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1	Summary	2
2	Description of outcomes	3
	2.1 Nonlinear differential operators	3
	2.2 Behavior and performance bounds	3
	2.3 Variational performance bound verification	4
	2.4 Tight performance bound characterization	6
	2.5 Controller synthesis to achieve specific performance bounds	8
	2.6 Numerical methods / tools	9
	2.7 References	9
3	Publications	10
	3.1 Refereed articles (published)	10
	3.2 Refereed articles (in press or review)	11
	3.3 Invited presentations	11
4	Personnel supported by the grant	12
5	Honors and awards received	12
6	Transitions	12
7	New discoveries	12
8	AFRL point of contact	12
9	Acknowledgment / Disclaimer	12

Report Documentation Page

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14. ABSTRACT

Highly complex behavior is common in both the natural and technological world. Nonlinear differential operators play an essential role in enabling accurate modeling and prediction of this behavior. Nonlinear systems theory provides a mathematical framework for the analysis and design of networks of these operators, thereby providing the foundation for scientists and engineers to understand and control this highly complex behavior. This project is primarily concerned with the development of analysis and computational tools that can accurately characterize the performance of specific classes of nonlinear differential operators in capturing specific behavioral properties of interest. A secondary concern is the development of controller synthesis tools that enable the design of networks of differential operators so as to yield specific behavioral properties. At the completion of this project after three years of funding, outcomes of this project include the development of new theoretical and computational tools for performance bound verification, tight performance bound characterization, and controller synthesis for representative behavioral properties. Integral-input-to-integral-output, integral-input-to-output, and input-to-state stability properties have been specifically considered. Substantial effort has been invested in the development of tools for the numerical approximation of solutions to the attendant optimization and optimal control problems. This includes computational tools utilizing approximating Markov chain methods and max-plus methods. The research undertaken is documented in 26 scholarly publications (17 published or accepted to be published, 9 in press or in review), and communicated via numerous presentations (including 14 invited) at internationally renowned meetings and academic institutions.

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1 Summary

Highly complex behavior is common in both the natural and technological world. Nonlinear differential operators play an essential role in enabling accurate modeling and prediction of this behavior. Nonlinear systems theory provides a mathematical framework for the analysis and design of networks of these operators, thereby providing the foundation for scientists and engineers to understand and control this highly complex behavior. This project is primarily concerned with the development of analysis and computational tools that can accurately characterize the performance of specific classes of nonlinear differential operators in capturing specific behavioral properties of interest. A secondary concern is the development of controller synthesis tools that enable the design of networks of differential operators so as to yield specific behavioral properties. At the completion of this project after three years of funding, outcomes of this project include the development of new theoretical and computational tools for performance bound verification, tight performance bound characterization, and controller synthesis for representative behavioral properties. Integral-input-to-integral-output, integral-input-to-output, and input-to-state stability properties have been specifically considered. Substantial effort has been invested in the development of tools for the numerical approximation of solutions to the attendant optimization and optimal control problems. This includes computational tools utilizing approximating Markov chain methods and max-plus methods. The research undertaken is documented in 26 scholarly publications (17 published or accepted to be published, 9 in press or in review), and communicated via numerous presentations (including 14 invited) at internationally renowned meetings and academic institutions. These outcomes are summarized in Table 1. Details of publications / presentations cited therein are listed in Section 3.

Tools	Scholarly publications / invited presentations
Performance bound verification	[A1–A3, A6, A9], [B3, B4, B8, B9], [T3, T5–T7, T9, T11, T12, T14]
Tight performance bound characterization	[A1, A5, A16], [B1, B2, B5], [T5, T7, T11, T12, T14]
Controller synthesis	[A4, A12, A13, A15], [B8], [T10]
Computation	[A2, A3, A7, A8, A10, A13, A15], [B2, B3, B5–B7, B9], [T1–T12, T14]
Other	[A11, A14, A17], [T13]

Table 1: Scholarly publications / invited presentations.

2 Description of outcomes

The primary outcome of this project [R4] is the development of analysis and computational tools for the characterization of performance of nonlinear differential operators with respect to specific behavioral properties. A secondary outcome is the development of control design tools for facilitating the design of networks of nonlinear differential operators to achieve specific behaviors. Specific project outcomes include the development of tools for (i) performance bound verification, (ii) tight performance bound characterization, (iii) controller synthesis to achieve specific performance bounds, and (iv) numerical computations for facilitating (i)-(iii). Specific properties classified as integral-input-to-output stability, integral-input-to-state stability, and input-to-state stability have been considered in detail. Ongoing efforts directed towards (iv) continue via the investigation and development of idempotent methods for the efficient solution of worst-case analysis and optimal control problems. These efforts form the basis of a recently funded three year AFOSR / AOARD project [R5].

2.1 Nonlinear differential operators

Throughout this project, a specific class of nonlinear differential operators is considered. These operators consist of systems of non-homogenous nonlinear ordinary differential equations together with an output map. Elements Σ in this class of operators map inputs w to outputs z , as illustrated in Figure 1, via the explicit form

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t)) w(t), & x(0) = x_o, \\ z(t) = h(x(t)), \end{cases} \quad (1)$$

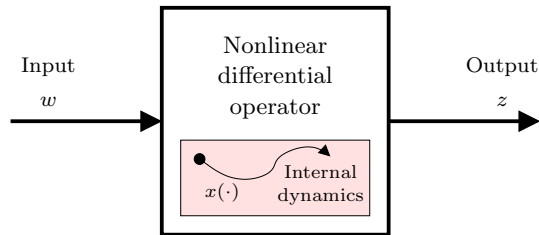


Figure 1: A nonlinear differential operator with input / output pair (w, z) and internal state x .

in which $x(\cdot)$ denotes a dynamically evolving internal state trajectory, and f, g, h are functions with appropriate regularity. It is well known that operators of this form can exhibit extremely complex and localized dynamical behavior, characterized by the existence of sources, sinks, saddles, periodic orbits, chaos, and so on. Numerous stability properties have been formalized in the literature (for example, [R1,R6,R9–R13], [A3]) to classify this behavior, typically via finite gain properties formulated in terms of the operator inputs, outputs, and internal states.

2.2 Behavior and performance bounds

Operators of the form of (1) may be classified according to specific behavioral properties by certifying whether corresponding input-to-output or input-to-state finite gain inequalities hold. Specific finite gain inequalities of interest include integral-input-to-integral-output stability (iIiOS, e.g. [R9], [B8], [A3]) and integral-input-to-output stability (iIOS, e.g. [R9], [A5], [B2]), given respectively by

$$\|z\|_{\mathcal{L}_2[0,t]}^2 \leq \beta(|x_o|) + \gamma \left(\|w\|_{\mathcal{L}_2[0,t]}^2 \right), \quad (2)$$

$$|z(t)| \leq \beta(|x_o|, t) + \gamma \left(\|w\|_{\mathcal{L}_p[0,t]} \right), \quad (3)$$

where β, γ are comparison functions [R12]. Finite gain inequalities of this form can be used to prescribe input-output or input-state constraints on the dynamics of the nonlinear differential operators of

interest. The comparison functions β, γ involved are not unique, and any specific comparison function need not represent a tight bound in the respective finite gain property. Indeed, a set Π^Σ of comparison function bounds compatible with operator Σ and a specific finite gain property may be defined. For example, in the case of iLiOSS property (2) considered in [A1], this set is defined explicitly by

$$\Pi^\Sigma \doteq \left\{ (\beta, \gamma) \in \overline{\mathcal{K}} \times \overline{\mathcal{K}} \mid \begin{array}{l} \text{iLiOSS property (2) holds for all} \\ w \in \mathcal{L}_2[0, t], t \in \mathbb{R}_{\geq 0}, x_o \in \mathbb{R}^n \end{array} \right\}, \quad (4)$$

where $\overline{\mathcal{K}}$ denotes a class of functions $\phi : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ that are right continuous at 0, non-decreasing, and satisfy $\phi(0) = 0$. Using this notation, it is evident from (2) that $(\beta + \epsilon, \gamma + \delta) \in \Pi^\Sigma$ for any $(\beta, \gamma) \in \Pi^\Sigma$, $\epsilon, \delta \in \overline{\mathcal{K}}$. In this context, performance bound verification for an operator Σ concerns the problem of determining whether a specific pair (β, γ) of comparison functions resides in the set Π^Σ . To this end, it is convenient to expand this notation further. In particular, the set of transient bounds $\mathcal{B}^\Sigma(\gamma) \subset \overline{\mathcal{K}}$ compatible with a given gain bound $\gamma \in \overline{\mathcal{K}}$, or the set of gain bounds $\mathcal{G}^\Sigma(\beta) \subset \overline{\mathcal{K}}$ compatible with a given transient bound $\beta \in \overline{\mathcal{K}}$, may be defined in terms of the set Π^Σ of (4) by

$$\mathcal{B}^\Sigma(\gamma) \doteq \left\{ \beta \in \overline{\mathcal{K}} \mid (\beta, \gamma) \in \Pi^\Sigma \right\}, \quad (5)$$

$$\mathcal{G}^\Sigma(\beta) \doteq \left\{ \gamma \in \overline{\mathcal{K}} \mid (\beta, \gamma) \in \Pi^\Sigma \right\}. \quad (6)$$

Hence, performance bound verification may be posed with respect to a fixed (or known) gain or transient bound. For example, with $\gamma \in \overline{\mathcal{K}}$ fixed, inspection of (4) and (5) indicates that $(\beta, \gamma) \in \Pi^\Sigma$ if and only if $\beta \in \mathcal{B}^\Sigma(\gamma)$. Similarly, tight performance bounds for operator Σ may be characterized in terms of these set-valued objects \mathcal{B}^Σ and \mathcal{G}^Σ . In particular, the tightest possible transient bound $\beta_*^\gamma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ for a given gain bound $\gamma \in \overline{\mathcal{K}}$, and the tightest possible gain bound $\gamma_*^\beta : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ for a given transient bound $\beta \in \overline{\mathcal{K}}$, are given by

$$\beta_*^\gamma(s) \doteq \inf \left\{ \beta(s) \mid \beta \in \mathcal{B}^\Sigma(\gamma) \right\}, \quad (7)$$

$$\gamma_*^\beta(s) \doteq \inf \left\{ \gamma(s) \mid \gamma \in \mathcal{G}^\Sigma(\beta) \right\}. \quad (8)$$

These tight bounds demonstrably reside in $\overline{\mathcal{K}}$, see [A1], so that $(\beta_*^\gamma, \gamma), (\beta, \gamma_*^\beta) \in \Pi^\Sigma$. That is, these tight bounds satisfy the corresponding finite gain property (in this case, iLiOS (2)). Consequently, these tight performance bounds may be used to summarize the behavior of specific operators with respect to specific finite gain properties. For example, two operators satisfying the iLiOS property (2) may be compared from the point of view of input-to-output gain via their respective tightest possible gain bounds γ_*^β for the same transient bound $\beta \in \overline{\mathcal{K}}$. Such tight bounds are of crucial importance in reducing conservatism in small-gain based feedback design [B8].

2.3 Variational performance bound verification

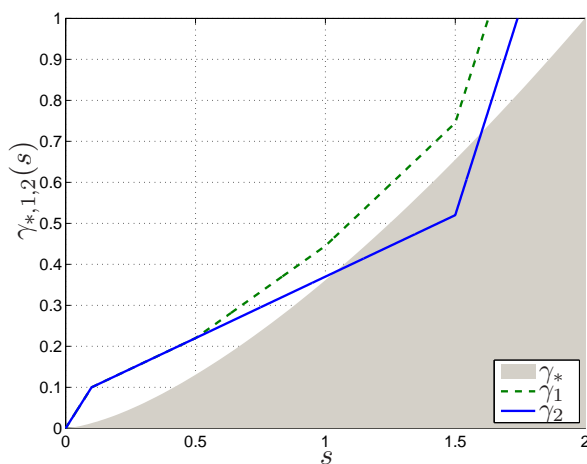
Verification describes a mechanism for testing the compatibility of a specific nonlinear differential operator with a specific behavioral property [A3]. In the context of the iLiOS property (2) described above, verification involves the confirmation (or otherwise) that a specific pair of comparison functions (β, γ) resides in the set Π^Σ of all such pairs for a specific operator Σ (see (4)). Mechanisms for verifying this set membership for iLiOS and other behavioral properties developed in this project appeal to dissipative systems theory [R14]. In particular, connections between specific behavioral properties and corresponding dissipation properties are established. These dissipation properties are equivalent to the existence of a solution to a corresponding Hamilton Jacobi Bellman partial differential equation

(HJB PDE) or dynamic programming principle (DPP) that is specific to each property. For example, in the case of the iliOS property (2) above (with γ differentiable), the nonlinear differential operator Σ satisfies property (2) if and only if an augmented differential operator Σ_a is dissipative [A3] with supply rate r , where

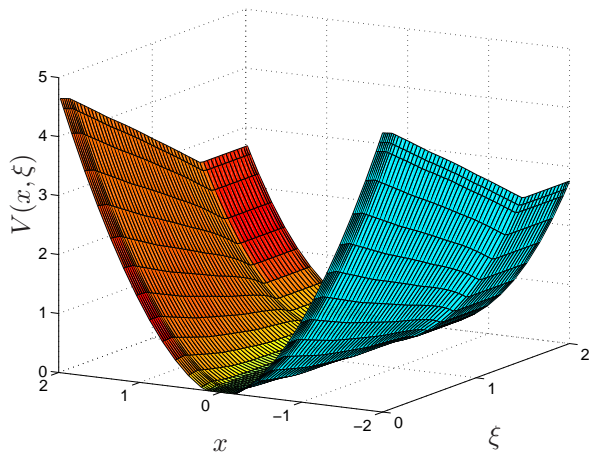
$$\Sigma_a : \begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} f(x(t)) + g(x(t))w(t) \\ |w(t)|^2 \end{bmatrix}, & \begin{bmatrix} x(0) \\ \xi(0) \end{bmatrix} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \\ \begin{bmatrix} z(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} h(x(t)) \\ \xi(t) \end{bmatrix}, & r\left(w, \begin{bmatrix} z \\ \eta \end{bmatrix}\right) \doteq \gamma'(\eta) |w|^2 - |z|^2. \end{cases} \quad (9)$$

It has been shown in [A3,A6,A9] that this dissipation property holds if and only if a viscosity supersolution V of the HJB PDE

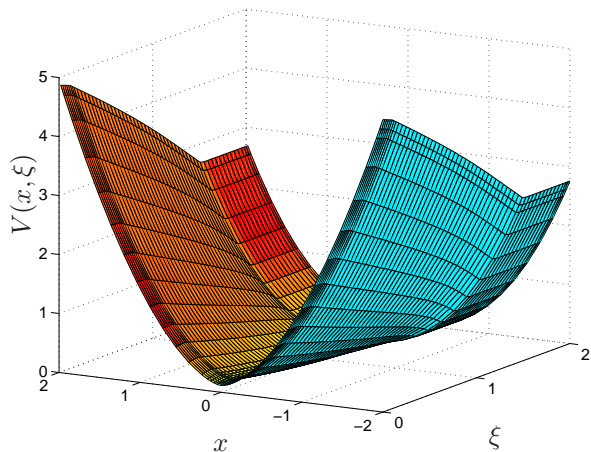
$$0 = H(x, \xi, \nabla_x V(x, \xi), \nabla_\xi V(x, \xi)) \quad (10)$$



(a) Performance bound candidates γ_1 and γ_2 .



(b) Solution of HJB PDE (10) for γ_1 .



(c) Solution of HJB PDE (10) for γ_2 .

Figure 2: Performance bound verification via solution of HJB PDE (10).

exists [A3], [R2], in which H denotes the Hamiltonian

$$H(x, \xi, p, q) \doteq -|h(x)|^2 - \langle p, f(x) \rangle - \sup_{w \in \mathbb{R}^s} \{ \langle p, g(x) w \rangle + (q - \gamma'(\xi)) |w|^2 \}. \quad (11)$$

Consequently, the iliOS property (2) may be verified by solving a specific HJB PDE [A3]. In particular, (2) holds with a specific comparison function bound pair (β, γ) if the HJB PDE (10) (which is parameterized by comparison function γ) admits a non-negative solution V that satisfies

$$V(x, 0) \leq \beta(|x|) \quad (12)$$

for all $x \in \mathbb{R}^n$. As a simple illustration, verification of the iliOS property (2) may be considered for the nonlinear differential operator [A3]

$$\Sigma : \begin{cases} f(x) \doteq \begin{cases} -\frac{3}{5}(x+1) + 1 & x < -1, \\ -x^{\frac{3}{5}} & x \in [-1, 1], \\ -\frac{3}{5}(x-1) - 1 & x > 1, \end{cases} \\ g(x) \doteq 1, \\ h(x) \doteq x. \end{cases} \quad (13)$$

Two candidate performance bounds (γ_1 valid and γ_2 invalid) for the comparison function γ in (2) are illustrated in Figure 2(a). (The tightest possible such bound γ_* is also illustrated for comparison, see Section 2.4.) These candidate bounds are tested via computation of the solution V of the HJB PDE (10), with the respective solutions illustrated in Figure 2(b) and (c). Candidate γ_2 is rendered invalid by strict positivity of the HJB PDE solution at the origin [A3]. However, candidate γ_1 is valid for any comparison function $\beta \in \overline{\mathcal{K}}$ satisfying (12), for example, any quadratic function $\beta(x) \doteq \alpha |x|^2$ for $\alpha \in \mathbb{R}_{>0}$ sufficiently large. Performance bound verification for integral input-to-state stability (iISS, [R9]) and related behavioural properties may be considered in a similar way, see [A5, A7], [B2, B9].

2.4 Tight performance bound characterization

Performance characterization with respect to the aforementioned behavioural properties involves the determination of tight bounds for either or both comparison functions. This project has developed methods for the variational characterization of tight performance bounds for behavioral properties including iliOS, iISS, and ISS, see [A1, A2, A5, A7, A10, A16], [B1–B3, B5, B9]. In the case of the iliOS property (2), these tight bounds are defined by β_*^γ and γ_*^β of (7) and (8) respectively. In order to facilitate computation of these tight bounds, variational characterizations for β_*^γ and γ_*^β have been developed [A1, A5, A16], [B1, B2, B5]. These characterizations are given by

$$\gamma_*^\beta(s) \doteq \sup_{x \in \mathbb{R}^n} \left\{ \widehat{W}(x, \sqrt{s}) - \beta(|x|) \right\}, \quad (14)$$

$$\beta_*^\gamma(s) \doteq \sup_{|x| \leq s} \sup_{\xi \in \mathbb{R}_{\geq 0}} \left\{ \widehat{W}(x, \sqrt{\xi}) - \gamma(\xi) \right\}, \quad (15)$$

where the underlying function $\widehat{W} : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ is the value of a specific optimization problem [A1, A16]. In particular,

$$\widehat{W}(x, \xi) \doteq \sup_{t \geq 0} \sup_{v \in \mathcal{L}_2[0, t]} \left\{ \|z\|_{\mathcal{L}_2[0, t]}^2 \mid z = (\Sigma \circ \mathcal{S}) v \right\}, \quad (16)$$

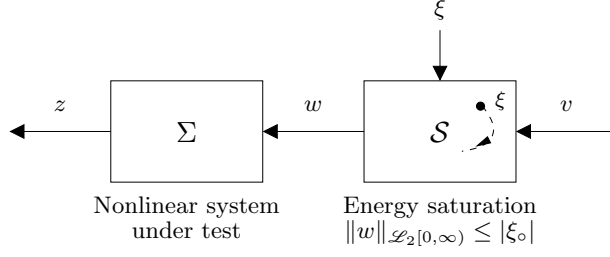


Figure 3: Operator cascade [A1, A6] used in tight performance bound characterization for iliOS (2).

where Σ denotes the nonlinear differential operator of (1) under test, \mathcal{S} denotes an “energy saturation” operator [A1, A16], and $\Sigma \circ \mathcal{S}$ denotes a cascade of these two operators, see Figure 3. The specific operator \mathcal{S} employed in [A1, A16] is given explicitly by

$$\mathcal{S} : \begin{cases} \dot{\xi}(t) = -\xi(t)|v(t)|^2, & \xi(0) = \xi, \\ w(t) = \sqrt{2}\xi(t)v(t). \end{cases} \quad (17)$$

(This operator implements the input energy constraint $\|w\|_{\mathcal{L}_2[0,t]} \leq \xi$.) Lower bounds for γ_*^β and β_*^γ also prove useful. These lower bounds, denoted by γ_* and β_* , are defined (respectively) in terms of \widehat{W} of (16) by

$$\gamma_*(s) \doteq \widehat{W}(0, \sqrt{s}), \quad (18)$$

$$\beta_*(s) \doteq \sup_{|x| \leq s} \widehat{W}(x, 0). \quad (19)$$

In view of bounds (14), (15), (18), (19), the key to computation of tight performance bounds is computation of the value function \widehat{W} of (16). Application of dynamic programming [R3] to (16) reveals [A1] that \widehat{W} is a viscosity solution of the HJB PDE

$$0 = \widehat{H}(x, \xi, \nabla_x \widehat{W}(x, \xi), \nabla_\xi \widehat{W}(x, \xi)) \quad (20)$$

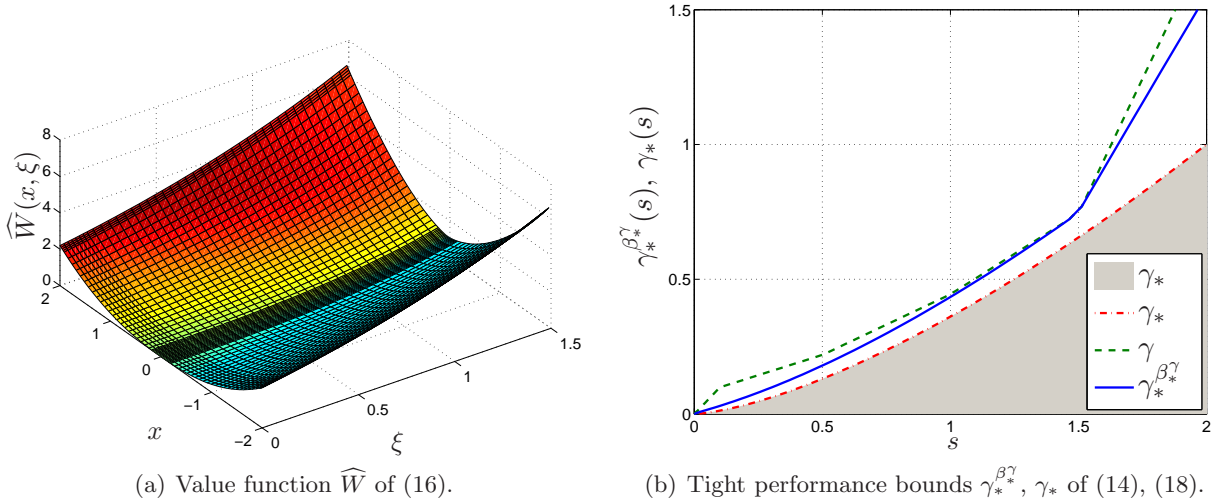


Figure 4: Tight performance bound characterization via solution of HJB PDE (20).

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}_{\neq 0}$, subject to the boundary condition $\widehat{W}(\cdot, 0) = \|\Sigma w_0\|_{\mathcal{L}_2[0, \infty)}^2$ parameterized by the initial internal state of operator Σ . Here, w_0 denotes the zero input to operator Σ , while \widehat{H} is the Hamiltonian

$$\widehat{H}(x, \xi, p, q) \doteq -|h(x)|^2 - \langle p, f(x) \rangle - \sup_{v \in \mathbb{R}^m} \left\{ \langle p, \sqrt{2} \xi g(x) v \rangle - q \xi |v|^2 \right\}. \quad (21)$$

Hence, the HJB PDE (20) in concert with the variational performance bound verification results of Section 2.3 can be used to compute the value \widehat{W} of (16), and hence the tight performance bounds of (14), (15), (18), (19). This approach may be demonstrated [A1] via the simple nonlinear differential operator (13). In particular, Figure 4(a) illustrates an approximate solution of the HJB PDE (20) obtained via an approximating Markov chain method, see [A1], [R7]. Figure 4(b) illustrates the obtained tight performance bounds $\gamma_*^{\beta_*^\gamma}$ and γ_* of (14) and (18). Note that in the former case, the tight gain bound obtained is computed via the steps $\gamma \rightarrow \beta_*^\gamma \rightarrow \gamma_*^{\beta_*^\gamma}$. Note that both of these steps use the same approximation for \widehat{W} of (16), so that (20) need only be solved once. Here, $\gamma \in \overline{\mathcal{K}}$ is an arbitrary (but valid) comparison function bound for the gain γ in (2).

Further connections between performance bound verification and the characterization of tight performance bounds can also be established. For example, it may be shown [A9] that the respective solutions V and \widehat{W} of the HJB PDEs (10) and (20) can be related for a given gain bound $\gamma \in \overline{\mathcal{K}}$ via

$$V(x, \xi) = \sup_{\rho \geq 0} \left\{ \widehat{W}(x, \sqrt{\rho}) - \gamma(\xi + \rho) \right\} + \gamma(\xi) \quad (22)$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Furthermore, the verification condition (12) indicates that the tightest possible transient bound β_*^γ given $\gamma \in \overline{\mathcal{K}}$ (as defined by (7)) can be represented equivalently by

$$\beta_*^\gamma(s) = \sup_{|x| \leq s} V(x, 0) \quad (23)$$

for all $s \in \mathbb{R}_{> 0}$. Indeed, it may be seen that combining (22) and (23) immediately yields the variational characterization (15).

2.5 Controller synthesis to achieve specific performance bounds

The behavior of nonlinear differential operators may be manipulated via the introduction of feedback to achieve specific performance bounds. By exploiting the connection between dissipation and a specific behavioral property, c.f. performance verification (see Section 2.3 above), an optimal control problem may be posed whose solution implements that specific behavioral property in closed-loop. The solution of this optimal control problem may be synthesized via solution of an attendant Hamilton-Jacobi-Bellman-Isaacs (HJBI) PDE. Controller synthesis to achieve the specific iLiOS property (2) in this way is investigated in [A4]. In considering feedback interconnections of nonlinear differential operators, approaches to small-gain based control design to achieve specific behavioral properties such as iLiOS (2) may also be developed. These approaches utilize well-known small-gain conditions, formulated in terms of comparison functions derived from behavioral properties for individual open-loop operators, to infer performance bounds for the closed-loop interconnections. These inferences concern the performance of feedback interconnections of operators [A4], [B8], and that of networks of operators [A12, A13, A15]. In applying these small-gain conditions, tighter performance bounds for the component operators leads to tighter performance guarantees for the interconnections. Consequently, small-gain based design can benefit directly from the characterization of tight performance bounds of the component operators summarized in Section 2.4.

2.6 Numerical methods / tools

HJB PDEs of the form (10) and (20) rarely admit explicit solutions. Where explicit solutions do exist, they typically correspond to very simple operators. For example, an explicit solution $\widehat{W} : \mathbb{R}^2 \mapsto \mathbb{R}$ of the tight performance bounds HJB PDE (20) does exist in the case of scalar linear operators Σ , see [B1]. By virtue of (22), this means that an explicit solution $V : \mathbb{R} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ of the performance verification HJB PDE (10) often also exists (depending on the choice of $\gamma \in \overline{\mathcal{K}}$). However, even in this seemingly simple case, these solutions take a non-trivial form [B1]. This is due to the fact that while the operator Σ is linear, the required cascade operator $\Sigma \circ \mathcal{S}$ in (16) remains nonlinear. In general, this inherent nonlinearity acts to further complicate any nonlinear dynamics in Σ , rendering explicit solutions to these HJB PDEs rare indeed. Consequently, numerical methods are crucial to the applicability of the performance verification and characterization tools developed. The approaches to numerical computation considered in this project [R4] may be classified as targeting either the HJB PDE in question, or the corresponding dynamic programming principle. In particular, approximating Markov chain methods [R7] and preliminary max-plus (idempotent) methods [R8] have been developed for solving specific performance verification and tight performance bound characterization problems. Respectively, these Markov chain and max-plus methods are documented in [A1, A3–A6, A16], [B2] and [A2, A7, A8, A10], [B3, B6, B7, B9].

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- [R14] J.C. Willems. Dissipative dynamical systems. *Arch. Rational Mechanics & Analysis*, 45:321–393, 1972.

3 Publications

3.1 Refereed articles (published)

- [A1] H. Zhang and P.M. Dower. Performance bounds for nonlinear systems with a nonlinear \mathcal{L}_2 -gain property. *International Journal of Control*, 85(9):1293–1312, 2012.
- [A2] H. Zhang and P.M. Dower. An improved max-plus eigenvector method for the approximation of nonlinear \mathcal{L}_2 -gain bounds. In *proc. 20th Int. Symposium on Mathematical Theory of Networks and Systems, MTNS'2012 (Melbourne, Australia)*, 2012.
- [A3] P.M. Dower, H. Zhang, and C.M. Kellett. Nonlinear \mathcal{L}_2 -gain verification for nonlinear systems. *Systems & Control Letters*, 61(4):563–572, 2012.
- [A4] H. Zhang, P.M. Dower, and C.M. Kellett. State feedback controller synthesis to achieve a nonlinear \mathcal{L}_2 -gain property. In *Proc. 18th IFAC World Congress (Milan, Italy)*, pages 11061–11066. Elsevier, 2011.
- [A5] H. Zhang and P.M. Dower. Minimum integral input-to-state stability bounds. In *Proc. Australian Control Conference (Melbourne, Australia)*, pages 76–81, 2011.
- [A6] P.M. Dower, H. Zhang, and C.M. Kellett. A verification theorem for nonlinear systems with nonlinear \mathcal{L}_2 -gain. In *Proc. Australian Control Conference (Melbourne, Australia)*, pages 272–277, 2011.
- [A7] P.M. Dower and H. Zhang. Deterministic optimal stopping via a max-plus method. In *Proc. Australian Control Conference (Melbourne, Australia)*, pages 248–253, 2011.
- [A8] P.M. Dower and W.M. McEneaney. A max-plus based fundamental solution for a class of infinite dimensional Riccati equations. In *Proc. Joint 50th IEEE Conference on Decision and Control and European Control Conference (Orlando FL, USA)*, pages 615–620, 2011.
- [A9] H. Zhang, P.M. Dower, and C.M. Kellett. A bounded real lemma for nonlinear \mathcal{L}_2 -gain. In *Proc. 49th IEEE Conference on Decision and Control (Atlanta GA, USA)*, pages 2729–2734. IEEE, 2010.
- [A10] H. Zhang and P.M. Dower. A max-plus approach to the approximation of transient bounds for systems with nonlinear \mathcal{L}_2 -gain. In *Proc. 19th Int. Symposium on Mathematical Theory of Networks and Systems, MTNS'2010 (Budapest, Hungary)*, pages 1303–1309, 2010.
- [A11] B.S. Ruffer, C.M. Kellett, P.M. Dower, and S.R. Weller. Belief propagation as a dynamical system: The linear case and open problems. *IET Control Theory & Applications*, 4(7):1188–1200, 2010.
- [A12] B.S. Ruffer, C.M. Kellett, and P.M. Dower. On copositive Lyapunov functions for a class of monotone systems. In *Proc. 19th Int. Symposium on Mathematical Theory of Networks and Systems, MTNS'2010 (Budapest, Hungary)*, pages 863–870, 2010.
- [A13] B.S. Ruffer, H. Ito, and P.M. Dower. Computing asymptotic gains of large-scale interconnections. In *Proc. 49th IEEE Conference on Decision and Control (Atlanta GA, USA)*, pages 7413–7418. IEEE, 2010.
- [A14] B.S. Ruffer, P.M. Dower, C.M. Kellett, and S.R. Weller. On robust stability of the belief propagation algorithm for LDPC decoding. In *proc. 19th Int. Symposium on Mathematical Theory of Networks and Systems, MTNS'2010 (Budapest, Hungary)*, pages 1723–1728, 2010.
- [A15] B.S. Ruffer, P.M. Dower, and H. Ito. Computational comparison principles in large-scale system stability analysis. In *proc. 10th Annual Conference on Control Systems (Kumamoto, Japan)*, 2010.
- [A16] P.M. Dower, H. Zhang, and C.M. Kellett. Nonlinear \mathcal{L}_2 -gain analysis via a cascade. In *Proc. 49th IEEE Conference on Decision and Control (Atlanta GA, USA)*, pages 6242–6247. IEEE, 2010.
- [A17] P.M. Dower and P.M. Farrell. Small-signal power transient modelling and regulation in Raman

amplified optical fibre links. *International Journal of Control*, 83(10):1975–1993, 2010.

3.2 Refereed articles (in press or review)

(This research is ongoing and is funded via AFOSR / AOARD grant [R5].)

- [B1] H. Zhang and P.M. Dower. An explicit solution to a class of constrained optimal control problems. In *press*, *Australian Control Conference (Sydney)*, 2012.
- [B2] H. Zhang and P.M. Dower. Computation of tight integral input-to-state stability bounds for nonlinear systems. In *press*, *Systems & Control Letters*, 26 pages, 2012.
- [B3] H. Zhang and P.M. Dower. Approximation of nonlinear \mathcal{L}_2 -gain bounds via a max-plus method. In *press*, *IEEE Conference on Decision & Control (Maui, USA)*, 2012.
- [B4] C.M. Kellett and P.M. Dower. A generalization of input-to-state stability. In *press*, *IEEE Conference on Decision & Control (Maui, USA)*, 2012.
- [B5] P.M. Dower and H. Zhang. Tight comparison function bounds for nonlinear systems with a nonlinear \mathcal{L}_2 -gain property. In *press*, *Australian Control Conference (Sydney)*, 2012.
- [B6] P.M. Dower and W.M. McEneaney. A max-plus method for the optimal control of a diffusion equation. In *press*, *proc. IEEE Conference on Decision & Control (Maui, USA)*, 2012.
- [B7] P.M. Dower and W.M. McEneaney. A max-plus based fundamental solution for a class of infinite dimensional integro-differential Riccati equations. *Submitted to SIAM J. Control & Optimization*, 29 pages, 2012.
- [B8] P.M. Dower, C.M. Kellett, and H. Zhang. A weak \mathcal{L}_2 -gain property for nonlinear systems. In *press*, *proc. IEEE Conference on Decision & Control (Maui, USA)*, 2012.
- [B9] P.M. Dower. An approximation arising in max-plus based optimal stopping. In *press*, *Australian Control Conference (Sydney)*, 2012.

3.3 Invited presentations

- [T1] P.M. Dower. Idempotent methods for worst case analysis and optimal control. In *AFOSR Workshop on Data-to-Decisions & Autonomy (Melbourne VIC, Australia)*, July 2012.
- [T2] P.M. Dower. Idempotent methods for worst case analysis and optimal control. In *AFOSR Dynamics & Control Program Review (Arlington VA, USA)*, August 2012.
- [T3] P.M. Dower and H. Zhang. Optimal stopping via a max-plus method. In *Australian Control Conference (Melbourne VIC, Australia)*, November 2011.
- [T4] P.M. Dower and W.M. McEneaney. A max-plus approach to solving a class of infinite dimensional riccati equations. In *2nd AFOSR Monterey Workshop on Computational Control (Monterey CA, USA)*, November 2011.
- [T5] P.M. Dower. Characterization and computation of performance bounds for nonlinear systems. In *AFOSR Dynamics & Control Program Review (Arlington VA, USA)*, June 2011.
- [T6] P.M. Dower. Numerical solution of a class of deterministic optimal stopping problems via a max-plus affine power method. In *SIAM CT'11 (Baltimore MD, USA)*, July 2011.
- [T7] P.M. Dower. Numerical computation of transient and gain bounds for systems with finite nonlinear \mathcal{L}_p -gain. In *SIAM CT'11 (Baltimore MD, USA)*, July 2011.
- [T8] P.M. Dower and W.M. McEneaney. A max-plus based fundamental solution for a class of infinite dimensional riccati equations. In *IEEE Conference on Decision & Control (Orlando FL, USA)*, December 2011.
- [T9] H. Zhang and P.M. Dower. A max-plus approach to the approximation of transient bounds for systems with nonlinear \mathcal{L}_2 -gain. In *Conference on Mathematical Theory of Networks and Systems - MTNS (Budapest, Hungary)*, July 2010.

- [T10] H. Ito B. Ruffer and P.M. Dower. Computing asymptotic gains of large scale interconnections. In *IEEE Conference on Decision and Control (Atlanta GA, USA)*, December 2010.
- [T11] P.M. Dower. Nonlinear \mathcal{L}_2 -gain and computational methods. In *AFOSR Dynamics & Control Program Review (Arlington VA, USA)*, August 2010.
- [T12] P.M. Dower. Variational characterizations of performance for nonlinear systems. In *1st AFOSR Monterey Workshop on Computational Control (Monterey CA, USA)*, November 2009.
- [T13] P.M. Dower. Dynamic power regulation in raman amplified optical fibre links. In *Department of Electrical Engineering, National Sun Yat-Sen University (Kaohsiung, Taiwan)*, December 2009.
- [T14] P.M. Dower. Gain approximation via dynamic programming. In *AFOSR Dynamics & Control Program Review (Washington DC, USA)*, August 2009.

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Huan Zhang Research Fellow, University of Melbourne, Australia (2010 – 2012)
 Peter M. Dower Senior Lecturer, University of Melbourne, Australia (2009 – 2012)

5 Honors and awards received

2012 IET Control Theory and Applications Premium Award for best paper [A11] in the last two years.

6 Transitions

Transitions are limited to the scholarly publications and invited presentations, see Section 3.

7 New discoveries

New discoveries are summarized in Sections 1 and 2.

8 AFRL point of contact

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