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ADDITIONS TO HILL'S THEORY OF SATURN

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1. Introduction. A comparison[1] between Hill's Tables of Saturn and the orbit recently obtained by numerical integration[2] indicated errors in the Tables that seemed to be connected with the secular perturbations and with the great inequality. It is of course possible that the errors are the consequence of some numerical mistake, either in Hill's theory or in the Tables. L. E. Cunningham[3] has indeed discovered a mistake at the very beginning of the theory, which affects the whole of it, but which probably is not large enough to account for the errors in the Tables. There may be other mistakes, as yet undiscovered; and a complete numerical verification of Hill's theory is much to be desired. But it also seems worth-while to inquire whether Hill's theory may be incomplete by reason of his overlooking portions of the disturbing forces that might produce important perturbations; this is the subject of the present article.

The cube of the mass of Jupiter is about 10^{-9} and the product of the masses of Jupiter and Uranus is 5.10^{-8}; thus, since the ratio of the mean distances is not very different in the two cases, it may be supposed that the perturbations factored by the latter number are at least as important as those factored by the former. However, there is less difficulty with small divisors in the case of Uranus. But whereas Hill has calculated the third-order perturbations by Jupiter with some approach to completeness, he has confined attention in the other case to two classes of terms: the secular variations of the coefficients of the first-order periodic perturbations, and the periodic perturbations containing the mean anomalies of three planets (Jupiter, Saturn, and Uranus) in their arguments. He has omitted the second-order contributions of Uranus to the secular perturbations proper and to the great inequality. I shall treat these terms here. Since it is not my present purpose to complete Hill's theory, which would require a good deal of labor, but only to determine whether appreciable corrections are needed, I aim only at a precision of $0''.01$ in the great inequality, and $0''.01$ per century in the secular terms.

2. Secular perturbations. To the order of precision aimed at, the only appreciable terms are given by Hill's formula on page 457 (page references are to *Astr. Papers Am. Ephemeris*, 4, or Hill's *Collected Works*, 3), from which the final product may be omitted:

$$\delta T' = \frac{dT'}{dg'} n'\delta z' + r'\frac{dT'}{dr'} \nu' + \frac{dT'}{dg''} n''\delta z'' + r''\frac{dT'}{dr''} \nu''.$$

The single accent refers to Saturn and the double accent to Uranus. The derivatives with respect to g' and g'' are obtained by direct differentiation of T' as given on page 130, where Uranus has one accent and Saturn none; the other two derivatives are given on page 463. The first-order perturbations $n'\delta z'$ and ν' are given on page 138, and I have taken $n''\delta z''$ and ν'' from *Astr. Papers* 7, 297, where again Uranus has one accent and Saturn none. The terms of $\delta T'$ that are sought are those with arguments cos γ' and sin γ', neither g' nor g'' being present explicitly; such terms can arise only from multiplication of a term in the derivative having $\pm\gamma' + i''g'' + i'g'$ in its argument by a term in the first-order perturbations having $i''g'' + i'g'$ in its argument. In Table I are given all of the separate coefficients

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TABLE I. CONTRIBUTIONS TO SECULAR TERMS

Arg. $\kappa'\gamma'+i''g''+i'g'$			$\frac{dT'}{dg'}$		$r'\frac{dT'}{dr'}$		$\frac{dT'}{dg''}$		$r''\frac{dT'}{dr''}$	
κ'	i''	i'	cos	sin	cos	sin	cos	sin	cos	sin
−1	+2	−1	+6	+3	+4	−10	−12	−5	− 8	+20
−1	+3	−1	+1	+1	+3	− 1	− 2	−3	− 4	+ 2
−1	+3	−2	+4	−6	−9	− 6	− 7	+9	+14	+10

Arg. $i''g''+i'g'$		$n'\delta z'$		ν'		$n''\delta z''$		ν''	
i''	i'	cos	sin	cos	sin	cos	sin	cos	sin
+2	−1	− 1	+ 8	+2	0	+16	−138	−61	−9
+3	−1	−18	+24	−1	− 1	+72	− 97	+ 9	+7
+3	−2	+22	−12	−6	−11	+ 2	− 1	− 1	−2

that seemingly can affect the result. The coefficients are in whole seconds of arc. Making the multiplications and additions, and remembering to multiply by arc 1″, gives

$$\delta T' = +0''.0029 \cos \gamma' + 0''.0088 \sin \gamma'.$$

Integrating,

$$\delta W' = \int \delta T' n' dt$$
$$= +0''.0029\, n't \cos \gamma' + 0''.0088\, n't \sin \gamma'.$$

Changing γ' into g',

$$\overline{\delta W'} = +0''.0029\, n't \cos g' + 0''.0088\, n't \sin g'.$$

Finally,

$$n'\delta^2 z' = \int \overline{\delta W'} n' dt$$
$$= -0''.0088\, n't \cos g' + 0''.0029\, n't \sin g'$$
$$= -0''.19\, T \cos g' + 0''.06\, T \sin g',$$

where the unit of T is the Julian century.

3. The great inequality. The contribution of Uranus to the great inequality arises in precisely the same fashion as the terms calculated on pages 480–486. The complete calculation of all these terms involving the mean anomalies of three planets, g'', g', and g, in their arguments would include the case $i'' = 0$, and of this class of terms the contribution to the great inequality has $0g'' + 5g' - 2g$ in its argument. It is worthy of note that while the usual second-order contribution given on page 335 contains the fourth power of the integrating divisor, the contribution here calculated contains only the third power, owing to the special way in which it arises. I shall calculate also the term containing $4g' - 2g$ in its argument, which may be called the evection of the great inequality, expressing as it does the long-period perturbation of the eccentricity and perihelion.

The only portion of $\delta T'$ that can contribute anything appreciable is

$$\delta T' = \frac{dT'}{dg'} n'\delta z' + r' \frac{dT'}{dr'} \nu'.$$

The derivatives have the same meaning as before, but $n'\delta z'$ and ν' here denote the first-order perturbations of Saturn by Jupiter, given on page 106. In Table II are given the appreciable

TABLE II. CONTRIBUTIONS TO GREAT INEQUALITY

Arg. $\kappa'\gamma'+i''g''+i'g'$			$\frac{dT'}{dg'}$		$r'\frac{dT'}{dr'}$	
κ'	i''	i'	cos	sin	cos	sin
0	0	−1	−0.130	−0.044	−0.167	+0.205
+1	0	−1	+1.664	−0.002	−0.008	−2.904
+1	0	0	....	....	+0.282	+0.331

Arg. $i'g'+ig$		$n'\delta z'$		ν'	
i'	i	cos	sin	cos	sin
+4	−2	− 651	+ 81	+39	+317
+5	−2	−2418	−1029	+24	− 46

contributions. By multiplication and addition there is obtained

$$\delta T' = +0''.00034 \cos (5g' - 2g)$$
$$-0''.00024 \sin (5g' - 2g)$$
$$-0''.0103 \cos (\gamma' + 4g' - 2g)$$
$$-0''.0041 \sin (\gamma' + 4g' - 2g)$$
$$-0''.0049 \cos (-\gamma' + 5g' - 2g)$$
$$+0''.0005 \sin (-\gamma' + 5g' - 2g).$$

The integrating factors are given on page 93, and the same process as before yields

$$\delta W' = +0''.0102 \sin (5g' - 2g)$$
$$+0''.0072 \cos (5g' - 2g)$$
$$+0''.0103 \sin (\gamma' + 4g' - 2g)$$
$$-0''.0041 \cos (\gamma' + 4g' - 2g)$$
$$-0''.15 \sin (-\gamma' + 5g' - 2g)$$
$$-0''.02 \cos (-\gamma' + 5g' - 2g),$$

whence

$$\overline{\delta W'} = +0''.0205 \sin (5g' - 2g)$$
$$+0''.0031 \cos (5g' - 2g)$$
$$-0''.15 \sin (4g' - 2g)$$
$$-0''.02 \cos (4g' - 2g),$$

and

$$n'\delta^2 z' = -0''.61 \cos (5g' - 2g) + 0''.09 \sin (5g' - 2g) - 0''.15 \cos (4g' - 2g) + 0''.02 \sin (4g' - 2g).$$

4. Conclusion. It is established that Hill's theory of Saturn requires additions that are large enough to be of practical importance. The use of these additions does not result in appreciable reduction of the discrepancies between Hill's Tables and the numerical-integration orbit, the sum of the squares of the residuals being reduced only from 356 to 335. It seems likely that the additions here determined are more important than any other omissions in the theory; if this is the case then a very strong presumption is created that numerical mistakes are present either in the theory or in Hill's Tables of Saturn, or both.

REFERENCES

1. *A. J.* **56**, 35, 1951.
2. *Astr. Pap. Am. Ephemeris* **12**, 1951.
3. *Astr. Pap. Am. Ephemeris* **11**, 503, 1950.

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