+\mathrm{g}_z\right]^2 & -\left[1+\mathrm{g}_z\right]\\ \left[1+\mathrm{g}_z\right] & \alpha\left[1+\mathrm{g}_z\right]^2\end{pmatrix}\vec{\mathrm{H}}-\gamma\begin{pmatrix}\alpha\left[1+\mathrm{g}_z\right]\left[\Lambda+\mathrm{h}_z+\Theta\mathrm{g}_z\right] & -\left[\Lambda+\mathrm{h}_z+\Theta\mathrm{g}_z-\alpha\mathrm{G}\otimes\mathrm{H}\right]\\ \Lambda+\mathrm{h}_z+\Theta\mathrm{g}_z-\alpha\mathrm{G}\otimes\mathrm{H} & \alpha\left[1+\mathrm{g}_z\right]\left[\Lambda+\mathrm{h}_z+\Theta\mathrm{g}_z\right]\end{pmatrix}\vec{\mathrm{G}}$$

$$=\gamma\begin{pmatrix}\alpha\left[1-\mathrm{G}^2\right] & -\sqrt{1-\mathrm{G}^2}\\ \sqrt{1-\mathrm{G}^2} & \alpha\left[1-\mathrm{G}^2\right]\end{pmatrix}\vec{\mathrm{H}}-\gamma\begin{pmatrix}\alpha\sqrt{1-\mathrm{G}^2}\left[\Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-\mathrm{G}^2}\right] & -\left[\Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-\mathrm{G}^2}-\alpha\mathrm{G}\otimes\mathrm{H}\right]\\ \Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-\mathrm{G}^2}-\alpha\mathrm{G}\otimes\mathrm{H} & \alpha\sqrt{1-\mathrm{G}^2}\left[\Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-\mathrm{G}^2}\right]\end{pmatrix}\vec{\mathrm{G}}\quad(16)$$

$$\equiv\gamma\tilde{Z}\,\vec{\mathrm{H}}-\gamma\tilde{R}\,\vec{\mathrm{G}}$$

Here $g_x^2 + g_y^2 \equiv G^2$ and $G \otimes H \equiv \hat{z} \cdot \vec{G} \times \vec{H} = g_x h_y - g_y h_x$. There is also an equation for the longitudinal part (*z*-component):

$$\left(1+\alpha^2\right)\frac{\partial}{\partial t} g_z + \gamma\alpha\left(\vec{G}\cdot\vec{H} - \Theta G^2\right) g_z = \gamma\left\{G \otimes H - \alpha\vec{G}\cdot\vec{H} + \alpha\left[\Lambda + h_z\right]G^2\right\} \tag{17}$$

where $\vec{G}\cdot\vec{H} \equiv g_x h_x + g_y h_y$, but the algebraic constraint on the length of the vector $\vec{g}$ makes it redundant.

Up to now, all the dynamic variables g_x, g_y, g_z and h_x, h_y, h_z have been real. In the spirit of the Holstein-Primakoff transformation (*4*), let us define the following (complex) "circular" basis for the transverse variables using the matrix

$$\tilde{C} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \Rightarrow \begin{cases} \vec{G} = \tilde{C}\vec{G} = \begin{pmatrix} g_x + i g_y \\ g_x - i g_y \end{pmatrix} = \begin{pmatrix} \mathcal{G} \\ \mathcal{G}^* \end{pmatrix} \\ \vec{H} = \tilde{C}\vec{H} = \begin{pmatrix} h_x + i h_y \\ h_x - i h_y \end{pmatrix} = \begin{pmatrix} H_c \\ H_c{}^* \end{pmatrix} \end{cases} \tag{18}$$

In this notation,

$$\begin{aligned} G \otimes H = g_x h_y - g_y h_x &= \frac{1}{4i}\left(\mathcal{G} + \mathcal{G}^*\right)\left(H_c - H_c{}^*\right) - \frac{1}{4i}\left(\mathcal{G} - \mathcal{G}^*\right)\left(H_c + H_c{}^*\right) \\ &= \frac{1}{2i}\left(\mathcal{G}^* H_c - \mathcal{G} H_c{}^*\right) = \mathrm{Im}\left(\mathcal{G}^* H_c\right) \end{aligned} . \tag{19}$$

The matrices $\tilde{Z}$ and $\tilde{R}$ in equation 3 are both of the form $\tilde{T}\left[A,B\right] = \begin{pmatrix} A\alpha & -B \\ B & A\alpha \end{pmatrix}$, which can be diagonalized by the similarity transformation defined by $\tilde{C}$:

$$\tilde{C}\tilde{T}\left[A,B\right]\tilde{C}^{-1} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} A\alpha & -B \\ B & A\alpha \end{pmatrix}\frac{1}{2i}\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \alpha A + iB & 0 \\ 0 & \alpha A - iB \end{pmatrix} \tag{20}$$

leading to the following expression for the source vector in the circular basis:

$$\begin{pmatrix} \hat{S}_+ \\ \hat{S}_- \end{pmatrix} = \begin{pmatrix} \hat{S}_x + i\hat{S}_y \\ \hat{S}_x - i\hat{S}_y \end{pmatrix} = \gamma\tilde{\Xi}\vec{H} - \gamma\tilde{\Gamma}\vec{G} \tag{21}$$

where the matrices

$$\tilde{\Xi} = \begin{pmatrix} \sqrt{1-|\mathcal{G}|^2}\left[\alpha\sqrt{1-|\mathcal{G}|^2}+i\right] & 0 \\ 0 & \sqrt{1-|\mathcal{G}|^2}\left[\alpha\sqrt{1-|\mathcal{G}|^2}-i\right] \end{pmatrix} \tag{22}$$

and

$$\tilde{\Pi} = \begin{pmatrix} \alpha\sqrt{1-|\mathcal{G}|^2}\left[\Lambda-\Theta+h_z+\Theta\sqrt{1-|\mathcal{G}|^2}\right] & 0 \\ +i\left[\Lambda-\Theta+h_z+\Theta\sqrt{1-|\mathcal{G}|^2}-\alpha\,\mathrm{Im}\left(\mathcal{G}^* H_c\right)\right] & \\ 0 & \alpha\sqrt{1-|\mathcal{G}|^2}\left[\Lambda-\Theta+h_z+\Theta\sqrt{1-|\mathcal{G}|^2}\right] \\ & -i\left[\Lambda-\Theta+h_z+\Theta\sqrt{1-|\mathcal{G}|^2}-\alpha\,\mathrm{Im}\left(\mathcal{G}^* H_c\right)\right] \end{pmatrix} \tag{23}$$

are both diagonal. Since the nonzero elements of these matrices are complex conjugates of one another, the system reduces to a single complex-valued equation

$$\left(1+\alpha^2\right)\frac{\partial}{\partial t}\mathcal{G}+\gamma\begin{pmatrix}\alpha\sqrt{1-|\mathcal{G}|^2}\left[\Lambda-\Theta+h_z+\Theta\sqrt{1-|\mathcal{G}|^2}\right] \\ +i\left[\Lambda-\Theta+h_z+\Theta\sqrt{1-|\mathcal{G}|^2}-\alpha\,\mathrm{Im}\left(\mathcal{G}^* H_c\right)\right]\end{pmatrix}\mathcal{G}=\gamma\sqrt{1-|\mathcal{G}|^2}\left(\alpha\sqrt{1-|\mathcal{G}|^2}+i\right)H_c \tag{24}$$

where $\mathcal{G}$ and $\mathcal{G}^*$ are "magnon" variables (5). Note that the quantities $\sqrt{1-|\mathcal{G}|^2}$ and $\mathrm{Im}\left(\mathcal{G}^* H_c\right)$ are real.

A straightforward power-series expansion of this equation up to third order in $\mathcal{G}$ leads to the following perturbation equations:

$$\begin{aligned}
&(\alpha-i)\frac{\partial}{\partial t}\mathcal{G}_1+\gamma\Lambda\mathcal{G}_1=\gamma H_c \\
&(\alpha-i)\frac{\partial}{\partial t}\mathcal{G}_2+\gamma\Lambda\mathcal{G}_2=-\gamma h_z\mathcal{G}_1 \\
&(\alpha-i)\frac{\partial}{\partial t}\mathcal{G}_3+\gamma\Lambda\mathcal{G}_3=\frac{\gamma}{\alpha+i}\begin{Bmatrix}-(\alpha+i)h_z\mathcal{G}_2-\frac{1}{2}(3\alpha+i)\left|\mathcal{G}_1\right|^2 H_c+\alpha\frac{1}{2}{\mathcal{G}_1}^2{H_c}^* \\ +\frac{1}{2}\left(\alpha\Lambda+[\alpha+i]\Theta\right)\left|\mathcal{G}_1\right|^2\mathcal{G}_1\end{Bmatrix}
\end{aligned} \tag{25}$$

The corresponding z-component equations are

$$\mathrm{g_z} = -1+\sqrt{1-|\mathcal{G}|^2} = -1+\sqrt{1-\mathcal{G}^*\mathcal{G}} \approx -\frac{1}{2}\mathcal{G}_1{}^*\mathcal{G}_1 - \frac{1}{2}\left(\mathcal{G}_1{}^*\mathcal{G}_2 + \mathcal{G}_2{}^*\mathcal{G}_1\right)+\cdots$$

$$\Rightarrow \begin{cases} \mathrm{g_{z1}} = 0 \\ \mathrm{g_{z2}} = -\frac{1}{2}\mathcal{G}_1{}^*\mathcal{G}_1 \\ \mathrm{g_{z3}} = -\frac{1}{2}\left(\mathcal{G}_1{}^*\mathcal{G}_2 + \mathcal{G}_2{}^*\mathcal{G}_1\right) \end{cases} \tag{26}$$

Note that according to these expressions an incident field with only h_z nonzero, i.e., polarized along the direction of static magnetization, cannot generate a *linear* response. This is in keeping with elementary considerations.

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3. First-order Solution

To solve the equation

$$(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_1 + \gamma\Lambda\mathcal{G}_1 = \gamma H_c \tag{27}$$

we make the following decomposition: let

$$\tilde{M} = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}$$

Then

$$\left(1+\alpha^2\right)\partial_t\begin{pmatrix}\mathrm{g}_x\\ \mathrm{g}_y\end{pmatrix} + \omega_B\tilde{M}\begin{pmatrix}\mathrm{g}_x\\ \mathrm{g}_y\end{pmatrix} = \gamma\tilde{M}\begin{pmatrix}S_x\\ S_y\end{pmatrix} \Rightarrow \left(1+\alpha^2\right)\partial_t\vec{\mathrm{g}} + \omega_B\tilde{M}\vec{\mathrm{g}} = \gamma\tilde{M}\vec{S}$$

$$\tilde{C} = \begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix} \Rightarrow \tilde{C}\begin{pmatrix}\mathrm{g}_x\\ \mathrm{g}_y\end{pmatrix} \square \begin{pmatrix}g_{c+}\\ g_{c-}\end{pmatrix} \Rightarrow \begin{cases}\begin{pmatrix}\mathrm{g}_x\\ \mathrm{g}_y\end{pmatrix} = \tilde{C}^{-1}\begin{pmatrix}g_{c+}\\ g_{c-}\end{pmatrix} = \frac{1}{2i}\begin{pmatrix}i & i\\ 1 & -1\end{pmatrix}\begin{pmatrix}g_{c+}\\ g_{c-}\end{pmatrix} \\ \begin{pmatrix}S_x\\ S_y\end{pmatrix} = \tilde{C}^{-1}\begin{pmatrix}S_{c+}\\ S_{c-}\end{pmatrix} = \frac{1}{2i}\begin{pmatrix}i & i\\ 1 & -1\end{pmatrix}\begin{pmatrix}S_{c+}\\ S_{c-}\end{pmatrix}\end{cases} \tag{28}$$

$$\Rightarrow \left(1+\alpha^2\right)\partial_t\tilde{C}^{-1}\begin{pmatrix}g_{c+}\\ g_{c-}\end{pmatrix} + \omega_B\tilde{M}\tilde{C}^{-1}\begin{pmatrix}g_{c+}\\ g_{c-}\end{pmatrix} = \gamma\tilde{M}\tilde{C}^{-1}\begin{pmatrix}S_{c+}\\ S_{c-}\end{pmatrix}$$

$$\Rightarrow \left(1+\alpha^2\right)\partial_t\begin{pmatrix}g_{c+}\\ g_{c-}\end{pmatrix} + \omega_B\tilde{C}\tilde{M}\tilde{C}^{-1}\begin{pmatrix}g_{c+}\\ g_{c-}\end{pmatrix} = \gamma\tilde{C}\tilde{M}\tilde{C}^{-1}\begin{pmatrix}S_{c+}\\ S_{c-}\end{pmatrix}$$

$$\tilde{C}\tilde{M}\tilde{C}^{-1}=\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\begin{pmatrix}\alpha & -1\\ 1 & \alpha\end{pmatrix}\frac{1}{2i}\begin{pmatrix}i & i\\ 1 & -1\end{pmatrix}=\begin{pmatrix}\alpha+i & -1+i\alpha\\ \alpha-i & -1-i\alpha\end{pmatrix}\frac{1}{2i}\begin{pmatrix}i & i\\ 1 & -1\end{pmatrix}$$

$$=\frac{1}{2i}\begin{pmatrix}i\alpha-1-1+i\alpha & i\alpha-1+1-i\alpha\\ i\alpha+1-1-i\alpha & i\alpha+1+1+i\alpha\end{pmatrix}=\frac{1}{2i}\begin{pmatrix}2i\alpha-2 & 0\\ 0 & 2i\alpha+2\end{pmatrix}=\begin{pmatrix}\alpha+i & 0\\ 0 & \alpha-i\end{pmatrix}$$

$$\Rightarrow\left(1+\alpha^2\right)\partial_t\begin{pmatrix}g_{C+}\\ g_{C-}\end{pmatrix}+\omega_B\begin{pmatrix}[\alpha+i]g_{C+}\\ [\alpha-i]g_{C-}\end{pmatrix}=\gamma\begin{pmatrix}[\alpha+i]S_{C+}\\ [\alpha-i]S_{C-}\end{pmatrix} \tag{29}$$

$$\Rightarrow\begin{cases}\left(\partial_t+\dfrac{\omega_B}{\alpha-i}\right)g_{C+}=\dfrac{\gamma}{\alpha-i}S_{C+}\\ \left(\partial_t+\dfrac{\omega_B}{\alpha+i}\right)g_{C-}=\dfrac{\gamma}{\alpha+i}S_{C-}\end{cases}$$

The following is complex notation for non-Hermitian quantities:

$$\begin{pmatrix}S_x\\ S_y\end{pmatrix}=\begin{pmatrix}S_{xc}\\ S_{yc}\end{pmatrix}\cos\omega t+\begin{pmatrix}S_{xs}\\ S_{ys}\end{pmatrix}\sin\omega t=\begin{pmatrix}S_{xc}\\ S_{yc}\end{pmatrix}\frac{1}{2}\left[e^{i\omega t}+e^{-i\omega t}\right]+\begin{pmatrix}S_{xs}\\ S_{ys}\end{pmatrix}\frac{-i}{2}\left[e^{i\omega t}-e^{-i\omega t}\right]$$

$$=\frac{1}{2}\begin{pmatrix}S_{xc}+iS_{xs}\\ S_{yc}+iS_{ys}\end{pmatrix}e^{-i\omega t}+\begin{pmatrix}S_{xc}-iS_{xs}\\ S_{yc}-iS_{ys}\end{pmatrix}e^{i\omega t}\square\begin{pmatrix}S_{x+}\\ S_{y+}\end{pmatrix}e^{-i\omega t}+\begin{pmatrix}S_{x-}\\ S_{y-}\end{pmatrix}e^{i\omega t}$$

$$\Rightarrow\begin{pmatrix}S_{C+}\\ S_{C-}\end{pmatrix}\square\begin{pmatrix}S_{x+}+iS_{y+}\\ S_{x+}-iS_{y+}\end{pmatrix}e^{-i\omega t}+\begin{pmatrix}S_{x-}+iS_{y-}\\ S_{x-}-iS_{y-}\end{pmatrix}e^{i\omega t}=\begin{pmatrix}S_{xc}+iS_{xs}+iS_{yc}-S_{ys}\\ S_{xc}+iS_{xs}-iS_{yc}+S_{ys}\end{pmatrix}e^{-i\omega t}+\begin{pmatrix}S_{xc}-iS_{xs}+iS_{yc}+S_{ys}\\ S_{xc}-iS_{xs}-iS_{yc}-S_{ys}\end{pmatrix}e^{i\omega t}$$

$$=\begin{pmatrix}S_{xc}-S_{ys}+i\left[S_{xs}+S_{yc}\right]\\ S_{xc}+S_{ys}+i\left[S_{xs}-S_{yc}\right]\end{pmatrix}e^{-i\omega t}+\begin{pmatrix}S_{xc}+S_{ys}-i\left[S_{xs}-S_{yc}\right]\\ S_{xc}-S_{ys}-i\left[S_{xs}+S_{yc}\right]\end{pmatrix}e^{i\omega t}\square\begin{pmatrix}S_+\\ S_-\end{pmatrix}e^{-i\omega t}+\begin{pmatrix}S_-{}^*\\ S_+{}^*\end{pmatrix}e^{i\omega t}$$

$$\Rightarrow\begin{cases}\left(\partial_t+\dfrac{\omega_B}{\alpha-i}\right)g_{C+}=\dfrac{\gamma}{\alpha-i}\left(S_+e^{-i\omega t}+S_-{}^*e^{i\omega t}\right)\\ \left(\partial_t+\dfrac{\omega_B}{\alpha+i}\right)g_{C-}=\dfrac{\gamma}{\alpha+i}\left(S_-e^{-i\omega t}+S_+{}^*e^{i\omega t}\right)\end{cases} \tag{30}$$

The following is the transient solution:

$$p_{\pm} = \frac{\omega_B}{\alpha \mp i} \Rightarrow \begin{cases} g_{C+}(t) = A_+ e^{-p_+ t} + B_+ e^{-i\omega t} + C_+ e^{i\omega t} \\ g_{C-}(t) = A_- e^{-p_- t} + B_- e^{-i\omega t} + C_- e^{i\omega t} \end{cases}$$

$$\left(\partial_t + p_+\right) g_{C+} = \left(p_+ - i\omega\right) B_+ e^{-i\omega t} + \left(p_+ + i\omega\right) C_+ e^{i\omega t} = \frac{\gamma}{\alpha - i}\left(S_+ e^{-i\omega t} + S_-{}^* e^{i\omega t}\right)$$

$$\Rightarrow \begin{cases} B_+ = \dfrac{\gamma}{\alpha - i}\dfrac{S_+}{p_+ - i\omega} = \dfrac{\gamma S_+}{\omega_B - i\omega(\alpha - i)} = \dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega} \\ C_+ = \dfrac{\gamma}{\alpha - i}\dfrac{S_-{}^*}{p_+ + i\omega} = \dfrac{\gamma S_-{}^*}{\omega_B + i\omega(\alpha - i)} = \dfrac{\gamma S_-{}^*}{\omega_B + \omega + i\alpha\omega} \end{cases}$$

$$\left(\partial_t + p_-\right) g_{C-} = \left(p_- - i\omega\right) B_- e^{-i\omega t} + \left(p_- + i\omega\right) C_- e^{i\omega t} = \frac{\gamma}{\alpha + i}\left(S_- e^{-i\omega t} + S_+{}^* e^{i\omega t}\right)$$

$$\Rightarrow \begin{cases} B_- = \dfrac{\gamma}{\alpha + i}\dfrac{S_-}{p_- - i\omega} = \dfrac{\gamma S_-}{\omega_B - i\omega(\alpha + i)} = \dfrac{\gamma S_-}{\omega_B + \omega - i\alpha\omega} \\ C_- = \dfrac{\gamma}{\alpha + i}\dfrac{S_+{}^*}{p_- + i\omega} = \dfrac{\gamma S_+{}^*}{\omega_B + i\omega(\alpha + i)} = \dfrac{\gamma S_+{}^*}{\omega_B - \omega + i\alpha\omega} \end{cases} \tag{31}$$

For the boundary condition at $t = 0$,

$$g_{C\pm}(0) \equiv g_{C\pm 0} = A_\pm + B_\pm + C_\pm \Rightarrow A_\pm = g_{C\pm 0} - B_\pm - C_\pm$$

$$\Rightarrow \begin{cases} g_{C+}(t) = g_{C+0} e^{-p_+ t} + \dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega}\left(e^{-i\omega t} - e^{-p_+ t}\right) + \dfrac{\gamma S_-{}^*}{\omega_B + \omega + i\alpha\omega}\left(e^{i\omega t} - e^{-p_+ t}\right) \\ g_{C-}(t) = g_{C-0} e^{-p_- t} + \dfrac{\gamma S_-}{\omega_B + \omega - i\alpha\omega}\left(e^{-i\omega t} - e^{-p_- t}\right) + \dfrac{\gamma S_+{}^*}{\omega_B - \omega + i\alpha\omega}\left(e^{i\omega t} - e^{-p_- t}\right) \end{cases} \tag{32}$$

$$\begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left[g_{c+} + g_{c-} \right] \\ \frac{1}{2i}\left[g_{c+} - g_{c-} \right] \end{pmatrix}$$

$$p_- = p_+^{\,*} \Rightarrow g_{c-} = g_{c-0} e^{-p_- t} + \frac{\gamma}{\alpha + i} \frac{S_-}{p_- - i\omega} \left(e^{-i\omega t} - e^{-p_- t} \right) + \frac{\gamma}{\alpha + i} \frac{S_+^{\,*}}{p_- + i\omega} \left(e^{i\omega t} - e^{-p_- t} \right)$$

$$= g_{c-0} e^{-p_+^{\,*} t} + \left(\frac{\gamma}{\alpha - i} \right)^* \frac{S_-}{p_+^{\,*} - i\omega} \left(e^{-i\omega t} - e^{-p_+^{\,*} t} \right) + \left(\frac{\gamma}{\alpha - i} \right)^* \frac{S_+^{\,*}}{p_+^{\,*} + i\omega} \left(e^{i\omega t} - e^{-p_+^{\,*} t} \right)$$

$$= g_{c-0} e^{-p_+^{\,*} t} + \left(\frac{\gamma}{\alpha - i} \right)^* \frac{S_-}{p_+^{\,*} - i\omega} \left(e^{i\omega t} - e^{-p_+ t} \right)^* + \left(\frac{\gamma}{\alpha - i} \right)^* \frac{S_+^{\,*}}{p_+^{\,*} + i\omega} \left(e^{-i\omega t} - e^{-p_+ t} \right)^*$$

$$= g_{c-0} \left\{ e^{-p_+ t} \right\}^* + \left(\frac{\gamma}{\alpha - i} \right)^* \left\{ \frac{S_-^{\,*}}{p_+ + i\omega} \left(e^{i\omega t} - e^{-p_+ t} \right) + \frac{S_+}{p_+ - i\omega} \left(e^{-i\omega t} - e^{-p_+ t} \right) \right\}^* \tag{33}$$

$$g_{c-0} = g_{x0} - i g_{y0} = g_{c+0}^{\,*}$$

$$\Rightarrow g_{c-} = \left\{ g_{c+0} e^{-p_+ t} + \frac{\gamma}{\alpha - i} \frac{S_-^{\,*}}{p_+ + i\omega} \left(e^{i\omega t} - e^{-p_+ t} \right) + \frac{\gamma}{\alpha - i} \frac{S_+}{p_+ - i\omega} \left(e^{-i\omega t} - e^{-p_+ t} \right) \right\}^* = g_{c+}^{\,*} \tag{34}$$

$$\Rightarrow \begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\left[\left(g_{x0} + i g_{y0} \right) e^{-p_+ t} + \frac{\gamma}{\alpha - i} \frac{S_-^{\,*}}{p_+ + i\omega} \left(e^{i\omega t} - e^{-p_+ t} \right) + \frac{\gamma}{\alpha - i} \frac{S_+}{p_+ - i\omega} \left(e^{-i\omega t} - e^{-p_+ t} \right) \right] \\ \mathrm{Im}\left[\left(g_{x0} + i g_{y0} \right) e^{-p_+ t} + \frac{\gamma}{\alpha - i} \frac{S_-^{\,*}}{p_+ + i\omega} \left(e^{i\omega t} - e^{-p_+ t} \right) + \frac{\gamma}{\alpha - i} \frac{S_+}{p_+ - i\omega} \left(e^{-i\omega t} - e^{-p_+ t} \right) \right] \end{pmatrix}$$

$$\text{Large t} \Rightarrow \begin{cases} g_{c+}(t) \to \dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega} e^{-i\omega t} + \dfrac{\gamma S_-^{\,*}}{\omega_B + \omega + i\alpha\omega} e^{i\omega t} \\ g_{c-}(t) \to \dfrac{\gamma S_-}{\omega_B + \omega - i\alpha\omega} e^{-i\omega t} + \dfrac{\gamma S_+^{\,*}}{\omega_B - \omega + i\alpha\omega} e^{i\omega t} \end{cases}$$

$$\Rightarrow \begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\left[\dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega} e^{-i\omega t} + \dfrac{\gamma S_-^{\,*}}{\omega_B + \omega + i\alpha\omega} e^{i\omega t} \right] \\ \mathrm{Im}\left[\dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega} e^{-i\omega t} + \dfrac{\gamma S_-^{\,*}}{\omega_B + \omega + i\alpha\omega} e^{i\omega t} \right] \end{pmatrix} \tag{35}$$

The linear source terms are

$$\begin{pmatrix} S_{\mathrm{x}} \\ S_{\mathrm{y}} \end{pmatrix} = \begin{pmatrix} \mathrm{h}_x \\ \mathrm{h}_y \end{pmatrix} = \begin{pmatrix} \mathrm{h}_{\mathrm{xc}} \\ \mathrm{h}_{\mathrm{yc}} \end{pmatrix} \cos\omega t + \begin{pmatrix} \mathrm{h}_{\mathrm{xs}} \\ \mathrm{h}_{\mathrm{ys}} \end{pmatrix} \sin\omega t = \frac{1}{2}\left\{ \begin{pmatrix} \mathrm{h}_{\mathrm{xc}} + i\mathrm{h}_{\mathrm{xs}} \\ \mathrm{h}_{\mathrm{yc}} + i\mathrm{h}_{\mathrm{ys}} \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} \mathrm{h}_{\mathrm{xc}} - i\mathrm{h}_{\mathrm{xs}} \\ \mathrm{h}_{\mathrm{yc}} - i\mathrm{h}_{\mathrm{ys}} \end{pmatrix} e^{i\omega t} \right\}$$

$$\Rightarrow \begin{pmatrix} h_x \\ h_y \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} h_x^{\,*} \\ h_y^{\,*} \end{pmatrix} e^{i\omega t} \Rightarrow \begin{pmatrix} S_{c+} \\ S_{c-} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \left[\begin{pmatrix} h_x \\ h_y \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} h_x^{\,*} \\ h_y^{\,*} \end{pmatrix} e^{i\omega t} \right]$$

$$= \begin{pmatrix} h_x + ih_y \\ h_x - ih_y \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} h_x^{\,*} + ih_y^{\,*} \\ h_x^{\,*} - ih_y^{\,*} \end{pmatrix} e^{i\omega t} \Rightarrow \begin{pmatrix} H_+ \\ H_- \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} H_-^{\,*} \\ H_+^{\,*} \end{pmatrix} e^{i\omega t} \tag{36}$$

$$\mathcal{H}_x = \sqrt{\mathrm{h}_{\mathrm{xc}}^{\,2} + \mathrm{h}_{\mathrm{xs}}^{\,2}}\ ,\ \mathcal{H}_y = \sqrt{\mathrm{h}_{\mathrm{yc}}^{\,2} + \mathrm{h}_{\mathrm{ys}}^{\,2}} \Rightarrow \begin{cases} h_x = \mathcal{H}_x e^{-i\theta_x} \\ h_y = \mathcal{H}_y e^{-i\theta_y} \end{cases}$$

The two-frequency solution is

$$\begin{pmatrix} \mathrm{g}_x + i\mathrm{g}_y \\ \mathrm{g}_x - i\mathrm{g}_y \end{pmatrix} = \begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_1^{\,*} \end{pmatrix}$$

$$\Rightarrow \begin{cases} \left(\partial_t + \dfrac{\omega_B}{\alpha - i}\right) \mathcal{G}_{1\mathrm{s}} = \dfrac{\gamma}{\alpha - i}\left(H_+ e^{-i\omega_{\mathrm{s}} t} + H_-^{\,*} e^{i\omega_{\mathrm{s}} t} \right) \\ \left(\partial_t + \dfrac{\omega_B}{\alpha - i}\right) \mathcal{G}_{1\mathrm{p}} = \dfrac{\gamma}{\alpha - i}\left(H_+ e^{-i\omega_{\mathrm{p}} t} + H_-^{\,*} e^{i\omega_{\mathrm{p}} t} \right) \end{cases}$$

$$\Rightarrow \begin{cases} \mathcal{G}_{1\mathrm{s}} = \dfrac{\gamma\left(h_{sx} + ih_{sy}\right)}{\omega_B - \omega_{\mathrm{s}} - i\alpha\omega_{\mathrm{s}}} e^{-i\omega_{\mathrm{s}} t} + \dfrac{\gamma\left(h_{sx} - ih_{sy}\right)^*}{\omega_B + \omega_{\mathrm{s}} + i\alpha\omega_{\mathrm{s}}} e^{i\omega_{\mathrm{s}} t} \\ \mathcal{G}_{1\mathrm{p}} = \dfrac{\gamma\left(h_{\mathrm{p}x} + ih_{\mathrm{p}y}\right)_+}{\omega_B - \omega_{\mathrm{p}} - i\alpha\omega_{\mathrm{p}}} e^{-i\omega_{\mathrm{p}} t} + \dfrac{\gamma\left(h_{\mathrm{p}x} - ih_{\mathrm{p}y}\right)^*}{\omega_B + \omega_{\mathrm{p}} + i\alpha\omega_{\mathrm{p}}} e^{i\omega_{\mathrm{p}} t} \end{cases} \tag{37}$$

4. Second-order Solution

The solutions to

$$(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_2 + \gamma\Lambda\mathcal{G}_2 = -\gamma \mathrm{h}_z \mathcal{G}_1 \tag{38}$$

require the first-order solutions in the forms

$$\begin{aligned} \mathcal{G}_{1a,b} &= \Gamma_{a,b+} H_{a,b+} e^{-i\omega_{a,b} t} + \Gamma_{a,b-}{}^{*} H_{a,b-}{}^{*} e^{+i\omega_{a,b} t} \\ \mathcal{G}_{1a,b}{}^{*} &= \Gamma_{a,b-} H_{a,b-} e^{-i\omega_{a,b} t} + \Gamma_{a,b+}{}^{*} H_{a,b+}{}^{*} e^{+i\omega_{a,b} t} \end{aligned} \tag{39}$$

which gives the following expressions for the source terms:

$$-\gamma \mathrm{h}_z \mathcal{G}_1 = \left\{ \mathrm{Re}\left(h_{az} e^{-i\omega_a t} \right) + \mathrm{Re}\left(h_{bz} e^{-i\omega_b t} \right) \right\} \cdot \left\{ \mathcal{G}_{1a} + \mathcal{G}_{1b} \right\}$$

$$= -\frac{\gamma}{2} \left\{ h_{az} e^{-i\omega_a t} + h_{bz} e^{-i\omega_b t} + h_{az}{}^{*} e^{i\omega_a t} + h_{bz}{}^{*} e^{i\omega_b t} \right\} \left\{ \begin{array}{l} \Gamma_{a+} H_{a+} e^{-i\omega_a t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} e^{+i\omega_a t} \\ + \Gamma_{b+} H_{b+} e^{-i\omega_b t} + \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_b t} \end{array} \right\}$$

$$= -\frac{\gamma}{2} \left[\begin{array}{l} h_{az} \Gamma_{a+} H_{a+} e^{-2i\omega_a t} + h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az} \Gamma_{b+} H_{b+} e^{-i\omega_{sum} t} + h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{-i\omega_{dif} t} \\ + h_{bz} \Gamma_{a+} H_{a+} e^{-i\omega_{sum} t} + h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} e^{+i\omega_{dif} t} + h_{bz} \Gamma_{b+} H_{b+} e^{-2i\omega_b t} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} \\ + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} e^{+2i\omega_a t} + h_{az}{}^{*} \Gamma_{b+} H_{b+} e^{+i\omega_{dif} t} + h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_{sum} t} \\ + h_{bz}{}^{*} \Gamma_{a+} H_{a+} e^{-i\omega_{dif} t} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} e^{+i\omega_{sum} t} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} + h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+2i\omega_b t} \end{array} \right] \tag{40}$$

$$= -\frac{\gamma}{2} \left[\begin{array}{l} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-2i\omega_a t} + \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+2i\omega_a t} \\ + h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \\ + \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_{sum} t} + \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_{sum} t} \\ + \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_{dif} t} + \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_{dif} t} \\ + h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+2i\omega_b t} + h_{bz} \Gamma_{b+} H_{b+} e^{-2i\omega_b t} \end{array} \right] \tag{41}$$

where $\omega_{sum} = \omega_a + \omega_b$ and $\omega_{dif} = \omega_a - \omega_b$ are the sum and difference frequencies. The basic equation for the second-order perturbation $\mathcal{G}_2$ is

$$(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_2 + \omega_B \mathcal{G}_2 = -\frac{\gamma}{2}\left\{\begin{array}{l}\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-2i\omega_a t} + \left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+2i\omega_a t} \\ + h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+} \\ + \left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{sum}t} + \left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_{sum}t} \\ + \left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{dif}t} + \left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_{dif}t} \\ + h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+2i\omega_b t} + h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\end{array}\right\}$$

$$\Rightarrow \mathcal{G}_2 = -\left[\begin{array}{l}\Gamma_{2a+}\Gamma_{a+}h_{az}H_{a+}e^{-2i\omega_a t} + \Gamma_{2a-}\Gamma_{a-}{}^{*}h_{az}{}^{*}H_{a-}{}^{*}e^{+2i\omega_a t} \\ + \Gamma_0\left(h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+}\right) \\ + \Gamma_{2b+}h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t} + \Gamma_{2b-}h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+2i\omega_b t} \\ + \Gamma_{sum+}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{sum}t} + \Gamma_{sum-}\left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_{sum}t} \\ + \Gamma_{dif+}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{dif}t} + \Gamma_{dif-}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_{dif}t}\end{array}\right] \quad (42)$$

where

$$\Gamma_0 = \frac{\gamma}{2}\frac{1}{\omega_B} \qquad \Gamma_{2a\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp 2\omega_a - i\alpha\left(2\omega_a\right)}$$

$$\Gamma_{2b\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp 2\omega_b - i\alpha\left(2\omega_b\right)} \qquad \Gamma_{s\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp \omega_{sum} - i\alpha\left(\omega_{sum}\right)} \quad (43)$$

$$\Gamma_{d\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp \omega_{dif} - i\alpha\left(\omega_{dif}\right)}$$

are new resonances associated with the various mixing products. For the *z*-component, we have

$$-\frac{1}{2}\left|\mathcal{G}_1\right|^2=-\frac{1}{2}\left|\mathcal{G}_{1a}+\mathcal{G}_{1b}\right|^2=-\frac{1}{2}\left\{\left|\mathcal{G}_{1a}\right|^2+\left|\mathcal{G}_{1b}\right|^2+2\,\mathrm{Re}\left(\mathcal{G}_{1a}{}^{*}\mathcal{G}_{1b}\right)\right\}$$

$$=-\frac{1}{2}\left\{\begin{array}{l}\left|\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\right|^2+\left|\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\right|^2\\ +2\,\mathrm{Re}\left(\left[\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\right]^{*}\left[\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\right]\right)\end{array}\right\}$$

$$=-\frac{1}{2}\left\{\begin{array}{l}\left|\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\right|^2+\left|\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\right|^2\\ +2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}e^{-i\omega_a t}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+i\omega_a t}\right]\left[\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\right]\right)\end{array}\right\}$$

$$=-\frac{1}{2}\left\{\begin{array}{l}\left|\Gamma_{a+}H_{a+}\right|^2+\left|\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right|^2+2\,\mathrm{Re}\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right)\\ +\left|\Gamma_{b+}H_{b+}\right|^2+\left|\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right|^2+2\,\mathrm{Re}\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right)\\ +2\,\mathrm{Re}\left(\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}e^{-i\omega_{sum}t}\right)+2\,\mathrm{Re}\left(\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\omega_{dif}t}\right)\\ +2\,\mathrm{Re}\left(\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_{dif}t}\right)+2\,\mathrm{Re}\left(\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_{sum}t}\right)\end{array}\right\}$$

$$\mathrm{Re}\left(\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_{sum}t}\right)=\mathrm{Re}\left(\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}e^{-i\omega_{sum}t}\right)$$

$$\mathrm{Re}\left(\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_{dif}t}\right)=\mathrm{Re}\left(\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{-i\omega_{dif}t}\right)$$

$$\Rightarrow \mathrm{g}_{z2}=-\frac{1}{2}\left\{\begin{array}{l}\left|\Gamma_{a+}H_{a+}\right|^2+\left|\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right|^2+2\,\mathrm{Re}\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right)\\ +\left|\Gamma_{b+}H_{b+}\right|^2+\left|\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right|^2+2\,\mathrm{Re}\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right)\\ +2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_{sum}t}\right)\\ +2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_{dif}t}\right)\end{array}\right\} \tag{44}$$

Two frequencies of special interest to remote sensing are the sum and difference frequencies. The magnetization amplitudes at these frequencies are given by

$$\mathcal{G}_{2sum} = -\left[\Gamma_{sum+}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{sum}t} + \Gamma_{sum-}\left(h_{az}\Gamma_{b-}H_{b-} + h_{bz}\Gamma_{a-}H_{a-}\right)^{*} e^{+i\omega_{sum}t}\right]$$
$$\mathrm{g}_{2sum,x} = \mathrm{Re}\,\mathcal{G}_{2sum} \qquad (45)$$
$$\mathrm{g}_{2sum,y} = \mathrm{Im}\,\mathcal{G}_{2sum}$$
$$\mathrm{g}_{2sum,z} = -\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_{sum}t}\right)$$

and

$$\mathcal{G}_{2dif} = -\left[\Gamma_{dif+}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{dif}t} + \Gamma_{dif-}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_{dif}t}\right]$$
$$\mathrm{g}_{2dif,x} = \mathrm{Re}\,\mathcal{G}_{2dif} \qquad (46)$$
$$\mathrm{g}_{2dif,y} = \mathrm{Im}\,\mathcal{G}_{2dif}$$
$$\mathrm{g}_{2dif,z} = -\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_{dif}t}\right)$$

---

5. Third-order Solution

The time-domain equation of the third-order system has a complicated source term:

$$\left(\alpha - i\right)\frac{\partial}{\partial t}\mathcal{G}_3 + \omega_B\mathcal{G}_3 = \frac{\gamma}{\alpha + i}\left\{\begin{array}{l} -\frac{1}{2}\left(3\alpha + i\right)\left|\mathcal{G}_1\right|^2\mathcal{H} + \alpha\frac{1}{2}\mathcal{G}_1{}^2\mathcal{H}^{*} \\ +\frac{1}{2}\left(\alpha\Lambda + \left[\alpha + i\right]\Theta\right)\left|\mathcal{G}_1\right|^2\mathcal{G}_1 - \left(\alpha + i\right)\mathrm{h}_z\mathcal{G}_2 \end{array}\right\} \qquad (47)$$
$$\square\ \frac{\gamma}{2}\left\{-\left(1 + \frac{2\alpha}{\alpha + i}\right)\mathbb{S}_1 + \frac{\alpha}{\alpha + i}\mathbb{S}_2 + \left(\frac{\alpha}{\alpha + i}\Lambda + \Theta\right)\mathbb{S}_3 - 2\mathbb{S}_4\right\}$$

To evaluate $\mathcal{G}_3$, we must first express the various products in multi-frequency form. This leads to the following linear inhomogeneous equation:

$$(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_3 + \omega_B \mathcal{G}_3 = \frac{\gamma}{2}\begin{pmatrix} S_{3,a+}e^{-i\omega_a t} + S_{3,a-}e^{+i\omega_a t} + S_{3,b+}e^{-i\omega_b t} + S_{3,b-}e^{+i\omega_b t} \\ +S_{3,3a+}e^{-3i\omega_a t} + S_{3,3a-}e^{+3i\omega_a t} + S_{3,3b+}e^{-3i\omega_b t} + S_{3,3b-}e^{+3i\omega_b t} \\ +S_{3,2b+a,+}e^{-i\left(2\omega_b+\omega_a\right)t} + S_{3,2b+a,-}e^{+i\left(2\omega_b+\omega_a\right)t} \\ +S_{3,2a+b,+}e^{-i\left(2\omega_a+\omega_b\right)t} + S_{3,2a+b,-}e^{+i\left(2\omega_a+\omega_b\right)t} \\ +S_{3,2b-a,+}e^{-i\left(2\omega_b-\omega_a\right)t} + S_{3,2b-a,-}e^{+i\left(2\omega_b-\omega_a\right)t} \\ +S_{3,2a-b,+}e^{-i\left(2\omega_a-\omega_b\right)t} + S_{3,2a-b,-}e^{+i\left(2\omega_a-\omega_b\right)t} \end{pmatrix} \quad (48)$$

which is easily solved:

$$\mathcal{G}_3 = \begin{pmatrix} \Gamma_{a+}S_{3,a+}e^{-i\omega_a t} + \Gamma_{a-}S_{3,a-}e^{+i\omega_a t} + \Gamma_{b+}S_{3,b+}e^{-i\omega_b t} + \Gamma_{b-}S_{3,b-}e^{+i\omega_b t} \\ +\Gamma_{3a+}S_{3,3a+}e^{-3i\omega_a t} + \Gamma_{3a-}S_{3,3a-}e^{+3i\omega_a t} + \Gamma_{3b+}S_{3,3b+}e^{-3i\omega_b t} + \Gamma_{3b-}S_{3,3b-}e^{+3i\omega_b t} \\ +\Gamma_{2b+a+}S_{3,2b+a,+}e^{-i\left(2\omega_b+\omega_a\right)t} + \Gamma_{2b+a,-}S_{3,2b+a,-}e^{+i\left(2\omega_b+\omega_a\right)t} \\ +\Gamma_{2a+b+}S_{3,2a+b,+}e^{-i\left(2\omega_a+\omega_b\right)t} + \Gamma_{2a+b,-}S_{3,2a+b,-}e^{+i\left(2\omega_a+\omega_b\right)t} \\ +\Gamma_{2b-a+}S_{3,2b-a,+}e^{-i\left(2\omega_b-\omega_a\right)t} + \Gamma_{2b-a,-}S_{3,2b-a,-}e^{+i\left(2\omega_b-\omega_a\right)t} \\ +\Gamma_{2amb+}S_{3,2a-b,+}e^{-i\left(2\omega_a-\omega_b\right)t} + \Gamma_{2a-b,-}S_{3,2a-b,-}e^{+i\left(2\omega_a-\omega_b\right)t} \end{pmatrix} \quad (49)$$

where

$$
\begin{aligned}
\Gamma_{a\pm} &= \frac{\gamma}{2}\frac{1}{\omega_B \mp \omega_a - i\alpha\omega_a} & \Gamma_{b\pm} &= \frac{\gamma}{2}\frac{1}{\omega_B \mp \omega_b - i\alpha\omega_b} \\
\Gamma_{3a\pm} &= \frac{\gamma}{2}\frac{1}{\omega_B \mp 3\omega_a - i\alpha(3\omega_a)} & \Gamma_{3b\pm} &= \frac{\gamma}{2}\frac{1}{\omega_B \mp 3\omega_b - i\alpha(3\omega_b)} \\
\Gamma_{2b+a,\pm} &= \frac{\gamma}{2}\frac{1}{\omega_B \mp (2\omega_b + \omega_a) - i\alpha(2\omega_b + \omega_a)} & \Gamma_{2b-a,\pm} &= \frac{\gamma}{2}\frac{1}{\omega_B \mp (2\omega_b - \omega_a) - i\alpha(2\omega_b - \omega_a)} \\
\Gamma_{2a+b,\pm} &= \frac{\gamma}{2}\frac{1}{\omega_B \mp (2\omega_a + \omega_b) - i\alpha(2\omega_a + \omega_b)} & \Gamma_{2a-b,\pm} &= \frac{\gamma}{2}\frac{1}{\omega_B \mp (2\omega_a - \omega_b) - i\alpha(2\omega_a - \omega_b)}
\end{aligned}
\tag{50}
$$

Note that the third-order source terms give rise to three distinct effects: frequency tripling, amplitude-dependence of the response functions at the drive frequencies, and intermodulation products.

6. Grouping Third-order Magnetization Terms

The only terms of interest here are the intermod terms, which are color-coded to make the algebra easier to follow.

$$
\begin{aligned}
\mathbb{S}_1 &= \left|\mathcal{G}_1\right|^2 \mathcal{H} \\
&= \left\{\begin{array}{l}
\left|\Gamma_{a+}H_{a+}\right|^2 + \left|\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right|^2 + 2\,\mathrm{Re}\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right) \\
+\left|\Gamma_{b+}H_{b+}\right|^2 + \left|\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right|^2 + 2\,\mathrm{Re}\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right) \\
+2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_s t}\right) \\
+2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_d t}\right)
\end{array}\right\} \\
&\quad \left\{\frac{1}{2}\left\{\begin{array}{l}
H_{a+}e^{-i\omega_a t} + H_{a-}{}^{*}e^{+i\omega_a t} \\
+H_{b+}e^{-i\omega_b t} + H_{b-}{}^{*}e^{+i\omega_b t}
\end{array}\right\}\right.
\end{aligned}
\tag{51}
$$

$$
\square\ \frac{1}{2}\sum_{m=1}^{5} \mathrm{X}_m
$$

$$
\mathrm{X}_1 = \left\{\left|\Gamma_{a+}H_{a+}\right|^2 + \left|\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right|^2 + \left|\Gamma_{b+}H_{b+}\right|^2 + \left|\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right|^2\right\}\left\{H_{a+}e^{-i\omega_a t} + H_{a-}{}^{*}e^{+i\omega_a t} + H_{b+}e^{-i\omega_b t} + H_{b-}{}^{*}e^{+i\omega_b t}\right\} \tag{52}
$$

$$X_2 = 2\,\mathrm{Re}\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right)\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$=\left\{\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\right\}\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$=\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}H_{a-}{}^{*}e^{+i\omega_a t}$$

$$+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}H_{b+}e^{-i\omega_b t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}H_{b-}{}^{*}e^{+i\omega_b t}$$

$$+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}H_{a-}{}^{*}e^{+i\omega_a t}$$

$$+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}H_{b+}e^{-i\omega_b t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}H_{b-}{}^{*}e^{+i\omega_b t}$$

$$=\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{a+}e^{-3i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{a-}{}^{*}e^{-i\omega_a t}$$

$$+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{b+}e^{-i\left(2\omega_a+\omega_b\right)t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{b-}{}^{*}e^{-i\left(2\omega_a-\omega_b\right)t}$$

$$+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{a+}e^{+i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{a-}{}^{*}e^{+3i\omega_a t}$$

$$+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{b+}e^{+i\left(2\omega_a-\omega_b\right)t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t} \qquad (53)$$

$$X_3 = 2\,\mathrm{Re}\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right)\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$=\left\{\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\right\}\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$=\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}H_{a+}e^{-i\omega_a t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t}$$

$$+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}$$

$$+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}H_{a+}e^{-i\omega_a t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t}$$

$$+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}$$

$$=\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{a+}e^{-i\left(2\omega_b+\omega_a\right)t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{a-}{}^{*}e^{-i\left(2\omega_b-\omega_a\right)t}$$

$$+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{b+}e^{-3i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{b-}{}^{*}e^{-i\omega_b t}$$

$$+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{a+}e^{+i\left(2\omega_b-\omega_a\right)t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}$$

$$+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{b+}e^{+i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{b-}{}^{*}e^{+3i\omega_b t} \qquad (54)$$

$$
\begin{aligned}
X_4 &= 2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_s t}\right)\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}\\
&=\left\{\begin{array}{l}\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\\ +\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\end{array}\right\}\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}\\
&=\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}H_{a+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}H_{b+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}H_{a+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}H_{b+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
&=\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{a+}e^{-i\left(2\omega_a+\omega_b\right)t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{a-}{}^{*}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{b+}e^{-i\left(2\omega_b+\omega_a\right)t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{b-}{}^{*}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{a+}e^{+i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{a-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{b+}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{b-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}
\end{aligned}
\tag{56}
$$

$$X_5 = 2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_d t}\right)\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$=\left\{\begin{array}{l}\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\\ +\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\end{array}\right\}\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$\begin{aligned}
&=\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}H_{a+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}H_{b+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}H_{a+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}H_{b+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}
\end{aligned}\qquad (57)$$

$$\begin{aligned}
&=\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{a+}e^{-i\left(2\omega_a-\omega_b\right)t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{a-}{}^{*}e^{+i\omega_b t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{b+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{b-}{}^{*}e^{+i\left(2\omega_b-\omega_a\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{a+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{a-}{}^{*}e^{+i\left(2\omega_a-\omega_b\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{b+}e^{-i\left(2\omega_b-\omega_a\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{b-}{}^{*}e^{+i\omega_a t}
\end{aligned}\qquad (58)$$

$$\mathbb{S}_2 = \mathcal{G}_1{}^2\mathcal{H}^* = \begin{Bmatrix} \Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^*H_{a-}{}^*e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_b t} \end{Bmatrix}^2 \frac{1}{2}\begin{Bmatrix} H_{a+}e^{-i\omega_a t} + H_{a-}{}^*e^{+i\omega_a t} \\ +H_{b+}e^{-i\omega_b t} + H_{b-}{}^*e^{+i\omega_b t} \end{Bmatrix}^*$$

$$= \frac{1}{2}\begin{Bmatrix} \Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^*H_{a-}{}^*e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_b t} \end{Bmatrix}^2 \begin{Bmatrix} H_{a-}e^{-i\omega_a t} + H_{a+}{}^*e^{+i\omega_a t} \\ +H_{b-}e^{-i\omega_b t} + H_{b+}{}^*e^{+i\omega_b t} \end{Bmatrix}$$

$$= \frac{1}{2}\begin{Bmatrix} \Gamma_{a+}{}^2H_{a+}{}^2e^{-2i\omega_a t} + 2\Gamma_{a+}H_{a+}\Gamma_{a-}{}^*H_{a-}{}^* + \Gamma_{a-}{}^{*2}H_{a-}{}^{*2}e^{+2i\omega_a t} \\ +2\left(\Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^*H_{a-}{}^*e^{+i\omega_a t}\right)\left(\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_b t}\right) \\ +\Gamma_{b+}{}^2H_{b+}{}^2e^{-2i\omega_b t} + 2\Gamma_{b-}{}^*H_{b-}{}^*\Gamma_{b+}H_{b+} + \Gamma_{b-}{}^{*2}H_{b-}{}^{*2}e^{+2i\omega_b t} \end{Bmatrix} \begin{Bmatrix} H_{a-}e^{-i\omega_a t} + H_{a+}{}^*e^{+i\omega_a t} \\ +H_{b-}e^{-i\omega_b t} + H_{b+}{}^*e^{+i\omega_b t} \end{Bmatrix}$$

$$= \frac{1}{2}\begin{Bmatrix} \Gamma_{a+}{}^2H_{a+}{}^2e^{-2i\omega_a t} + 2\Gamma_{a+}H_{a+}\Gamma_{a-}{}^*H_{a-}{}^* + \Gamma_{a-}{}^{*2}H_{a-}{}^{*2}e^{+2i\omega_a t} \\ +2\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}e^{-i\omega_s t} + 2\Gamma_{a+}H_{a+}\Gamma_{b-}{}^*H_{b-}{}^*e^{-i\omega_d t} \\ +2\Gamma_{a-}{}^*H_{a-}{}^*\Gamma_{b+}H_{b+}e^{+i\omega_d t} + 2\Gamma_{a-}{}^*H_{a-}{}^*\Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_s t} \\ +\Gamma_{b+}{}^2H_{b+}{}^2e^{-2i\omega_b t} + 2\Gamma_{b-}{}^*H_{b-}{}^*\Gamma_{b+}H_{b+} + \Gamma_{b-}{}^{*2}H_{b-}{}^{*2}e^{+2i\omega_b t} \end{Bmatrix} \begin{Bmatrix} H_{a-}e^{-i\omega_a t} + H_{a+}{}^*e^{+i\omega_a t} \\ +H_{b-}e^{-i\omega_b t} + H_{b+}{}^*e^{+i\omega_b t} \end{Bmatrix} \quad (59)$$

$$= \frac{1}{2}\begin{Bmatrix} \Gamma_{a+}{}^2H_{a+}{}^2e^{-2i\omega_a t} + 2\Gamma_{a+}H_{a+}\Gamma_{a-}{}^*H_{a-}{}^* + \Gamma_{a-}{}^{*2}H_{a-}{}^{*2}e^{+2i\omega_a t} \\ +2\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}e^{-i\omega_s t} + 2\Gamma_{a+}H_{a+}\Gamma_{b-}{}^*H_{b-}{}^*e^{-i\omega_d t} \\ +2\Gamma_{a-}{}^*H_{a-}{}^*\Gamma_{b+}H_{b+}e^{+i\omega_d t} + 2\Gamma_{a-}{}^*H_{a-}{}^*\Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_s t} \\ +\Gamma_{b+}{}^2H_{b+}{}^2e^{-2i\omega_b t} + 2\Gamma_{b-}{}^*H_{b-}{}^*\Gamma_{b+}H_{b+} + \Gamma_{b-}{}^{*2}H_{b-}{}^{*2}e^{+2i\omega_b t} \end{Bmatrix} \begin{Bmatrix} H_{a-}e^{-i\omega_a t} + H_{a+}{}^*e^{+i\omega_a t} \\ +H_{b-}e^{-i\omega_b t} + H_{b+}{}^*e^{+i\omega_b t} \end{Bmatrix} \quad (60)$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{
\begin{array}{l}
\Gamma_{a+}{}^{2} H_{a+}{}^{2} e^{-2i\omega_a t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right] \\
+\left(2\Gamma_{a+} H_{a+} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + 2\Gamma_{b-}{}^{*} H_{b-}{}^{*} \Gamma_{b+} H_{b+} \right)\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right] \\
+\Gamma_{a-}{}^{*2} H_{a-}{}^{*2} e^{+2i\omega_a t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right] \\
+2\Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_s t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right] \\
+2\Gamma_{a+} H_{a+} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{-i\omega_d t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right] \\
+2\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} e^{+i\omega_d t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right] \\
+2\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_s t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right] \\
+\Gamma_{b+}{}^{2} H_{b+}{}^{2} e^{-2i\omega_b t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right] \\
+\Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right]
\end{array}
\right\} \qquad (61) \\
&= \frac{1}{2} \sum_{m=1}^{9} \Upsilon_m
\end{aligned}
$$

$$
\Upsilon_1 = \left(2\Gamma_{a+} H_{a+} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + 2\Gamma_{b-}{}^{*} H_{b-}{}^{*} \Gamma_{b+} H_{b+} \right)\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right] \qquad (62)
$$

$$\Upsilon_2 = \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$

$$= \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} H_{a-} e^{-i\omega_a t} + \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} H_{a+}{}^* e^{+i\omega_a t}$$

$$+\Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} H_{b-} e^{-i\omega_b t} + \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} H_{b+}{}^* e^{+i\omega_b t}$$

$$= \Gamma_{a+}{}^2 H_{a+}{}^2 H_{a-} e^{-3i\omega_a t} + \Gamma_{a+}{}^2 H_{a+}{}^2 H_{a+}{}^* e^{-i\omega_a t}$$

$$+\Gamma_{a+}{}^2 H_{a+}{}^2 H_{b-} e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{a+}{}^2 H_{a+}{}^2 H_{b+}{}^* e^{-i\left(2\omega_a - \omega_b\right)t}$$

$$\Upsilon_3 = \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} e^{+2i\omega_a t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$

$$= \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{a-} e^{+i\omega_a t} + \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{a+}{}^* e^{+3i\omega_a t}$$

$$+\Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{b-} e^{+i\left(2\omega_a - \omega_b\right)t} + \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{b+}{}^* e^{+i\left(2\omega_a + \omega_b\right)t} \qquad (63)$$

$$\Upsilon_4 = \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_s t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$

$$= \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_a t} e^{-i\omega_b t} H_{a-} e^{-i\omega_a t} + \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_a t} e^{-i\omega_b t} H_{a+}{}^* e^{+i\omega_a t}$$

$$+\Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_a t} e^{-i\omega_b t} H_{b-} e^{-i\omega_b t} + \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_a t} e^{-i\omega_b t} H_{b+}{}^* e^{+i\omega_b t}$$

$$= \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{a-} e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{a+}{}^* e^{-i\omega_b t}$$

$$+\Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{b-} e^{-i\left(2\omega_b + \omega_a\right)t} + \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{b+}{}^* e^{-i\omega_a t}$$

$$\Upsilon_5 = \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_d t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$

$$= \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_a t} e^{+i\omega_b t} H_{a-} e^{-i\omega_a t} + \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_a t} e^{+i\omega_b t} H_{a+}{}^* e^{+i\omega_a t}$$

$$+\Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_a t} e^{+i\omega_b t} H_{b-} e^{-i\omega_b t} + \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_a t} e^{+i\omega_b t} H_{b+}{}^* e^{+i\omega_b t}$$

$$= \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* H_{a-} e^{-i\left(2\omega_a - \omega_b\right)t} + \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* H_{a+}{}^* e^{+i\omega_b t} \qquad (64)$$

$$+\Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* H_{b-} e^{-i\omega_a t} + \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* H_{b+}{}^* e^{+i\left(2\omega_b - \omega_a\right)t}$$

$$\Upsilon_6 = \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} e^{+i\omega_d t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right]$$

$$= \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} e^{+i\omega_a t} e^{-i\omega_b t} H_{a-} e^{-i\omega_a t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} e^{+i\omega_a t} e^{-i\omega_b t} H_{a+}{}^{*} e^{+i\omega_a t}$$

$$+ \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} e^{+i\omega_a t} e^{-i\omega_b t} H_{b-} e^{-i\omega_b t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} e^{+i\omega_a t} e^{-i\omega_b t} H_{b+}{}^{*} e^{+i\omega_b t}$$

$$= \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} H_{a-} e^{-i\omega_b t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} H_{a+}{}^{*} e^{+i\left(2\omega_a - \omega_b\right)t}$$

$$+ \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} H_{b-} e^{-i\left(2\omega_b - \omega_a\right)t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} H_{b+}{}^{*} e^{+i\omega_a t}$$

$$\Upsilon_7 = \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_s t}\left[H_{a-} e^{-i\omega_a t} + H_{a+}{}^{*} e^{+i\omega_a t} + H_{b-} e^{-i\omega_b t} + H_{b+}{}^{*} e^{+i\omega_b t} \right]$$

$$= \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_a t} e^{+i\omega_b t} H_{a-} e^{-i\omega_a t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_a t} e^{+i\omega_b t} H_{a+}{}^{*} e^{+i\omega_a t}$$

$$+ \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_a t} e^{+i\omega_b t} H_{b-} e^{-i\omega_b t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_a t} e^{+i\omega_b t} H_{b+}{}^{*} e^{+i\omega_b t}$$

$$= \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} H_{a-} e^{+i\omega_b t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} H_{a+}{}^{*} e^{+i\left(2\omega_a + \omega_b\right)t} \qquad (65)$$

$$+ \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} H_{b-} e^{+i\omega_a t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} H_{b+}{}^{*} e^{+i\left(2\omega_b + \omega_a\right)t}$$

$$\Upsilon_8 = \Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$

$$= \Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} H_{a-} e^{-i\omega_a t} + \Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} H_{a+}{}^* e^{+i\omega_a t}$$

$$+\Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} H_{b-} e^{-i\omega_b t} + \Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} H_{b+}{}^* e^{+i\omega_b t}$$

$$= \Gamma_{b+}{}^2 H_{b+}{}^2 H_{a-} e^{-i\left(2\omega_b + \omega_a\right)t} + \Gamma_{b+}{}^2 H_{b+}{}^2 H_{a+}{}^* e^{-i\left(2\omega_b - \omega_a\right)t}$$

$$+\Gamma_{b+}{}^2 H_{b+}{}^2 H_{b-} e^{-3i\omega_b t} + \Gamma_{b+}{}^2 H_{b+}{}^2 H_{b+}{}^* e^{-i\omega_b t}$$

$$\Upsilon_9 = \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$

$$= \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} H_{a-} e^{-i\omega_a t} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} H_{a+}{}^* e^{+i\omega_a t}$$

$$+\Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} H_{b-} e^{-i\omega_b t} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} H_{b+}{}^* e^{+i\omega_b t}$$

$$= \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{a-}{}^{+i\left(2\omega_b - \omega_a\right)t} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{a+}{}^* e^{+i\left(2\omega_b + \omega_a\right)t} \qquad (66)$$

$$+\Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{b-} e^{+i\omega_b t} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{b+}{}^* e^{+3i\omega_b t}$$

$$\mathbb{S}_3 = \left|\mathcal{G}_1\right|^2 \mathcal{G}$$

$$= \begin{Bmatrix} \left|\Gamma_{a+}H_{a+}\right|^2 + \left|\Gamma_{a-}{}^* H_{a-}{}^*\right|^2 + 2\,\mathrm{Re}\left(\Gamma_{a-}{}^* H_{a-}{}^* \Gamma_{a+} H_{a+} e^{-2i\omega_a t}\right) \\ +\left|\Gamma_{b+}H_{b+}\right|^2 + \left|\Gamma_{b-}{}^* H_{b-}{}^*\right|^2 + 2\,\mathrm{Re}\left(\Gamma_{b-}{}^* H_{b-}{}^* \Gamma_{b+} H_{b+} e^{-2i\omega_b t}\right) \\ +2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_S t}\right) \\ +2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^* H_{b-}{}^* + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^* H_{b+}{}^*\right]e^{-i\omega_d t}\right) \end{Bmatrix} \begin{Bmatrix} \Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^* H_{a-}{}^* e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^* H_{b-}{}^* e^{+i\omega_b t} \end{Bmatrix} \qquad (67)$$

$$\square \sum_{m=1}^{5} \mathrm{M}_m$$

$$\mathrm{M}_1 = \left\{\left|\Gamma_{a+}H_{a+}\right|^2 + \left|\Gamma_{a-}{}^* H_{a-}{}^*\right|^2 + \left|\Gamma_{b+}H_{b+}\right|^2 + \left|\Gamma_{b-}{}^* H_{b-}{}^*\right|^2\right\} \begin{Bmatrix} \Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^* H_{a-}{}^* e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^* H_{b-}{}^* e^{+i\omega_b t} \end{Bmatrix} \qquad (68)$$

$$
\begin{aligned}
M_2 &= 2\,\mathrm{Re}\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right)\left\{\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}+\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\right\}\\
&=\left\{\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\right\}\left\{\begin{matrix}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{matrix}\right\}\\
&=\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&=\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{a+}H_{a+}e^{-3i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{-i\omega_a t}\\
&+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}e^{-i\left(2\omega_a+\omega_b\right)t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\left(2\omega_a-\omega_b\right)t}\\
&+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{a+}H_{a+}e^{+i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+3i\omega_a t}\\
&+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}e^{+i\left(2\omega_a-\omega_b\right)t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
M_3 &= 2\,\mathrm{Re}\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right)\left\{\begin{matrix}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{matrix}\right\}\\
&=\left\{\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\right\}\left\{\begin{matrix}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{matrix}\right\}\\
&=\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&=\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{a+}H_{a+}e^{-i\left(2\omega_b+\omega_a\right)t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{-i\left(2\omega_b-\omega_a\right)t}\\
&+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{b+}H_{b+}e^{-3i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\omega_b t}\\
&+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{a+}H_{a+}e^{+i\left(2\omega_b-\omega_a\right)t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}\\
&+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+3i\omega_b t}
\end{aligned}
\tag{70}
$$

$$M_4 = 2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_s t}\right)\begin{Bmatrix}\Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{Bmatrix}$$

$$= \begin{Bmatrix}\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\\ +\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\end{Bmatrix}\begin{Bmatrix}\Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{Bmatrix}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}
\end{aligned} \tag{71}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{a+}H_{a+}e^{-i\left(2\omega_a+\omega_b\right)t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b+\omega_a\right)t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{a+}H_{a+}e^{+i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{b+}H_{b+}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}
\end{aligned} \tag{72}$$

$$M_5 = 2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_d t}\right)\begin{Bmatrix}\Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{Bmatrix}$$

$$= \begin{Bmatrix}\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t} \\ +\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\end{Bmatrix}\begin{Bmatrix}\Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{Bmatrix}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}
\end{aligned} \tag{73}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{a+}H_{a+}e^{-i\left(2\omega_a - \omega_b\right)t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_b t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{b+}H_{b+}e^{-i\omega_a t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b - \omega_a\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{a+}H_{a+}e^{-i\omega_b t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a - \omega_b\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b - \omega_a\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_a t}
\end{aligned} \tag{74}$$

and

$$\mathbb{S}_4 = \mathrm{h}_z \mathcal{G}_2 = \mathrm{Re}\left\{ h_{az} e^{-i\omega_a t} + h_{bz} e^{-i\omega_b t} \right\} \cdot \mathcal{G}_2$$

$$= \frac{1}{2} \begin{pmatrix} h_{az} e^{-i\omega_a t} + h_{bz} e^{-i\omega_b t} \\ + h_{az}^{*} e^{i\omega_a t} + h_{bz}^{*} e^{i\omega_b t} \end{pmatrix} \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_{0} S_{0} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix}$$

$$= \frac{1}{2} \left(\begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_{0} S_{0} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix} h_{az} e^{-i\omega_a t} \right.$$
$$+ \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_{0} S_{0} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix} h_{bz} e^{-i\omega_b t}$$
$$+ \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_{0} S_{0} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix} h_{az}^{*} e^{i\omega_a t}$$
$$\left. + \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_{0} S_{0} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix} h_{bz}^{*} e^{i\omega_b t} \right) \tag{75}$$

$$\frac{1}{2} \left(\begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} h_{az} e^{-i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} h_{az} e^{-i\omega_a t} + \Gamma_{0} S_{0} h_{az} e^{-i\omega_a t} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} h_{az} e^{-i\omega_a t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} h_{az} e^{-i\omega_a t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} h_{az} e^{-i\omega_a t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} h_{az} e^{-i\omega_a t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} h_{az} e^{-i\omega_a t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} h_{az} e^{-i\omega_a t} \end{bmatrix} \right.$$
$$+ \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} h_{bz} e^{-i\omega_b t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} h_{bz} e^{-i\omega_b t} + \Gamma_{0} S_{0} h_{bz} e^{-i\omega_b t} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} h_{bz} e^{-i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} h_{bz} e^{-i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} h_{bz} e^{-i\omega_b t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} h_{bz} e^{-i\omega_b t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} h_{bz} e^{-i\omega_b t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} h_{bz} e^{-i\omega_b t} \end{bmatrix}$$
$$+ \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} h_{az}^{*} e^{i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} h_{az}^{*} e^{i\omega_a t} + \Gamma_{0} S_{0} h_{az}^{*} e^{i\omega_a t} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} h_{az}^{*} e^{i\omega_a t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} h_{az}^{*} e^{i\omega_a t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} h_{az}^{*} e^{i\omega_a t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} h_{az}^{*} e^{i\omega_a t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} h_{az}^{*} e^{i\omega_a t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} h_{az}^{*} e^{i\omega_a t} \end{bmatrix}$$
$$\left. + \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} h_{bz}^{*} e^{i\omega_b t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} h_{bz}^{*} e^{i\omega_b t} + \Gamma_{0} S_{0} h_{bz}^{*} e^{i\omega_b t} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} h_{bz}^{*} e^{i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} h_{bz}^{*} e^{i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} h_{bz}^{*} e^{i\omega_b t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} h_{bz}^{*} e^{i\omega_b t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} h_{bz}^{*} e^{i\omega_b t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} h_{bz}^{*} e^{i\omega_b t} \end{bmatrix} \right) \tag{76}$$

$$
\begin{aligned}
&= \frac{1}{2}\Bigg(\Big[\Gamma_{2a+} S_{2a+} h_{az} e^{-3i\omega_a t} + \Gamma_{2a-} S_{2a-} h_{az} e^{+i\omega_a t} + \Gamma_{0} S_{0} h_{az} e^{-i\omega_a t} + \Gamma_{2b+} S_{2b+} h_{az} e^{-i(2\omega_b+\omega_a)t} + \Gamma_{2b-} S_{2b-} h_{az} e^{+i(2\omega_b-\omega_a)t} \\
&\quad + \Gamma_{s+} S_{s+} h_{az} e^{-i(2\omega_a+\omega_b)t} + \Gamma_{s-} S_{s-} h_{az} e^{+i\omega_b t} + \Gamma_{d+} S_{d+} h_{az} e^{-i(2\omega_a-\omega_b)t} + \Gamma_{d-} S_{d-} h_{az} e^{-i\omega_b t} \Big] \\
&\quad + \Big[\Gamma_{2a+} S_{2a+} h_{bz} e^{-i(2\omega_a+\omega_b)t} + \Gamma_{2a-} S_{2a-} h_{bz} e^{+i(2\omega_a-\omega_b)t} + \Gamma_{0} S_{0} h_{bz} e^{-i\omega_b t} + \Gamma_{2b+} S_{2b+} h_{bz} e^{-3i\omega_b t} + \Gamma_{2b-} S_{2b-} h_{bz} e^{+i\omega_b t} \\
&\quad + \Gamma_{s+} S_{s+} h_{bz} e^{-i(2\omega_b+\omega_a)t} + \Gamma_{s-} S_{s-} h_{bz} e^{+i\omega_a t} + \Gamma_{d+} S_{d+} h_{bz} e^{-i\omega_a t} + \Gamma_{d-} S_{d-} h_{bz} e^{-i(2\omega_b-\omega_a)t} \Big] \\
&\quad + \Big[\Gamma_{2a+} S_{2a+} h_{az}^{*} e^{-i\omega_a t} + \Gamma_{2a-} S_{2a-} h_{az}^{*} e^{3i\omega_a t} + \Gamma_{0} S_{0} h_{az}^{*} e^{i\omega_a t} + \Gamma_{2b+} S_{2b+} h_{az}^{*} e^{-i(2\omega_b-\omega_a)t} + \Gamma_{2b-} S_{2b-} h_{az}^{*} e^{+i(2\omega_b+\omega_a)t} \\
&\quad + \Gamma_{s+} S_{s+} h_{az}^{*} e^{-i\omega_b t} + \Gamma_{s-} S_{s-} h_{az}^{*} e^{+i(2\omega_a+\omega_b)t} + \Gamma_{d+} S_{d+} h_{az}^{*} e^{i\omega_b t} + \Gamma_{d-} S_{d-} h_{az}^{*} e^{+i(2\omega_a-\omega_b)t} \Big] \\
&\quad + \Big[\Gamma_{2a+} S_{2a+} h_{bz}^{*} e^{-i(2\omega_a-\omega_b)t} + \Gamma_{2a-} S_{2a-} h_{bz}^{*} e^{+i(2\omega_a+\omega_b)t} + \Gamma_{0} S_{0} h_{bz}^{*} e^{i\omega_b t} + \Gamma_{2b+} S_{2b+} h_{bz}^{*} e^{-i\omega_b t} + \Gamma_{2b-} S_{2b-} h_{bz}^{*} e^{3i\omega_b t} \\
&\quad + \Gamma_{s+} S_{s+} h_{bz}^{*} e^{-i\omega_a t} + \Gamma_{s-} S_{s-} h_{bz}^{*} e^{+i(2\omega_b+\omega_a)t} + \Gamma_{d+} S_{d+} h_{bz}^{*} e^{+i(2\omega_b-\omega_a)t} + \Gamma_{d-} S_{d-} h_{bz}^{*} e^{+i\omega_a t} \Big] \Bigg)
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
&= \frac{1}{2}\Bigg(\Gamma_{2a+} S_{2a+} h_{az} e^{-3i\omega_a t} + \Gamma_{2b+} S_{2b+} h_{bz} e^{-3i\omega_b t} + \Gamma_{2a-} S_{2a-} h_{az}^{*} e^{3i\omega_a t} + \Gamma_{2b-} S_{2b-} h_{bz}^{*} e^{3i\omega_b t} \\
&\quad + \Big[\Gamma_{2a-} S_{2a-} h_{az} + \Gamma_{s-} S_{s-} h_{bz} + \Gamma_{d-} S_{d-} h_{bz}^{*} + \Gamma_{0} S_{0} h_{az}^{*} \Big] e^{+i\omega_a t} \\
&\quad + \Big[\Gamma_{0} S_{0} h_{az} + \Gamma_{d+} S_{d+} h_{bz} + \Gamma_{s+} S_{s+} h_{bz}^{*} + \Gamma_{2a+} S_{2a+} h_{az}^{*} \Big] e^{-i\omega_a t} \\
&\quad + \Big[\Gamma_{d+} S_{d+} h_{az}^{*} + \Gamma_{0} S_{0} h_{bz}^{*} + \Gamma_{2b-} S_{2b-} h_{bz} + \Gamma_{s-} S_{s-} h_{az} \Big] e^{+i\omega_b t} \\
&\quad + \Big[\Gamma_{d-} S_{d-} h_{az} + \Gamma_{0} S_{0} h_{bz} + \Gamma_{2b+} S_{2b+} h_{bz}^{*} + \Gamma_{s+} S_{s+} h_{az}^{*} \Big] e^{-i\omega_b t} \\
&\quad + \Big[\Gamma_{2b+} S_{2b+} h_{az} + \Gamma_{s+} S_{s+} h_{bz} \Big] e^{-i(2\omega_b+\omega_a)t} + \Big[\Gamma_{2b-} S_{2b-} h_{az}^{*} + \Gamma_{s-} S_{s-} h_{bz}^{*} \Big] e^{+i(2\omega_b+\omega_a)t} \\
&\quad + \Big[\Gamma_{s+} S_{s+} h_{az} + \Gamma_{2a+} S_{2a+} h_{bz} \Big] e^{-i(2\omega_a+\omega_b)t} + \Big[\Gamma_{s-} S_{s-} h_{az}^{*} + \Gamma_{2a-} S_{2a-} h_{bz}^{*} \Big] e^{+i(2\omega_a+\omega_b)t} \\
&\quad + \Big[\Gamma_{d+} S_{d+} h_{az} + \Gamma_{2a+} S_{2a+} h_{bz}^{*} \Big] e^{-i(2\omega_a-\omega_b)t} + \Big[\Gamma_{2a-} S_{2a-} h_{bz} + \Gamma_{d-} S_{d-} h_{az}^{*} \Big] e^{+i(2\omega_a-\omega_b)t} \\
&\quad + \Big[\Gamma_{d-} S_{d-} h_{bz} + \Gamma_{2b+} S_{2b+} h_{az}^{*} \Big] e^{-i(2\omega_b-\omega_a)t} + \Big[\Gamma_{2b-} S_{2b-} h_{az} + \Gamma_{d+} S_{d+} h_{bz}^{*} \Big] e^{+i(2\omega_b-\omega_a)t} \Bigg)
\end{aligned} \tag{78}
$$

$$
\begin{aligned}
\mathbb{S}_1\big|_{\mathrm{IM}} = \frac{1}{2}\Big\{ & \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{a+} H_{a+} H_{b+} e^{-i\left(2\omega_a+\omega_b\right)t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{a+} H_{a+} H_{b-}{}^{*} e^{-i\left(2\omega_a-\omega_b\right)t} \\
& + \Gamma_{a-} H_{a-} \Gamma_{a+}{}^{*} H_{a+}{}^{*} H_{b+} e^{+i\left(2\omega_a-\omega_b\right)t} + \Gamma_{a-} H_{a-} \Gamma_{a+}{}^{*} H_{a+}{}^{*} H_{b-}{}^{*} e^{+i\left(2\omega_a+\omega_b\right)t} \\
& + \Gamma_{b-}{}^{*} H_{b-}{}^{*} \Gamma_{b+} H_{b+} H_{a+} e^{-i\left(2\omega_b+\omega_a\right)t} + \Gamma_{b-}{}^{*} H_{b-}{}^{*} \Gamma_{b+} H_{b+} H_{a-}{}^{*} e^{-i\left(2\omega_b-\omega_a\right)t} \\
& + \Gamma_{b-} H_{b-} \Gamma_{b+}{}^{*} H_{b+}{}^{*} H_{a+} e^{+i\left(2\omega_b-\omega_a\right)t} + \Gamma_{b-} H_{b-} \Gamma_{b+}{}^{*} H_{b+}{}^{*} H_{a-}{}^{*} e^{+i\left(2\omega_b+\omega_a\right)t} \\
& + \left[\Gamma_{a-} H_{a-} \Gamma_{b+} H_{b+} + \Gamma_{a+} H_{a+} \Gamma_{b-} H_{b-}\right] H_{a+} e^{-i\left(2\omega_a+\omega_b\right)t} \\
& + \left[\Gamma_{a-} H_{a-} \Gamma_{b+} H_{b+} + \Gamma_{a+} H_{a+} \Gamma_{b-} H_{b-}\right] H_{b+} e^{-i\left(2\omega_b+\omega_a\right)t} \\
& + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+}{}^{*} H_{b+}{}^{*} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*}\right] H_{a-}{}^{*} e^{+i\left(2\omega_a+\omega_b\right)t} \\
& + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+}{}^{*} H_{b+}{}^{*} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*}\right] H_{b-}{}^{*} e^{+i\left(2\omega_b+\omega_a\right)t} \\
& + \left[\Gamma_{a-} H_{a-} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{a+} H_{a+} \Gamma_{b+}{}^{*} H_{b+}{}^{*}\right] H_{a+} e^{-i\left(2\omega_a-\omega_b\right)t} \\
& + \left[\Gamma_{a-} H_{a-} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{a+} H_{a+} \Gamma_{b+}{}^{*} H_{b+}{}^{*}\right] H_{b-}{}^{*} e^{+i\left(2\omega_b-\omega_a\right)t} \\
& + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-} H_{b-} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b+} H_{b+}\right] H_{a-}{}^{*} e^{+i\left(2\omega_a-\omega_b\right)t} \\
& + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-} H_{b-} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b+} H_{b+}\right] H_{b+} e^{-i\left(2\omega_b-\omega_a\right)t} \Big\}
\end{aligned}
\tag{79}
$$

$$
=\frac{1}{2}\left\{\begin{array}{l}
\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{b+}+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{a+}\right)e^{-i\left(2\omega_a+\omega_b\right)t}\\
+\left(\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{b-}{}^{*}+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{a-}{}^{*}\right)e^{+i\left(2\omega_a+\omega_b\right)t}\\
+\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{b-}{}^{*}+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{a+}\right)e^{-i\left(2\omega_a-\omega_b\right)t}\\
+\left(\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{b+}+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{a-}{}^{*}\right)e^{+i\left(2\omega_a-\omega_b\right)t}\\
+\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{a+}+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{b+}\right)e^{-i\left(2\omega_b+\omega_a\right)t}\\
+\left(\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{a-}{}^{*}+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{b-}{}^{*}\right)e^{+i\left(2\omega_b+\omega_a\right)t}\\
+\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{a-}{}^{*}+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{b+}\right)e^{-i\left(2\omega_b-\omega_a\right)t}\\
+\left(\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{a+}+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{b-}{}^{*}\right)e^{+i\left(2\omega_b-\omega_a\right)t}
\end{array}\right\} \quad (80)
$$

$$
\mathbb{S}_2\Big|_{\mathrm{IM}}=\frac{1}{2}\left\{\begin{array}{l}
\Gamma_{a+}{}^{2}H_{a+}{}^{2}H_{b+}e^{-i\left(2\omega_a+\omega_b\right)t}+\Gamma_{a+}{}^{2}H_{a+}{}^{2}H_{b-}{}^{*}e^{-i\left(2\omega_a-\omega_b\right)t}\\
+\Gamma_{a-}{}^{*2}H_{a-}{}^{*2}H_{b+}e^{+i\left(2\omega_a-\omega_b\right)t}+\Gamma_{a-}{}^{*2}H_{a-}{}^{*2}H_{b-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}\\
+\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}H_{a+}e^{-i\left(2\omega_a+\omega_b\right)t}+\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}H_{b+}e^{-i\left(2\omega_b+\omega_a\right)t}\\
+\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{a+}e^{-i\left(2\omega_a-\omega_b\right)t}+\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b-\omega_a\right)t}\\
+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{a-}{}^{*}e^{+i\left(2\omega_a-\omega_b\right)t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{b+}e^{-i\left(2\omega_b-\omega_a\right)t}\\
+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}\\
+\Gamma_{b+}{}^{2}H_{b+}{}^{2}H_{a+}e^{-i\left(2\omega_b+\omega_a\right)t}+\Gamma_{b+}{}^{2}H_{b+}{}^{2}H_{a-}{}^{*}e^{-i\left(2\omega_b-\omega_a\right)t}\\
+\Gamma_{b-}{}^{*2}H_{b-}{}^{*2}H_{a+}e^{+i\left(2\omega_b-\omega_a\right)t}+\Gamma_{b-}{}^{*2}H_{b-}{}^{*2}H_{a-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}
\end{array}\right\} \quad (81)
$$

$$
= \frac{1}{2}\left\{\begin{array}{l}
\left[\Gamma_{a+}{}^{2} H_{a+}{}^{2} H_{b+} + \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{a+}\right] e^{-i\left(2\omega_a + \omega_b\right)t} \\
+\left[\Gamma_{a+}{}^{2} H_{a+}{}^{2} H_{b-}{}^{*} + \Gamma_{a+} H_{a+} \Gamma_{b-}{}^{*} H_{b-}{}^{*} H_{a+}\right] e^{-i\left(2\omega_a - \omega_b\right)t} \\
+\left[\Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{b+} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} H_{a-}{}^{*}\right] e^{+i\left(2\omega_a - \omega_b\right)t} \\
+\left[\Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{b-}{}^{*} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} H_{a-}{}^{*}\right] e^{+i\left(2\omega_a + \omega_b\right)t} \\
+\left[\Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{b+} + \Gamma_{b+}{}^{2} H_{b+}{}^{2} H_{a+}\right] e^{-i\left(2\omega_b + \omega_a\right)t} \\
+\left[\Gamma_{a+} H_{a+} \Gamma_{b-}{}^{*} H_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{a+}\right] e^{+i\left(2\omega_b - \omega_a\right)t} \\
+\left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} H_{b+} + \Gamma_{b+}{}^{2} H_{b+}{}^{2} H_{a-}{}^{*}\right] e^{-i\left(2\omega_b - \omega_a\right)t} \\
+\left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{a-}{}^{*}\right] e^{+i\left(2\omega_b + \omega_a\right)t}
\end{array}\right\} \qquad (82)
$$

$$
\begin{aligned}
\mathbb{S}_3\big|_{\mathrm{IM}} &= \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}e^{-i\left(2\omega_a+\omega_b\right)t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\left(2\omega_a-\omega_b\right)t} \\
&+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}e^{+i\left(2\omega_a-\omega_b\right)t} + \Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t} \\
&+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{a+}H_{a+}e^{-i\left(2\omega_b+\omega_a\right)t} + \Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{-i\left(2\omega_b-\omega_a\right)t} \\
&+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{a+}H_{a+}e^{+i\left(2\omega_b-\omega_a\right)t} + \Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{a+}H_{a+}e^{-i\left(2\omega_a+\omega_b\right)t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b+\omega_a\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t} \\
&\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{a+}H_{a+}e^{-i\left(2\omega_a-\omega_b\right)t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b-\omega_a\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a-\omega_b\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b-\omega_a\right)t}
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
&= \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} + \Gamma_{a-} H_{a-} \Gamma_{b+} H_{b+} + \Gamma_{a+} H_{a+} \Gamma_{b-} H_{b-} \right] \Gamma_{a+} H_{a+} e^{-i\left(2\omega_a + \omega_b\right)t} \\
&+ \left[\Gamma_{a-} H_{a-} \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+}{}^{*} H_{b+}{}^{*} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} \right] \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right] e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+ \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{a-} H_{a-} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{a+} H_{a+} \Gamma_{b+}{}^{*} H_{b+}{}^{*} \right] \Gamma_{a+} H_{a+} e^{-i\left(2\omega_a - \omega_b\right)t} \\
&+ \left[\Gamma_{a-} H_{a-} \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b+} H_{b+} + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-} H_{b-} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b+} H_{b+} \right] \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right] e^{+i\left(2\omega_a - \omega_b\right)t} \\
&+ \left[\Gamma_{b-}{}^{*} H_{b-}{}^{*} \Gamma_{a+} H_{a+} + \Gamma_{a-} H_{a-} \Gamma_{b+} H_{b+} + \Gamma_{a+} H_{a+} \Gamma_{b-} H_{b-} \right] \Gamma_{b+} H_{b+} e^{-i\left(2\omega_b + \omega_a\right)t} \\
&+ \left[\Gamma_{b-} H_{b-} \Gamma_{b+}{}^{*} H_{b+}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+}{}^{*} H_{b+}{}^{*} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} \right] \Gamma_{b-}{}^{*} H_{b-}{}^{*} \right] e^{+i\left(2\omega_b + \omega_a\right)t} \\
&+ \left[\Gamma_{b-}{}^{*} H_{b-}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-} H_{b-} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b+} H_{b+} \right] \Gamma_{b+} H_{b+} e^{-i\left(2\omega_b - \omega_a\right)t} \\
&+ \left[\Gamma_{b-} H_{b-} \Gamma_{b+}{}^{*} H_{b+}{}^{*} \Gamma_{a+} H_{a+} + \left[\Gamma_{a-} H_{a-} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{a+} H_{a+} \Gamma_{b+}{}^{*} H_{b+}{}^{*} \right] \Gamma_{b-}{}^{*} H_{b-}{}^{*} \right] e^{+i\left(2\omega_b - \omega_a\right)t}
\end{aligned}
$$

$$
\mathbb{S}_4\Big|_{\mathrm{IM}} = \frac{1}{2} \begin{pmatrix}
+\left[\Gamma_{2b+} S_{2b+} h_{az} + \Gamma_{sum+} S_{sum+} h_{bz} \right] e^{-i\left(2\omega_b + \omega_a\right)t} & +\left[\Gamma_{2b-} S_{2b-} h_{az}{}^{*} + \Gamma_{sum-} S_{sum-} h_{bz}{}^{*} \right] e^{+i\left(2\omega_b + \omega_a\right)t} \\
+\left[\Gamma_{2a+} S_{2a+} h_{bz} + \Gamma_{sum+} S_{sum+} h_{az} + \right] e^{-i\left(2\omega_a + \omega_b\right)t} & +\left[\Gamma_{2a-} S_{2a-} h_{bz}{}^{*} + \Gamma_{sum-} S_{sum-} h_{az}{}^{*} \right] e^{+i\left(2\omega_a + \omega_b\right)t} \\
+\left[\Gamma_{2a+} S_{2a+} h_{bz}{}^{*} + \Gamma_{dif+} S_{dif+} h_{az} \right] e^{-i\left(2\omega_a - \omega_b\right)t} & +\left[\Gamma_{2a-} S_{2a-} h_{bz} + \Gamma_{dif-} S_{dif-} h_{az}{}^{*} \right] e^{+i\left(2\omega_a - \omega_b\right)t} \\
+\left[\Gamma_{2b+} S_{2b+} h_{az}{}^{*} + \Gamma_{dif-} S_{dif-} h_{bz} \right] e^{-i\left(2\omega_b - \omega_a\right)t} & +\left[\Gamma_{2b-} S_{2b-} h_{az} + \Gamma_{dif+} S_{dif+} h_{bz}{}^{*} \right] e^{+i\left(2\omega_b - \omega_a\right)t}
\end{pmatrix} \tag{84}
$$

$$
\left[\begin{array}{l}
S_{2a+}e^{-2i\omega_a t}+S_{2a-}e^{+2i\omega_a t}+S_0+S_{2b+}e^{-2i\omega_b t}+S_{2b-}e^{+2i\omega_b t} \\
+S_{sum+}e^{-i\omega_{sum} t}+S_{sum-}e^{+i\omega_{sum} t}+S_{dif+}e^{-i\omega_{dif} t}+S_{dif-}e^{+i\omega_{dif} t}
\end{array}\right]
$$

$$
=\left\{\begin{array}{l}
\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-2i\omega_a t}+\left(h_{az}^{*}\Gamma_{a-}^{*}H_{a-}^{*}\right)e^{+2i\omega_a t} \\
+h_{az}\Gamma_{a-}^{*}H_{a-}^{*}+h_{bz}\Gamma_{b-}^{*}H_{b-}^{*}+h_{az}^{*}\Gamma_{a+}H_{a+}+h_{bz}^{*}\Gamma_{b+}H_{b+} \\
+h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}+h_{bz}^{*}\Gamma_{b-}^{*}H_{b-}^{*}e^{+2i\omega_b t} \\
+\left(h_{az}\Gamma_{b+}H_{b+}+h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{sum} t}+\left(h_{az}^{*}\Gamma_{b-}^{*}H_{b-}^{*}+h_{bz}^{*}\Gamma_{a-}^{*}H_{a-}^{*}\right)e^{+i\omega_{sum} t} \\
+\left(h_{az}\Gamma_{b-}^{*}H_{b-}^{*}+h_{bz}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{dif} t}+\left(h_{bz}\Gamma_{a-}^{*}H_{a-}^{*}+h_{az}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_{dif} t}
\end{array}\right\}
$$

$$
\Rightarrow\left\{\begin{array}{c}
S_{2a+}=\left(h_{az}\Gamma_{a+}H_{a+}\right) \\
S_{2a-}=\left(h_{az}^{*}\Gamma_{a-}^{*}H_{a-}^{*}\right) \\
S_0=\left(h_{az}\Gamma_{a-}^{*}H_{a-}^{*}+h_{bz}\Gamma_{b-}^{*}H_{b-}^{*}+h_{az}^{*}\Gamma_{a+}H_{a+}+h_{bz}^{*}\Gamma_{b+}H_{b+}\right) \\
S_{2b+}=\left(h_{bz}\Gamma_{b+}H_{b+}\right) \\
S_{2b-}=\left(h_{bz}^{*}\Gamma_{b-}^{*}H_{b-}^{*}\right) \\
S_{sum+}=\left(h_{az}\Gamma_{b+}H_{b+}+h_{bz}\Gamma_{a+}H_{a+}\right) \\
S_{sum-}=\left(h_{az}^{*}\Gamma_{b-}^{*}H_{b-}^{*}+h_{bz}^{*}\Gamma_{a-}^{*}H_{a-}^{*}\right) \\
S_{dif+}=\left(h_{az}\Gamma_{b-}^{*}H_{b-}^{*}+h_{bz}^{*}\Gamma_{a+}H_{a+}\right) \\
S_{dif-}=\left(h_{bz}\Gamma_{a-}^{*}H_{a-}^{*}+h_{az}^{*}\Gamma_{b+}H_{b+}\right)
\end{array}\right. \qquad (85)
$$

$$\Rightarrow \mathbb{S}_4\big|_{\mathrm{IM}} = \frac{1}{2}\left(\begin{array}{l}
\left[\Gamma_{sum+}\left(h_{bz}\Gamma_{a+}H_{a+}+h_{az}\Gamma_{b+}H_{b+}\right)h_{bz}+\Gamma_{2b+}\left(h_{bz}\Gamma_{b+}H_{b+}\right)h_{az}\right]e^{-i\left(2\omega_b+\omega_a\right)t} \\
+\left[\Gamma_{sum-}\left(h_{bz}^{*}\Gamma_{a-}^{*}H_{a-}^{*}+h_{az}^{*}\Gamma_{b-}^{*}H_{b-}^{*}\right)h_{bz}^{*}+\Gamma_{2b-}\left(h_{bz}^{*}\Gamma_{b-}^{*}H_{b-}^{*}\right)h_{az}^{*}\right]e^{+i\left(2\omega_b+\omega_a\right)t} \\
+\left[\Gamma_{sum+}\left(h_{az}\Gamma_{b+}H_{b+}+h_{bz}\Gamma_{a+}H_{a+}\right)h_{az}+\Gamma_{2a+}\left(h_{az}\Gamma_{a+}H_{a+}\right)h_{bz}\right]e^{-i\left(2\omega_a+\omega_b\right)t} \\
+\left[\Gamma_{sum-}\left(h_{az}^{*}\Gamma_{b-}^{*}H_{b-}^{*}+h_{bz}^{*}\Gamma_{a-}^{*}H_{a-}^{*}\right)h_{az}^{*}+\Gamma_{2a-}\left(h_{az}^{*}\Gamma_{a-}^{*}H_{a-}^{*}\right)h_{bz}^{*}\right]e^{+i\left(2\omega_a+\omega_b\right)t} \\
+\left[\Gamma_{dif+}\left(h_{az}\Gamma_{b-}^{*}H_{b-}^{*}+h_{bz}^{*}\Gamma_{a+}H_{a+}\right)h_{az}+\Gamma_{2a+}\left(h_{az}\Gamma_{a+}H_{a+}\right)h_{bz}^{*}\right]e^{-i\left(2\omega_a-\omega_b\right)t} \\
+\left[\Gamma_{dif-}\left(h_{az}^{*}\Gamma_{b+}H_{b+}+h_{bz}\Gamma_{a-}^{*}H_{a-}^{*}\right)h_{az}^{*}+\Gamma_{2a-}\left(h_{az}^{*}\Gamma_{a-}^{*}H_{a-}^{*}\right)h_{bz}\right]e^{+i\left(2\omega_a-\omega_b\right)t} \\
+\left[\Gamma_{dif-}\left(h_{bz}\Gamma_{a-}^{*}H_{a-}^{*}+h_{az}^{*}\Gamma_{b+}H_{b+}\right)h_{bz}+\Gamma_{2b+}\left(h_{bz}\Gamma_{b+}H_{b+}\right)h_{az}^{*}\right]e^{-i\left(2\omega_b-\omega_a\right)t} \\
+\left[\Gamma_{dif+}\left(h_{bz}^{*}\Gamma_{a+}H_{a+}+h_{az}\Gamma_{b-}^{*}H_{b-}^{*}\right)h_{bz}^{*}+\Gamma_{2b-}\left(h_{bz}^{*}\Gamma_{b-}^{*}H_{b-}^{*}\right)h_{az}\right]e^{+i\left(2\omega_b-\omega_a\right)t}
\end{array}\right) \tag{86}$$

For the *z*-component, we have

$$g_{z3} = -\frac{1}{2}\left(\mathcal{G}_1^{*}\mathcal{G}_2+\mathcal{G}_1\mathcal{G}_2^{*}\right) = -\mathrm{Re}\left(\mathcal{G}_1^{*}\mathcal{G}_2\right)$$

$$= +\mathrm{Re}\left(\left\{\begin{array}{l}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}^{*}H_{a-}^{*}e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}^{*}H_{b-}^{*}e^{+i\omega_b t}\end{array}\right\}^{*}\mathcal{G}_2\right) = +\mathrm{Re}\left(\begin{array}{l}\Gamma_{a+}^{*}H_{a+}^{*}e^{+i\omega_a t}\mathcal{G}_2+\Gamma_{a-}H_{a-}e^{-i\omega_a t}\mathcal{G}_2 \\ +\Gamma_{b+}^{*}H_{b+}^{*}e^{+i\omega_b t}\mathcal{G}_2+\Gamma_{b-}H_{b-}e^{-i\omega_b t}\mathcal{G}_2\end{array}\right) \tag{87}$$

$$= \mathrm{Re}\sum_{m=1}^{4}\Xi_m$$

$$
\begin{aligned}
\Xi_1 = \Gamma_{a+}{}^{*} H_{a+}{}^{*} e^{+i\omega_a t} \Bigg[& \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-2i\omega_a t} + \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+2i\omega_a t} \\
& + h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \\
& + \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_s t} + \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_s t} \\
& + \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_d t} + \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_d t} \\
& + h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+2i\omega_b t} + h_{bz} \Gamma_{b+} H_{b+} e^{-2i\omega_b t}
\end{aligned}
$$

$$
\begin{aligned}
= {} & \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{+i\omega_a t} e^{-2i\omega_a t} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_a t} e^{+2i\omega_a t} \\
& + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_a t} \\
& + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{+i\omega_a t} e^{-i\omega_s t} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_a t} e^{+i\omega_s t} \\
& + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{+i\omega_a t} e^{-i\omega_d t} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_a t} e^{+i\omega_d t} \\
& + \Gamma_{a+}{}^{*} H_{a+}{}^{*} h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_a t} e^{+2i\omega_b t} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} h_{bz} \Gamma_{b+} H_{b+} e^{+i\omega_a t} e^{-2i\omega_b t}
\end{aligned}
$$

$$
\begin{aligned}
= {} & \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-i\omega_a t} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+3i\omega_a t} \\
& + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_a t} \\
& + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_b t} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\left(2\omega_a + \omega_b\right)t} \\
& + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{+i\omega_b t} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\left(2\omega_a - \omega_b\right)t} \\
& + \Gamma_{a+}{}^{*} H_{a+}{}^{*} h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\left(2\omega_b + \omega_a\right)t} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} h_{bz} \Gamma_{b+} H_{b+} e^{-i\left(2\omega_b - \omega_a\right)t}
\end{aligned}
\tag{88}
$$

$$\Xi_2 = \Gamma_{a-} H_{a-} e^{-i\omega_a t} \left[\begin{array}{l} \left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-2i\omega_a t} + \left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+2i\omega_a t} \\ + h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+} \\ + \left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_s t} + \left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_s t} \\ + \left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_d t} + \left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_d t} \\ + h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+2i\omega_b t} + h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t} \end{array} \right]$$

$$= \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-i\omega_a t}e^{-2i\omega_a t} + \Gamma_{a-}H_{a-}\left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{-i\omega_a t}e^{+2i\omega_a t}$$
$$+ \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+}\right)e^{-i\omega_a t}$$
$$+ \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_a t}e^{-i\omega_s t} + \Gamma_{a-}H_{a-}\left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{-i\omega_a t}e^{+i\omega_s t}$$
$$+ \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_a t}e^{-i\omega_d t} + \Gamma_{a-}H_{a-}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{-i\omega_a t}e^{+i\omega_d t}$$
$$+ \Gamma_{a-}H_{a-}h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\omega_a t}e^{+2i\omega_b t} + \Gamma_{a-}H_{a-}h_{bz}\Gamma_{b+}H_{b+}e^{-i\omega_a t}e^{-2i\omega_b t}$$

$$= \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-3i\omega_a t} + \Gamma_{a-}H_{a-}\left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_a t} \qquad (89)$$
$$+ \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+}\right)e^{-i\omega_a t}$$
$$+ \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\left(2\omega_a+\omega_b\right)t} + \Gamma_{a-}H_{a-}\left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_b t}$$
$$+ \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\left(2\omega_a-\omega_b\right)t} + \Gamma_{a-}H_{a-}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{-i\omega_b t}$$
$$+ \Gamma_{a-}H_{a-}h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b-\omega_a\right)t} + \Gamma_{a-}H_{a-}h_{bz}\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b+\omega_a\right)t}$$

$$\Xi_3 = \Gamma_{b+}{}^{*} H_{b+}{}^{*} e^{+i\omega_b t} \left[\begin{array}{l} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-2i\omega_a t} + \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+2i\omega_a t} \\ + h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \\ + \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_s t} + \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_s t} \\ + \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_d t} + \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_d t} \\ + h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+2i\omega_b t} + h_{bz} \Gamma_{b+} H_{b+} e^{-2i\omega_b t} \end{array} \right]$$

$$= \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{+i\omega_b t} e^{-2i\omega_a t} + \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_b t} e^{+2i\omega_a t}$$
$$+ \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_b t}$$
$$+ \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{+i\omega_b t} e^{-i\omega_s t} + \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_b t} e^{+i\omega_s t}$$
$$+ \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{+i\omega_b t} e^{-i\omega_d t} + \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_b t} e^{+i\omega_d t}$$
$$+ \Gamma_{b+}{}^{*} H_{b+}{}^{*} h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_b t} e^{+2i\omega_b t} + \Gamma_{b+}{}^{*} H_{b+}{}^{*} h_{bz} \Gamma_{b+} H_{b+} e^{+i\omega_b t} e^{-2i\omega_b t}$$

$$= \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-i\left(2\omega_a + \omega_b\right) t} + \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\left(2\omega_a + \omega_b\right) t}$$
$$+ \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_b t}$$
$$+ \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_a t} + \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\left(2\omega_b + \omega_a\right) t}$$
$$+ \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{+i\left(2\omega_b - \omega_a\right) t} + \Gamma_{b+}{}^{*} H_{b+}{}^{*} \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_a t}$$
$$+ \Gamma_{b+}{}^{*} H_{b+}{}^{*} h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+3i\omega_b t} + \Gamma_{b+}{}^{*} H_{b+}{}^{*} h_{bz} \Gamma_{b+} H_{b+} e^{-i\omega_b t} \qquad (90)$$

$$\Xi_4 = \Gamma_{b-} H_{b-} e^{-i\omega_b t} \left[\begin{array}{l} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-2i\omega_a t} + \left(h_{az}^{*} \Gamma_{a-}^{*} H_{a-}^{*} \right) e^{+2i\omega_a t} \\ + h_{az} \Gamma_{a-}^{*} H_{a-}^{*} + h_{bz} \Gamma_{b-}^{*} H_{b-}^{*} + h_{az}^{*} \Gamma_{a+} H_{a+} + h_{bz}^{*} \Gamma_{b+} H_{b+} \\ + \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_s t} + \left(h_{az}^{*} \Gamma_{b-}^{*} H_{b-}^{*} + h_{bz}^{*} \Gamma_{a-}^{*} H_{a-}^{*} \right) e^{+i\omega_s t} \\ + \left(h_{az} \Gamma_{b-}^{*} H_{b-}^{*} + h_{bz}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_d t} + \left(h_{bz} \Gamma_{a-}^{*} H_{a-}^{*} + h_{az}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_d t} \\ + h_{bz}^{*} \Gamma_{b-}^{*} H_{b-}^{*} e^{+2i\omega_b t} + h_{bz} \Gamma_{b+} H_{b+} e^{-2i\omega_b t} \end{array} \right]$$

$$= \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-i\omega_b t} e^{-2i\omega_a t} + \Gamma_{b-} H_{b-} \left(h_{az}^{*} \Gamma_{a-}^{*} H_{a-}^{*} \right) e^{-i\omega_b t} e^{+2i\omega_a t}$$
$$+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{a-}^{*} H_{a-}^{*} + h_{bz} \Gamma_{b-}^{*} H_{b-}^{*} + h_{az}^{*} \Gamma_{a+} H_{a+} + h_{bz}^{*} \Gamma_{b+} H_{b+} \right) e^{-i\omega_b t}$$
$$+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_b t} e^{-i\omega_s t} + \Gamma_{b-} H_{b-} \left(h_{az}^{*} \Gamma_{b-}^{*} H_{b-}^{*} + h_{bz}^{*} \Gamma_{a-}^{*} H_{a-}^{*} \right) e^{-i\omega_b t} e^{+i\omega_s t}$$
$$+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{b-}^{*} H_{b-}^{*} + h_{bz}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_b t} e^{-i\omega_d t} + \Gamma_{b-} H_{b-} \left(h_{bz} \Gamma_{a-}^{*} H_{a-}^{*} + h_{az}^{*} \Gamma_{b+} H_{b+} \right) e^{-i\omega_b t} e^{+i\omega_d t}$$
$$+ \Gamma_{b-} H_{b-} h_{bz}^{*} \Gamma_{b-}^{*} H_{b-}^{*} e^{-i\omega_b t} e^{+2i\omega_b t} + \Gamma_{b-} H_{b-} h_{bz} \Gamma_{b+} H_{b+} e^{-i\omega_b t} e^{-2i\omega_b t}$$

$$= \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{b-} H_{b-} \left(h_{az}^{*} \Gamma_{a-}^{*} H_{a-}^{*} \right) e^{+i\left(2\omega_a + \omega_b\right)t}$$
$$+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{a-}^{*} H_{a-}^{*} + h_{bz} \Gamma_{b-}^{*} H_{b-}^{*} + h_{az}^{*} \Gamma_{a+} H_{a+} + h_{bz}^{*} \Gamma_{b+} H_{b+} \right) e^{-i\omega_b t}$$
$$+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\left(2\omega_b + \omega_a\right)t} + \Gamma_{b-} H_{b-} \left(h_{az}^{*} \Gamma_{b-}^{*} H_{b-}^{*} + h_{bz}^{*} \Gamma_{a-}^{*} H_{a-}^{*} \right) e^{+i\omega_a t}$$
$$+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{b-}^{*} H_{b-}^{*} + h_{bz}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_a t} + \Gamma_{b-} H_{b-} \left(h_{bz} \Gamma_{a-}^{*} H_{a-}^{*} + h_{az}^{*} \Gamma_{b+} H_{b+} \right) e^{-i\left(2\omega_b - \omega_a\right)t}$$
$$+ \Gamma_{b-} H_{b-} h_{bz}^{*} \Gamma_{b-}^{*} H_{b-}^{*} e^{+i\omega_b t} + \Gamma_{b-} H_{b-} h_{bz} \Gamma_{b+} H_{b+} e^{-3i\omega_b t} \qquad (91)$$

7. Conclusion

This report was written with the intention of archiving the results of lengthy calculations needed in order to evaluate the feasibility of remote detection of magnetic systems by FMR. More specifically, these calculations make it possible to calculate the induced magnetization within a ferromagnetic body illuminated by electromagnetic waves at levels of excitation high enough to give rise to a nonlinear response in the body and induce it to radiate at new frequencies that are detected at a remote receiving antenna. The results tabulated here contribute to the study of nonlinear radar and target signatures in general.

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