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# Pseudospectral Optimal Control: Hidden Properties and Flight Results

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## Abstract

The main goal of this joint project between the Naval Postgraduate School and the University of California, Santa Cruz, was to analyze the mathematical proprieties of pseudospectral (PS) optimal control theory. The secondary goal of our project was to exploit the new properties to develop or enhance computationally efficient control algorithms. These goals were motivated by a need to generate real-time solutions to constrained nonlinear control problems that plays a central role in low-cost operations of high-performance space systems. In this project, we developed a general framework for PS optimal control theory based on discretization over an arbitrary (i.e. non-Gaussian) grid. Our analysis on dual consistency revealed some hidden properties of PS methods; and led to a new PS scheme utilizing primal-only conditions. These new developments enhance the algorithmic efficiency that is crucial for real-time applications. During this research period, we also got an opportunity to apply our ideas to the time-optimal reorientation of a NASA spacecraft in orbit; and performed a historic command and control of the Transition Region and Coronal Explorer (TRACE) space telescope.

## Introduction

High performance of space systems demands full considerations of the nonlinearity and stringent constraints. Optimal control plays a central role in designing efficient feedback control algorithms for such complicated high-performance space systems. As a means to address the challenges of future space systems, we address the problem of generating a real-time solution to the following generic constrained nonlinear optimal control problem:

$$(B)\left\{\begin{array}{lrl} \textsf{Minimize} & J[x(\cdot),u(\cdot)] = & E(x(-1),x(1)) + \displaystyle\int_{-1}^{1} F(x(t),u(t))\,dt \\ \textsf{Subject to} & \dot{x}(t) = & f(x(t),u(t)) \\ & e(x(-1),x(1)) = & 0 \\ & h(x(t),u(t)) \leq & 0 \end{array}\right.$$

where $F:\mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\mapsto\mathbb{R}$, $E:\mathbb{R}^{N_x}\times\mathbb{R}^{N_x}\mapsto\mathbb{R}$, $f:\mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\mapsto\mathbb{R}^{N_x}$, $e:\mathbb{R}^{N_x}\times\mathbb{R}^{N_x}\mapsto\mathbb{R}^{N_e}$, $h:\mathbb{R}^{N_x}\times\mathbb{R}^{N_u}\mapsto\mathbb{R}^{N_h}$, and $N$ with the appropriate subscript is some given natural number.

Over the last decade, pseudospectral (PS) methods have emerged as one of the most efficient numerical methods for solving constrained nonlinear optimal control problems, as evident from its current widespread use, include experimental demonstrations and flight operations. Based on different orthogonal polynomials and Gaussian grids, various PS optimal control methods can be developed. In PS methods, the grid is typically determined by the roots or the extrema of orthogonal functions. Since different orthogonal polynomials have different properties with respect to interpolation, integration, and discretization of the optimal control problem, the approximations of the states and costates based on these various grids could have different outcomes. For example, Chebyshev PS method offers rapid node computation and derivative computation via a fast Chebyshev differentiation scheme that is similar to an FFT computation. However, the non-unity weight function associated with Chebyshev polynomials creates difficulties in constructing a proper dual space. As a mean to analyze the effects of different orthogonal polynomials and grids on solving optimal control problems, we focus on developing PS methods over arbitrary grids for Problem B. Such research can provides a unified framework for the analysis of different PS methods. It also provides a way to compare performances among different PS methods and suggest guidelines for choosing the proper grids and discretization approaches for constrained optimal control problems.

## Major Accomplishments of The Project

### PS Method over Arbitrary Grids

Because Problem B has boundary conditions, we choose $t_0 = -1$ and $t_N = 1$ to be the boundary points. We further choose an arbitrary grid between the two end points, and denote it as $-1 < t_1 < t_2 < \cdots < t_{N-1} < 1$. In Ref.[12], we developed a PS discretization of Problem B that generates a sequence of finite dimensional optimization problems given by,

$$(B^N)\begin{cases} \textsf{Minimize} & \bar{J}^N(\bar{x},\bar{u}) = \sum_{k=0}^{N} F(\bar{x}^k,\bar{u}^k)w_k + E(\bar{x}^0,\bar{x}^N) \\ \textsf{Subject to} & \sum_{j=0}^{N} D_{kj}\bar{x}^j = f(\bar{x}^k,\bar{u}^k) \\ & e(\bar{x}^0,\bar{x}^N) = 0 \\ & h(\bar{x}^k,\bar{u}^k) \le 0 \\ & k = 0,1,\ldots,N \end{cases}$$

In Problem $B^N$, $D$ is a square differentiation matrix and $w_k$ are quadrature weights defined by

$$D_{ij} = \dot{\phi}_j(t_i), \qquad w_k = \int_{-1}^{1} \phi_k(t)dt$$

where $\phi_j(t)$ are the Lagrange polynomials given as

$$\phi_j(t) = \frac{g_N(t)}{g'_N(t_j)(t-t_j)}, \; g_N(t) = \prod_{j=0}^{N}(t-t_j).$$

Note that both $w_k$ and $D$ are constants that depend on the grid points only. Therefore, the performance of the PS methods is determined by the choice of the grid. This general formulation helps us to analyze the effects of grid selection. For example, the feasibility and the consistency of PS methods over arbitrary grids can be guaranteed under certain conditions. One of them is that the weights, $w_k$, $k = 0,1,\ldots,N$, need to be positive for all $k$. When a uniform grid is chosen, when $N > 11$, at least one of the weights is negative. Therefore, uniform grid is not an appropriate choice, as convergence is not guaranteed. This point is demonstrated by way of a linear optimal control problem in Ref.[12].

## Dual Consistency

For Problem $B^N$ to converge to Problem $B$, the discretization scheme must commute with the dualization process. That is the KKT conditions of Problem $B^N$ must preserve the structure of the continuous necessary conditions. For dual consistency analysis, consider a distilled scalar unconstrained optimal control problem with no cost function. The costate, $\lambda(t)$, must satisfy the adjoint equation $\dot{\lambda} = -\frac{\partial f(x,u)}{\partial x}\lambda$, whose PS discretization over arbitrary grids is

$$D\bar{\lambda} \;=\; -\Lambda(\bar{x},\bar{u})\cdot\bar{\lambda} \tag{1}$$

where $\bar{\lambda} = [\bar{\lambda}_0, \cdots, \bar{\lambda}_N]^T$ and $\Lambda(\bar{x},\bar{u})$ is a diagonal matrix with the ith diagonal entries be $\frac{\partial f}{\partial x}(\bar{x}^i,\bar{u}^i)$. On the other hand, the discretized state equation $D\bar{x} = \bar{f}(\bar{x},\bar{u})$ generates the KKT conditions

$$D^T\hat{\lambda} \;=\; \Lambda(\bar{x},\bar{u})\cdot\hat{\lambda} \tag{2}$$

where $\hat{\lambda}$ are KKT multipliers. Clearly, $\hat{\lambda}$ is not the discretization of the continuous costate, since $D^T$ is not a differentiation operator. However, it can be shown that differentiation matrix defined on arbitrary grid can always be factorized into $D = M^{-1}S$ with $M$ and $S$ be square matrices defined by

$$M_{i,j} \;=\; \sum_{k=0}^{N}\phi_i(\tau_k^l)\phi_j(\tau_k^l)w_k^l, \qquad S_{i,j} \;=\; \sum_{k=0}^{N}\phi_i(\tau_k^l)\dot{\phi}_j(\tau_k^l)w_k^l$$

where $\tau_k^l$ are Legendre-Gauss-Lobatto (LGL) nodes and $w_k^l$ are LGL weights. Matrix $M$ is always positive definite, and satisfies Sylvester-like equation

$$MD + D^TM \;=\; \Delta \tag{3}$$

with all entries of $\Delta$ be zero except for $\Delta_{00} = -1$ and $\Delta_{NN} = 1$. With the help of (3), the KKT system (2) is transferred into

$$D\cdot(M^{-1}\hat{\lambda}) \;=\; -\left[M^{-1}\Lambda(\bar{x},\bar{u})M\right]\cdot(M^{-1}\hat{\lambda}) + M^{-1}\Delta\cdot(M^{-1}\hat{\lambda}) \tag{4}$$

Since the discrete differentiation is applied to $M^{-1}\hat{\lambda}$, the scaled KKT multiplier, $M^{-1}\hat{\lambda}$, serves as the discretization of the continuous costate. This is the first hidden property of PS optimal control methods: *for PS methods over arbitrary grid distribution, costate is formed by the multiplication of adjoint weight matrix M and the KKT multipliers.* For standard LGL PS method, since $M$ is a diagonal matrix formed by LGL weights, we recover the standard approach for costate computation, i.e., scale the KKT multipliers by LGL weights.

From (4), it is also clear that to maintain the structure of the vector field in the adjoint equation, we need $M^{-1}\Lambda(\bar{x},\bar{u})M = \Lambda(\bar{x},\bar{u})$. For nonlinear problems, this condition holds *if and only if $M$ is diagonal*. It is another hidden property of PS methods been discovered. From the definition of $M$ and the property of Lagrange interpolating polynomial, $M$ in general cannot be diagonal for arbitrary grid. It is diagonal only if the grid, $t_k$, coincide with LGL points, $\tau_k$, i.e., LGL grid is adopted. It explains the superior performance of LGL PS method. In fact, the diagonal property of M is deteriorated as the grid distribution is further away from the set of LGL points. Shown in Fig.1 are the absolute value of off-diagonal entries of matrix $M$ with LGL grid, CGL grid and uniform grid. It is clear that for CGL grid, off-diagonal entries are small implying that $M$ is approximately diagonal. For uniform grid, $M$ is far away from being diagonal. Since all Gaussian quadrature grids share similar distribution, such Gaussian grids maintain the diagonal property of $M$ approximately. Our results also explain the connection between the Clenshaw-Curtis weights and the covector mapping theorem for the Chebyshev PS method developed in Ref.[2]. That is, for a CGL grid, M is not diagonal but is well-approximated by a diagonal matrix composed of the Clenshaw-Curtis weights.

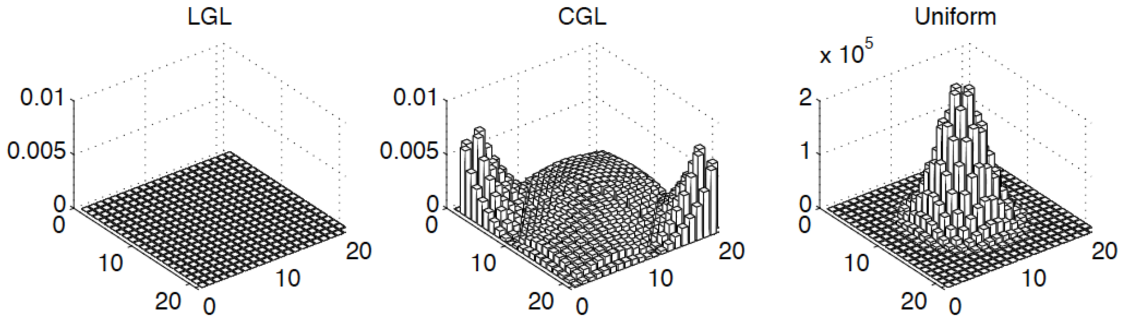


Figure 1: Magnitude of off-diagonal entries of matrix $M$ with different types of grids ($N = 20$).

## Primal-only Closure Conditions

The last term in (4) relates to the transversality conditions. For LGL PS method, utilizing the diagonal property of $M$, a set of primal-dual closure conditions are introduced to imposed right transversality condition. This approach was generalized to Chebyshev PS methods with $M$ be approximated by a diagonal matrix composed of the Clenshaw-Curtis weights. For arbitrary grid PS methods, to handle the last term in (4) and the transversality conditions, we modify Problem $B^N$ as follows: introduce new primal decision variables $\hat{x}^0$ and $\hat{x}^N$, which represent almost the same quantities as $\bar{x}^0$ and $\bar{x}^N$. In the discretization of the event cost, $E(x(-1), x(1))$, and event conditions, $e(-1, 1) = 0$, we use $\hat{x}^0$ and $\hat{x}^N$ instead of $\bar{x}^0$ and $\bar{x}^N$. Therefore, the discrete cost and the event conditions are

$$\begin{aligned} \bar{J}^N &= \sum_{k=0}^{N} F(\bar{x}^k, \bar{u}^k) w_k + E(\hat{x}^0, \hat{x}^N) \\ 0 &= e(\hat{x}^0, \hat{x}^N) \end{aligned}$$

The discretization of the dynamical equations is modified as

$$D\bar{x} = \bar{f}(\bar{x}, \bar{u}) + (\bar{x}^0 - \hat{x}^0) r_0 + (\bar{x}^N - \hat{x}^N) r_N$$

where $\bar{f}(\bar{x}, \bar{u}) = [f(\bar{x}^0, \bar{u}^0), \cdots, f(\bar{x}^N, \bar{u}^N)]^T$. $r_0$ and $r_N$ are the first and the last column of $M^{-1}$. It can be shown that $r_0$ and $r_N$ are explicitly given by

$$\begin{aligned} r_{0,k} &= \frac{(-1)^{N+1}}{2} \dot{L}_N(t_k)(t-1) \\ r_{N,k} &= \frac{1}{2} \dot{L}_N(t_k)(t+1), \qquad k = 0, 1, \cdots, N \end{aligned}$$

where $L_N(t)$ is the Nth order Legendre polynomial. Such modification generates a modified Problem $B^N$ that enables dualization to commute with discretization. This new scheme facilitates enhancing the algorithmic efficiency of the underlying spectral algorithm by generating a full-rank KKT system. The proposed primal-only closure condition is easy to implement and applicable to different PS methods including Legendre and Chebyshev PS methods as special cases.

## Flight Results

During this research period, we also got an opportunity to apply our ideas to the time-optimal reorientation of a NASA spacecraft in orbit. On August 2010, we led a team of engineers and operators to perform a historic command and control of the Transition Region and Coronal Explorer (TRACE) space telescope. Prior to our flight test, no spacecraft had ever performed a minimum-time attitude maneuver. This is, in part, because of the well-known difficulties in solving the minimum-time spacecraft attitude control problem. Our techniques allow an operator to perform verification and validation of the command signals without a need to understand PS optimal control theory. Because TRACE is actuated by reaction wheels, the resulting dynamics are not only non-Eulerian but also, the control space is non-convex and state-dependent. The well-known difficulties in solving such problems were easily circumvented by our PS theory. The mathematical implications as well as the engineering results acquired from the telemetry are being analyzed now with the goal of publications in the coming years.

### Other Advancements

During this research period, we also extended our previous work on feasibility, consistency and convergence analysis. When the Legendre PS method is used for feedback linearizable systems, we showed in Ref.[3] that the rate of convergence is at least $\frac{1}{N^{2m/3-1}}$, where $m$ is defined by the smoothness of the optimal trajectory; i.e. an appropriate Sobolev space. If the optimal control is $C^\infty$, then the convergence rate can be made faster than any given polynomial rate. Relative to our earlier results, we also weakened the assumptions in the theorems we proved. In this research project, we also analyzed the performance of a PS algorithm for generating real-time solution of infinite-horizon optimal control problems. We proved the closed-loop stability under box constraints on states and controls. Such results are important for real-time space control applications.

## Ongoing Work and Future Research Plans

This is the final year of a three-year project. The new developed PS computational scheme paves the way to construct more efficient algorithms for solving optimal control problems, for example, multiscale PS methods for dynamical systems with different timescales. Many space applications involve both slow and fast dynamics. The nature of such systems with mixed time-constants suggests a possible efficient discretization using multi-resolution for different subsystems. The new developed arbitrary grid PS method provides flexibility in designing different grids for different subsystems. We will explore this idea, together with a covector mapping theorem for costate computation, to develop multiscale PS methods.

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